THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
Midterm solution

1. a. For $c \in \{a, b\}, |c| \le 1$,

$$\begin{aligned}\int_R^\infty |f(x+ic)|dx &\le \int_R^\infty \frac{A}{1+x^2+c^2}dx \\ &\le \int_R^\infty \frac{A}{x^2}dx \text{ if } R > 0 \\ &\le \frac{A}{R} \to 0 \text{ as } R \to +\infty\end{aligned}$$

$$\begin{aligned}\int_a^b |f(R+iy)|dy &\le \int_a^b \frac{A}{1+x^2+y^2}dy \\ &\le \int_a^b \frac{A}{x^2}dx \text{ if } R > 0 \\ &\le \frac{A(b-a)}{R^2} \\ &\le \frac{2A}{R^2} \to 0 \text{ as } R \to +\infty\end{aligned}$$

The result then follows by summing.

b. Let Γ be the rectangular contour with corners R, $R - \operatorname{sgn}(\xi)i$, $-R - \operatorname{sgn}(\xi)i$ and $-R$. By Cauchy's theorem, since f is holomorphic, $\int_\Gamma f(z)dz = 0$. Letting $R \to \infty$, part (a) (applied on f and $-f$ with $a = -1$ and $b = 1$) shows $\hat{f}(\xi) = \int_{\mathbb{R}-\operatorname{sgn}(\xi)i} f(z)e^{-2\pi iz\xi}dx$. Then

$$|\hat{f}(\xi)| \le \int_\mathbb{R} \frac{A}{1+x^2+1^2}e^{-2\pi \operatorname{sgn}(\xi)\xi}dx \le \int_\mathbb{R} \frac{A}{1+x^2}dx e^{-2\pi|\xi|} = Ce^{-2\pi|\xi|}.$$

2. a. Let $B_t = B(0,t)$. Fix $z \in \mathbb{D} \setminus \{0\}$. Let $r < |z|/3 < |z| < R$. Then by Cauchy's theorem applied on $B_R \setminus B_r$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_R - \partial B_r} \frac{f(w)}{w-z}dw. \tag{1}$$

On ∂B_R, ~~$|z/w| < 1 - |z|/R < 1$~~, **[EDIT:** $|z/w| < |z|/R < 1$] hence

$$\int_{\partial B_R} \frac{f(w)}{w-z}dw = \int_{\partial B_R} \frac{f(w)}{w}\sum (z/w)^n dw = \sum_{n=0}^\infty \int_{\partial B_R} \frac{f(w)}{w^{n+1}}dw z^n.$$

The interchange of the order of summation and integration follows by uniform convergence of the geometric series, since f is bounded on ∂B_R and the tail is bounded by $|z/w|^N/(1-|z/w|) \leq (1-|z|/R)^N/(|z|/R)$. Similarly,

$$\int_{\partial B_r} \frac{f(w)}{w-z} dw = -\sum_{n=0}^{\infty} \int_{\partial B_r} f(w) w^n dw \frac{1}{z^{n+1}}.$$

The desired equation then follows with

$$c_n = \begin{cases} \frac{1}{2\pi i} \int_{\partial B_R} \frac{f(w)}{w^{n+1}} dw & \text{if } n \geq 0 \\ \frac{1}{2\pi i} \int_{\partial B_r} \frac{f(w)}{w^{n+1}} dw & \text{if } n < 0 \end{cases}$$

Cauchy's theorem applied on the $B_R \backslash B_r$ then shows $\frac{1}{2\pi i} \int_{\partial B_r} \frac{f(w)}{w^{n+1}} dw = \frac{1}{2\pi i} \int_{\partial B_R} \frac{f(w)}{w^{n+1}} dw$, and hence

$$c_n = \frac{1}{2\pi i} \int_{\partial B_R} \frac{f(w)}{w^{n+1}} dw,$$

which is independent of z.

b. i. Consider (1), which holds for a *fixed* R with $0 < |z| < R$. Since $|f(z)| \leq \frac{A}{|z-z_0|^{1-\varepsilon}}$,

$$\left| \frac{1}{2\pi i} \int_{\partial B_r} \frac{f(w)}{w-z} dw \right| \leq \frac{1}{2\pi} \frac{A}{r^{1-\varepsilon}} \frac{1}{(2/3)|z-z_0|} (2\pi r)$$
$$= \frac{3A}{2|z-z_0|} r^{\varepsilon} \to 0 \text{ as } r \to 0$$

Then $f(z) = \frac{1}{2\pi i} \int_{\partial B_R} \frac{f(w)}{w-z} dw$, where the right-hand side is holomorphic by differentiation under integral sign. This is justified because the domain is a fixed compact set and the integrand is C^1.

