THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2017-2018 semester 2 MATH2010
week 3 tutorial

1 Curves

1.1 Formulae

For a regular curve parametrized by $\mathbf{x}(t)$ with $t \in (t_0, t_1)$,
the arclength is defined by $s = \int_{t_0}^{t_1} \|\mathbf{x}'\|$,
and the curvature is defined as $\kappa = \|\frac{d}{ds}\frac{\mathbf{x}'}{\|\mathbf{x}'\|}\| = \left\|\left(\frac{\mathbf{x}'}{\|\mathbf{x}'\|}\right)'\right\|/\|\mathbf{x}'\|$.

Direct computation shows for curves of the form $r = r(\theta)$, the arclength is $\int \sqrt{r^2 + r'^2}$, and the curvature is $\frac{|r^2+2r'^2-rr''|}{(r^2+r'^2)^{3/2}}$.

1.2 Examples

Archimedean spiral $(r = k\theta)$

arclength: $\frac{k}{2}\left[\theta\sqrt{\theta^2+1} + \log(\theta + \sqrt{\theta^2+1})\right]_{\theta_0}^{\theta_1}$

curvature: $\frac{1}{k}\frac{2+\theta^2}{(1+\theta^2)^{3/2}}$

properties: can be used for angle trisection and circle squaring [1]

Logarithmic spiral $(r = ke^{a\theta})$ [2]

arclength: $k\frac{\sqrt{a^2+1}}{a}[e^{a\theta}]_{\theta_0}^{\theta_1}$

curvature: $\frac{1}{k\sqrt{1+a^2}e^{a\theta}}$

properties:

- infinite winding around the origin but finite length
- unique self-similar spirals
- unique curves whose tangents mark constant angles with the radial lines

Conics $r = \frac{a}{1-\varepsilon\cos(\theta-\phi)}$ (note: $(r;\theta)$ with $r < 0$ coincides with the point $(-r;\theta+\pi)$)

arclength: not elementary

curvature: $\frac{1}{a\left[\left(\frac{\varepsilon s}{1-\varepsilon c}\right)^2+1\right]^{3/2}}$, where $s = \sin(\theta-\phi)$ and $c = \cos(\theta-\phi)$

properties:

- ε is the eccentricity of the conic. The curve is, respectively, a circle, an ellipse, a parabola, and a hyperbola if $\varepsilon = 0$, $0 < \varepsilon < 1$, $\varepsilon = 1$, and $\varepsilon > 1$.
- The origin is a focus of the conic.

and one more... $r = 1 - \sin\theta + \frac{1}{6}\frac{|\cos\theta - \cos 3\theta|}{\sin\theta}$

arclength: unlikely to be elementary (at least it is hopelessly tedious to solve by hand.)

curvature: $\frac{3}{2}\frac{|54-54\sin\theta\mp 33\cos\theta\pm 36\sin 2\theta - 9\cos 2\theta \pm 9\cos 3\theta - \cos 4\theta|}{(41/2 - 18\sin\theta \mp 9\cos\theta \pm 6\sin 2\theta \mp 3\cos 3\theta + 3/2\cos 4\theta)^{3/2}}$, where the upper and lower signs in $\pm$ and $\mp$ are taken when θ lies in, respectively, the right half-plane and the left half-plane. (by WolframAlpha)

2 Topology

Can you construct a set in $\mathbb{R}^n$ ...

- that is open but not closed?
- that is closed but not open?
- that is neither open nor closed?
- that is both open and closed? (try to find all such sets, if any)
- that is nonempty but has empty interior?
- that has nonempty intersection with every ball but has an empty interior?
- that has empty interior, but the interior of the complement of whose exterior is nonempty?

References

[1] Weisstein, Eric W. "Archimedes' Spiral." From *MathWorld* –A Wolfram Web Resource. http://mathworld.wolfram.com/ArchimedesSpiral.html

[2] Weisstein, Eric W. "Logarithmic Spiral." From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/LogarithmicSpiral.html