THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Homework 4 solution

5.10 By (5.2b), the orders of both functions are 1 (cos is a sum of exp). Then by Hadamard factorisation theorem and symmetric grouping of factors,

$$f(z) = e^z - 1 = e^{Az+B} z \prod_{n\neq 0} (1 - \frac{z}{2\pi i n}) e^{z/(2\pi i n)} = e^{Az+B} z \prod_{n>0} (1 + \frac{z^2}{(2\pi n)^2})$$

$$g(z) = \cos \pi z = e^{Cz+D} \prod_{n\in\mathbb{Z}} (1 - \frac{z}{n+1/2}) e^{z/(n+1/2)} = e^{Cz+D} \prod_{n>0} (1 - \frac{z^2}{(n-1/2)^2})$$

Considering $f'(0) = g(0) = 1$ shows $e^B = e^D = 1$ and considering the constant terms in $(f'/f)(0)$ and $(g'/g)(0)$shows $A = 1/2$ and $C = 0$.

5.11 Suppose f misses a and b, and $a \neq b$. By Hadamard's factorisation theorem, since $f - a$ has no zero, $f(z) = e^{P(z)}$ for some polynomial P. By fundamental theorem of algebra, P is surjective on $\mathbb{C}$, and hence $(f - a)(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. In particular, $b - a = f(z) - a$ for some z, contradictory to the assumption that f misses b.

5.14 We prove the contrapositive. Suppose f has finitely many zeros $a_1, ..., a_k$ Then f/Q has no zero for the polynomial $Q(z) = \prod(z - a_i)$, and hence $f = Qe^P$ for some polynomial P. Then the order of f is the degree of P, and hence is integral.

5.15 By Weierstrass factorisation theorem, there exist holomorphic f and g such that $\{a_n\}$ and $\{b_n\}$ are the set of zeros of f and g respectively. Then $h = f/g$ is a meromorphic function that vanishes exactly at $\{a_n\}$ and has poles exactly at $\{b_n\}$. Now, let φ be a meromorphic function with zeros $\{\tilde{a}_n\}$ and poles $\{\tilde{b}_n\}$. Then φ/h is entire without zeros if h is defined with $a_n = \tilde{a}_n$ and $b_n = \tilde{b}_n$. Then by taking log, $\varphi/h = e^\psi$ for some entire ψ. Then $\varphi = (e^\psi f)/g$, where $e^\psi f$ and g are entire.

6.1 By the factorisation of $1/\Gamma$, $\Gamma(s) = e^{-\gamma s} \frac{1}{s} \prod_{n>0} \frac{n}{n+s} e^{s/n}$ Since $e^{-\gamma s} = \lim_N e^{s(\log N - \sum_1^N 1/n)}$,

$$\begin{aligned}
\Gamma(s) &= \lim_N e^{s(\log N - \sum_1^N 1/n)} \frac{1}{s} \prod_1^N \frac{n}{n+s} e^{s/n} \\
&= \lim_N \frac{e^{s \log N} N!}{s(s+1)...(s+N)} \\
&= \lim_N \frac{N^s N!}{s(s+1)...(s+N)}
\end{aligned}$$

6.4 Since $f^{(n)}(z) = \alpha(\alpha+1)...(\alpha+n-1)(1-z)^{-(\alpha+n)}$,

$$\begin{aligned}\lim_n a_n(\alpha)/(n^{\alpha-1}/\Gamma(\alpha)) &= \Gamma(\alpha)\lim_n \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!n^{\alpha-1}}\\ &= \lim_n \frac{n^\alpha n!}{\alpha(\alpha+1)...(\alpha+n)} \lim_n \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!n^{\alpha-1}}\\ &= \lim \frac{n}{\alpha+n}\\ &= 1\end{aligned}$$

6.5 Since $\overline{\Gamma(\bar{z})}$ is meromorphic and agrees on the positive real axis with $\Gamma(z)$, $\Gamma(\bar{z}) = \overline{\Gamma(z)}$.

