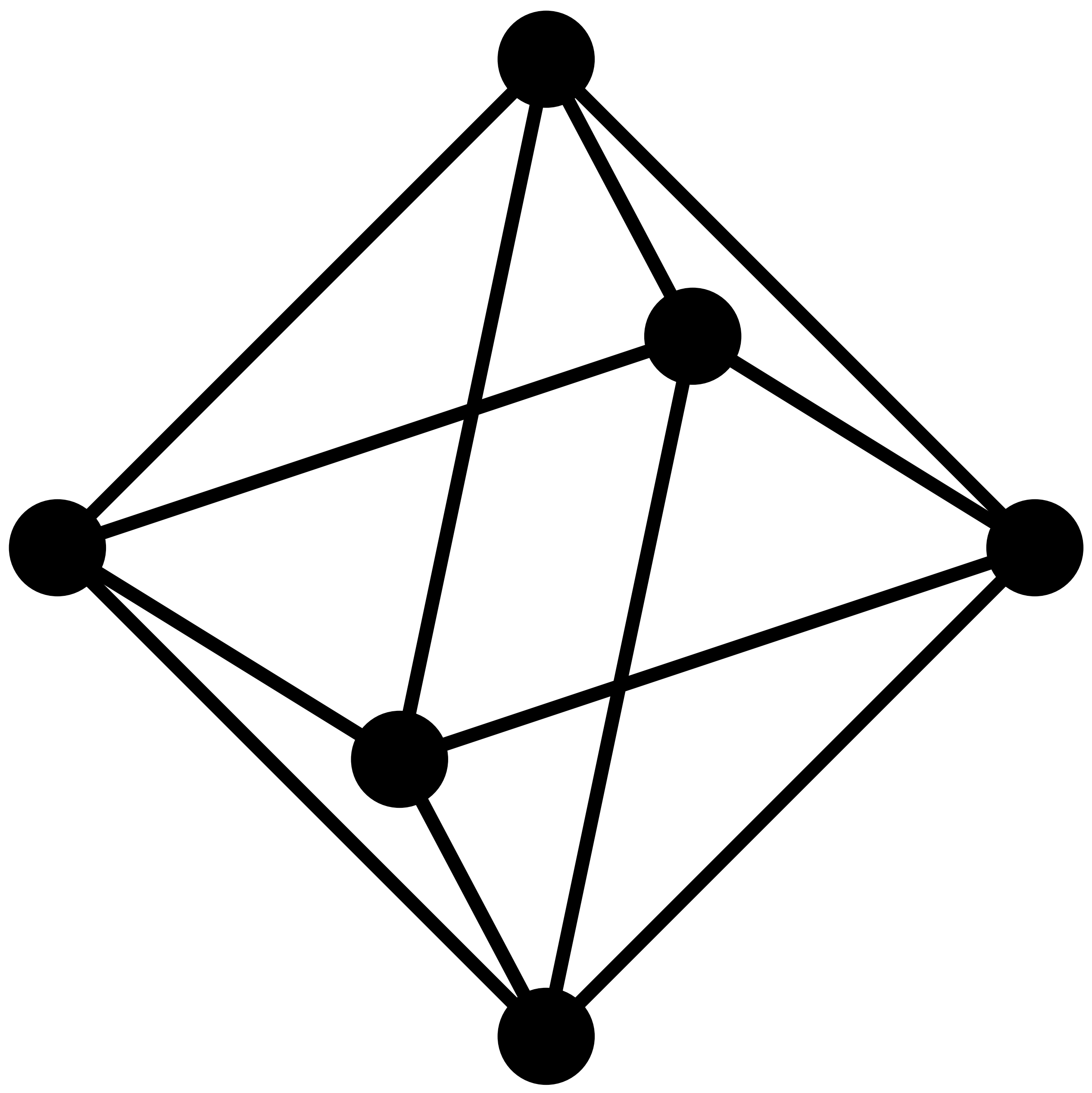

Random Topology

Higher-Order Connectivity from Preferential Attachment

joint with Gennady Samorodnitsky, Christina Lee Yu, Rangy He, Avhan Misra

Chunyin Siu
Stanford University
siuc@stanford.edu

About Me…

About Me…

Random Topology

Topological Data Analysis

About Me…

Random Topology

Topological Data Analysis

Neuroimaging Analysis

Random Topology

Random Topology

Topological Data Analysis

Neuroimaging Analysis

Random Topology

Percolation

Random Topology

Topological Data Analysis

Neuroimaging Analysis

Percolation…

even though this talk is not about percolation

Percolation…

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ThorRagnarok
https://wallhere.com/en/wallpaper/28421

