THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Homework 2 solution

2.11 Cauchy integral formula gives

$$\frac{1}{2\pi}\int f(w)\frac{w}{w-z}dt = f(z),$$

where $w = Re^{it}$. Let $\zeta = R^2/\bar{z}$. Then $|\zeta| > R$, and hence $f(w)/(w-\zeta)$ is holomorphic on $B(0,R)$. Then $\frac{1}{2\pi}\int_0^{2\pi} f(w)\frac{\bar{z}}{\bar{z}-\bar{w}}dt = \frac{1}{2\pi i}\int_{\partial B(0,R)}\frac{f(w)}{w-\zeta}dw = 0$. Note $\frac{\bar{z}}{\bar{z}-\bar{w}} = 1 + \overline{\left(\frac{w}{z-w}\right)}$ Summing the two equations gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{2\pi} f(w)\left(\frac{w}{z-w} + 1 + \overline{\left(\frac{w}{z-w}\right)}\right)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(w)\Re\left(\frac{2w}{z-w}+1\right)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(w)\Re\left(\frac{z+w}{z-w}\right)dt \end{aligned}$$

Part b follows from direct computation.

2.12

a. Define $v(z) = \int_0^z(-u_y dx + u_x dy)$, which is well defined by simple-connectedness of $\mathbb{D}$. Then $f = u + iv$ satisfies Cauchy-Riemann equations and hence is holomorphic. Then $\Re f = u$. For uniqueness, since the difference of the two holomorphic function has a constant real part (namely 0), the difference is a constant, and hence the imaginary part is also determined up to a constant.

b. Apply the formula in 2.11(a) on f defined in (a). Consider the real part.

3.11

a. By mean-value property of the harmonic function $\log|1-z|$, the integral is $2\pi a \log|1-0| = 0$.

b. By dominated convergence theorem, it suffices to dominate $\log|1-ae^{i\theta}|$ for $|\theta| \le \pi$ as $a \to 1^-$. It is claimed that

$$f(\theta) = \begin{cases} |\log|\theta/2|| & \text{if } |\theta| < \varepsilon \\ \log|1-e^{i\theta}| + \eta & \text{otherwise} \end{cases}$$

is a such a dominator for suitable $\varepsilon, \eta > 0$. By integration by parts, it is integrable. It remains to show it indeed dominates the functions. For $|\theta| \ge \varepsilon$ and a suitable ε, $|ae^{i\theta} - 1| \ge |\sin\theta| \ge |\theta|/2$, and hence $|\log|1-a_n e^{i\theta}|| \le |\log|\theta/2||$. For $|\theta| \ge \varepsilon$, $\log|re^{i\theta}|$ is uniformly continuous on $\overline{B(0,1) \setminus \{|\arg z| < \varepsilon\}}$, the convergence is uniform, and hence $\log|1-e^{i\theta}| + \eta$ eventually dominates the functions.

3.19

1. $E = \{u(x) = \max u\}$ is closed by continuity. It is also open: if $u(x_0) = \max u$, then by mean-value property, $u(x) = \max u$ for $x \in \partial B(x_0, \rho)$, and hence by letting ρ vary, for $x \in B(x_0, r)$. Then by connectedness, E is either the whole set, in which case u is constant; or the empty set, in which case u does not attain the maximum.
2. By compactness, the maximum is attained on $\bar{\Omega}$. Since it is not attained on Ω, it is attained on $\bar{\Omega} \setminus \Omega$. The result then follows.

3.20

a. claim: $f(z) = \frac{1}{\pi R^2} \int_{B(z,R)} f$ if f is holomorphic on $B(z, R)$.

Parametrizing Cauchy integral formula gives $f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + \rho e^{it}) dt$. Since $dA = \rho d\rho d\theta$, the claim then follows by multiplying by ρ and integrating wrt ρ from 0 to R.

