**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**2018-2019 semester 2 MATH2020**
**Tutorial 10**

The relationship between the Laplacian operator, harmonic functions and mean-value property on $\mathbb{R}^2$ is discussed. Recall that the Laplacian operator is defined by

$$\Delta = \sum_i \partial_{ii},$$

and on $\mathbb{R}^2$,

$$\Delta = \partial_{xx} + \partial_{yy}.$$

Harmonic functions are functions $u$ that vanish identically when acted upon by the Laplacian operator, in symbols,

$$\Delta u \equiv 0.$$

For instance, $u(x,y) = \log\sqrt{x^2+y^2}$ may be readily verified to be harmonic. Chapter 2.2 of [Eva10] and Chapter 2 of [GT00] are good references for the theory of harmonic functions.

# 1 Results from Differential Calculus

Two results about the Laplacian operator and harmonic functions obtainable by differential calculus are highlighted below. They will serve as the starting point of the investigation below. Their details and implications may be found in the appendix. They were discussed in the tutorial in MATH2010 in semester 2 of 2017-2018.

**discretization** $\Delta u(x) = \lim_{h\to 0} \frac{2n}{h^2}\Big[\Big(\frac{1}{2n}\sum_{\substack{\sigma\in\{1,-1\}\\ 1\le i\le n}} u(x+\sigma h e_i)\Big) - u(x)\Big]$

**weak maximum principle** Suppose $u$ is harmonic on the unit ball and is continuous up to the boundary. Then $u$ attains its maximum on the boundary, i.e. there exists an $x_0$ on the boundary such that for every $x \in B(0,1)$, $u(x) \le u(x_0)$.

**uniqueness for boundary value problem** Two harmonic functions on the unit ball that are continuous up to the boundary are equal if they agree on the boundary.

Note that weak maximum principle does *not* rule out the possibility that the maximum is attained in the interior as well. Indeed, the maximum is attained everywhere for constant functions, which are harmonic.

# 2 Interpretation of the Laplacian Operator and Mean-Value Property

This section is based on a technical lemma derived with Green's Theorem. The discretization shows that the Laplacian operator is basically the average deviation $u(x)$ from $u(x_0)$, with the deviation measured along coordinate axes. Corollary 2 will rephrase this idea without the use of coordinate exes. This interpretation allows handling the Laplacian, and hence harmonicity, by considering the mean-value of the function on discs or circumferences. In particular, Corollary 3 shows a smooth function is harmonic iff its function value is the mean-value on discs. Two further consequences are then derived. Below, integral with a horizontal bar means average, i.e. integrate and then divide by the size of the domain.

**Lemma 1.** For a $C^2$ function $u$ on $\mathbb{R}^2$, $⨍_{B(x,r)} \Delta u dA = \frac{2}{r}\frac{d}{d\rho}\big|_{\rho=r} ⨍_{\partial B(x,\rho)} uds$. (integral with a bar means average, i.e. integrate and then divide by the size of the domain)

*Proof.* Observe that $\Delta u = \nabla \cdot \nabla u$. Then by Green's theorem,

$$\begin{aligned}
\int_{B(x,r)} \Delta u dA &= \int_{B(x,r)} \nabla \cdot \nabla u dA \\
&= \int_{\partial B(x,r)} \nabla u \cdot \nu ds \quad \text{(Green's Theorem, } \nu \text{ is the normal)} \\
&= \int_{\partial B(x,r)} \partial_\nu u ds \\
&= \int_0^{2\pi} \partial_\nu u(x + r \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}) r d\theta \\
&= \int_0^{2\pi} \partial_\rho|_{\rho=r} u(x + \rho \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}) r d\theta \quad (\text{since } \nu = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}) \\
&= \partial_\rho|_{\rho=r} \int_0^{2\pi} u(x + \rho \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}) r d\theta \\
&= \partial_\rho|_{\rho=r} \int_{\partial B(x,r)} uds
\end{aligned}$$

The result then follows from dividing both sides by $\pi r^2$. □

This lemma gives an interpretation of the Laplacian as the second order instantaneous rate of change of the mean values on spheres, and neglecting the second-order scaling factor, it is the average deviation from the function value. More precisely, we have the following corollary.