Percolation…

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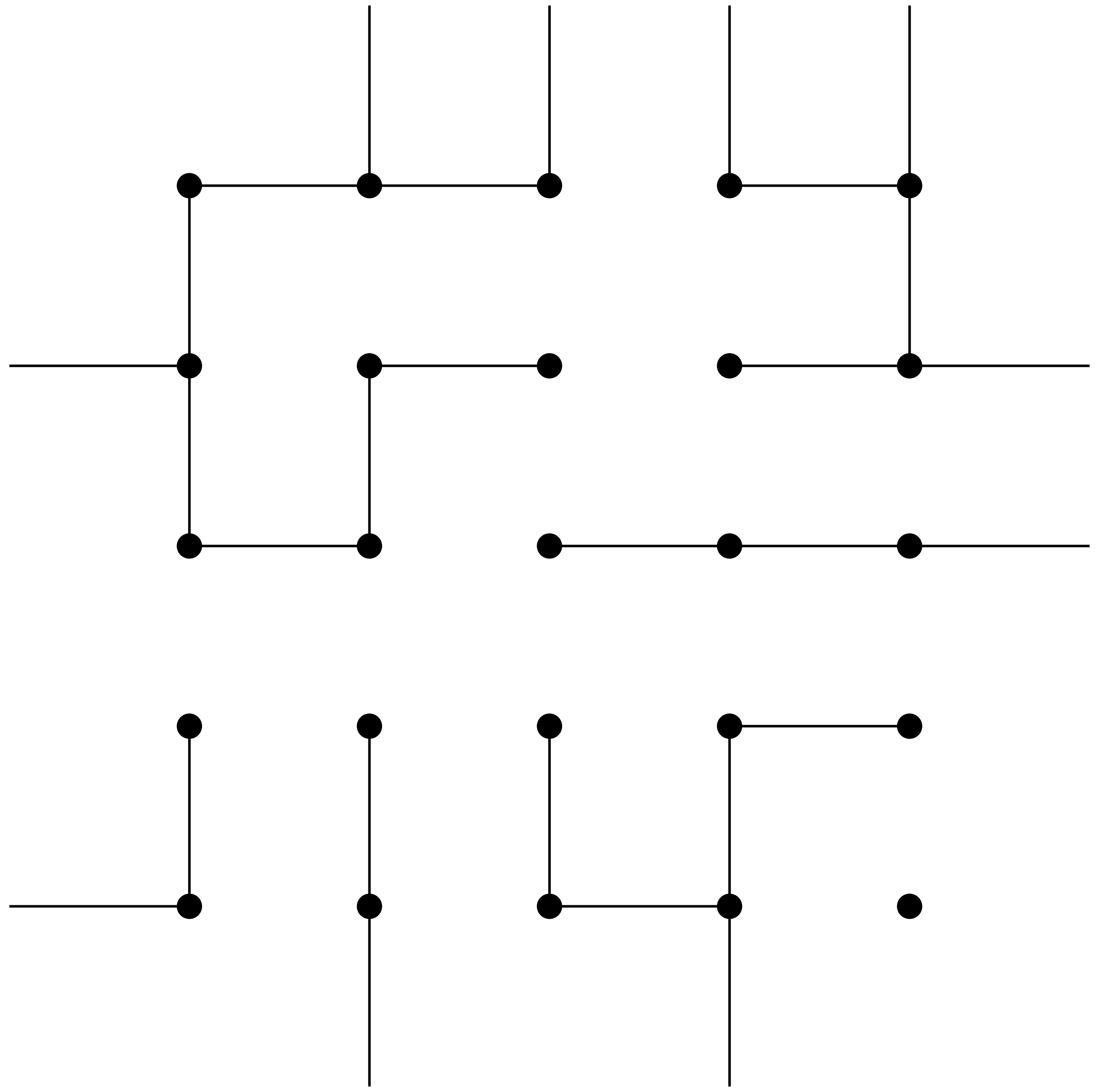
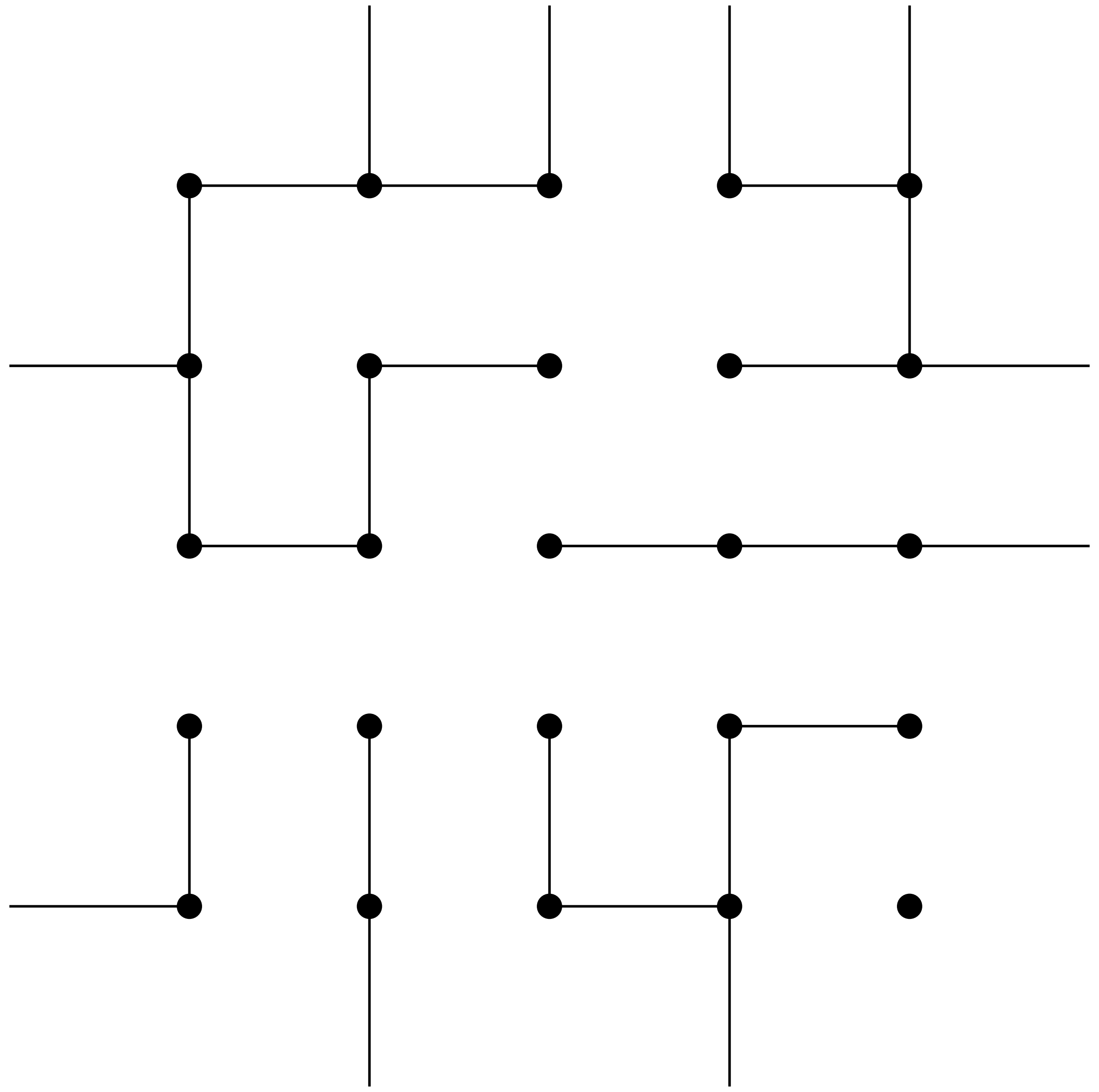