By considering $z \in B(z_0, s)$ and $R = r - s$, $\|f\|_{L^\infty(D_s(z_0))} \le \frac{1}{\pi(r-s)^2} \|f\|_{L^2(B(z_0,r))}$.

b. Cover the compact set with finitely many $B(z_i, r_i)$'s such that $B(z_i, 2r_i) \subseteq U$. It suffices to show uniform convergence on each $B(z_i, r_i)$. Since $\{f_n\}$ is Cauchy in $L^2(U)$, it is Cauchy in $L^2(B(z_i, 2r_i))$, and hence by part (a), in $L^\infty(B(z_i, r_i))$. The result then follows from the completeness of L^∞ and Morera's theorem.

5.2

1. Since $\log R = o(R^s)$ as $R \to \infty$ for every $s > 0$, $n \log|z| \le |z|^s$, and hence $|z^n| \le \exp(|z|^n)$ for large $|z|$ for every $s > 0$. Therefore, the order is 0.
2. Since $|\exp(bz^n)| = \exp(\Re(bz^n))$, where $u = \Re(bz^n)$ is a polynomial of degree at most n, and hence $u = o(|z|^{n+\varepsilon})$ for every $\varepsilon > 0$, and hence the order is at most n.

   Putting $z_R = \overline{b/|b|}^{1/n} R$, such that $f(z_R) = \exp(|b|R^n) > 0$. Taking log shows it is impossible that $f(z_R) \le A \exp(BR^{n-\varepsilon})$, so the order is indeed n.
3. Consider positive z and take log. The order is ∞.

5.3 Let $t = \Im\tau$.

Note that $-n^2 t + 2|n||z| \le -\frac{1}{2} n^2 t$ if $n \ge 4|z|/t$. Then

$$\begin{aligned}
|\Theta(z|\tau)| &\le \sum e^{-\pi n^2 t} e^{2\pi|n||z|} \\
&\le \sum_{n \ge 4|z|/t} e^{-\frac{1}{2}\pi n^2 t} + \sum_{n < 4|z|/t} e^{-\pi n^2 t} e^{2\pi|n||z|} \\
&\le \sum e^{-\frac{1}{2}\pi n^2 t} + \sum_{n < 4|z|/t} e^{-\pi n^2 t} e^{(8\pi/t)|z|^2} \\
&\le \sum e^{-\frac{1}{2}\pi n^2 t} + \sum e^{-\pi n^2 t} e^{(8\pi/t)|z|^2} \\
&= C + C e^{(8\pi/t)|z|^2} \\
&\le C e^{C|z|^2}
\end{aligned}$$

Therefore, the order is at most 2. To show equality, observe that $\Theta(z + \tau|\tau) = e^{-\pi i(\tau + 2z)} \Theta(z|\tau)$. Then $\Theta(z + k\tau|\tau) = e^{-\pi i(2kz + k^2\tau)} \Theta(z|\tau)$. Now, if k is large enough, $|z + k\tau| \le 2k|\tau|$, and in particular, if $z \in \mathbb{R}$,

$$|\Theta(z + k\tau|\tau)| = e^{-\pi k^2 \Im\tau} |\Theta(z|\tau)| \ge e^{-\pi|z+k\tau|^2 \Im\tau/(4|\tau|^2)} |\Theta(z|\tau)|.$$

The result then follows if $\Theta(\cdot|\tau)$ is not identically zero, and hence $\Theta(z|\tau) \neq 0$ for some real z. Since $\Theta(\cdot|\tau)$ is a Fourier series on $\mathbb{R}$ with nonzero coefficients, it is not identically zero. The result then follows.

