THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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week 3 tutorial

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness.

1 Complex differential operators

The differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ simplifies notation for handling holomorphic functions. Below is a brief presentation of their abstract definition as eigenvectors of the multiplicative action of the imaginary unit in the tangent space. The consideration of these eigenvectors is traced from the potential discrepancy of the action of the imaginary units, and their actions on functions are discussed.

It is hoped that this note helps ease the reader's uneasiness in handling these operators, but in any case, these operators are mostly formal devices for simplifying notations, and from this perspective, efficient application is more important than abstract understanding.

Elementary knowledge of manifold theory is assumed. The main reference is Chapter 1.2 of Huybrechts' *Complex Geometry*. The reader is referred to Chapter 3 of Lee's *Introduction to Smooth Manifolds* for a review on manifold theory.

1.1 Two complex versions of the tangent space

Let U be a nonempty open set in $\mathbb{C}$. Recall (or note) that partial derivatives may be identified with tangent vectors along which the partial derivatives are directional derivatives, for instance $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ may be identified with e_1 and e_2. The definitions of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ arise from the compatibility consideration of the two actions (nonstandard terminology) of the imaginary units.

The complexification of the tangent plane can be achieved simply by taking formal complex linear combination of tangent vectors. Explicitly, this gives

$$(T_pU)_{\mathbb{C}} = \{u + iv : u, v \in T_pU\},$$

On the other hand, the geometric action J of rotation by $\pi/2$ on the real tangent space T_pU also defines a complex vector space (T_pU, J), where $iv = Jv$. This is well defined as long as $J^2 = -\operatorname{id}$.

Clearly, the actions of i on $(T_pU)_{\mathbb{C}}$ and (T_pU, J) are different. For instance, in (T_pU, J), $Je_1 = e_2 = 0e_1 + 1e_2$, which is different from $ie_1 = ie_1 + 0e_2$ in $(T_pU)_{\mathbb{C}}$. The two are compatible, though, on the eigenspace $T_p^{1,0}U$ of J wrt i, where $Jv = iv$, and they are off only by a sign on the eigenspace $T_p^{0,1}U$ wrt $-i$, where $Jv = -iv$. Since $J^2 = -\operatorname{id}$, $\pm i$ are all the eigenvalues of J, hence this gives a decomposition of $(T_pU)_{\mathbb{C}}$ into $T_p^{1,0}U \oplus T_p^{0,1}U$. Note further that conjugation is a conjugate-isomorphism between the two factors.

Now, (T_pU, J) is complex-isomorphic to $T_p^{1,0}U$ and conjugate-isomorphic to $T_p^{0,1}U$ via projection composed with inclusion, or more explicitly, $Pv = \frac{1}{2}(v - iJv)$ and $\bar{P}v = \frac{1}{2}(v + iJv)$ respectively. Now, all complex-analytical computation on (T_pU, J) can be carried out in $T_p^{1,0}U$.

Let $E = Pe_1$. It will be shown that $E = \frac{\partial}{\partial z}$, or rather, its dual E^* wrt the basis $\{E, \bar{E}\}$ is

$$E^* = dz = dx + idy,$$

where $z : \mathbb{R}^2 \to \mathbb{C}$ is the map defined by

$$z(x, y) = x + iy.$$

To this end, it suffices to show $E^*e_1 = 1$ and $E^*Je_1 = i$. Since $Pv = (E^*v)E$, $E = Pe_1 = E^*e_1E$, and hence $E^*e_1 = 1$. Plugging in E and $\bar{E}$ shows $E^*J = iE^*$, and the result follows.

