THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2017-2018 semester 2 MATH2010C
Quiz 1 solution

1. (a) From the equation of $\mathcal{P}$, $v = (1, -3, 2)$ is a normal to $\mathcal{P}$, and hence is perpendicular to $\mathcal{P}$. Therefore,

$$\mathcal{L} : x = A + tv = (7 + t, 7 - 3t, -4 + 2t)$$

where $t \in \mathbb{R}$ is a parametric form of $\mathcal{L}$.

(b) By Pythagoras theorem, B is the intersection of $\mathcal{P}$ and $\mathcal{L}$. Let $B = (7+t, 7-3t, -4+2t)$. Plugging this into the equation of $\mathcal{P}$ gives $(7+t) - 3(7-3t) + 2(-4+2t) = 6$, and hence $t = 2$. Therefore, $B = (9, 1, 0)$.

2. (a) $\begin{cases} x(\theta) & = r(\theta)\cos\theta = \sin(\theta + \pi/4)\cos\theta \\ y(\theta) & = r(\theta)\sin\theta = \sin(\theta + \pi/4)\sin\theta \end{cases}$

(b) Recall that the arclength is $\int_0^{\pi/2} \|\mathbf{x}'(\theta)\| d\theta = \int_0^{\pi/2} \sqrt{(x'(\theta))^2 + (y'(\theta))^2} d\theta$.
By product rule,

$$\begin{cases} x'(\theta) & = \cos(\theta + \pi/4)\cos\theta - \sin(\theta + \pi/4)\sin\theta \\ y'(\theta) & = \cos(\theta + \pi/4)\sin\theta + \sin(\theta + \pi/4)\cos\theta \end{cases}$$

By the formula for $(a \pm b)^2$,

$$\begin{cases} (x'(\theta))^2 = & \cos^2(\theta + \pi/4)\cos^2\theta \\ & \quad - 2\cos(\theta + \pi/4)\cos\theta\sin(\theta + \pi/4)\sin\theta \\ & \quad + \sin^2(\theta + \pi/4)\sin^2\theta \\ (y'(\theta))^2 = & \cos^2(\theta + \pi/4)\sin^2\theta \\ & \quad + 2\cos(\theta + \pi/4)\sin\theta\sin(\theta + \pi/4)\cos\theta \\ & \quad + \sin^2(\theta + \pi/4)\cos^2\theta \end{cases}$$

Note that the cross terms of x'^2 and y'^2 are off by a sign (one is minus, and one is plus), so they cancel upon summing. Then

$$\begin{aligned} & (x'(\theta))^2 + (y'(\theta))^2 \\ & = \cos^2(\theta + \pi/4)\cos^2\theta + \sin^2(\theta + \pi/4)\sin^2\theta \\ & \quad + \cos^2(\theta + \pi/4)\sin^2\theta + \sin^2(\theta + \pi/4)\cos^2\theta \\ & = \cos^2(\theta + \pi/4)(\cos^2\theta + \sin^2\theta) + \sin^2(\theta + \pi/4)(\cos^2\theta + \sin^2\theta) \\ & = \cos^2(\theta + \pi/4) + \sin^2(\theta + \pi/4) \\ & = 1 \end{aligned}$$

Therefore,

$$\text{arclength} = \int_0^{\pi/2} \sqrt{1} d\theta = \pi/2$$

Remark. The formula $s = \int \sqrt{1 + (f'(x))^2} dx$ is only valid for the graph of $y = f(x)$, because the parametrization $(x, y) = (t, f(t))$ gives $\|\mathbf{x}'\| = 1 + f'^2$.

Remark. A number of students observed that the curve is (part of) a circle, as can be seen from

$$x^2 + y^2 = r^2 = r\sin(\theta + \pi/4) = \frac{1}{\sqrt{2}} r(\sin\theta + \cos\theta) = \frac{1}{\sqrt{2}}(x + y)$$

With the observation that (i) the starting and ending points of the curve are an diameter apart, (ii) the curve does not wind around the circle more than once, and (iii) the curve does not backtrack ((ii) and (iii) follows from the injectivity of the curve), it follows directly that the arclength is the radius time π. However, this approach cannot give expressions of x and y in terms of θ.

3. (a) a sphere of radius $\sqrt{c}$
   (b) Note that $f(\mathbf{x}(t)) = c$. Differentiating both sides gives $2(xx' + yy' + zz') = 0$. Observe that LHS is $2\mathbf{x} \cdot \mathbf{x}'$. The vanishing of the dot product implies $\mathbf{x}$ and its tangent $\mathbf{x}'$ are perpendicular.

4. (a) Consider the path $(2+t, 1+mt)$, which tends to $(2, 1)$ as $t \to 0$. On this path, $\frac{x-y-1}{x-2y} = \frac{t-mt}{t-2mt} = \frac{1-m}{1-2m}$. Therefore, the limit along this path is $\frac{1-m}{1-2m}$, which changes as m varies. Therefore, the limit does not exist by path test.
   (b) Since the cosine is always bounded between -1 and 1, $-|x| \leq x\cos\ln(x^2+y^2) \leq |x|$. Since $\lim_{(x,y)\to(0,0)} \pm|x| = 0$, by squeeze theorem, $\lim_{(x,y)\to(0,0)} x\cos\ln(x^2 + y^2) = 0$.

   Remark. It is NOT true that $-x \leq x\cos\ln(x^2 + y^2) \leq x$, because x and the cosine may be negative.

5. (a) Letting $\mathbf{a} = \mathbf{x}-\mathbf{y}$ and $\mathbf{b} = \mathbf{y}$, $\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\|+\|\mathbf{b}\|$ implies $\|\mathbf{x}\| \leq \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}\|$, hence

   $$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

   By the symmetery of $\mathbf{x}$ and $\mathbf{y}$, this also shows

   $$-(\|\mathbf{x}\| - \|\mathbf{y}\|) = \|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| = \|\mathbf{x} - \mathbf{y}\|$$

   Since $\|\mathbf{x}-\mathbf{y}\|$ bounds both $\|\mathbf{x}\| - \|\mathbf{y}\|$ and its negative, it bounds it in absolute value, i.e.

   $$\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \leq \|\mathbf{x} - \mathbf{y}\|$$

   (b) Fix $\varepsilon > 0$. Let $\delta = \varepsilon$. If $0 < \|\mathbf{x}-\mathbf{a}\| < \delta$, then $\big|\|\mathbf{x}\| - \|\mathbf{a}\|\big| \leq \|\mathbf{x}-\mathbf{a}\| < \delta = \varepsilon$. Therefore, $\lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{x}\| = \|\mathbf{a}\|$.