THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
week 12 tutorial

1 Pre-re-mid-term

1. Suppose f is holomorphic on $\mathbb{C} \setminus \{4, 0, 6, i\}$, and the singularities $4, 0, 6, i$ are *not* removable. Find the radius of convergence of the power series expansion of f centered at the following points:

   (a) 1

   (b) $2 + i$

   (c) $2 - i$

   (d) $9 - 4i$

2. Give an example of a function that has an essential singularity at the origin. Explain how you would calculate the residue of the function at the origin, by

   (a) Laurent expansion; and

   (b) contour integration.

3. Let Ω be an open connected subset of $\mathbb{C}$.

   (a) Suppose f is holomorphic on Ω, and suppose $\int_\gamma f(z)dz = 0$ for any smooth closed curve γ contained in Ω. Show that f has a primitive on Ω, i.e. there exists a holomorphic function F on Ω such that $F' = f$.

   (b) Suppose f is holomorphic on Ω, $f(z) \neq 0$ for all $z \in \Omega$, and suppose $\int_\gamma f'(z)/f(z)dz = 0$ for any smooth closed curve γ contained in Ω. Show that there exists a holomorphic g on Ω such that $e^{g(z)} = f(z)$ for all $z \in \Omega$. Hence show that for any positive integer m, there exists a holomorphic function h on Ω such that $h(z)^m = f(z)$ for all $z \in \Omega$. (g and h are called a logarithm and an m-th root of f on Ω, respectively.)

   (c) Show that the assumption that $\int_\gamma f(z)dz = 0$ for all smooth closed curve $\gamma \subset \Omega$ cannot be removed in part (a). Also, identify a topological condition on Ω, so that the assumption that $\int_\gamma f(z)dz = 0$ for all smooth closed curve $\gamma \subset \Omega$ is satisfied by any holomorphic function f on Ω.

2 Riemann map for simply-connected proper domains with sufficiently smooth boundary

The following is an adaptation of Proposition 27.3 in [1].

Let Ω be a simply-connected proper (i.e. not the whole $\mathbb{C}$) domain in $\mathbb{C}$. Suppose the Laplace's equation with Dirichlet boundary data is solvable on Ω. Then the Riemann map $f : \Omega \to \mathbb{D}$ can be constructed by solving PDEs as follows. Such a construction is amenable to numerical computation.

The main idea is as follows. We would like to construct $\log |f|$, which is harmonic and vanishes on $\partial\Omega$. However, the singularity at $a = f^{-1}(0)$ (or from another perspective, the branch cut) needs to be handled with care. Consider $g = f/(z-a)$, which vanishes nowhere, and hence $\log g$ is well defined and everywhere finite. Then $u = \Re \log |g|$ is a harmonic function and $u|\partial\Omega = -\log |z-a|$, which by the setup, can be solved.

Now, to construct f, fix $a \in \Omega$, and solve for the harmonic function u with boundary value $-\log |z-a|$, as well as its harmonic conjugate v. Define $f(z) = (z-a)\exp(u+iv)$. It remains to show f is the desired conformal map.

f is clearly holomorphic, and by maximum principle, f maps Ω into $\mathbb{D}$. By the defining formula, since exp never vanishes, f has a simple zero at a and no other zeros. Then by argument principle, f attains every point in $\mathbb{D}$ exactly once, and hence is bijective.

The Riemann maps thus constructed for two simple domains are visualized in Figure 1. The triangulation algorithm [2] is used to generate the meshes.

More numerically sophisticated methods for computing conformal maps from surfaces to a planar or simple domain may be found in [3, 4].

References

[1] O. Forster, *Non-compact Riemann Surfaces* in Lectures on Riemann Surfaces, Grad. Texts Math., 81, Springer, New York, 1981, pp. 175-235.

[2] J. R. Shewchuk, *Triangle: Engineering a 2D Quality Mesh Generator and Delaunay Triangulator*, in Applied Computational Geometry: Towards Geometric Engineering, M.C. Lin and D. Manocha, eds., Springer-Verlag, Berlin, 1996, pp. 203-222.