1.2 Mapping, Function and Holomorphy

Consider a C^1 function $f : U \to \mathbb{C}$. It may be viewed as a complex-valued function, or a mapping into $\mathbb{R}^2$. The former leads to consideration of the differential form df, whereas the latter leads to the differential Df. To relate the two, a postcomposition with $z(x, y) = x + iy$ converts the mapping into a function, and hence $df = D(z \circ f) = dzDf$. Similarly, $d\bar{f} = d\bar{z}Df$, and hence we have the matrix representation of Df in the basis of $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$, namely

$$[Df] = \begin{bmatrix} \frac{\partial}{\partial z}f & \frac{\partial}{\partial \bar{z}}f \\ \frac{\partial}{\partial z}\bar{f} & \frac{\partial}{\partial \bar{z}}\bar{f} \end{bmatrix},$$

where $\frac{\partial}{\partial z}f = df(\frac{\partial}{\partial z})$, and the rest is similar.

f is holomorphic iff it is, infinitesimally, a multiplication. This can be characterised by commutation of the differential of the map and the action of the imaginary unit, in symbol,

$$DfJ = JDf.$$

From the perspective of differential form, this is equivalent to

$$\frac{\partial}{\partial \bar{z}}f = 0.$$

To see this, first note that $\frac{\partial}{\partial \bar{z}}f = df\frac{\partial}{\partial \bar{z}} = dzDf\frac{\partial}{\partial \bar{z}} = E^*Df\bar{E}$. Suppose the differential commutes with J, then $JDf\bar{E} = DfJ\bar{E} = -iDf\bar{E}$, and hence $Df\bar{E} \in T_{f(p)}^{0,1}\mathbb{C}$. Conversely, if $E^*Df\bar{E} = 0$, then $JDf\bar{E} = -iDf\bar{E} = DfJ\bar{E}$, conjugating, and observing Df and J are real, gives $JDfE = DfJE$ as well, and hence $DfJ = JDf$. The result then follows.

1.3 Connection with advanced calculus

Since $df = \partial_z f dz + \partial_{\bar{z}} f d\bar{z}$, f is holomorphic iff fdz is closed. This reduces Morera's theorem and Cauchy-Goursat theorem to advanced calculus under the assumption of C^1, and once Cauchy integral formula for infinitesimal circles is known by direct computation.

Morera's theorem If $f(z)dz$ is exact, then it is closed.

Cauchy-Goursat theorem If $f(z)dz$ is closed, then $\int_{\partial\Omega} f(z)dz = 0$.

Calculus of residue Apply Cauchy-Goursat theorem on the complement of discs at singularities, and the integral on the circles are given by direct computation.

For integration, a change of basis computation shows $dz \wedge d\bar{z} = -2idx \wedge dy$, hence

$$\iint f dx \wedge dy = -2i \iint f dz \wedge d\bar{z}.$$

For change of variables, by the matrix representation of Df above,

$$\iint_{f(\Omega)} gdudv = \iint_{\Omega} (g \circ f)||\partial_z f|^2 - |\partial_{\bar{z}} f|^2|dxdy.$$

2 Meromorphic function

One of the most important facts about an isolated singularity is that the local behavior is completely described by the Laurent series. A lot of properties becomes transparent upon inspection of the Laurent series.

Proposition 1. If $f : \mathbb{D} \setminus \{0\}$ is holomorphic, then $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ for some (c_n). The convergence is absolute and uniform on compact sets.

Proof. Fix $z \in \mathbb{D} \setminus \{0\}$. Let $r < |z|$. Then by Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_1 - \partial B_r} \frac{f(w)}{w - z} dw. \tag{1}$$

Just as the case of holomorphic functions, we would like to express $\frac{1}{w-z}$ in series form $\frac{1}{w}\sum(z/w)^n$ or $\frac{1}{z}\sum(w/z)^n$, and pull the sum out of the intergal. This is justified by Fubini's theorem as a power series is absolutely convergent, and the other factors of the integral as well as the domain of integration are bounded. On ∂B_1, $|z/w| < 1$, hence

$$\int_{\partial B_1} \frac{f(w)}{w - z} dw = \int_{\partial B_1} \frac{f(w)}{w} \sum (z/w)^n dw = \sum_{n=0}^{\infty} \int_{\partial B_1} \frac{f(w)}{w^{n+1}} dw z^n.$$

Similarly,

$$\int_{\partial B_r} \frac{f(w)}{w - z} dw = -\sum_{n=0}^{\infty} \int_{\partial B_r} f(w) w^n dw \frac{1}{z^{n+1}}.$$

Since $f(w)w^n$ is holomorphic on $\mathbb{D} \setminus \{0\}$, Cauchy-Goursat theorem shows $\int_{\partial B_1 - \partial B_r} f(w)w^n dw = 0$, and hence $\int_{\partial B_r} f(w)w^n dw = \int_{\partial B_1} f(w)w^n dw$. The desired equation then follows with

$$c_n = \frac{1}{2\pi i} \int_{\partial B_1} \frac{f(w)}{w^{n+1}} dw.$$

□

Exercise 2. Why is the convergence absolute and uniform on compact sets?

