# THE CHINESE UNIVERSITY OF HONG KONG
## Department of Mathematics
## 2018-2019 semester 1 MATH4060
## Homework 1 solution

**1.8** Note that for $f = u + iv$,

$$M \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} = \begin{bmatrix} \partial_z f & \partial_{\bar z} f \\ \partial_z \bar f & \partial_{\bar z} \bar f \end{bmatrix} M = \begin{bmatrix} \partial_x f & \partial_y f \\ \partial_x \bar f & \partial_y \bar f \end{bmatrix},$$

where $M = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$.

Then

$$\begin{aligned}
\begin{bmatrix} \partial_z h & \partial_{\bar z} h \\ \partial_z \bar h & \partial_{\bar z} \bar h \end{bmatrix} &= M \begin{bmatrix} \partial_x \Re h & \partial_y \Re h \\ \partial_x \Im h & \partial_y \Im h \end{bmatrix} M^{-1} \\
&= M \left( \begin{bmatrix} \partial_u \Re g & \partial_v \Re g \\ \partial_u \Im g & \partial_v \Im g \end{bmatrix} \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} \right) M^{-1} \\
&= M \left( M^{-1} \begin{bmatrix} \partial_z g & \partial_{\bar z} g \\ \partial_z \bar g & \partial_{\bar z} \bar g \end{bmatrix} M \right) \left( M^{-1} \begin{bmatrix} \partial_z f & \partial_{\bar z} f \\ \partial_z \bar f & \partial_{\bar z} \bar f \end{bmatrix} M \right) M^{-1} \\
&= \begin{bmatrix} \partial_z g & \partial_{\bar z} g \\ \partial_z \bar g & \partial_{\bar z} \bar g \end{bmatrix} \begin{bmatrix} \partial_z f & \partial_{\bar z} f \\ \partial_z \bar f & \partial_{\bar z} \bar f \end{bmatrix}.
\end{aligned}$$

The result follows by considering the first row.

**1.13** If either the real part or the imaginary part is constant, then so is its harmonic conjugate, and hence the function itself. By maximum principle, constant modulus implies constant function value.

Alternatively, constant modulus means $\log f$ (defined locally) has a constant real part, and hence $\log f$, and hence $f$ is constant.

Alternatively, it suffices to show $\partial_z f = 0$. Constancy of $|f|^2$ implies $\bar f = (|f|^2)/f$ is holormophic, hence $\overline{\partial_z f} = \partial_{\bar z} \bar f = 0$, and hence $\partial_z f = 0$. The result then follows.

**1.14**

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n = \sum_{n=M}^{N} a_n (B_n - B_{n-1}) &= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1} \\
&= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} (a_n - a_{n-1}) B_n - \sum_{n=M-1}^{N-1} a_n B_n \\
&= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N} (a_{n+1} - a_n) B_n.
\end{aligned}$$

**1.19** For $\sum n z^n$, the terms do not converge to 0. For $\sum z^n/n^2$, absolute convergence implies convergence. For $\sum z^n/n$, use 1.14 with $a_n = 1/n$ and $b_n = z^n$, and hence $B_k = \frac{z - z^{k+1}}{1-z}$.

**1.21** Absolute convergence renders the summation order immaterial.

a. For each $n \in \mathbb{N}$,

$$\frac{z^{2^n}}{1 - z^{2^{n+1}}} = \sum_{k=0}^{\infty} z^{2^n + 2^{n+1}k}$$

For each $m \in \mathbb{N}$, $m = 2^n \times p$ for some $n \in \mathbb{N}$, $p$ odd, and the representation is unique. Thus,

$$\frac{z}{1-z} = \sum_{m=1}^{\infty} z^m = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} z^{2^n+2^{n+1}k} = \sum_{n=0}^{\infty} \frac{z^{2^n}}{1-z^{2^{n+1}}}.$$

b. As in (a), we would like to show $\sum_{k\geq 0}\sum_{\ell\geq 1} 2^k(-1)^{\ell+1}z^{2^k\ell} = \sum_{n\geq 1} z^n$. It suffices to show

$$\sum_{k\geq 0, \ell\geq 0, 2^k\ell = n} (-1)^{l+1}2^k = 1.$$

Let $n = 2^p q$, where $q$ is odd. Then

$$\begin{aligned}\sum_{k\geq 0,\ell\geq 0,2^k\ell=n} (-1)^{l+1}2^k &= \sum_{0\leq k\leq p} 2^k(-1)^{2^{p-k}q+1} \\ &= 2^p - \sum_{0\leq k\leq p} 2^k \text{...pull out term } p\text{, note the sign} \\ &= 1\end{aligned}$$

**1.22** Suppose not. Then $\frac{z}{1-z} = \sum_1^n \frac{z^{a_i}}{1-z^{d_i}}$ on the unit disc, and by identity theorem, the whole complex plane (as meromorphic functions). RHS has a pole at the $e^{2\pi i/\max d_i}$ but LHS does not. Alternatively, multiply by the product of the denominators to obtain a polynomial equation, which has finitely many roots, and hence cannot have the entire unit disc in the solution space.