Percolation…

even though this talk is not about percolation

ThorRagnarok
https://wallhere.com/en/wallpaper/28421

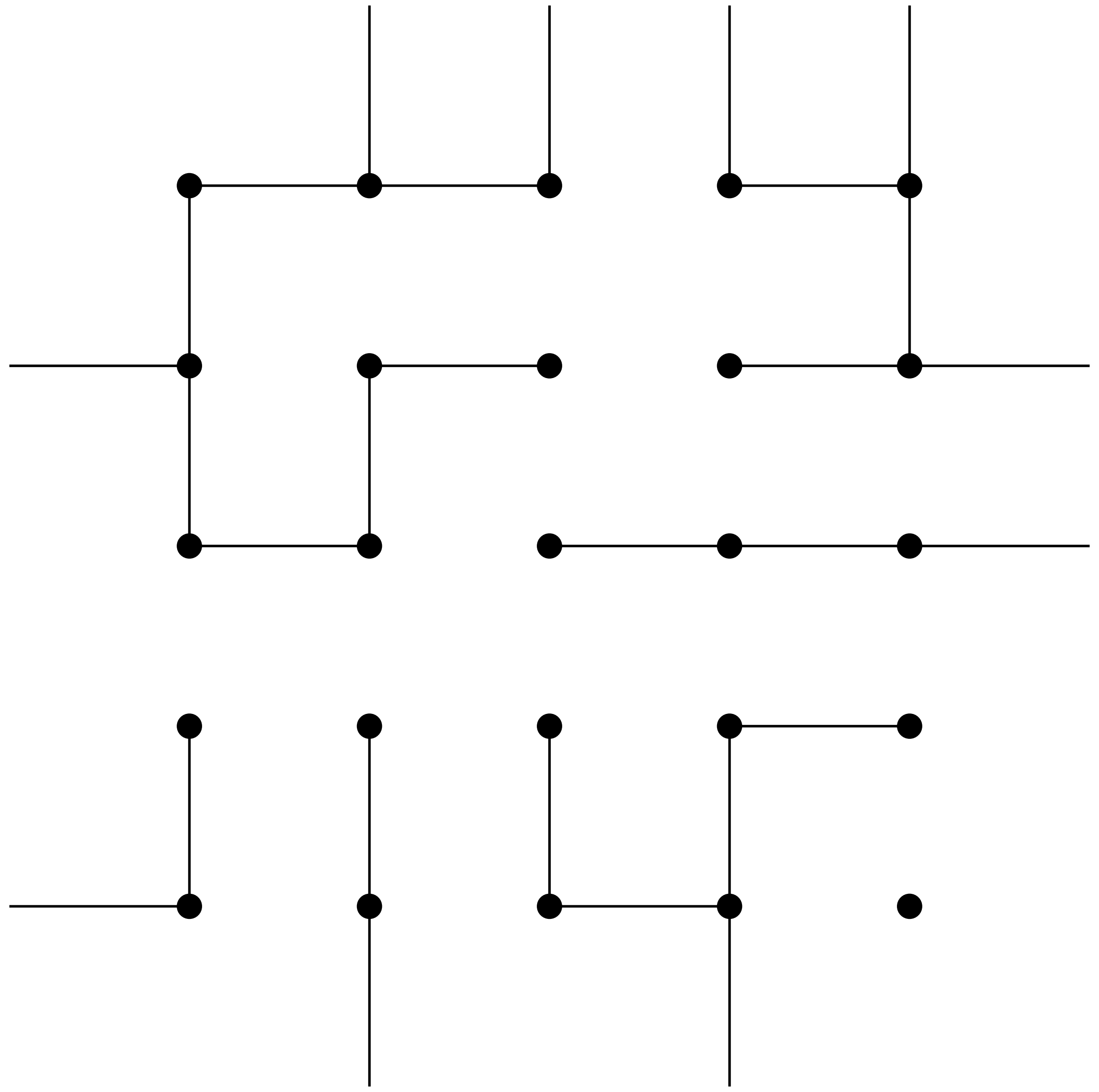


- Connectivity

Percolation…

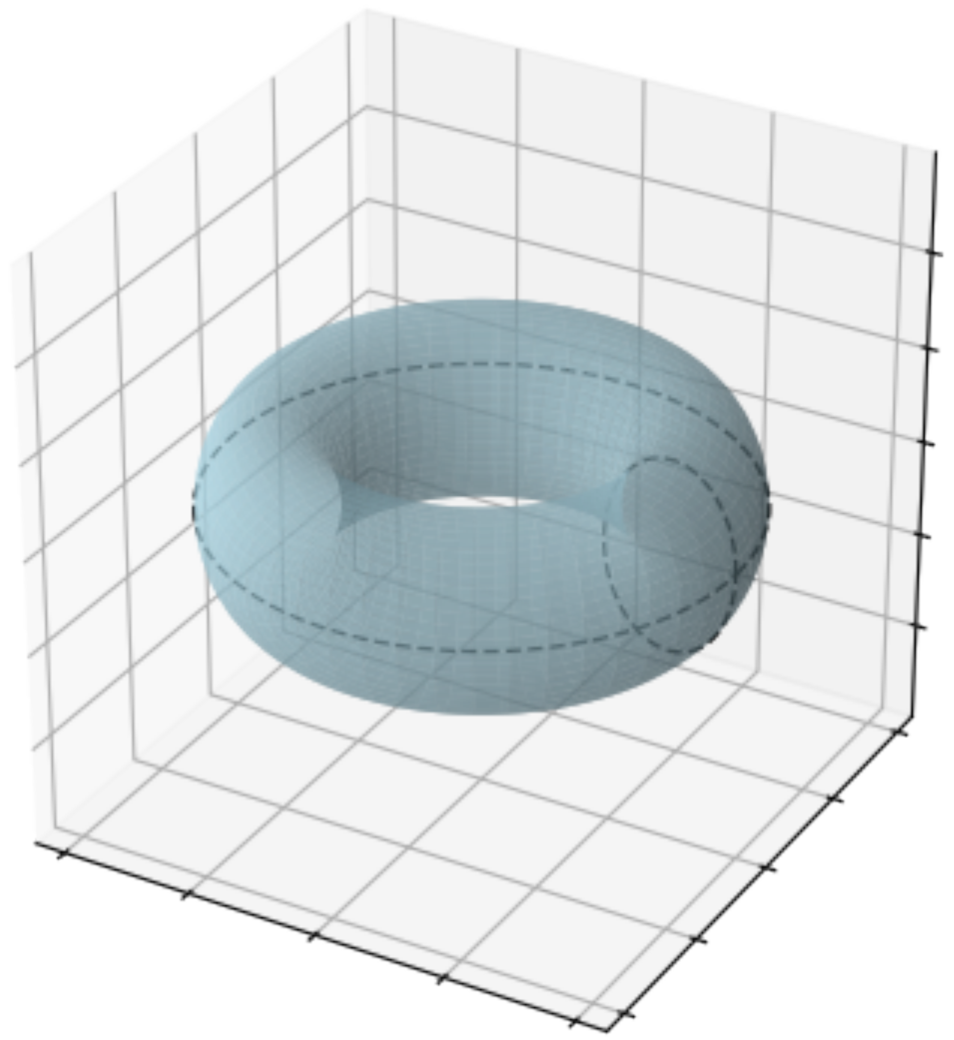
even though this talk is not about percolation