The result then follows from the following chain of equations.

$$\begin{aligned}|\Gamma(1/2+it)|^2 &= \Gamma(1/2+it)\overline{\Gamma(1/2-it)}\\ &= \frac{\pi}{\sin\pi(1/2+it)}\\ &= \frac{\pi}{\cosh\pi t}.......(\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y)\end{aligned}$$

6.7 a. Proceed hinted. The new bounds are then $0 < u < \infty$ and $0 < r < 1$ and the Jacobian is $\left|\frac{\partial(s,t)}{\partial(u,r)}\right| = u$, and hence

$$\Gamma(\alpha+\beta) = \int_0^\infty e^{-u}u^{\alpha+\beta-1}du \int_0^1 r^{\alpha-1}(1-r)^{\beta-1}dr = \Gamma(\alpha+\beta)\int_0^1 r^{\alpha-1}(1-r)^{\beta-1}dr.$$

The result then follows by the change of variable $t = 1-r$.

b. In the defining integral of B, do the change of variable $u = 1/(1-t) - 1$ to change the domain of integration from $(0,1)$ to $(0,\infty)$. Then $1-t = 1/(u+1)$, $t = u/u+1$ and $dt = du/(u+1)^2$. This gives

$$B(\alpha,\beta) = \int_0^\infty \frac{u^{\beta-1}}{(u+1)^{\alpha+\beta}}du.$$

The result from interchanging α and β because part (a) implies $B(\alpha,\beta) = B(\beta,\alpha)$.

6.10 a. Consider the holomorphic function $f(w) = e^{-w}w^{z-1}$ on $\{w : \Re w > 0, \Im w > 0\}$ as hinted. Note $|f(w)| \le e^{-\Re w}|w|^{u-1}$, where $u = \Re z \in (0,1)$.

The integral on the small arc is bounded by

$$\varepsilon^u \int_0^{\pi/2} e^{-\varepsilon\cos\theta}d\theta \le (\pi/2)\varepsilon^u \to 0$$

as $\varepsilon \to 0$.

The integral on the large arc is bounded by

$$R^u \int_0^{\pi/2} e^{-R\cos\theta}d\theta \le R^u \int_0^{\pi/2} e^{-R(1-(2/\pi)\theta)}d\theta = R^{u-1}\int_0^R e^{-t}dt \le R^{u-1} \to 0$$

as $R \to \infty$.

Therefore, $\int_0^\infty e^{-it} t^{z-1} dt = i^{-z} \int_0^\infty e^{-t} t^{z-1} dt = e^{\pi i z/2}\Gamma(z)$. Conjugating and replacing z by $\bar{z}$ (note $\bar{z}$ still lies in the vertical strip between 0 and 1) shows $\int_0^\infty e^{it} t^{z-1} dt = e^{-\pi i z/2}\Gamma(z)$. The result then follows by taking linear combinations.

b. The equations follows by putting $z = 0$ and $z = 1/2$ into the second equation in (a). It remains to show the equation holds on $|\Re z| < 1$. Right-hand side is clearly holomorphic because the pole of Γ at 0 cancels with that of the zero of sin. Left-hand isde is holomorphic on $-1 + \varepsilon < \Re z < -\varepsilon$ by Morera's theorem, (break the integral into one on $(0, 1)$ and $(1, \infty)$, where on the former the integrand is bounded by $t^{\Re z}$ as $|\sin t| \le |t|$).

