THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 2 MATH2020
Tutorial 12

1 Introduction to Complex Analysis

This tutorial is an introduction to complex analysis. The materials below are standard, and [Ahl79] and [SS03] are good references to elementary complex analysis.

Recall that $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$, where $i^2 = -1$. It is natural to identify $\mathbb{C}$ with $\mathbb{R}^2$ by $(x, y) \mapsto x + iy$. x and y are called the real and imaginary parts of $x + iy$ respectively.

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Denoting the real and imaginary parts of f by u and v (then $f(z) = u(z)+iv(z)$) Then f may be identified as a function from $\mathbb{R}^2$ to itself defined by $\tilde{f}(x, y) = (u(x + iy), v(x + iy))$.

$f : \mathbb{C} \to \mathbb{C}$ is said to be complex-differentiable at $z_0 \in \mathbb{C}$ iff

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0) \tag{1}$$

for some *complex number* $f'(z_0)$, which is called the complex derivative of f at z_0.

Recall that the multiplication by a complex number A amounts scaling by the modulus of A and rotating by the argument of A, and from the real point of view, letting $z = x + iy$ and $z_0 = x_0 + iy_0$, (1) reads

$$\tilde{f}(x, y) = \tilde{f}(x_0, y_0) + r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + o((x - x_0, y - y_0)), \tag{2}$$

which says $\tilde{f}$ is real-differentiable with Jacobian $r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Since the Jacobian is $\begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix}$, if f is complex-differentiable at $z_0 = x_0 + iy_0$, then the equations

$$\begin{cases} \partial_x u = r\cos\theta = \partial_y v \\ \partial_y u = -r\cos\theta = -\partial_x v \end{cases} \tag{3}$$

hold. This is known as the Cauchy-Riemann equation.

Example 1. The following functions are complex-differentiable.

1. $f(z) = z^2 + 1$ ((1) may be verified directly with $f'(z_0) = 2z_0$.)
2. $g(x + iy) = e^x(\cos y + i\sin y)$ ((3) may be verified)

Exercise 2. Show the followings hold.

1. If $f : \mathbb{C} \to \mathbb{C}$ is complex-differentiable, then the real part u of f is harmonic, i.e. $\Delta u = 0$.

2. If $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic, then there exists a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that (3) holds, and hence there exists a complex-differentiable function whose real part is u. (Hint: Consider the curl-free vector field $(-\partial_y u, \partial_x u)$.)

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function, and Γ be a C^1 curve on $\mathbb{C} = \mathbb{R}^2$. Let u and v be the real and imaginary parts of f and $x(t)$ and $y(t)$ be the real and imaginary parts of Γ One may define the line integral of f on Γ by

$$\int_\Gamma f(z)dz = \int_\Gamma f(z)z'(t)dt = \int (u(x(t)) + iv(x(t)))(x'(t) + iy'(t))dt.$$

Example 3. Let Γ be the positively oriented unit circle. Define f as in Example 1.

1. $\int_\Gamma f(z)dz = 0$

2. $\frac{1}{2\pi i}\int_\Gamma f(z)/zdz = f(0)$

The results in the Example above are not coincidental. Indeed, from the observation that

$$\begin{aligned}\int_\Gamma f(z)dz &= \int (u(x(t)) + iv(x(t)))(x'(t) + iy'(t))dt \\ &= \int_\Gamma (udx - vdy) + i\int_\Gamma (vdx + udy),\end{aligned}$$

where $(u, -v)$ and (v, u) are curl-free because of Cauchy-Riemann equation, the following theorems hold.

Theorem 4 (Goursat Theorem (C^1 version))**.** Under the setting presented before complex line integral is defined, if f is complex-differentiable and $\tilde{f}$ is C^1, and Γ is a simple closed curve, then $\int_\Gamma f(z)dz = 0$.

Theorem 5 (Cauchy Integral Formula (C^1 version))**.** Under the setting presented before complex line integral is defined, if f is complex-differentiable and $\tilde{f}$ is C^1, Γ is a simple closed curve, and z_0 lies in the interior of the domain bounded by Γ, then $\frac{1}{2\pi i}\int_\Gamma f(z)/(z - z_0)dz = f(z_0)$.

Remark. The assumptions of C^1 in the theorems above allow proofs by Green's Theorem. However, these assumptions are in fact redundant: the theorems still hold if the assumptions of C^1 is lifted, and f is merely assumed to be complex-differentiable. This more general theorem will be proven in a standard course on complex analysis.

2 Further Readings

Multivariate integral calculus is basically the capstone of calculus. The completion of this course is at the same time the completion of the toolbox of calculus, which may be readily applied to other areas of mathematics, theoretical and applied alike. Below, extensions to this course are surveyed for those who are interested in going further than the coursework materials.

First, it is not exactly true that the calculus toolbox has been completed. As discussed in the final tutorial, complex analysis exploits the complex structure of the plane for computation. In particular, *calculus of residue* is an important tool for evaluating improper integrals. As mentioned in the relevant part of the tutorial note, [Ahl79] and [SS03] are good references to elementary complex analysis.