ThorRagnarok
https://wallhere.com/en/wallpaper/28421

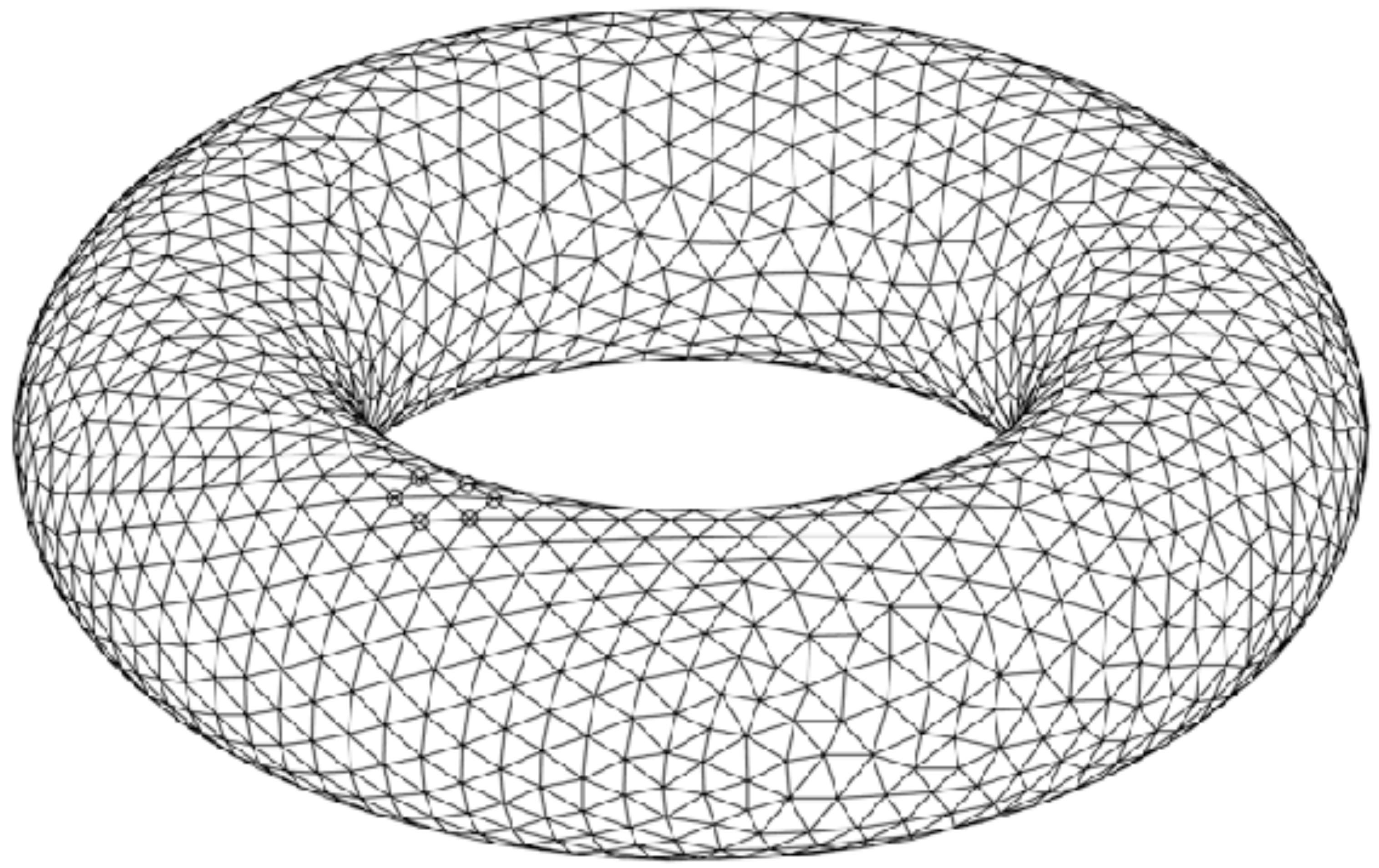


- Connectivity
- random fluctuations vs global structures

Topology



Andrey Yao



Maintenance Script (?)
https://handwiki.org/wiki/File:Torus-triang.png

The Forgotten Path…

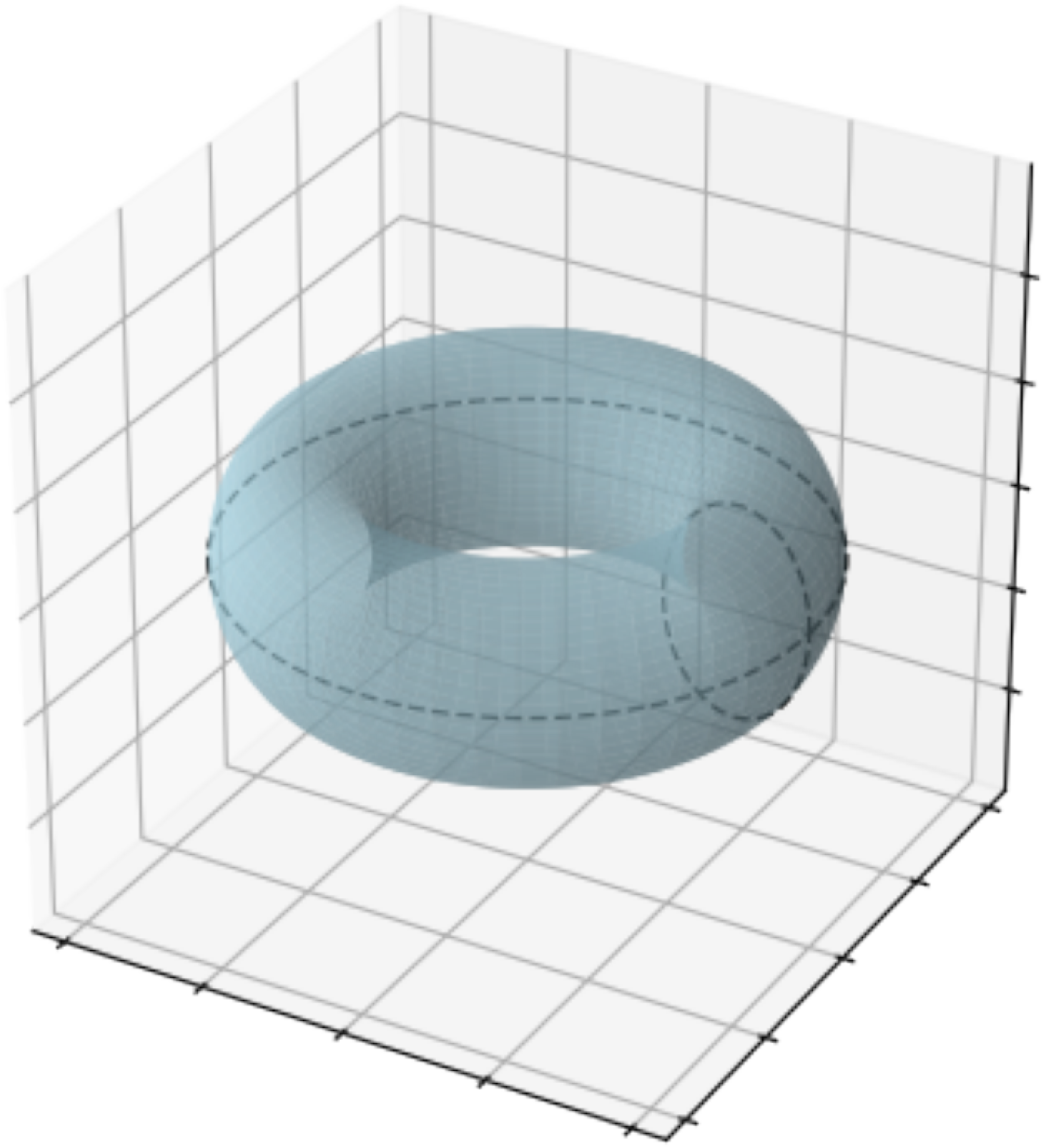
- Frisch and Hammersley (1963) Example 3:
  - Nearly all extant percolation theory deals with regular interconnecting structures, for lack of knowledge of how to define randomly irregular structures. Adventurous readers may care to rectify this deficiency by pioneering branches of mathematics that might be called stochastic geometry or statistical topology.
- Arnold’s 28th problems (1973):
  - Investigate the topology of thickened random points in Euclidean space

