THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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week 10 tutorial

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness.

A solution to the Weierstrass problem on the disc *without* the boundedness assumption is presented below. Some particular values of Γ and ζ were proven.

1 The Weierstrass problem and the Mittag-Leffler problem

This section is mainly based on Chapter VIII.4 of Conway's *Functions of One Complex Variable*. Proposition 1 is from exercise 3 and Proposition 2 is a special case of Theorem 4.2.

The Mittag-Leffler problem concerns the existence of meromorphic functions with prescribed *principal parts*. It has a similar form to the Weierstrass problem.

Weierstrass For every set $\{m_n\} \subseteq \mathbb{N}$, there exists a holomorphic function f that has m_n zeros at a_n, and no zero elsewhere.

Mittag-Leffler For every collection $\{g_n\}$ of meromorphic functions, with g_n defined on a neighbourhood of a_n and having a_n as the only pole, there exists a meromorphic function g whose set of poles is $\{a_n\}$, and whose principal part at a_n agrees with g_n.

The following proposition reduces the Weierstrass problem to the Mittag-Leffler problem.

Proposition 1. For a domain Ω in $\mathbb{C}$ and $\{a_n\} \in \Omega$, Mittag-Leffler's statement implies Weierstrass's.

Proof. Let $g_n(z) = m_n/(z - a_n)$ and g be the global meromorphic function given by Mittag Leffler's statement. Fix a point z_0 in the domain and define

$$f(z) = \exp \int_{z_0}^{z} g(w)dw.$$

The integral above is *not* independent of path, *but*, by Residue theorem, the discrepancy between two paths is always an integral multiple of $2\pi i$ (the fact that m_n's are integers is crucial here), which is annihilated by the exponential function, and hence f is well defined even though the integral is not.

f is clearly holomorphic and nonzero away from a_n. Near a_n, choosing z_1 near a_n and choosing a path through z_1 that stays near a_n afterwards and choosing an arbitrary branch of the logarithm shows $f(z) = f(z_1)\left(\frac{z-a_n}{z_1-a_n}\right)^{m_n}$. The result then follows.

□

Indeed, the Mittag-Leffler problem is always solvable on the disc, as the following proposition shows.

Proposition 2. Mittag-Leffler's statement on the disc is true as long as $\{a_n\}$ does not accumulate.

Proof. Let (R_m) be a strictly increasing sequence with $R_m \to 1$ and $\{R_m\} \cap \{|a_n|\} = \emptyset$. Let $f_m = \sum_{|a_n|<R_m} g_n$, which is a finite sum, by discreteness.

If f_m converges to some f_∞, then the limit will be the desired function. However, it does *not* in general. Nonetheless, observe that if f_m does converge, telescoping gives

$$f_\infty = f_m + \sum_{M \geq m} (f_{M+1} - f_M),$$

where each $f_{m+1} - f_m$ is holomorphic on a neighbourhood of $\overline{B(0, R_m)}$. To force convergence, it suffices to approximate each term in the sum by a holomorphic function, since subtraction by holomorphic functions do not change the principal parts.

More precisely, for each m, by Runge's approximation (This may be bypassed on the disc; see the remark after the proof for details.), let h_m be a globally holomorphic function (i.e. holomorphic on the disc) such that $\|f_{m+1} - f_m - h_m\|_{L^\infty(\overline{B(0,R_m)})} < \varepsilon/2^m$.

Then

$$\varphi_m = f_m + \left[\sum_{M \geq m} (f_{M+1} - f_M - h_M) \right]$$

is convergent on $\overline{B(0, R_m)}$ by construction, and has the correct poles and principal parts on $B(0, R_m)$. However, different φ_m's, say φ_m and $\varphi_{m'}$, do not agree on the intersection of their domains, namely $B(0, R_{\min(m,m')})$.

To enforce compatibility, a further holomorphic correction is needed: define

$$F_m = f_m + \left[\sum_{M \geq m} (f_{M+1} - f_M - h_M) \right] - (h_1 + \ldots + h_{m-1}).$$

Then F_m is convergent and has the correct poles and principal parts on $B(0, R_m)$ and different F_m defines the same function, and hence F_m defines the desired function.

□

Remark. In the proof above, Runge's approximation is invoked to provide a global holomorphic approximation to a local holomorphic function. In the particular case of the disc with $g_n(z) = \frac{m_n}{z-a_n}$, this can be done in a straight-forward manner. Since each $f_{M+1} - f_M$ is then a finite sum of g_n's, it suffices to approximate each g_n.