**2.10** No, uniform limits of holomorphic functions are holomorphic.

**2.13** There exists at least one $n$ such that $f^{(n)}(z) = 0$ for uncountably many $z$'s. By sigma-compactness and isolation of zeros, $f^{(n)} = 0$.

**2.15** Both $f$ and $1/f$ are holomorphic. Apply maximum principle.

**3.12** The residue at $-u$ is $(\pi \cot \pi z)'|_{z=-u} = -\pi^2 \csc^2 \pi u$. By L'Hopital's rule, the residue at $n$ is $\frac{1}{(u+n)^2}$. It remains to show the integral converges to 0. It suffices to show $\cot \pi z$ is bounded on the contours by a uniform constant.

Let $M = 1/4$. Then $e^{2\pi M} > 2$. For $|y| \leq M$, $|\cot \pi z| \leq \frac{e^{2\pi M}+1}{e^{2\pi M}-1} \leq 4$. For $|y| > M$, $|x| > \sqrt{R^2 - M^2} > R - M > N + 1/4$, so $|\sin \pi z| \geq |\sin \pi x \cosh \pi y| \geq \sin \pi/4$, and hence $|\cot z| \leq 2\cosh M \csc \pi/4$. The result then follows.

**3.14** Consider the behaviour at infinity by considering $f(1/z)$.

It is not a removable singularity, because the otherwise implies $f$ is bounded on $B(0,1)^C$, and hence on $\mathbb{C}$, and hence is constant, by Liouville property.

It is not an essential singularity because the otherwise implies the image of $B(0,1)^C$ is dense and hence intersects with that of $B(0,1)$, which, by open mapping theorem, is open.

By comparing the Taylor series of $f(z)$ and $f(1/z)$, it is a polynomial. Fundamental theorem of algebra implies $f$ is in fact linear.

**3.16** Rouche's theorem (roughly) says if the difference of two functions is small on the boundary, then it they have the same number of zeros. Let $L = \sup_{\overline{B(0,1)}} |g|$, and for $\varepsilon = 0$ and $r = 1$, define

$$M_\varepsilon^r = \min_{\partial B(z_\varepsilon, r)} |f_\varepsilon| > 0 \tag{1}$$

For (a), for $\varepsilon < M_0^1/(2L)$, the difference $\varepsilon g$ of $f_\varepsilon$ from $f = f_0$ is smaller than $f$ in modulus on $\partial B(0,1)$, hence $f_\varepsilon$ has a unique zero just as $f$ does.

For (b), fix a small $\varepsilon_0$ and a small $\eta > 0$. (a) implies (1) holds for $\varepsilon_0$ and $0 < r < 1 - |z_\varepsilon|$. Then for $|\varepsilon - \varepsilon_0| < M_{\varepsilon_0}^\eta/(2L)$, $z_\varepsilon \in B(z_{\varepsilon_0}, \eta)$. Continuity then follows.

Alternatively, by the generalized argument principle, for holomorphic $f$ and $g$, $\int_\Gamma g \cdot (f'/f) = \sum g(z_i)$, where $z_i$'s are zeros of $f$ enclosed in $\Gamma$, $z_\varepsilon = \frac{1}{2\pi i} \int_{\partial B(0,1)} z(f + \varepsilon g)'/(f + \varepsilon g)$, which is holomorphic in $\varepsilon$.

Alternatively, apply Implicit function theorem. Upon checking Cauchy-Riemann equation, this shows $z \mapsto z_\varepsilon$ is holomorphic as well.

**3.17** The image of the unit disc is open by open mapping theorem, and is relatively closed in the unit disc by compactness of the closed unit disc ($f(\partial B) \subseteq \partial B$ is used here). By connectedness, it suffices to show $f$ has a zero. This can be done by applying maximum principle on (a) $f$ and $1/f$, (b) $1/f$.