THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2017-2018 semester 2 MATH2010
week 7 tutorial

This tutorial is a differentiability freak show.

Recall

$$C^1 \text{ on a neighbourhood} \Rightarrow \text{differentiable} \Rightarrow \text{directional derivatives exist} + C^0$$

and of course

$$\text{directional derivatives} \Rightarrow \text{partial derivatives}$$

1. f that is C^0 with no directional derivatives at the origin
$$f(x,y) = \begin{cases} (x^2+y^2)^{1/2} \sin \frac{1}{(x^2+y^2)^{1/2}} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$
In fact, f is not differentiable at the origin along any path with a minimum positive speed. (how to prove this?)

2. g that has directional derivatives but not C^0 at the origin
$$g(x,y) = \begin{cases} \frac{x^2+y^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
However, if a function is differentiable along every path through the origin, then it is continuous (why?).

3. h that is C^0 and has directional derivatives along all directions, but not differentiable.
$$h(x,y) = \begin{cases} \frac{y^3}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

4. p that is differentiable but not C^1 at the origin
$$p(x,y) = \begin{cases} (x^2+y^2) \sin \frac{1}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

5. q that is C^0 and has partial derivatives but no directional derivatives at the origin
$$q(x) = \begin{cases} \frac{(xy)^2}{(x^2+y^2)^{3/2}} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$
(Can you come up with a function that satisfies the same conditions, but does not even have one-sided directional derivatives?)

References

[1] Gelbaum & Olmsted. (1964). Counterexamples in Analysis