Agenda

preferential attachment

Agenda

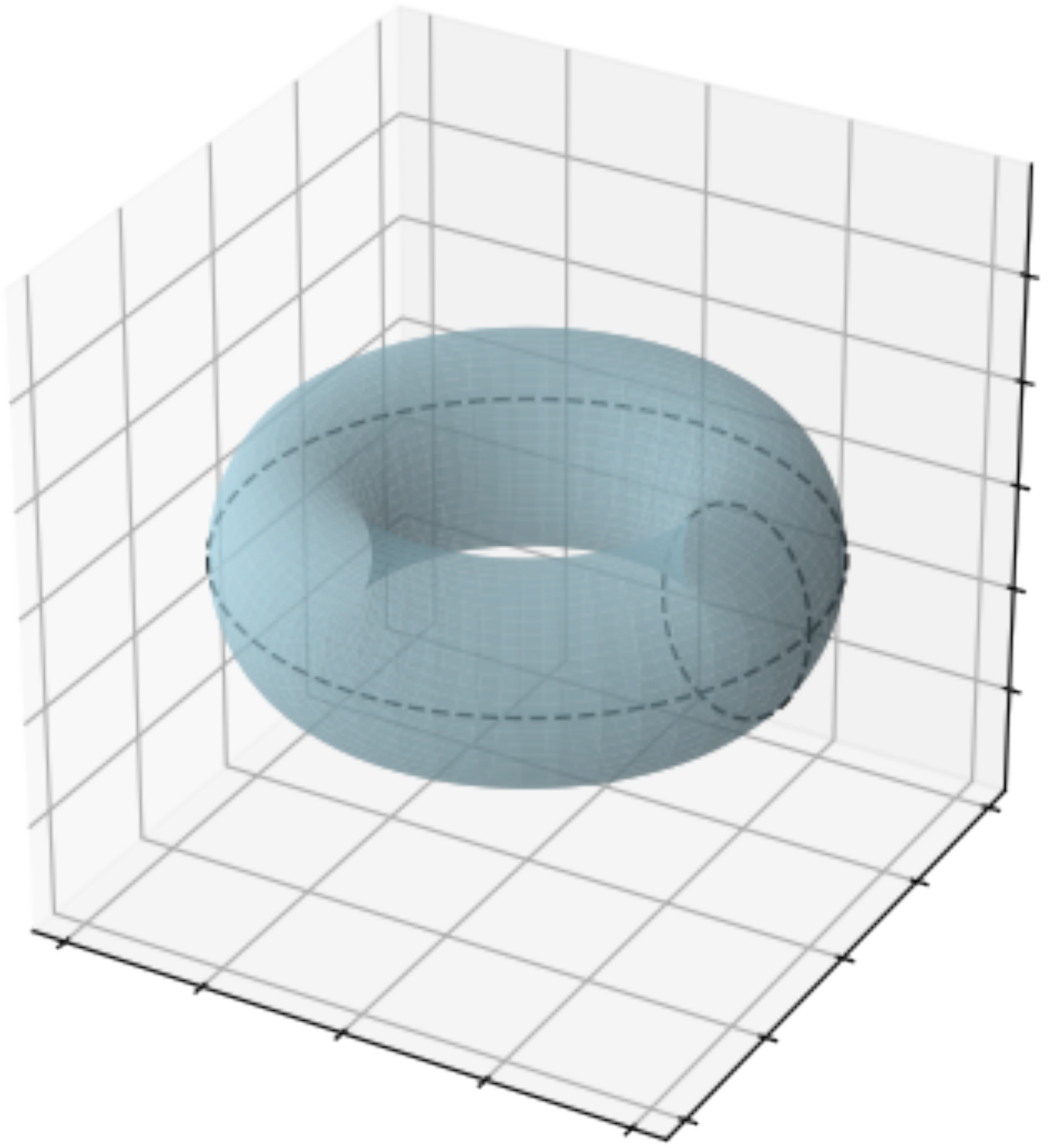
preferential attachment



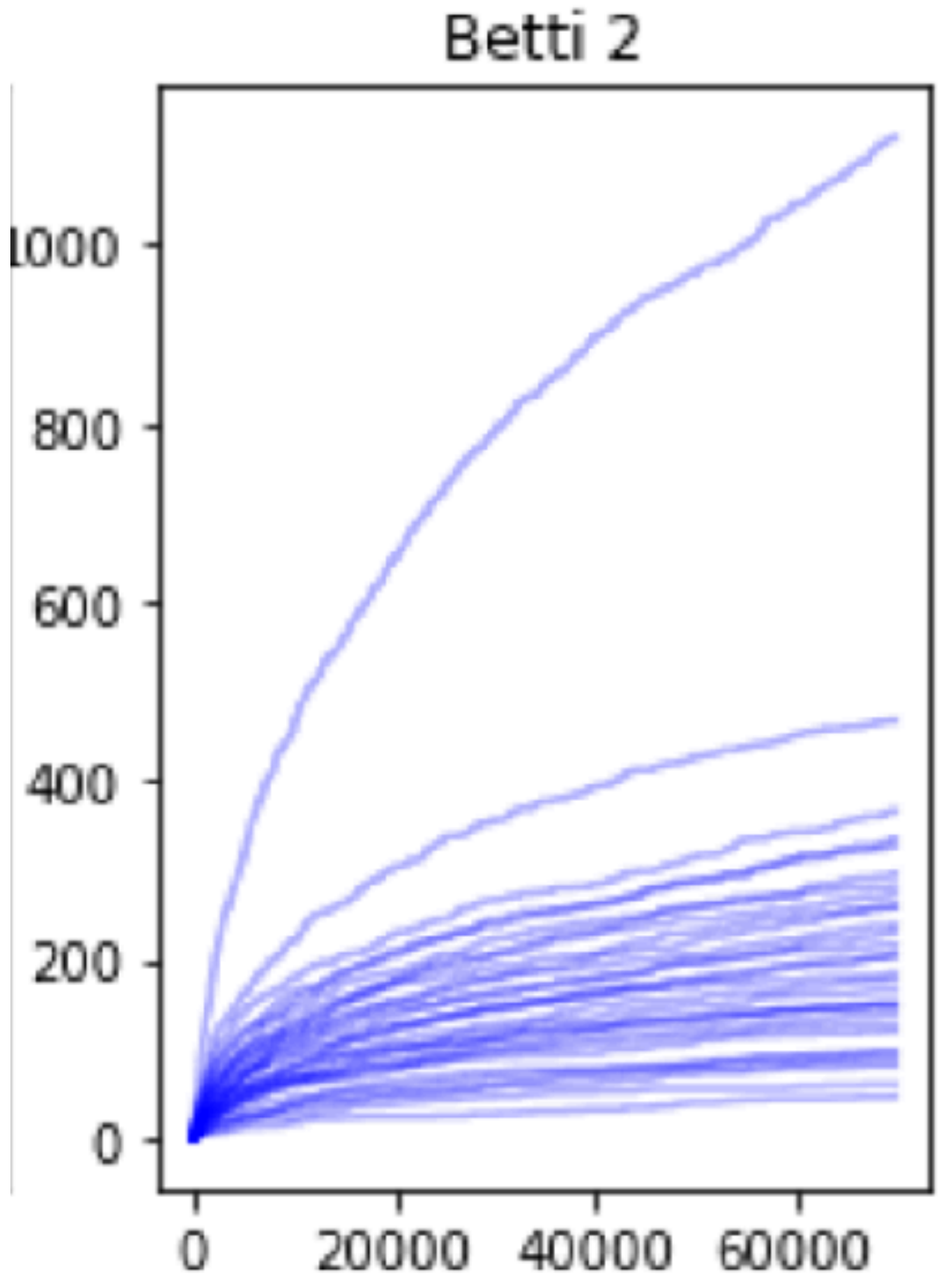
topology

Agenda

preferential attachment



topology



our result

I. Preferential Attachment

.....

So, preferential attachment…

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Highly connected hubs

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Highly connected hubs
- Dense core of hubs?

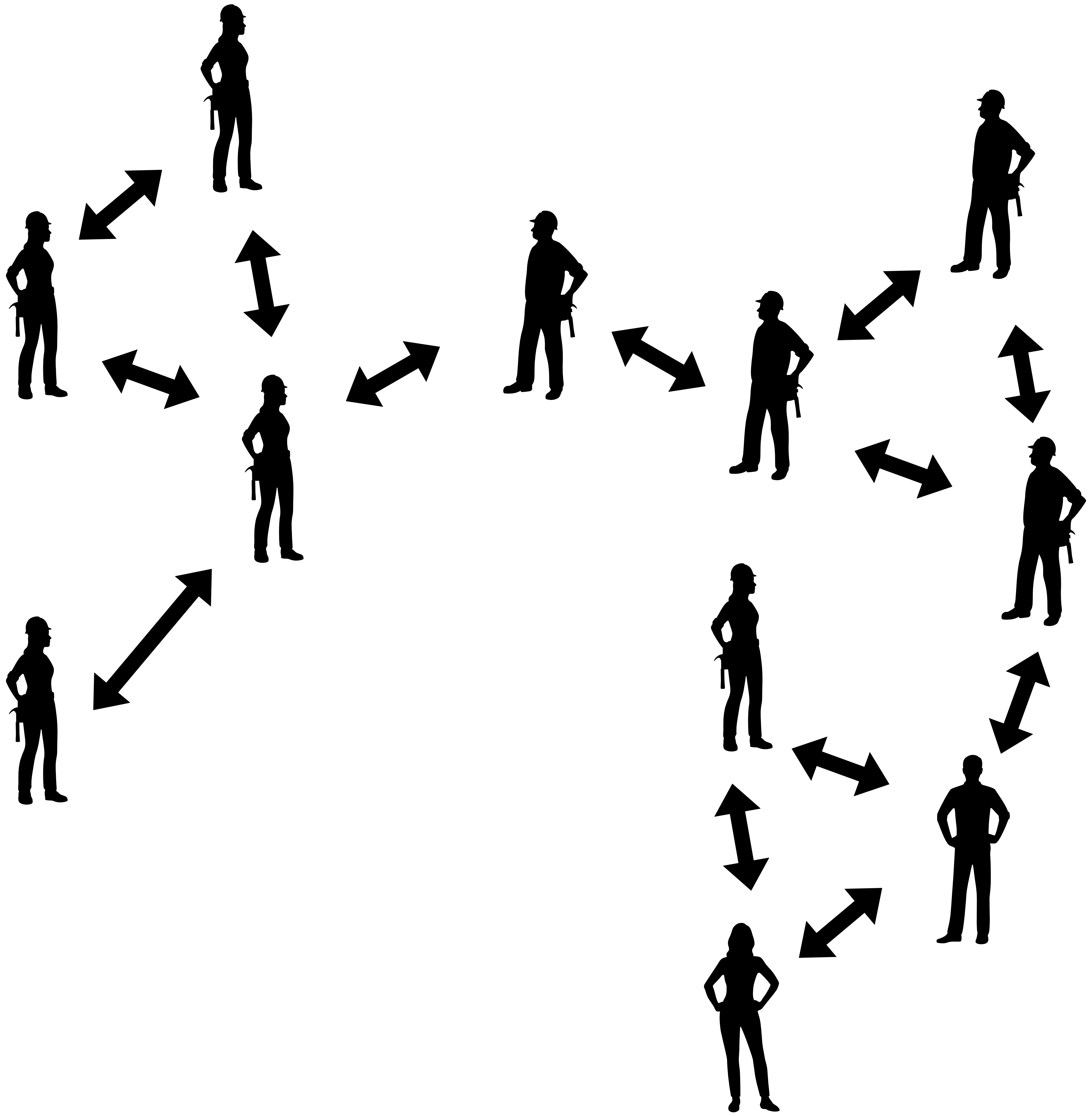
(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Preferential Attachment
[Albert and Barabasi 1999]