The theoretical foundation of integration theory belongs to the realm of *mathematical analysis* and *real analysis*. A Riemann-sum based theoretical foundation of integration was briefly mentioned in the course, and much of the theory was omitted. Mathematical analysis makes precise the notions of limit, differentiation, integration and infinite series, and all the logical gap is filled rigorously. [BS11] and [Spi67] are good introductory books to singlevariate analysis, while the classical textbook [Rud76] is a deeper, and harder, book on the subject. I do not know of any introductory textbook on analysis that delineates the theory of multivariate integration up to Fubini's theorem, though. ([Rud76]'s exposition on multivariate analysis is less recommendable than its singlevariate counterpart.) For most people, the rigorous study of multiple integration takes place in the setting of *measure theory*, a branch of real analysis. In measure theory, the whole notion of integration is redeveloped using Lebesgue sum rather than Riemann sum for better convergence properties. While Lebesgue integration is more useful than Riemann integration, one still ought to have mastered Riemann integration before studying Lebesgue integration for the sake of development of mathematical maturity. [Roy88] is a good introductory book to measure theory on $\mathbb{R}$, and [Rud86], a sequel to [Rud76] is a good reference for general measure theory and it includes the proof of an abstract Fubini's theorem.

Just as the study of equations is a natural extension of that of the four arithmetics, the study of *partial differential equations* (PDEs) is a natural extension of the study of calculus. Partial differential equations relate the changes of a quantity along different directions, and can be used in disciplines ranging from physics to biology and finance to model vastly different phenomena. We have had a taste of the theory of partial differential equations in our study of harmonic functions (solutions to $\sum \partial_{ii} u = 0$), and the theory of PDEs is so deep and wide that it is still a field of active research. [Eva10] is a good reference for the study of PDEs.

Studying geometry by parametrization opens up *manifold theory*. A manifold, as a generalization to surfaces, is an object that can be locally parametrized by a domain in $\mathbb{R}^n$. ("Local" is essential: recall that even the sphere cannot be globally parametrized, for instance, the parametrization by polar coordinates leaves out the north pole and south pole.) [Lee13] is a good introductory text to manifold theory. Among other important results about manifold, generalized Stoke's Theorem is proven in the book, and more importantly, the theory of *differential form*, the language in which Stoke's Theorem is written, is explained in detail.

Finally, *topological* consideration has been lurking in the background of our study of multivariate calculus. Simple connectedness (not having 1-dimensional holes), a topological property, is needed for the converses of certain theorems to hold. Topology may be thought of as qualitative geometry, and it studies properties that are stable against perturbation.

The numbers of holes is a topological property, for instance, waving a punctured sheet will not make the hole disappear, and stretching a rubber sheet (before it breaks) will not make it porous either. In relations to geometry, topology provides the coarsest means to classify "geometrical" objects, and topological assumptions are often needed in geometrical theorems. Topology is interesting in its own right as well, as qualitative information is sometimes enough. For instance, existence of solutions is qualitative in nature. *Knot theory* furnishes a more picturesque application.

Point-set topology, or general topology, is a dry pre-requisite of topology theory (actually of manifold theory as well). [Rud76] illustrates basic point-set topological concepts by discussing metric space topology. [Mun17] is a well-known introductory book on point-set topology. The theory of "holes", or more properly, *homology* and *cohomology*, belongs to algebraic topology, for which [Hat01] is a good (and free!) introduction. Cohomology may be computed by studying differential form. This is known as *de Rham cohomology*, which is discussed in [Lee13]. [Ada04] and [Liv93] are good introductory books on knot theory.

Finally, I wish you all the best in the exam as well as in your pan-mathematical career.

References

[Ada04] Colin Adams, *The knot book*, 1 ed., American Mathematical Society, Providence, RI, 2004.

[Ahl79] Lars V Ahlfor, *Complex analysis*, 3 ed., McGraw-Hill Education, New York, NY, 1979.

[BS11] Robert G. Bartle and Donald R. Sherbert, *Introduction to real analysis*, 4 ed., John Wiley Sons, Inc., Hoboken, NJ, 2011.

[Eva10] Lawrence C. Evans, *Partial differential equations*, 2 ed., American Mathematical Society, Providence, RI, 2010.

[Hat01] Allen Hatcher, *Algebraic topology*, 1 ed., Cambridge University Press, Cambridge, 2001.

[Lee13] John M. Lee, *Introduction to smooth manifolds*, 2 ed., Springer, New York, NY, 2013.

[Liv93] Charles Livingston, *Knot theory*, 1 ed., The Mathematical Association of America, Washington DC, 1993.

[Mun17] James R. Munkres, *Topology*, 2 ed., Pearson, London, 2017.

[Roy88] H.L. Royden, *Real analysis*, 3 ed., Macmillan Publishing Company, New York, NY, 1988.

[Rud76] Walter Rudin, *Principles of mathematical analysis*, 3 ed., McGraw-Hill Education, New York, NY, 1976.

[Rud86] ———, *Real and complex analysis*, 3 ed., McGraw-Hill Education, New York, NY, 1986.

[Spi67] Michael Spivak, *Calculus*, 1 ed., W.A. Benjamin Inc., London, 1967.

[SS03] Elias M Stein and Rami Shakarchi, *Complex analysis*, 1 ed., Princeton University Press, Princeton, NJ, 2003.