THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 6

We close the study of top-dimensional integration by discussing differentiation under integral sign. In multivariable differential calculus, the order of two differential operators may be exchanged if the function is smooth enough. In multivarable integral calculus, the order of two integrals may be exchanged if the function and the domain are nice enough. Differentiation under integral sign is about exchanging the order of a differential operator and an integral.

Proposition 1 (Differentiation Under Integral Sign (Smooth Version))**.** Let $X \subseteq \mathbb{R}^n$ be a closed and bounded set and I be an open interval. Suppose $f : I \times X$ is C^1. Then

$$\frac{d}{dt}\int_X f(t,x)dx = \int_X \frac{\partial}{\partial t} f(t,x)dx \tag{1}$$

Proof. Fix $t_0 \in I$.

$$\begin{aligned}
\frac{d}{dt}\int_X f(t,x)dx &= \frac{d}{dt}\left(\int_X f(t,x)dx - \int_X f(t_0,x)dx\right) \quad \left(\int_X f(t_0,x)dx \text{ is a constant.}\right)\\
&= \frac{d}{dt}\int_X (f(t,x) - f(t_0,x))dx \quad \text{(Fundamental theorem of calculus)}\\
&= \frac{d}{dt}\int_X \int_{t_0}^t \frac{\partial}{\partial t} f(t,x)dx\\
&= \frac{d}{dt}\int_{t_0}^t \int_X \frac{\partial}{\partial t} f(t,x)dx \quad \text{(Fubini's theorem)}\\
&= \int_X \frac{\partial}{\partial t} f(t,x)dx \quad \text{(Fundamental theorem of calculus)}
\end{aligned}$$

□

In fact, differentiation under integral sign holds in much greater generality. The following proposition is an example. The proof, however, is beyond the scope of this course, as it requires reconstructing the definition of integration by measure theory. The interested reader may learn this from Chapter 1 of [Rud86] and prove the proposition by dominated convergence theorem.

Proposition 2 (Differentiation Under Integral Sign (Measure-Theoretic Version))**.** Let X be a space where integration makes sense and I be an open interval. Let $f : I \times X$ be a function such that $\frac{\partial}{\partial t} f(t,x)$ is well defined for every (t,x) and is integrable for every t. If there exists g such that $\int_X g(x)dx$ is finite and for every (t,x), $\left|\frac{\partial}{\partial t} f(t,x)\right| \le g(x)$, then (1) holds.

A numerical and a theoretical applications are presented below.

Example 3 (Equation (2.1) of [Con])**.** For every nonnegative integer n, $\int_0^\infty e^{-t}t^n dt = n!$

Proof. This is often proven by repeated application of integration by parts, but we prove it by differentiation under integral sign for the purpose of illustration.

Note that for $x > 0$,

$$\int_0^\infty e^{-tx} dt = 1/x.$$

Differentiating n times gives

$$(-1)^n \int_0^\infty e^{-tx} t^n dt = \int_0^\infty \frac{\partial^n}{\partial x^n} e^{-tx} dt = \frac{d^n}{dx^n} \int_0^\infty e^{-tx} dt = \frac{d^n}{dx^n} 1/x = (-1)^n (n!) x^{-(n+1)}.$$

The result follows by putting $x = 1$. □

The theoretical application below concerns approximation of continuous functions by smooth functions. The analysis of smooth functions is relatively easy, since abundant tools from differential calculus are available: mean-value theorem, first-order optimality condition, Taylor expansion, etc. However, the class of smooth functions is in fact very small. Most continuous functions are not differentiable (Problem 38 of Chapter 7.8 of [Roy88]) . The following proposition justifies the sufficiency of the study of smooth functions, as it states that every continuous function, despite its nondifferentiability, can be *approximated* by smooth ones.

Proposition 4 (Mollification (Theorem 7.ii in Appendix C of [Eva10]))**.** Every continuous function $f : \mathbb{R} \to \mathbb{R}$ can be approximated by a sequence of infinitely differentiable functions f_n's, in the sense that for every $x \in \mathbb{R}$, and every positive error allowance ε, if n is sufficiently large, then $|f_n(x) - f(x)| < \varepsilon$.

Corollary 5. Every continuous function $f : \mathbb{R} \to \mathbb{R}$ can be approximated by a sequence of infinitely differentiable functions f_n's, in the sense that for every $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Proof. Consider $\varphi : \mathbb{R} \to \mathbb{R}$.

$$\varphi(x) = \begin{cases} C \exp(\frac{1}{|x|^2 - 1/2}) & \text{if } |x| < 1/2 \\ 0 & \text{otherwise} \end{cases},$$

where C is a constant such that $\int_{-\infty}^{\infty} \varphi = 1$. The following properties of φ may be readily verified.

- φ is symmetric
- $\varphi \geq 0$
- $\varphi(x) = 0$ for $|x| \geq 1/2$
- $\int_{-\infty}^{\infty} \varphi = 1$
- φ is infinitely differentiable (even at $\pm 1/2$)

Define

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)[n\varphi(ny)]dy = \int_{-\infty}^{\infty} f(y)[n\varphi(n(x-y))]dy,$$

where the second equality holds by substituting y by $x-y$. This is called the *convolution* of f with $n\varphi(n\cdot)$. $f_n(x)$ is the $n\varphi(n\cdot)$-weighted average of $f(y)$ with $y \in B(x, 1/n)$.

Since $\frac{d^m}{dx^m} f_n(x) = \int_{-\infty}^{\infty} f(x-y) \frac{d^m}{dx^m}[n\varphi(n(x-y))]dy$, where $\frac{d^m}{dx^m}[n\varphi(n(x-\cdot))]$ is continuous and vanishes outside of a bounded set, f_n is infinitely differentiable.

It remains to show that f_n approximates f in the stated sense.

$$\begin{aligned} |f_n(x) - f(x)| &\leq \int_{-\infty}^{\infty} |f(x) - f(y)|[n\varphi(n(x-y))]dy \\ &\leq n \int_{B(x,1/n)} |f(x) - f(y)|\varphi(n(x-y))dy \\ &\leq \left(\max_z \varphi(z)\right) \left(n \int_{B(x,1/n)} |f(x) - f(y)|dy\right). \end{aligned}$$

Now, since f is continuous, $|f(y) - f(x)| < \frac{\varepsilon}{2\max_z \varphi(z)}$ for $y \in B(x, 1/n)$ if n is sufficiently large, and hence the expression in the second bracket is at most $n\left(\frac{2}{n}\right)\left(\frac{\varepsilon}{2\max_z \varphi(z)}\right) = \frac{\varepsilon}{\max_z \varphi(z)}$. The result then follows.

□

References

[Con] Keith Conrad, *Differentiating under the integral sign*, `https://kconrad.math.uconn.edu/blurbs/analysis/diffunderint.pdf`.

[Eva10] Lawrence C. Evans, *Partial differential equations*, 2 ed., American Mathematical Society, Providence, RI, 2010.

[Roy88] H.L. Royden, *Real analysis*, 3 ed., Macmillan Publishing Company, New York, NY, 1988.

[Rud86] Walter Rudin, *Real and complex analysis*, 3 ed., McGraw-Hill Education, New York, NY, 1986.