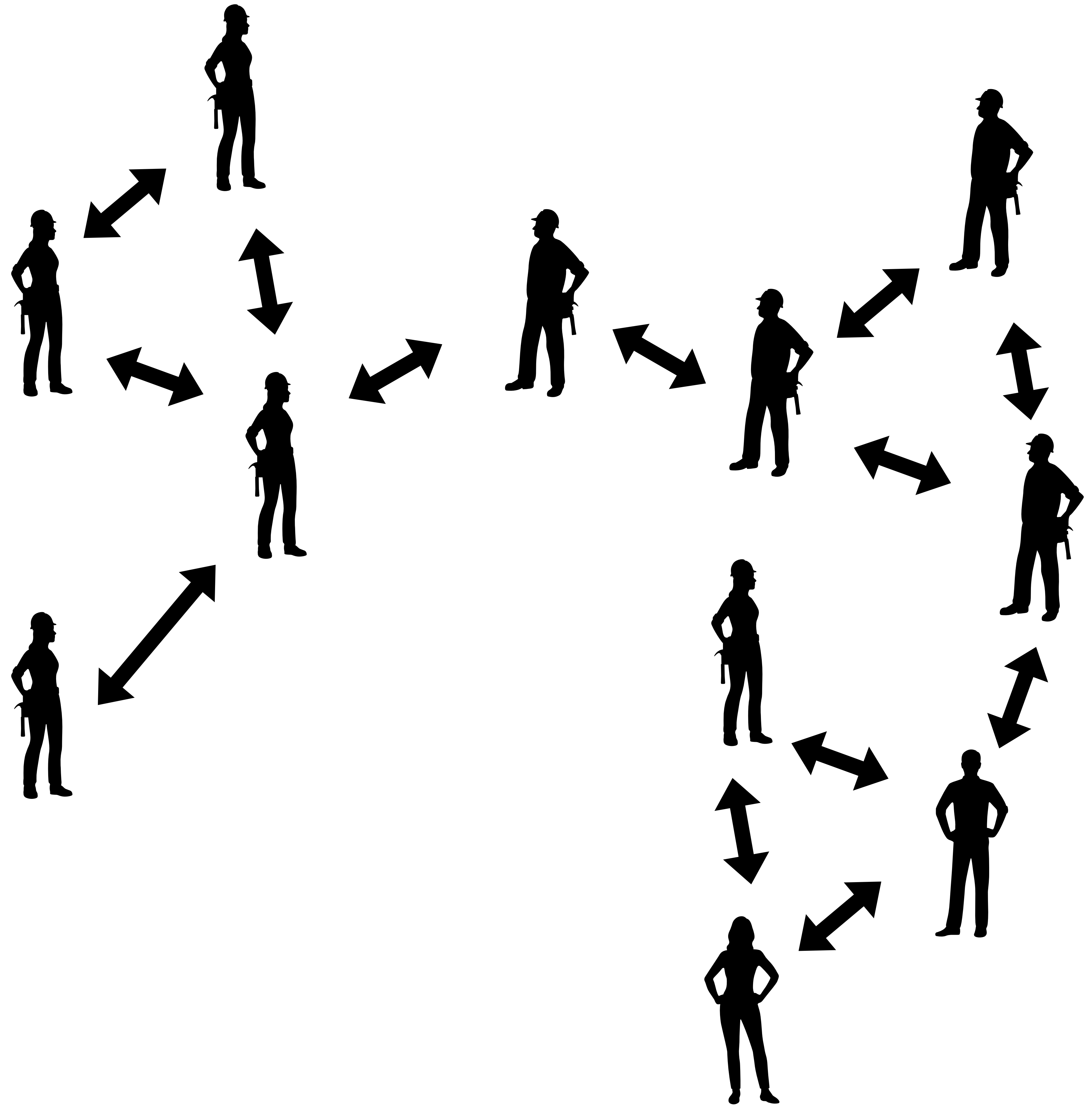
(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Preferential Attachment

[Albert and Barabasi 1999]

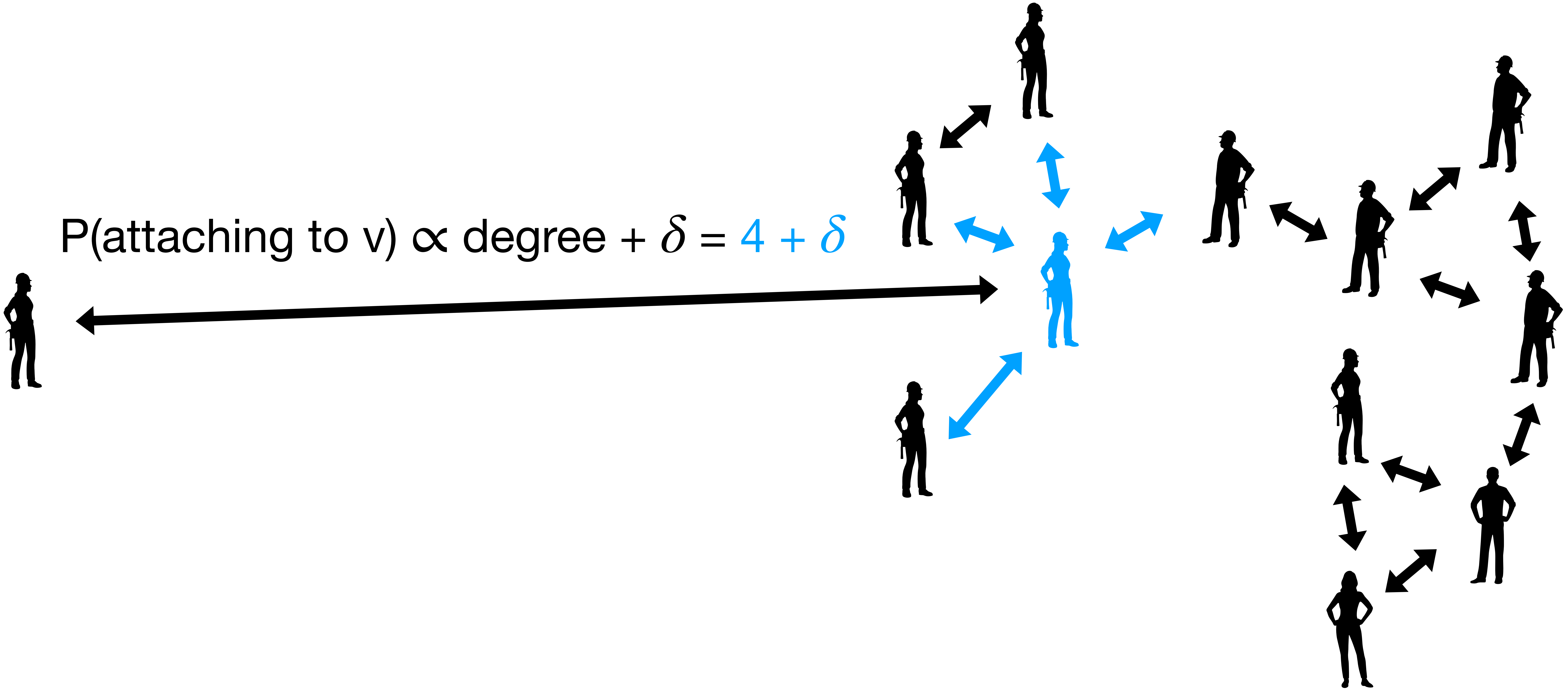


Preferential Attachment
[Albert and Barabasi 1999]