5.4 a. Fix z such that $|z|$ is large. Define F_1 and F_2 as in the hint, with N being the last integer such that $Nt - |z| \leq \frac{\log 2}{2\pi}$. Then if $|z|$ is large enough,

$$\frac{1-\varepsilon}{t}|z| \leq N < N+1 \leq \frac{1+\varepsilon}{t}|z|.$$

We first show that $|F_2(z)|$ is bounded between positive constants, or rather, its log is bounded.

$$\log|F_2(z)| = \sum_{n>N} \log|1 - e^{-2\pi nt}e^{2\pi iz}|.$$

Taylor expansion gives

$$\frac{1}{2}|w| \leq |\log(1-w)| \leq 2|w|$$

for $|w| < 1/2$, and indeed by the choice of N, $|e^{-2\pi nt}e^{2\pi iz}| \leq 1/2$, hence $\frac{1}{2}|G(z)| \leq \log|F_2(z)| \leq 2|G(z)|$, where

$$G(z) = \sum_{n>N} e^{-2\pi nt}e^{2\pi iz} = \frac{1}{1-e^{-2\pi t}}e^{-2\pi(N+1)t}e^{2\pi iz}.$$

Then by maximality of N, we have

$$\frac{e^{-2\pi t}}{2}\frac{1}{1-e^{-2\pi t}} \leq |G(z)| \leq \frac{1}{2}\frac{1}{1-e^{-2\pi t}}.$$

Combining,

$$\frac{e^{-2\pi t}}{4}\frac{1}{1-e^{-2\pi t}} \leq \log|F_2(z)| \leq \frac{1}{1-e^{-2\pi t}}.$$

Boundedness of $|F_2(z)|$ by positive constants then follows.

Now it suffices to consider F_1. Since $|e^{-2\pi nt}e^{2\pi iz}| \geq 1/2$, and hence

$$|1 - e^{-2\pi nt}e^{2\pi iz}| \leq 1 + e^{-2\pi nt}e^{2\pi|z|} \leq 3e^{2\pi|z|},$$

$$\begin{aligned}
|F_1(z)| &\leq \prod |1 - e^{-2\pi nt}e^{2\pi iz}| \\
&\leq 3^N e^{2\pi N|z|} \\
&\leq \exp\left(\frac{1+\varepsilon}{t}(|z|\log 3 + 2\pi|z|^2)\right) \\
&\leq \exp\left(\frac{1+2\varepsilon}{t}(2\pi|z|^2)\right) \ldots |z| \text{ sufficiently large}
\end{aligned}$$

Therefore, F is of order at most 2.

To show the order is indeed 2, let $z_k = 1/2 - kt$. Then $kt < |z_k| \le (1+\varepsilon)kt$ for k sufficiently large. Since

$$|1 - e^{-2\pi nt}e^{2\pi iz}| = 1 + e^{-2\pi nt}e^{2\pi kt} \ge e^{-2\pi nt}e^{2\pi kt},$$

$$\begin{aligned}
|F_1(z)| &\ge \prod_1^N e^{-2\pi nt}e^{2\pi kt} \\
&= \exp(-\pi N(N+1)t + 2\pi Nkt) \\
&\ge \exp(\pi\frac{1-\varepsilon}{t}|z_k|(-(1+\varepsilon)|z_k| + 2\frac{1}{1+\varepsilon}|z_k|))
\end{aligned}$$

Note that the argument of exp in the last line is a quadratic in $|z_k|$ with a positive exponent if ε is sufficiently small. Therefore, $|F_1(z_k)| \ge A\exp(B|z_k|^2)$, and the result follows.

b. A factor vanishes precisely when $i(z+2m) = nt$, so the function vanishes at $z = -int + m$.

For exponent $= -2$,

$$\begin{aligned}
\sum |z_k|^{-2} &= \sum\sum \frac{1}{(nt)^2 + m^2} \\
&\ge \sum_{n\ge 1}\sum_{m\ge nt} \frac{1}{2m^2} \\
&\ge \sum_{n\ge 1}\int_{nt}^{\infty} \frac{1}{x}dx \\
&= \sum_{n\ge 1} \frac{1}{nt} \\
&= \infty
\end{aligned}$$

For exponent < -2, it suffices to consider $m, n \ge 0$ by symmetry, and $m, n > 0$ since $\sum_{j=1}^{\infty} 1/j^2 < \infty$.

