THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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week 5 tutorial

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness.

Below is a brief introduction to properties of harmonic functions. Removable singularity theorem and Liouville's theorem for harmonic functions are proven by maximum principle and Poisson integral formula. The main reference is Chapter 2 of Gilbarg and Trudinger's *Elliptic Partial Differential Equations of Second Order*. Below, Ω always denotes a nonempty connected open set in $\mathbb{R}^2 = \mathbb{C}$.

1 Properties of Harmonic Functions

A C^2 function $u : \Omega \to \mathbb{R}$ is harmonic iff $\Delta u = u_{xx} + u_{yy} = 0$.

Harmonic functions and holomorphic functions are intimately related.

1. f is holomorphic iff $\partial_{\bar{z}} f = 0$, whereas u is harmonic iff $\partial_z \partial_{\bar{z}} u = 0$.

2. If f is holomorphic, then $\Re f$, $\Im f$ and $\log |f|$ are harmonic whenever finitely defined. If Ω is simply connected and u is harmonic, then $f = u + iv$, where $v = \int (u_x dy - u_y dx)$, is holomorphic, and $\log |e^f| = u$.

3. (Cauchy integral formul and mean-value property) If f is holmorphic, then
$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z,r)} \frac{f(w)}{w - z} dw.$$
If u is harmonic, then
$$u(z) = \frac{1}{2\pi r} \int_{\partial B(z,r)} u(w) dw = \frac{1}{2\pi} \int_{\partial B(z,r)} \frac{u(w)}{|w - z|} dw. \tag{1}$$

4. (strong maximum (modulus) principle) If a holomorphic f attains the maximum modulus in the interior, then it is constant. If a harmonic u attains the maximum in the interior, then it is constant.

5. (weak maximum (modulus) principle) The maximum modulus of a holomorphic function or a harmonic function on a bounded domain is attained on the boundary.

Mean-value property for harmonic function is more rigid than that for holomorphic function because the domain of integration in (1) cannot be any $\partial B(w, r)$ containing z. Indeed, the offset mean-value property is given by the more involved Poisson integral formula.

Proposition 1 (Poisson integral formula)**.** Suppose u is harmonic on a neighbourhood of $\overline{B(0,R)}$. Let $\varphi = u|\partial B(0,R)$. Then for $x \in B(0,R)$

$$u(x) = \int_{\partial B(0,R)} \varphi(y) P_2(x,y)dy, \tag{2}$$

where $P_n(x,y) = \frac{1}{|\partial B(0,R)|} \frac{R^2-|x|^2}{R^2} \left(\frac{R}{|x-y|}\right)^n$.

Conversely, if φ is a continuous function on $\partial B(0,R)$, then (2) defines a harmonic function on $B(0,R)$ whose continuous extension to $\partial B(0,R)$ exists and agrees with φ.

Corollary 2. Harmonic functions are smooth.

Below, we prove removable singularity theorem and Liouville's theorem for harmonic functions.

Proposition 3 (Removable singularity theorem)**.** Suppose u is harmonic on $B(0,r)\setminus\{0\}$. If $u(z) = o(\log|z|)$ as $z \to 0$, then u extends to a harmonic function on $B(0,r)$.

Proof. It suffices to show u agrees to $\tilde{u}$ defined by Poisson integral formula, which is a harmonic function on $B(0,r)$. Let $w = \tilde{u} - u$. Then $w(z) = o(\log|z|) = o(\log|z| - \log r)$. Note that both w and $\log|z| - \log r$ vanish on $\partial B(0,r)$. By maximum principle, for $\varepsilon > 0$, since $\pm w(z) + \varepsilon \log|z| \to -\infty$, $\sup_{B(0,r)\setminus\{0\}} \pm w + \varepsilon(\log|z| - \log r) \leq 0$. The result follows by letting $\varepsilon \to 0$. □

To prove Liouville's property, it is handy to have an estimate on the gradient.

Proposition 4 (gradient estimate)**.** Suppose u is harmonic on a neighbourhood of $\overline{B(0,R)}$. Then

$$|\partial_i u(0)| \leq \frac{n}{R}\|u\|_{L^\infty(B(0,R))}.$$

Remark. Repeated application of the gradient estimate shows harmonic functions are in fact analytic.

Proof. Apply differentiation under the integral sign on Poisson integral formula. □

Proposition 5 (Liouville's theorem)**.** If a harmonic function on $\mathbb{R}^2$ is bounded, then it is constant.

Proof. Let $R \to \infty$ in the gradient estimate. □

Exercise 6. Complete the following alternative proof of Liouville's theorem:

By Poisson integral formula, we have the following Harnack inequality for nonnegative harmonic u on $\mathbb{R}^2$

$$\frac{1}{(R+|x|)}\frac{R-|x|}{R}u(0) \leq u(x) \leq \frac{1}{(R-|x|)}\frac{R+|x|}{R}u(0).$$

Liouville's theorem for nonnegative functions then follows by letting $R \to \infty$ on the far right. The general case follows by translation.