6.12 a. For every positive integer k, applying $s\Gamma(s) = \Gamma(s+1)$ gives $\Gamma(-1/2 - k) = \frac{-2\sqrt{\pi}}{(-1/2-1)(-1/2-2)\ldots(-1/2-k)}$, and hence

$$|1/\Gamma(-1/2 - k)| \ge \frac{k!}{2\sqrt{\pi}}.$$

Since $|-1/2 - k| \le 2k$, $\frac{1/\Gamma(-1/2-k)}{e^{A|1/2-k|}} \ge \frac{k!}{2\sqrt{\pi}(e^{2A})^k} \to \infty$, and hence $1/\Gamma(s)$ is not $O(e^{A|s|})$.

b. By Hadamard's factorisation theorem, $F = e^P/\Gamma$ for some linear polynomial P, and hence $1/\Gamma = e^{-P}F$. If $F(z) = O(e^{C|z|})$, then so is $1/\Gamma$. The contradiction then follows.

6.14 a. fundamental theorem of calculus

b. Since Γ, and hence $\log\Gamma$ is eventually increasing on the positive real axis (because $\log\Gamma$ is convex (see Theorem 8.18c of Rudin's *Principles of Mathematical Analysis*), and $\Gamma(3) > \Gamma(2)$), $\log\Gamma(x) \le \int_x^{x+1} \log\Gamma \le \log\Gamma(x+1)$, or equivalently,

$$(x-1)\log(x-1)-(x-1)+c \le \int_{x-1}^{x} \log\Gamma \le \Gamma(x) \le \int_x^{x+1} \log\Gamma = x\log x - x + c,$$

Since for every $\alpha < 1$, for x sufficiently large, $(x-1)\log(x-1) - (x-1) + c \ge \alpha x \log(\alpha x) + o(x\log x) \ge \alpha x \log x + o(x \log x)$, and hence $\Gamma(x) \sim x\log x - x + c$, where $-x + c = O(x)$. The result then follows.

6.15

$$\begin{aligned} \int_0^\infty \frac{t^s}{e^t - 1}\frac{dt}{t} &= \int_0^\infty t^s \sum_1^\infty e^{-nt}\frac{dt}{t} \\ &= \sum_1^\infty \int_0^\infty (t/n)^s e^{-t}\frac{dt}{t} \\ &= \sum_1^\infty \frac{1}{n^s}\Gamma(s) \\ &= \zeta(s)\Gamma(s) \end{aligned}$$

The exchange of integral is justified by Fubini's theorem for real s because the all expressions are nonnegative, and hence for all s with $\Re s > 1$ because $|(t/n)^s| \leq (t/n)^{\Re s}$, which reduces to the real case. [There are two versions of Fubini's theorem. The first, not mentioned in the tutorial note, says if a function f is measurable and nonnegative, $\int_{\mathbb{R}^2} f dA = \int_{\mathbb{R}} \int_{\mathbb{R}} f dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f dy dx$. The second, mentioned in tutorial note 2, says the same conclusion holds if at least one of the three integrals is finite when f is replaced by $|f|$.]

6.16 Split the integral in 6.15 to one over $(0, 1)$ and $(1, \infty)$. The one on $(1, \infty)$ converges absolutely uniformly on compact sets, and hence by Morera's theorem defines a holomorphic function. For the one on $(0, 1)$, $\frac{t^s}{e^t-1}\frac{1}{t} = t^{s-2}\frac{t}{e^t-1} = t^{s-2}\sum_0^\infty c_n t^n$, where $\frac{t}{e^t-1} = \sum_0^\infty c_n t^n$, by holomorphy, converges uniformly on compact subsets of $B(0, 2\pi)$, and in particular, $(0, 1)$. Termwise integration, justified by uniform convergence, then shows $\int_0^1 \frac{t^s}{e^t-1}\frac{dt}{t} = \sum_0^\infty \frac{c_n}{n+s-1}$, whose poles are $1, 0, -1, -2, ...$, all of which except 1 cancel with the zeros of $\frac{1}{\Gamma(s)}$.

Therefore, $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 \frac{t^s}{e^t-1}\frac{dt}{t} + \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^s}{e^t-1}\frac{dt}{t}$, where the former term has a unique pole that is simple at 1 and the latter is entire.