THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 11

The role of curl-free and divergence-free vector fields in calculating line and surface integrals are investigated by a number of examples.

Recall from single-variable integral calculus that one of the simplest integration techniques is to observe whether the integrand is the derivative of some function, or in multivariable calculus terminology, whether the integrand is a gradient. Indeed, the fundamental theorem of line integral reduces the integral of the flow of a gradient field to the evaluation of a potential function at a finite number of points. Generalizing this idea (all gradient fields are curl-free) by Stoke's Theorem (or Green's Theorem, or Divergence Theorem), we would like to observe how integration of curl-free or divergence-free vector fields can be simplified.

Example 1. Let $F(x, y) = \frac{1}{x^2+4y^2}(-y, x)$. Let C be the positively oriented octagon centered at the origin with $(1, 0)$ as a vertex. Evaluate the flow of F about C.

Solution. Note that F is curl-free. Let Γ be the ellipse $x^2 + 4y^2 = 100$. Then Γ and C together bound a ring-shaped region Ω around the origin. Then applying the circulation form of Green's theorem on Ω (or a suitable decomposition thereof) shows the flow of F about C is the same as the one about Γ. Evaluating the flow about Γ with the parametrization $(10 \cos t, 5 \sin t)$ shows the desired value is π.

Example 2. Let F a be curl-free vector field on $\mathbb{R}^3 \setminus \{(s, 0, 0) : s \in \mathbb{R}\}$. Let C be the curve parametrized by $(\cos t, \sin t, \cos 2t)$, where $0 \leq t \leq 2\pi$. Show that $\int_C F \cdot dr = 0$.

Solution. This solution depends crucially on the pictures drawn in the tutorial. Note that C bounds a saddle-like surface that intersects with the x-axis at two points. Let S_1 be the complement of two small balls centered at these two points in the saddle-like surface. Then S_1 has three boundary components: one is C and the other are formed by the two removed balls. Applying Stoke's Theorem on S_1 shows the circulation about C is the sum I of the circulations about the other two boundary components of S_1 (when properly oriented). The two boundary components are in turn boundary components of a cylinder-like surface S_2 wrapping around the x-axis. Applying Stoke's Theorem again on S_2 shows $I = 0$. The result then follows.

Remark. Note that the in the above argument, the orientation of the small boundary components of S_1 needs to be determined carefully when applying Stoke's Theorem. An alternative solution that bypasses orientation consideration is left as an exercise. (Hint: Forgo S_1 and choose a different surface.)

Similar simplification can be made for evaluation of flux of divergence-free vector fields through surfaces.

Example 3. Evaluate the flux of $F(x, y, z) = (2xy + y^2, -y^2 + e^z, xy)$ through the upper unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z > 0\}$

Solution. Observe that F is divergence-free. Applying divergence theorem on the upper half ball $\{(x, y, z) : x^2 + y^2 + z^2 < 1, z > 0\}$ shows it suffices to compute the (downward) flux through the unit disc on the xy plane.

Computing in (planar) polar coordinates, or using oddity, shows the desired flux is 0.