THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 9

The intuition of the quantities $\partial_x M + \partial_y N$ and $\partial_x N - \partial_y M$ are illustrated as in Section 16.4 of [WH10]

Recall the two forms of Green's Theorem.

Theorem 1 (Green's Theorem)**.** Let Ω be a domain in $\mathbb{R}^2$. Suppose $\partial\Omega$ is a simple closed curve with smooth outward normal $\hat{\mathbf{n}}$. Let $\mathbf{F} = (M, N)$ be a vector field. Then

$$\int_{\partial\Omega} \mathbf{F} \cdot \hat{\mathbf{n}} ds = \int_{\Omega} (\partial_x M + \partial_x N) dA$$
$$\int_{\partial\Omega} \mathbf{F} \cdot \hat{\mathbf{T}} ds = \int_{\Omega} (\partial_x N - \partial_x M) dA$$

As illustrated in the lecture, the line integrals on the left-hand side may be interpreted as the flow along the boundary and flux across the boundary. Below, the integrands on the right-hand side are interpreted.

1 Flux out of a Point

Consider $\mathbf{F}$ as the velocity of a fluid. $\partial_x M + \partial_y N$ may be interpreted as the flux out of the point. Fix a point (x, y) and consider the flux out of the small rectangle R aligned with the coordinate axes with opposite vertices (x, y) and $(x + \Delta x, y + \Delta y)$. It suffices to consider the flux through the four edges. Since the vector field is smooth, and the rectangle is small, we may assume that the vector field is roughly constant on each edge.

edge	approximate flux
top edge	$\mathbf{F}(x, y + \Delta y) \cdot \hat{\mathbf{j}} \Delta x$
lower edge	$\mathbf{F}(x, y) \cdot (-\hat{\mathbf{j}}) \Delta x$
right edge	$\mathbf{F}(x + \Delta, y) \cdot \hat{\mathbf{i}} \Delta y$
left edge	$\mathbf{F}(x, y) \cdot (-\hat{\mathbf{i}}) \Delta x$

Summing gives

$$\begin{aligned} \text{flux} &\approx (\mathbf{F}(x, y + \Delta y) - \mathbf{F}(x, y)) \cdot \hat{\mathbf{j}} \Delta x + (\mathbf{F}(x + \Delta x, y) - \mathbf{F}(x, y)) \cdot \hat{\mathbf{i}} \Delta y \\ &= (N(x, y + \Delta y) - N(x, y)) \Delta x + (M(x + \Delta x, y) - M(x, y)) \Delta y \end{aligned}$$

Taylor approximating gives

$$N(x, y + \Delta y) - N(x, y) \approx \partial_y N \Delta y;$$

Similarly,

$$M(x + \Delta x, y) - M(x, y) \approx \partial_x M \Delta x.$$

Then

$$\begin{aligned} \text{flux} &\approx (\mathbf{F}(x, y+\Delta y) - \mathbf{F}(x,y)) \cdot \hat{\mathbf{j}}\Delta x + (\mathbf{F}(x+\Delta x, y) - \mathbf{F}(x,y)) \cdot \hat{\mathbf{i}}\Delta y \\ &= (\partial_x M + \partial_y N)\Delta x \Delta y, \end{aligned}$$

which justifies our interpretation of $\partial_x M + \partial_y N$ as the flux out of a point (into the point if negative).

This gives the normal form of Green's Theorem a natural interpretation as well. When the flux out of each point of Ω is summed up in the integral on the right-hand side (of the equations in the theorem above), the flux from a point $x \in \Omega$ to another point $y \in \Omega$ is cancelled out by that from y to x, and hence only the flux across the boundary remains, and it is the line integral on the left-hand side.

2 Circulation about $\hat{\mathbf{k}}$ at a Point

First, note that, by abuse of notation, $\partial_x N - \partial_y M$ may be written as the coefficient of $\hat{\mathbf{k}}$ in

$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{bmatrix},$$

since $\partial_z M = \partial_z N = 0$.

To interpret $\partial_x N - \partial_y M$ as the circulation about $\hat{\mathbf{k}}$, consider again the rectangle R in the last section and assume $\mathbf{F}$ is roughly constant on each edge. The circulation along each edge is approximated as follows.

edge	approximate flux
top edge	$\mathbf{F}(x, y+\Delta y) \cdot (-\hat{\mathbf{i}})\Delta x$
lower edge	$\mathbf{F}(x,y) \cdot \hat{\mathbf{i}}\Delta x$
right edge	$\mathbf{F}(x+\Delta, y) \cdot \hat{\mathbf{j}}\Delta y$
left edge	$\mathbf{F}(x,y) \cdot (-\hat{\mathbf{j}})\Delta x$

Summing, grouping terms with Δx and Δy and Taylor approximating as in the last section gives

$$\text{flux} \approx (\partial_x N - \partial_y M)\Delta x \Delta y,$$

Again, the tangential form of Green's function may be interpreted as saying, after the inner circulation has been cancelled out, only the boundary circulation remains.

We conclude with the observation that the interpretation of $\partial_x N - \partial_y M$ as the circulation sheds new light to a criterion of the existence of a potential, for which [Tie] is a good reference.

Recall that a vector field admits a potential only if $\partial_x N = \partial_y M$, or equivalently, $\partial_x N - \partial_y M = 0$, which may be interpreted as being free of rotation or circulation. The converse holds if the domain is simply-connected.

If $\mathbf{F}$ admits a potential f (i.e. it is the gradient of some function), then $\mathbf{F}$ is the direction of steepest ascent, and hence if $\mathbf{F}$ does circulate somewhere, then the graph of f will have a local Penrose staircase there, which is impossible. However, the non-existence of local Penrose staircase does not preclude the existence of a global one, and hence the full converse does not hold. Indeed, the Penrose staircase may hide in a hole of the domain, like the origin in relations to $\mathbb{R}^2 \setminus \{(0,0)\}$, and the converse says this is the only obstruction to the existence of a potential.

References

[Tie] Chris Tiee, *Why gradients must have zero curl*, `https://ccom.ucsd.edu/~ctiee/notes/grad_n_curl.pdf`.

[WH10] Maurice D. Weir and Joel Hass, *Thomas' calculus*, 12 ed., Pearson, Boston, MA, 2010.