**Corollary 2.** Let $u$ be a $C^2$ function $u$ on $\mathbb{R}^2$, and fix $x \in \mathbb{R}^2$. Define

$$f(r) = ⨍_{\partial B(x,r)} u.$$

Then $f$ is twice-differentiable at $r = 0$, with $f(0) = u(x)$, $f'(0) = 0$ and $f''(0) = \frac{1}{2}\Delta u(x)$.

*Proof.* Upon passing to limit, the lemma above shows $\frac{1}{2}\Delta u(x) = \lim_{r\to 0^+} \frac{1}{r} f'(r)$. Since the limit exists, $\lim_{r\to 0^+} f'(r) = 0$, and hence by mean-value theorem, $f'(0) = 0$. Then $\frac{1}{2}\Delta u(x) = \lim_{r\to 0^+} \frac{1}{r}(f'(r) - f'(0))$, which by definition is $f''(0)$. □

**Corollary 3.** Let $u$ be a $C^2$ function $u$ on $\mathbb{R}^2$. The followings are equivalent.

- $u$ is harmonic (resp subharmonic, superharmonic).
- $u(x) =$ (resp $\leq, \geq$) $\fint_{\partial B(x,r)} u ds$.
- $u(x) =$ (resp $\leq, \geq$) $\fint_{B(x,r)} u dA$.

*Proof.* The first equivalence follows from Lemma 1. Mean-value property on circumferences implies that on discs via integration. Differentiating gives a local converse, which by Corollary 2, implies $u$ is harmonic. $\square$

These characterisations are called mean-value properties / equation / inequalities. Two consequences consequences are shown below.

**Proposition 4** (Strong Maximum Principle)**.** Suppose $\Omega$ is bounded connected open set. If a nonconstant $u \in C(\bar{\Omega})$ satisfies the mean-value inequality for subharmonic functions, then it attains its maximum and only on the boundary.

*Proof.* Since the function is continuous and $\Omega$ is bounded, the maximum is attained. It suffices to show that the maximum is not attained in the interior. Suppose, for contradiction, that it does at some interior $x$. Then $\max u = u(x) \leq \fint_{\partial B(x,r)} u ds \leq \max u$, and hence the average over the circumference is $\max u$. Then it is impossible that $u$ has values strictly smaller than $\max u$ on the circumference. Letting the radius vary, this implies $u(y) = \max u$ on $B(x,r)$ whenever $B(x,r) \subseteq \Omega$. By connectedness, repeating this with points on the boundary of balls playing the role of the centers shows $u = \max u$ everywhere. $\square$

*Remark.* The strong maximum principle is stronger than the weak maximum principle because it rules out the possibility that the maximum is attained in the interior, unless the function is constant.

Specialising to harmonic functions, this equivalence for harmonic functions holds even in the class of continuous functions.

**Theorem 5.** *If a continuous function satisfies mean-value property for harmonic functions, then it is smooth, and hence harmonic.*

*Proof.* Let $\varphi_\varepsilon$ be a mollifier[1] supported on $B(0,\varepsilon)$ that is symmetric in the sense that $\varphi_\varepsilon(x) = \psi_\varepsilon(|x|)$. Mean-value property implies $u = u * \varphi_\varepsilon$ on $\Omega_\varepsilon = \{x \in \Omega : d(x, \Omega^C) > \varepsilon\}$,