$$\begin{aligned}
\sum_{n,m>0} \frac{1}{|(nt,m)|^{-2-\varepsilon}} &\le 1 + \sum_{n,m>0;(n,m)\ne(1,1)} \int_{(n-1)t}^{nt}\int_{m-1}^{m} \frac{1}{|(nt,m)|^{-2-\varepsilon}} \\
&\le 1 + \iint_{|(x,y)|>\delta} |(x,y)|^{-2-\varepsilon}dxdy \\
&\le 1 + \int_0^{2\pi}\int_\delta^\infty r^{-1-\varepsilon}drd\theta \\
&\le 1 + 2\pi\frac{1}{\varepsilon}\delta^{-\varepsilon} \\
&< \infty
\end{aligned}$$

5.5 Holomorphicity on $|z| \le M$ follows from Morera's theorem and Fubini's theorem. To show the order is at most $\alpha/(\alpha-1)$, by Young's inequality, which bounds cross terms,

$$|zt| \le C_\varepsilon |z|^{\alpha/(\alpha-1)} + \varepsilon |t|^\alpha,$$

(The basic Young's inequality says $|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q}$ if $p, q > 1$ and $1/p + 1/q = 1$. Replacing a and b by a/δ and δb, where δ is chosen according to ε, gives $|ab| \le C_\varepsilon |a|^p + \varepsilon |b|^q$.)

hence

$$|F_\alpha(z)| \le \int e^{-|t|^\alpha} e^{C|z|^{\alpha/(\alpha-1)}} e^{\varepsilon|z|^\alpha} dt \le e^{C|z|^{\alpha/(\alpha-1)}} \int e^{-(1-\varepsilon)|t|^\alpha} dt,$$

where the integral is a finite number independent of z.

For the reverse inequality, put $z = -iR$ so that the integrand is positive. Then the integral on $\mathbb{R}$ is bounded below by the integral on $[M-1, M]$, where $M = R^{1/(\alpha-1)}$. On this interval, the integrand is bounded below by $\exp(-M^\alpha + 2\pi RM) = \exp((2\pi - 1)R^{\alpha/(\alpha-1)})$ if R is large enough. The result then follows.

5.6 Plug in $z = 1/2$ into $\sin \pi z = \pi z \prod [1 - (z/n)^2]$.

5.7 a. Taylor approximating log at 1 quadratically gives $|\log(1+a_n) - a_n| \le C|a_n|^2$, so the by Cauchy criterion, convergence of either $\sum \log(1+a_n)$ or $\sum a_n$ implies that of the other whenever $\sum |a_n|^2$ converges.

b. Let $0 < x_n < 1$, $x_n \to 0$ but $\sum x_n^2 = \infty$, say, $x_n = 1/(n+1)^{(1-\varepsilon)/2}$. Let

$$a_m = \begin{cases} x_{m/2} & \text{if } m \text{ even} \\ -x_{(m+1)/2} & \text{if } m \text{ odd} \end{cases}.$$

$\sum a_m$ is convergent because it is an alternating sum with terms vanishing at infinity. The divergence of $\prod(1+a_m)$ follows from that of $\sum \log(1+a_m)$, which, by grouping each pair of terms, is $\sum \log(1 - x_n^2) < -\frac{1}{2}\sum x_n^2 = -\infty$.

c. $a_1 = -1$ and $a_n = 1$ for $n > 1$.

5.8 Fix z. Repeated application of the double-angle formula for sine gives

$$\frac{\sin z}{z} \Big/ \frac{\sin(z/2^n)}{z/2^n} = \cos(z/2)\cos(z/4)...\cos(z/2^n).$$

The result follows from letting $n \to \infty$.

5.9 Inductively, $\prod_0^L (1 + z^{2^k}) = \sum_0^{2^{L+1}-1} z^j$. The result follows from letting $n \to \infty$.