Let $a = 4$. Then $\left|\frac{z-a_n}{z-a}\right| < 2/3 < 1$ for $z \in \overline{B(0, R_m)}$, and hence

$$\frac{1}{m_n} g_n(z) = \frac{1}{z - a_n} = \frac{1}{(z-a) - (z_n - a)} = \frac{1}{z-a} \frac{1}{1 - \frac{z-a_n}{z-a}} = \frac{1}{z-a} \sum_{k \geq 0} \left(\frac{z - a_n}{z - a} \right)^k,$$

where the convergence is uniform on $\overline{B(0, R_m)}$. The result then follows as $\frac{1}{z-a}$ is holomorphic on the disc.

Corollary 3. Weierstrass's statement holds on $\mathbb{D}$ as long as $\{a_n\}$ does not accumulate.

The following question was mentioned in the tutorial.

Question 4. Suppose $\{a_n\} \subseteq \Omega$ and $\{w_n\} \subseteq \mathbb{D}$. Suppose the a_n's are distinct. Does there exist a holomorphic function f on Ω such that $f(a_n) = w_n$?

The answer is affirmative as long as Mittag-Leffler's statement holds, or as long as Runge's approximation is always possible, which is the case for domains in $\mathbb{C}$. The reason is that one then has a holomorphic function g with a simple zero at each a_n and a meromorphic h with principal part $\frac{w_n}{g'(a_n)} \frac{1}{(z-a_n)}$ at each a_n. Then $f = gh$ is the desired function. This is in fact the content of Theorem 15.13 in Rudin's *Real and Complex Analysis*.

2 Particular Values of Γ and ζ

Proposition 5. The area of the unit sphere $\mathbb{S}^{n-1}$ is $\frac{2\pi^{n/2}}{\Gamma(n/2)}$.

Proof. Consider $I = \int_{\mathbb{R}^n} e^{-\pi r^2} dV$. Recall that $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$, which can be shown by Fourier transform or by squaring and integrating with polar coordinates. Then $I = 1$. Computing in spherical coordinates, since $\frac{\partial}{\partial r}$ is the positive unit normal to the sphere, $dV = \omega_r \wedge dr = r^{n-1}\omega \wedge dr$, where ω_r and is the area form of the sphere of radius r, and $\omega = \omega_1$.

$$\begin{aligned} I &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}^{n-1}} e^{-\pi r^2} r^{n-1} \omega \right) dr \\ &= \text{area}(\mathbb{S}^{n-1}) \int_{\mathbb{R}^+} e^{-\pi r^2} r^{n-1} dr \\ &= \text{area}(\mathbb{S}^{n-1}) \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^+} e^{-t} t^{(n-1)/2} dt \\ &= \text{area}(\mathbb{S}^{n-1}) \frac{\Gamma(n/2)}{2\pi^{n/2}} \end{aligned}$$

The result then follows. □

The following proposition is from Problems 4-5 of Chapter 3 of Stein and Shakarchi's *Fourier Analysis*.

Let $f(z) = \frac{z}{e^z - 1}$. Define the Bernoulli numbers $B_n = f^{(n)}(0)$.

Proposition 6. $\zeta(2m) = \frac{(-4\pi^2)^m}{(2m)!} \frac{B_{2m}}{2}$ for integers $m \geq 1$.

Proof. By considering the Taylor expansion of $z = (e^z - 1) f(z)$, the Bernoulli numbers satisfy the following recursion.

$$B_n = \begin{cases} 1 & \text{if } n = 0 \\ -\frac{1}{n+1} \sum_0^{n-1} \binom{n+1}{k} B_k & \text{if } n > 0 \end{cases}$$

Since the odd part $(f(z) - f(-z))/2$ is simply $-z/2$, $B_n = 0$ for odd $n > 1$. Since $z \cot z = \frac{2iz}{e^{2iz-1}} + iz = f(2iz) + iz$, we have

$$z \cot z = 1 + \sum_{1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}.$$

On the other hand, pole matching shows

$$\begin{aligned} z \cot z &= 1 - 2z^2 \sum_{1}^{\infty} \frac{1}{z^2 - n^2\pi^2} \\ &= 1 - 2z^2 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(n\pi)^{2m+2}} z^{2m} \\ &= 1 - 2z^2 \sum_{m=0}^{\infty} \frac{1}{\pi^{2m+2}} \zeta(2m+2) z^{2m}. \end{aligned}$$

The result then follows from coefficient comparison.

□