Exercise 3. Is the Laurent series unique, i.e. is it possible that $\sum_{-\infty}^{\infty} c_n z^n = \sum_{-\infty}^{\infty} \tilde{c}_n z^n$ but $c_n \neq \tilde{c}_n$ for some n?

The type of the singularity is immediate from the Laurent series. If $c_n = 0$ for all negative n, then the singularity is removable. If $c_n = 0$ for sufficiently negative n, then the singularity is a pole. Otherwise, it is essential. The behavior of an essential singularity is not clear from this definition though. In fact, $f(B_r \setminus \{0\})$ is dense for every $r < 1$. To prove this, we need more knowledge about removable singularities though. We will follow the arguments in Chapter 3.3 of Stein and Shakarchi's *Complex Analysis*.

Lemma 4 (Riemann removable singularity theorem)**.** If $f : \mathbb{D} \setminus \{0\}$ is holomorphic and f is bounded, then f admits a holomorphic extension to $\mathbb{D}$.

Proof. Refer (1) and consider the integral on ∂B_r . As $r \to 0$, since f is bounded, $|w - z| \geq |z| - r$ is bounded away from 0, and the length of the circle converges to 0, $\int_{\partial B_r} \frac{f(w)}{w-z} dw \to 0$ as $r \to 0$.

Therefore, $f(z) = \frac{1}{2\pi i} \int_{\partial B_1} \frac{f(w)}{w-z} dw$, where the righ-hand side is holomorphic, either Morera's theorem or by differentiation under integral sign, which holds because the domain is compact and the integrand is continuously differentiable [1]. □

Proposition 5 (Weierstrass theorem on essential singularity)**.** If $f : \mathbb{D} \setminus \{0\}$ has an essential singularity at 0, then $f(B_r \setminus \{0\})$ is dense for $r < 1$.

Proof. We prove the contrapositive. Suppose it is not dense, then it is bounded away from a number, say w_0. Then $g = \frac{1}{f - w_0}$ is bounded near 0, and hence has a removable singularity. Therefore, $f = w_0 + \frac{1}{g}$. Taylor expanding g gives $g(z) = c_n z^n + c_{n+1} z^{n+1} + ... = z^n (c_n + c_{n+1} z + ...) = z^n h(z)$ for some $c_n \neq 0$, and hence holomorphic h with $h(0) \neq 0$. Therefore, $f(z) = w_0 + \frac{1}{z^n} \frac{1}{h(z)}$, where $1/h$ is holomorphic near 0. Now, if $n = 0$, then the singularity of f is removable; if $n > 0$, it is a pole. □

Exercise 6. Suppose $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ is holomorphic and $f(z) = O(|z|^{-1/2})$ as $z \to 0$. Show that the singularity at 0 is removable.

Proof. If the singularity is a pole, then $f(z) = \frac{1}{z^n} g(z)$ for some $n \geq 1$ and holomorphic g with $g(0) \neq 0$, and hence $|z|^{-n} = O(f(z))$. Then the assumption implies $|z|^{-n} = O(|z|^{-1/2})$, which is impossible.

If the singularity is essential, then the image of $f(B(0, r) \setminus \{0\})$ is dense, in particular, cannot be bounded, contradictory to the assumption that it is bounded by $C/r^{1/2}$. □

Remark. The key idea is that f and z^{-n} are essentially the same (in the sense that they are bounded by each other), while z^{-n} is too large to be $O(z^{-1/2})$.

Alternative proof. Again, we show $\int_{\partial B_r} \frac{f(w)}{w-z} dw \to 0$. Again, $|w - z|$ is bounded away from 0, $f = O(r^{-1/2})$ and the length of the arc is $O(r)$, and hence the integral is $O(1 \cdot r^{-1/2} \cdot r) = O(r^{1/2})$. The result then follows. □

[1] see the note for week 2 for comments on differentiation under integral sign