Preferential Attachment

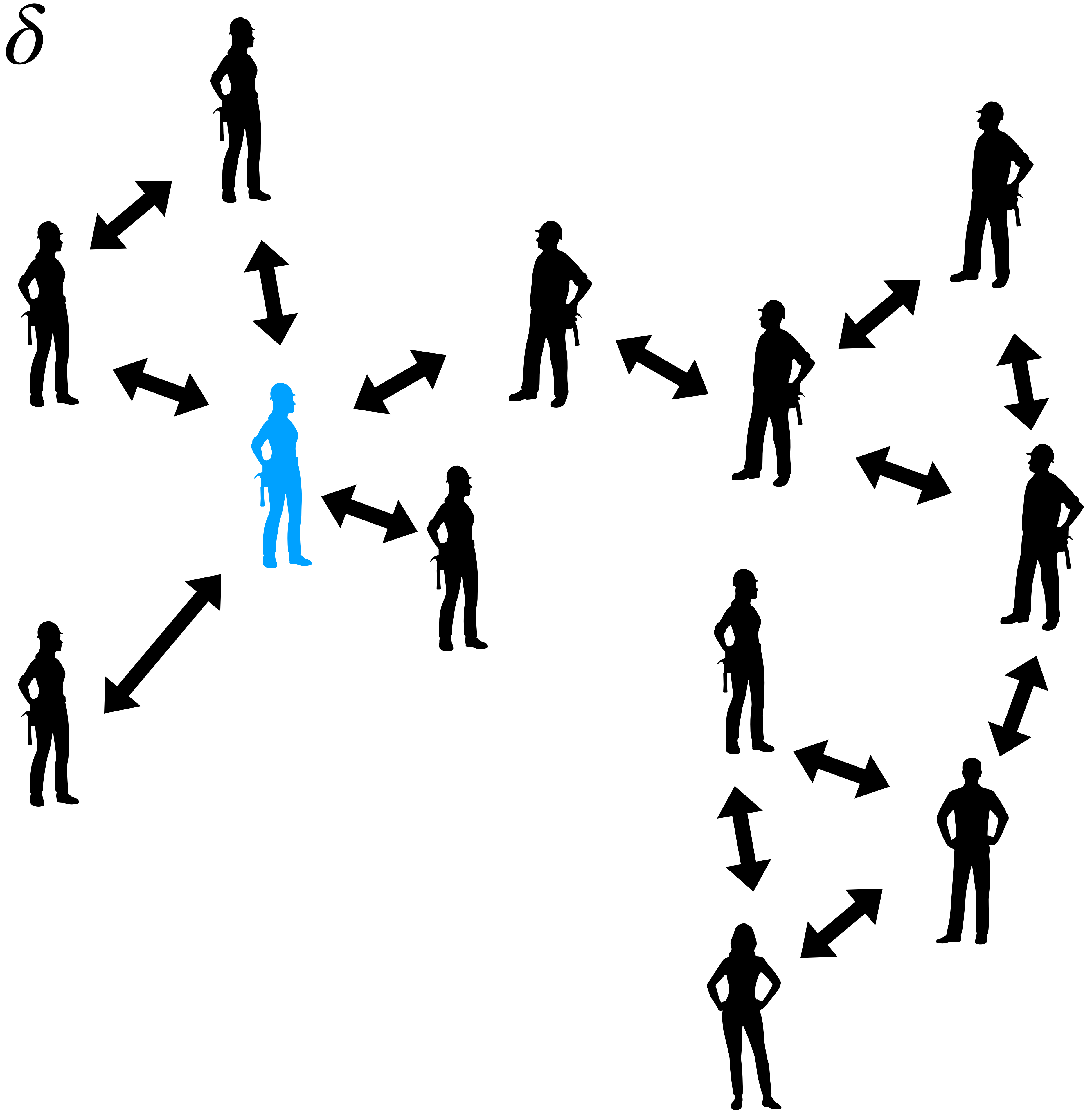
[Albert and Barabasi 1999]



Preferential Attachment

[Albert and Barabasi 1999]

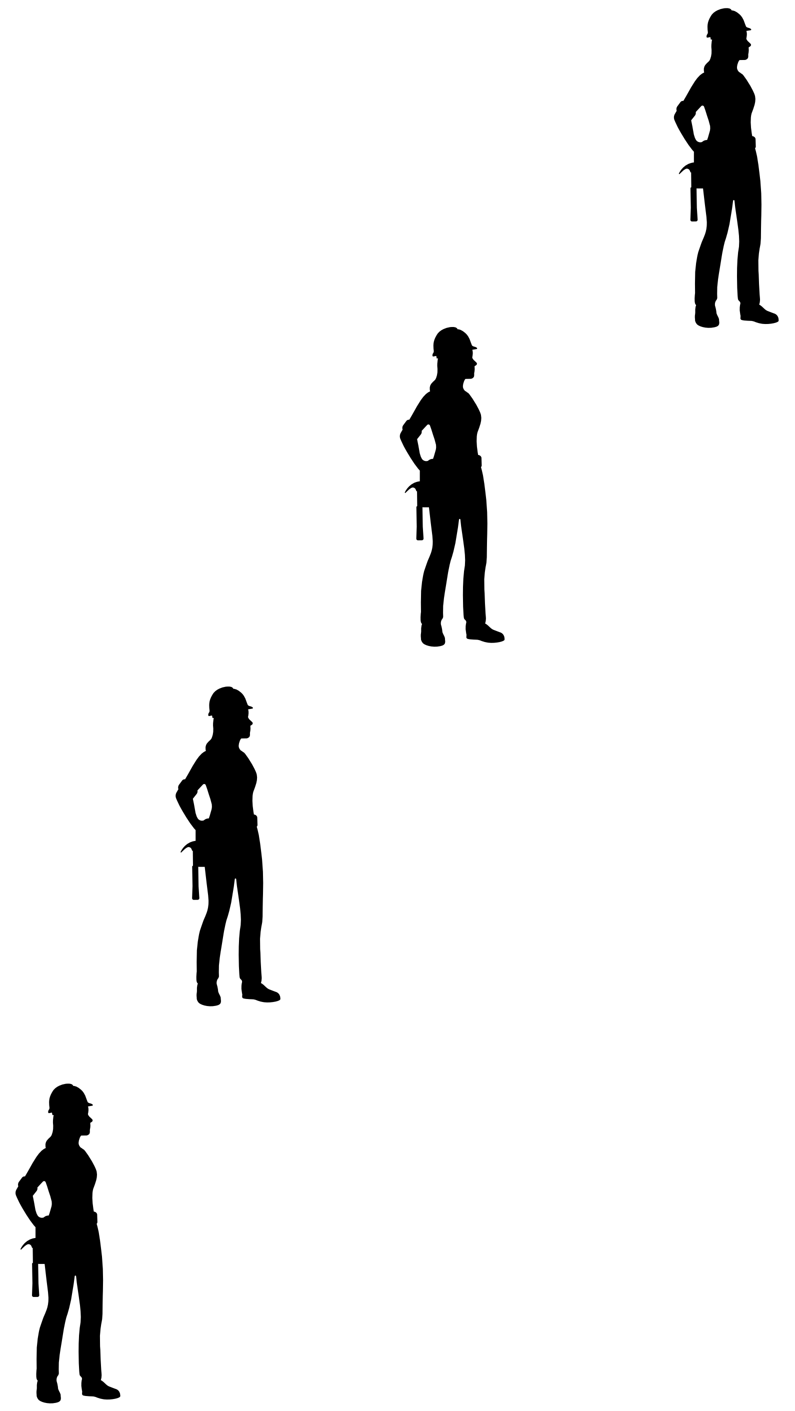
P(attaching to v) $\propto$ degree + a tuning parameter δ

