THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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week 5 tutorial

In this tutorial, higher order multivariable partial derivatives are used to study the Laplacian operator.

Definition 1. Let Ω be an open set in $\mathbb{R}^n$. For $u : \Omega \to \mathbb{R}$ with continuous second derivatives, define the Laplacian of u by

$$\Delta u = \sum_{i=1}^{n} D_{ii}u = D_{11}u + ... + D_{nn}u$$

Example 2. Let $\Omega = \mathbb{R}^2 \setminus \{0\}$ and $\varphi(x, y) = \log\sqrt{x^2 + y^2}$. Then $\Delta\varphi = 0$.

Proof.

$$\varphi_x(x, y) = \frac{x}{x^2 + y^2}$$
$$\varphi_{xx}(x, y) = \frac{y^2 - x^2}{x^2 + y^2}$$

By symmetry $\varphi_{yy}(x, y) = \frac{x^2-y^2}{x^2+y^2}$. Therefore, $\Delta\varphi(x, y) = \frac{y^2-x^2}{x^2+y^2} + \frac{x^2-y^2}{x^2+y^2} = 0$.

□

The Laplacian describes the average deviation $u(x)$ from $u(x_0)$ for x near x_0, as can be seen from the formulation below.

Proposition 3.

$$\Delta u(x) = \lim_{h\to 0} \frac{2n}{h^2}\Big[\big(\frac{1}{2n} \sum_{\substack{\sigma\in\{1,-1\}\\ 1\le i\le n}} u(x + \sigma h e_i)\big) - u(x)\Big]$$

Proof. It suffices to show the equivalent expression

$$\Delta u(x) = \lim_{h\to 0} \frac{1}{h^2} \sum [(u(x + he_i) - u(x)) + (u(x - he_i) - u(x))]$$

By the definition of partial deriviatives, this boils down to showing the following equation for single-variable functions v

$$v''(x) = \lim_{h\to 0} \frac{1}{h^2}[(v(x + h) - u(x)) + (u(x - h) - u(x))]$$

which indeed holds by L'Hopital's rule. □

Functions whose Laplacian vanishes identically are said to be *harmonic*. φ as defined in Example 2 is harmonic. Harmonic functions are useful in physics, chemistry and ecology because they describe the equilibrium distribution of such quantities as heat, concentration and population.

Proposition 4. φ is the only symmetric (the function value is a function of the norm) harmonic function on the plane, up to scaling and translation.

Proof. Let $\psi(x) = v(|x|)$ be a symmetric harmonic function on the plane.

$$\begin{aligned} D_{ii}\psi(x) &= D_i\left[v'(|x|)\frac{x_i}{|x|}\right] \\ &= \frac{(v''(|x|)\frac{x_i^2}{|x|} + v'(|x|))|x| - v'(|x|)\frac{x_i^2}{|x|}}{|x|^2} \\ &= \frac{(v''(|x|)|x| - v'(|x|))x_i^2 + v'(|x|)|x|^2}{|x|^3} \end{aligned}$$

Summing gives

$$\Delta\psi(x) = \frac{(v''(|x|)|x| - v'(|x|)) + 2v'(|x|)}{|x|} = \frac{v''(|x|)|x| + v'(|x|)}{|x|}$$

Harmonicity implies $rv''(r)+v'(r) = 0$. Observe that left-hand side is $(rv'(r))'$, and hence $rv'(r) = C_1$. Since $v'(r) = C_1/r$, integrating gives $v(r) = C_1 \log r + C_2$, and the result then follows. □

Remark. This proof can be generalised to determine symmetric harmonic functions on any dimension.

Harmonicity is a very rigid notion, in the sense that there are not too many harmonic functions; once certain properties are specified, for instance, symmetry as in the proposition above, the harmonic function may be uniquely determined. The rigidity of harmonicity can also be seen from Theorem 6 and 7. The maximum principle, which can be used to prove these results, is interesting and useful *per se*.

Theorem 5. [Maximum principle] Suppose u is harmonic on $B(0,1)$ and is continuous up to the boundary. Then u attains its maximum on the boundary, i.e. there exists an x_0 on the boundary such that for every $x \in B(0,1)$, $u(x) \leq u(x_0)$.

Proof. later. □

Theorem 6. Suppose u is harmonic on $B(0,1)$ and is continuous up to the boundary. If u vanishes on the boundary, then it is identically 0.

Proof. Maximum principle implies the maximum is attained on the boundary, and hence the maximum is 0. However, the same argument applied on $-u$ shows the minimum is $-0 = 0$. Therefore, u is identically 0. □

Corollary 7. Let $f : B(0,1) \to \mathbb{R}$ and $\phi : \partial B(0,1) \to \mathbb{R}$ be given. Then the following equation has at most one solution.

$$\begin{cases} \Delta u &= f \quad \text{on } \Omega \\ u &= \phi \quad \text{on } \partial\Omega \end{cases}$$

Proof. Let u and v be two solutions, it suffices to show that $u = v$. then $u - v$ is harmonic on Ω and vanishes on the boundary. The above theorem shows $u - v = 0$, and hence $u = v$. □

proof of Theorem 5 (Maximum principle). By single-variable calculus, the proposition is true if "harmonic" is replaced by "functions with strictly positive Laplacian", because the second derivative along each coordinate axis direction at an internal maximum is nonpositive, and hence so is their sum. (Try to write out the details of the paragraph.)

Now, suppose u is harmonic. Since $\Delta|x|^2 = 2n$ (n is the dimension of the space), for $\varepsilon > 0$, $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$ has a strictly positive Laplacian, and hence it attains its maximum on the boundary. Let M and M_ε be the maxima of u and u_ε *on the boundary.* It suffices to show M is the maximum of u on $\overline{B(0,1)}$. Since $|y|^2 = 1$ for y on the boundary, for x in the interior, $u(x) \leq u_\varepsilon(x) \leq M_\varepsilon = M + \varepsilon$. Letting $\varepsilon \to 0$, $u(x) \leq M$. The result then follows.

□

Further Reading

Evans. (2010). Partial Differential Equations (Second Edition).

Gilbarg & Trudinger. (1998). Elliptic Partial Differential Equations of Second Order.