[3] X. Gu, F. Luo and S.T. Yau, *Recent Advances in Computational Conformal Geometry*, Comm. Info. Syst. 9 (2009), pp. 163-196.

[4] B. Springborn, P. Schroder, U. Pinkall, *Conformal Equivalence of Triangle Meshes*, in ACM. Siggraph, 2008.

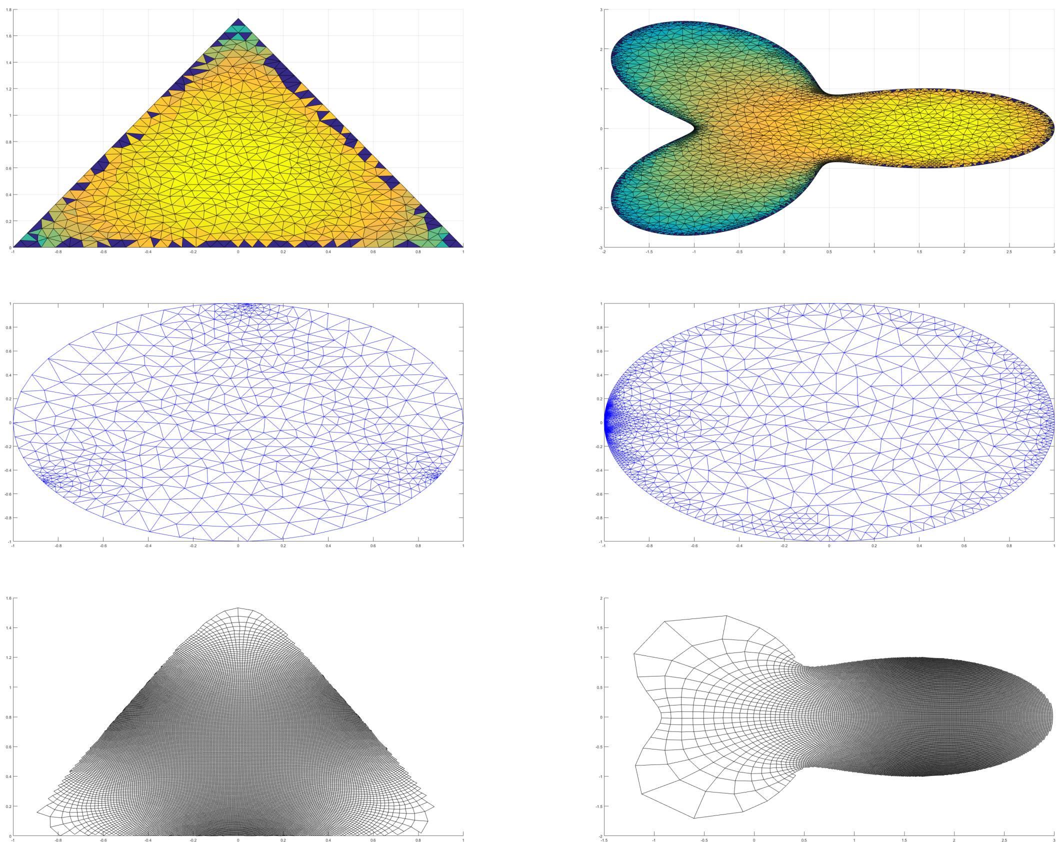

Figure 1: *(best viewed in color)* Riemann maps for a triangle and a trefoil leaf-shaped domain are shown. The top row shows a triangulation of the domains. The colors indicates the distance from the origin of the image in the disc of each point. The middle row shows the images of the triangulation in the disc. The bottom row shows the preimages in the domain of a grid in the disc. Their .fig and .png files are available at `https://drive.google.com/drive/folders/1cbVCXAQq6AqWC6QlTKwqpa4CONSp9Kar?usp=sharing`