ii. We prove the contrapositive. Suppose it is not dense, then it is bounded away from a number, say w_0. Then $g = \frac{1}{f-w_0}$ is bounded near 0, and hence has a removable singularity. Therefore, $f = w_0 + \frac{1}{g}$. Taylor expanding g gives $g(z) = c_n z^n + c_{n+1} z^{n+1} + \ldots = z^n (c_n + c_{n+1} z + \ldots) = z^n h(z)$ for some $c_n \neq 0$, and hence holomorphic h with $h(0) \neq 0$. Therefore, $f(z) = w_0 + \frac{1}{z^n} \frac{1}{h(z)}$, where $1/h$ is holomorphic near 0. Now, if $n = 0$, then the singularity of f is removable; if $n > 0$, it is a pole.

3. The image of the unit disc is open by open mapping theorem, and is relatively closed in the unit disc by compactness of the closed unit disc ($f(\partial B) \subseteq \partial B$ is used here). By connectedness, it suffices to show f has a zero. ~~This can be done by applying maximum principle on f and $1/f$.~~ **[EDIT: Suppose not. Applying maximum principle on f and $1/f$ shows $|f| \equiv 1$, and hence f attains the maximum modulus in the interior, and hence is constant. The contradiction then follows.]**

4. a. Let $f(z) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} \pi^{2n} z^n$. The series converges on $\mathbb{C}$ by root test. By direct inspection, $f(z^2) = \frac{\sin \pi z}{\pi z}$.

It follows that the set of zeros of $g(z) = zf(z)$ is precisely $\{n^2 : n \in \mathbb{Z}\}$.

~~For each z, choose one $z^{1/2}$. Then $f(z) = \frac{\sin \pi z^{1/2}}{\pi z^{1/2}}$, hence~~

~~$g(z) \leq \frac{1}{\pi}|z|^{1/2}\frac{|e^{\pi i z^{1/2}} - e^{\pi i z^{1/2}}|}{2} \leq \frac{1}{\pi}e^{\log z/2+\pi|z|^{1/2}} \leq \frac{1}{\pi}e^{(\pi+\varepsilon)|z|^{1/2}}$~~

[EDIT: For each z, there exists some $w \in \mathbb{C}$ such that $w^2 = z$, and hence $|w| = |z|^{1/2}$. Then $f(z) = \frac{\sin \pi w}{\pi w}$, and hence

$$|g(z)| \leq \frac{|z|}{\pi|w|}\frac{|e^{\pi i w} - e^{\pi i w}|}{2} \leq \frac{1}{\pi}e^{\log|z|/2+\pi|z|^{1/2}} \leq \frac{1}{\pi}e^{(\pi+\varepsilon)|z|^{1/2}}.$$

]

b. Note that $f = h/g$ away from zeros of g, so f is meromorphic. To show holomorphicity, it suffices to show f is continuous at zeros of g, the only potential singularities of f (then f is bounded near every singularity, and hence by removable singularity theorem, f has a holomorphic correction. Since this correction and f itself are both continuous extensions from $\{g \neq 0\}$, which, has a discete complement and hence is dense. Then by uniqueness of the continuous extension, f is equal to this holomorphic correction, and hence holomorphic).

Let a be a zero of g. Then $f(a) = 0$. Since $|f| = \sqrt{|g|}$ is continuous, $f(z) \to 0$ as $z \to a$, and hence f is continuous at a. The result then follows.

5. Suppose not. Dividing by z^m if necessary, assume $f(0) \neq 0$.

Choose a sequence R_n such that $(n-1)/n < R_n < n/(n+1)$ and f has no zero on $\partial B(0, R_n)$.

Since log is negative on (0, 1), Jensen's formula shows

$$\log|f(0)| = \frac{1}{2\pi R_n}\int_{\partial B(0,R_n)} \log|f| + \sum_{\substack{a \text{ zero of } f \\ |a|<R_n}} \log|\frac{a}{R_n}| \leq \log A + \sum_{m\leq n} \log|\frac{(m-1)/m}{R_n}|. \tag{2}$$

Telescoping gives

$$\begin{aligned}
\sum_{m\leq n} \log|\frac{(m-1)/m}{R_n}| &= \log \prod_{m\leq n} |\frac{(m-1)/m}{R_n}| \\
&= \log|\frac{1/n}{R_n^n}| \\
&\leq -\log n - \log\left[\left(\frac{n-1}{n}\right)^n\right] \\
&\to -\infty - \log(1/e) = -\infty \text{ as } n \to \infty
\end{aligned}$$

Therefore, the right-hand side of (2) tends to $-\infty$ as $n \to \infty$, contradictory to the constancy of the left-hand side. The result then follows.