THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 7

Starting from the study of line integral, the d's, e.g. dx, dy, dz, ds, etc, in an integral, which had mostly been a formal notation, will become increasingly important. An informal introduction to the d's will be discussed in this tutorial. While materials in this note is unlikely to appear in the examination, they are important in geometry theory.

The d's are called differential 1-forms. They are defined for the study of coordinate-free integration. This is the reason why its role has been marginal before the introduction of line integral: integration has been done in standard coordinates alone, and for line integral, there are no convenient standard coordinates.

In short, differential 1-forms are linear functionals on the space of tangent vectors (or their combinations), and they are basically the only things that can be integrated (in an coordinate-free manner). Below, the role of linear functionals is first discussed, followed by the identification of the functionals with the d's and the interpretation of flow and flux integral in terms of differential 1-forms. [Lee13] (Chapters 11, 14 and 16) is a good reference for the study of forms.

1 Integration of Linear Functional Fields

The integration of functional fields rather than functions will be illustrated by the following example.

Consider the graph γ of the function $\varphi : [0,1] \to \mathbb{R}$ defined by $\varphi(x) = y^2$, which is a curve in $\mathbb{R}^2$. Consider the "line integral" of the function $F(x,y) = x + y$ along the curve C. An naive approach to define the integral would be to integrate with respect to x, giving

$$I = \int_0^1 F(x, x^2)dx = \int_0^1 (x + x^2)dx = 5/6. \tag{1}$$

However, there is no reason to favor x over y, and integrating with respect to y gives

$$J = \int_0^1 F(\sqrt{y}, y)dy = \int_0^1 (\sqrt{y} + y)dy = 7/6 \neq 5/6 = I. \tag{2}$$

To investigate this discrepancy, it is instructive to examine the Riemann sums.

$$\begin{aligned} I &= \lim \sum F\left(\frac{i}{n}, \left(\frac{i}{n}\right)^2\right) \cdot \frac{1}{n} \\ &= \lim \sum F\left(\frac{i}{n}, \left(\frac{i}{n}\right)^2\right) \cdot 1 \cdot \frac{1}{n} \\ J &= \lim \sum F\left(\sqrt{\frac{i}{n}}, \frac{i}{n}\right) \cdot \frac{1}{n} \end{aligned} \tag{3}$$

For easier comparison, we may also discretize J with the tags used for I.

$$\begin{aligned} J &= \lim \sum F\left(\frac{i}{n}, \left(\frac{i}{n}\right)^2\right)\left[\left(\frac{i+1}{n}\right)^2 - \left(\frac{i}{n}\right)^2\right] \\ &= \lim \sum F\left(\frac{i}{n}, \left(\frac{i}{n}\right)^2\right)\left(2\frac{i}{n} + \frac{1}{n}\right)\cdot\frac{1}{n} \\ &= \lim \sum F\left(\frac{i}{n}, \left(\frac{i}{n}\right)^2\right)\cdot 2\frac{i}{n}\cdot\frac{1}{n}, \end{aligned} \tag{4}$$

where the last line follows because $1/n$ is very small and it vanishes in the limit. For a focused discussion, the reader is asked to take this by faith.

The following observations are made.

1. Comparing (3) and (4) shows that I and J differ by the factors 1 and $2(i/n)$.
2. Considering the parametrization $t \overset{\gamma}{\mapsto} (t, t^2)$ of the curve, 1 and $2(i/n)$ are respectively the x- and y-components of the (non-normalized) tangent vector $\gamma'(t) = (1, 2t)$ at $t = i/n$.
3. Taking the x- and y-components are instances of linear functionals, namely, $v \mapsto e_1 \cdot v$ and $v \mapsto e_2 \cdot v$.

The above observations show that linear functionals of tangent vectors keep track of both the heights and widths of the rectangles in Riemann sums so that they integrate to a consistent quantity. When $\gamma(t)$ moves faster on the curve as t varies, for the same increment $1/n$ of t, the increment along the curve is larger, and hence larger weights for the corresponding tags are needed in the Riemann sum. These weights should depend linearly on the tangent vector; if the speed doubles, the "infinitesimal distance" $\gamma(t)$ traverses also double, and hence so should the weight.

Furthermore, I and J are not really integrals of the function F along the curve, but rather, integrals of the F-scaled field of functionals of tangent vectors. In other words, the integrand is actually an assignment of a linear functional of tangent vectors to every point of the curve, rather than an assignment of a number to every point of the curve. For I, this functional field is $(\mathbf{x}, v) \mapsto F(\mathbf{x})e_1 \cdot v$; for J, the functional field is $(\mathbf{x}, v) \mapsto F(\mathbf{x})e_2 \cdot v$, where $(\mathbf{x}, v)$ denotes the tangent vector v based at a point $\mathbf{x}$.

In general, for a functional field ω, the integral of ω along the curve parametrized by $\gamma : I \to \mathbb{R}^n$ is $\int_\gamma \omega = \int_I \omega(\gamma(t), \gamma'(t))dt$. This may be verified to be independent of parametrization by chain rule, because the linear functional takes care of the scaling of the rectangles in Riemann sums.

2 Nomenclature of Differential 1-Forms

It remains to identify the d's. Indeed, recall that for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, $df(v) = \nabla f \cdot v$, and hence df is a linear functional. In particular, for $f(x, y) = x$,

$\nabla f = e_1$, so dx is the functional $v \mapsto e_1 \cdot v$. Similarly, dy is $v \mapsto e_2 \cdot v$. This justifies our notations in (1) and (2). Clearly, in $\mathbb{R}^2$, at every point, dx and dy span the space of functionals, and hence every functional field on $\mathbb{R}^2$ can be expressed as

$$\omega = M(x,y)dx + N(x,y)dy. \tag{5}$$

Sometimes, the notation may not be suggestive of the actual definition of the linear functional. For instance, ds is in fact $v \mapsto \hat{\mathbf{T}} \cdot v$, where $\hat{\mathbf{T}}$ is the unit tangent along the curve. Then when integrating along the curve γ, $\hat{\mathbf{T}} \cdot \gamma' = |\gamma'|$, because by definition $\hat{\mathbf{T}} = \gamma'/|\gamma'|$.

3 Differential 1-Forms and Vector Fields

(5) makes possible the identification of differential 1-forms and vector fields, simply by associating ω with the vector field $\mathbf{F} = (M, N)$.

Then for a curve parametrized by γ with unit tangent $\hat{\mathbf{T}}$ and unit normal $\hat{\mathbf{n}}$,

$$\int_\gamma \mathbf{F} \cdot \hat{\mathbf{T}} ds = \int_\gamma M dx + N dy$$
$$\int_\gamma \mathbf{F} \cdot \hat{\mathbf{n}} ds = \int \det[\mathbf{F}, \cdot]$$

They may be verified by using the ordered basis $\{\hat{\mathbf{n}}, \hat{\mathbf{T}}\}$ and the fact that $\gamma' \cdot \hat{\mathbf{n}} = 0$.

References

[Lee13] John M. Lee, *Introduction to smooth manifolds*, 2 ed., Springer, New York, NY, 2013.