[1] see Tutorial Note 6

and hence $u$ is smooth everywhere. More precisely,

$$\begin{aligned}
(u * \varphi_\varepsilon)(x) &= \int_{B(0,\varepsilon)} u(x+y)\varphi_\varepsilon(-y)dy \\
&= \int_0^\varepsilon \int_0^{2\pi} u(x + r \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix})\varphi_\varepsilon(-r\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix})rd\theta dr \\
&= \int_0^\varepsilon \int_0^{2\pi} u(x + r \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix})d\theta\psi_\varepsilon(r)rdr \quad \text{(symmetry)} \\
&= u(x)\int_0^\varepsilon \int_0^{2\pi} d\theta\psi_\varepsilon(r)rdr \\
&= u(x)\int_{B(0,\varepsilon)} \varphi_\varepsilon(y)dy \\
&= u(x)
\end{aligned}$$

□

# 3 Appendix: Details of Results about the Laplacian Operator from Differential Calculus

**Proposition 6.**

$$\Delta u(x) = \lim_{h\to 0} \frac{2n}{h^2}\left[\left(\frac{1}{2n}\sum_{\substack{\sigma\in\{1,-1\} \\ 1\le i\le n}} u(x+\sigma h e_i)\right) - u(x)\right]$$

*Proof.* It suffices to show the equivalent expression

$$\Delta u(x) = \lim_{h\to 0}\frac{1}{h^2}\sum[(u(x+he_i) - u(x)) + (u(x-he_i) - u(x))]$$

By the definition of partial deriviatives, this boils down to showing the following equation for single-variable functions $v$

$$v''(x) = \lim_{h\to 0}\frac{1}{h^2}[(v(x+h) - u(x)) + (u(x-h) - u(x))]$$

which indeed holds by L'Hopital's rule. □

**Theorem 7.** [Weak Maximum Principle] Suppose $u$ is harmonic on $B(0,1)$ and is continuous up to the boundary. Then $u$ attains its maximum on the boundary, i.e. there exists an $x_0$ on the boundary such that for every $x \in B(0,1)$, $u(x) \le u(x_0)$.

*Proof.* By single-variable calculus, the proposition is true if "harmonic" is replaced by "functions with strictly positive Laplacian", because the second derivative along each coordinate axis direction at an internal maximum is nonpositive, and hence so is their sum. (Try to write out the details of the paragraph.)

Now, suppose $u$ is harmonic. Since $\Delta|x|^2 = 2n$ ($n$ is the dimension of the space), for $\varepsilon > 0$, $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$ has a strictly positive Laplacian, and hence it attains its maximum on the boundary. Let $M$ and $M_\varepsilon$ be the maxima of $u$ and $u_\varepsilon$ *on the boundary.* It suffices to show $M$ is the maximum of $u$ on $\overline{B(0,1)}$. Since $|y|^2 = 1$ for $y$ on the boundary, for $x$ in the interior, $u(x) \leq u_\varepsilon(x) \leq M_\varepsilon = M + \varepsilon$. Letting $\varepsilon \to 0$, $u(x) \leq M$. The result then follows.

□

**Corollary 8.** Suppose $u$ is harmonic on $B(0,1)$ and is continuous up to the boundary. If $u$ vanishes on the boundary, then it is identically 0.

*Proof.* Maximum principle implies the maximum is attained on the boundary, and hence the maximum is 0. However, the same argument applied on $-u$ shows the minimum is $-0 = 0$. Therefore, $u$ is identically 0. □

**Corollary 9** (Uniqueness for Boundary Value Problem)**.** Let $f : B(0,1) \to \mathbb{R}$ and $\phi : \partial B(0,1) \to \mathbb{R}$ be given. Then the following equation has at most one solution.

$$\begin{cases} \Delta u &= f \quad \text{on } \Omega \\ u &= \phi \quad \text{on } \partial\Omega \end{cases}$$

*Proof.* Let $u$ and $v$ be two solutions, it suffices to show that $u = v$. then $u - v$ is harmonic on $\Omega$ and vanishes on the boundary. The above theorem shows $u - v = 0$, and hence $u = v$. □

# References


[Eva10] Lawrence C. Evans, *Partial differential equations*, 2 ed., American Mathematical Society, Providence, RI, 2010.

[GT00] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, 1 ed., American Mathematical Society, Providence, RI, 2000.