THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
week 6 tutorial

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness.

1 Difference between holomorphic functions and harmonic functions

One should note that there *are* properties that harmonic functions and holomorphic functions do *not* share.

Exercise 1. Determine the validity of the following claims, whose holomorphic analogues are true.

1. The product of two harmonic functions is harmonic.
2. If u is harmonic, then so is $\exp \circ u$.
3. Partial derivatives of a harmonic function are harmonic.
4. **(hard)** If a C^2 function u satisfies the mean-value property, i.e.
$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u$$
then it is harmonic. (Cf. If a continuous f satisfies Cauchy integral formula, then it is harmonic.)
5. **(hard)** If a harmonic function on the plane is $O(|z|^n)$, then it is a polynomial of degree at most n.
6. Zeros of a nonconstant harmonic function are isolated.
7. If a harmonic function defined on a connected domain vanishes on a nonempty open subset, then it vanishes on the domain.

(answer: F, F, T, T, T, F, T)

The following facts are useful for one of the questions.

Lemma 2. Suppose $u \in C^2(\Omega)$. $\fint_{B(x,r)} \Delta u dV = \frac{n}{r} \frac{d}{d\rho}\Big|_{\rho=r} \fint_{\partial B(x,\rho)} u dS$.

Proof. By divergence theorem,

$$\begin{aligned}\int_{B(x,r)} \Delta u dV &= \int_{\partial B(x,r)} \partial_\nu u dS \\ &= \int_{\partial B(0,1)} \frac{\partial}{\partial \rho}\Big|_{\rho=r} u(x+\rho\omega) r^{n-1} dS(\omega) \\ &= r^{n-1} \frac{\partial}{\partial \rho}\Big|_{\rho=r} \frac{1}{\rho^{n-1}} \int_{\partial B(0,\rho)} u dS\end{aligned}$$

The result then follows from dividing both sides by $\frac{s_n r^n}{n}$. □

Corollary 3. Suppose $u \in C^2(\Omega)$. Let $x \in \Omega$ and $f(r) = ⨏_{\partial B(x,r)} u$. Then f is twice-differentiable at $r = 0$, with $f(0) = u(x)$, $f'(0) = 0$ and $f''(0) = \frac{1}{n}\Delta u(x)$.

Proof. Upon passing to limit, the lemma above shows $\frac{1}{n}\Delta u(x) = \lim_{r\to 0^+} \frac{1}{r} f'(r)$. Since the limit exists, $\lim_{r\to 0^+} f'(r) = 0$, and hence by mean-value theorem, $f'(0) = 0$. Then $\frac{1}{n}\Delta u(x) = \lim_{r\to 0^+} \frac{1}{r}(f'(r) - f'(0))$, which by definition is $f''(0)$. □

2 Proof of Jensen's Formula by Harmonic Functions

Below, for a set $E \subseteq \mathbb{R}^n$, $⨏_E f = \frac{1}{|E|}\int_E f$, where $|E| = \int_E 1$.

Proposition 4 (Jensen's formula)**.** Let $f : \overline{B(0,R)} \to \mathbb{C}$ be holomorphic and $\{a_i\}$ be its set of zeros. Suppose f is nonzero on $\partial B(0,R)$ and at 0. Then

$$\log|f(0)| = \sum \log\left|\frac{a_i}{R}\right| + ⨏ \log|f|$$

Proof. Recall that, as the real part of $\log f$, $u = \log|f|$ is harmonic whenever finite, and the Green's function on $B(0,R)$ is $G(z) = \frac{1}{2\pi}\log\left|\frac{z}{R}\right|$.

Let $B_i = B(a_i, \varepsilon)$ and $\Omega = B(0,R) \setminus (B(0,\varepsilon) \cup \bigcup B_i)$. Let $\Phi = u\partial_\nu G - G\partial_\nu u$. By Green's identity and harmonicity,

$$\int_{\partial\Omega} \Phi = \int_\Omega (u\Delta G - G\Delta u) = 0$$

$\partial\Omega = \partial B(0,R) - \partial B(0,\varepsilon) - \sum \partial B_i$, and the integral on these domains are as follows.

$$\int_{\partial B(0,R)} \Phi = ⨏_{\partial B(0,R)} u - 0$$

$$\int_{\partial B(0,\varepsilon)} \Phi = u(0) - O(\varepsilon \log \varepsilon)$$

For the integral on B_i, note that $u = \log|g| + \sum \log|z - a_i|$ for some nonvanishing holomorphic g. Note that $\log g + \sum_{j\neq i} \log|z - a_j|$ do not contribute the the integral on

B_i, as it is smooth near z_i, and hence the integral is $O(\varepsilon)$. The remaining term $\log|z - a_i|$ now plays the role of G, and G the role of u, in the calculation above. Then

$$\int_{\partial B_i} \Phi = o(\varepsilon \log \varepsilon) - 2\pi G(z_i)$$

Combining everything and letting $\varepsilon \to 0$ gives the desired equation. $\square$