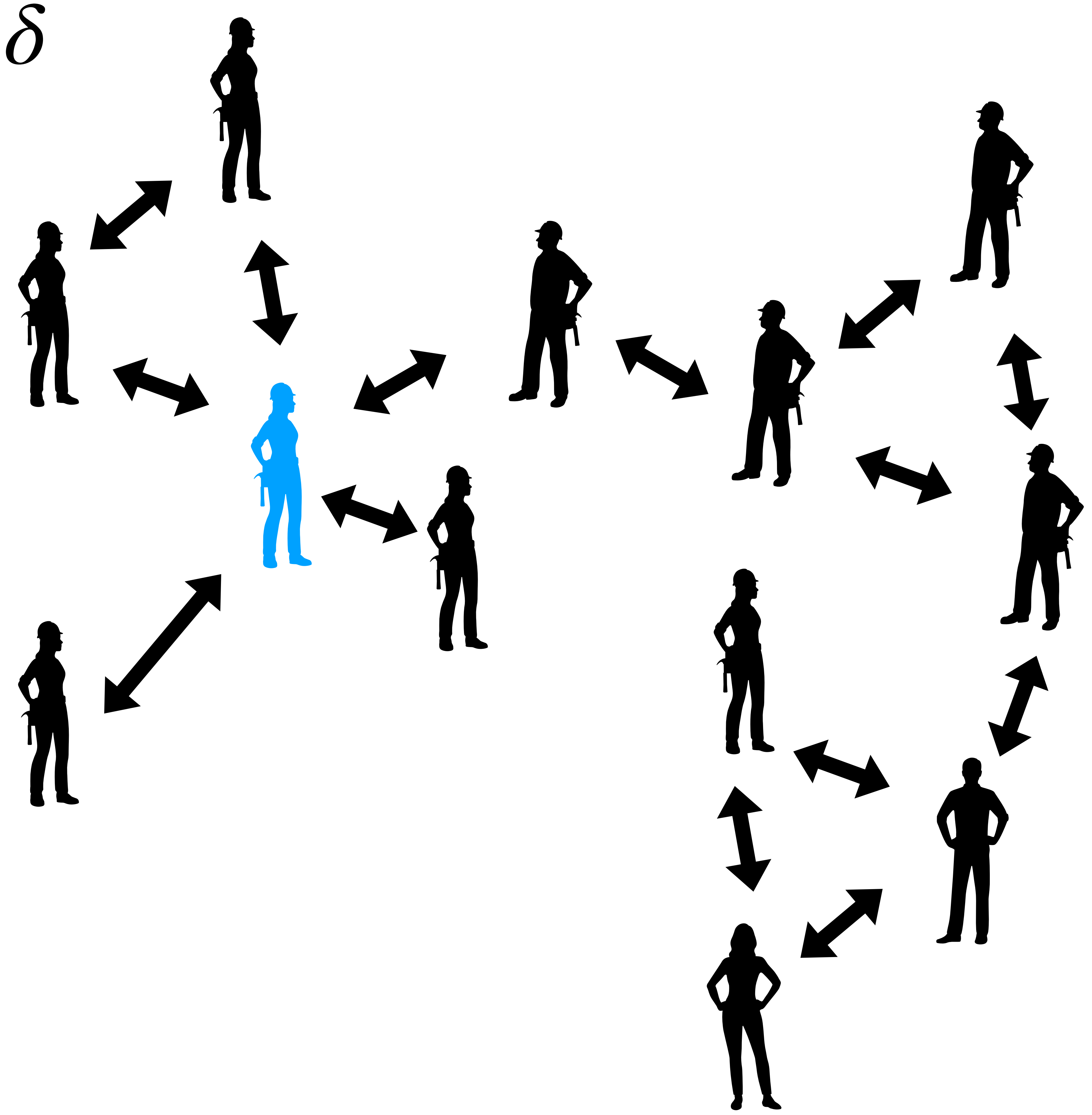


Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) $\propto$ degree + a tuning parameter δ





Preferential Attachment

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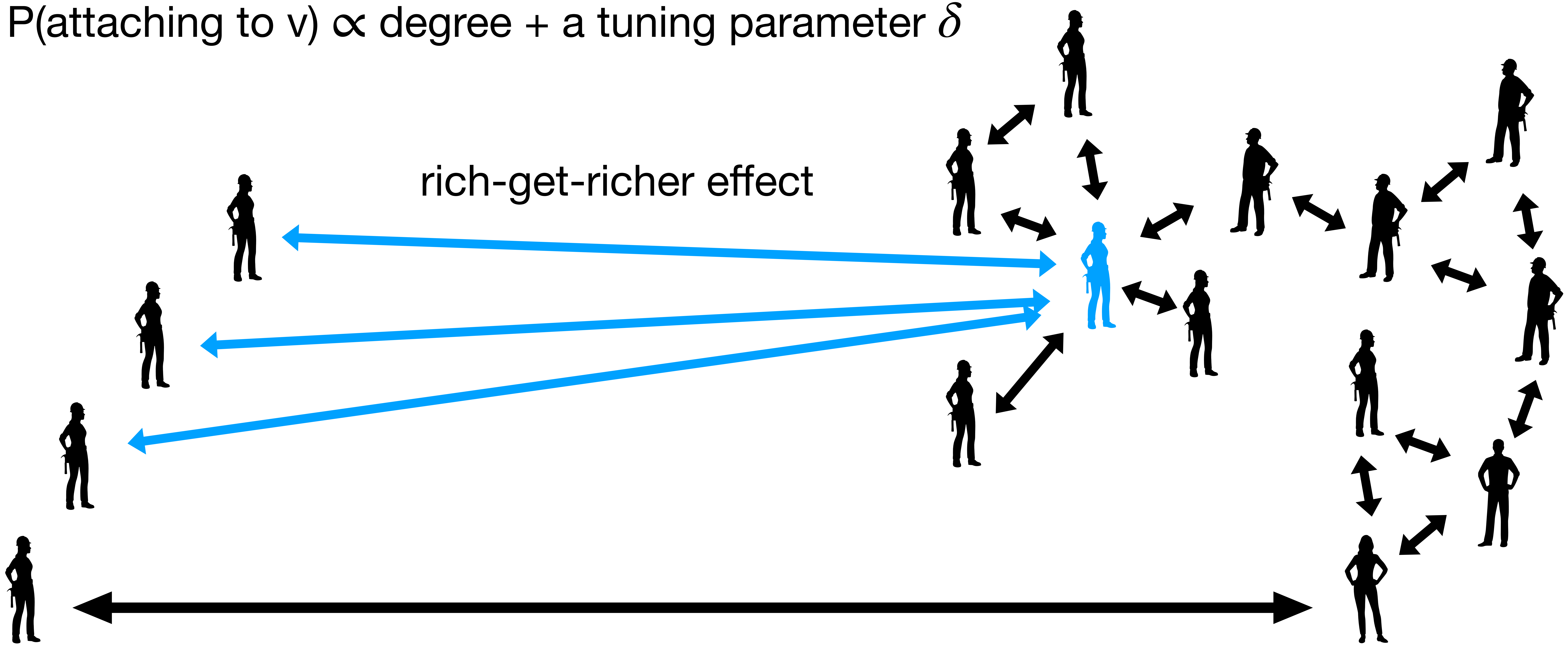


What do we know?

What do we know?

- Scale-freeness and Degree distribution
  [Barabasi and Albert 1999; Dorogovtsev, Mendes and Samukhin 2000; Krapivsky, Redner and Leyvraz 2000]

pmf of degree distribution

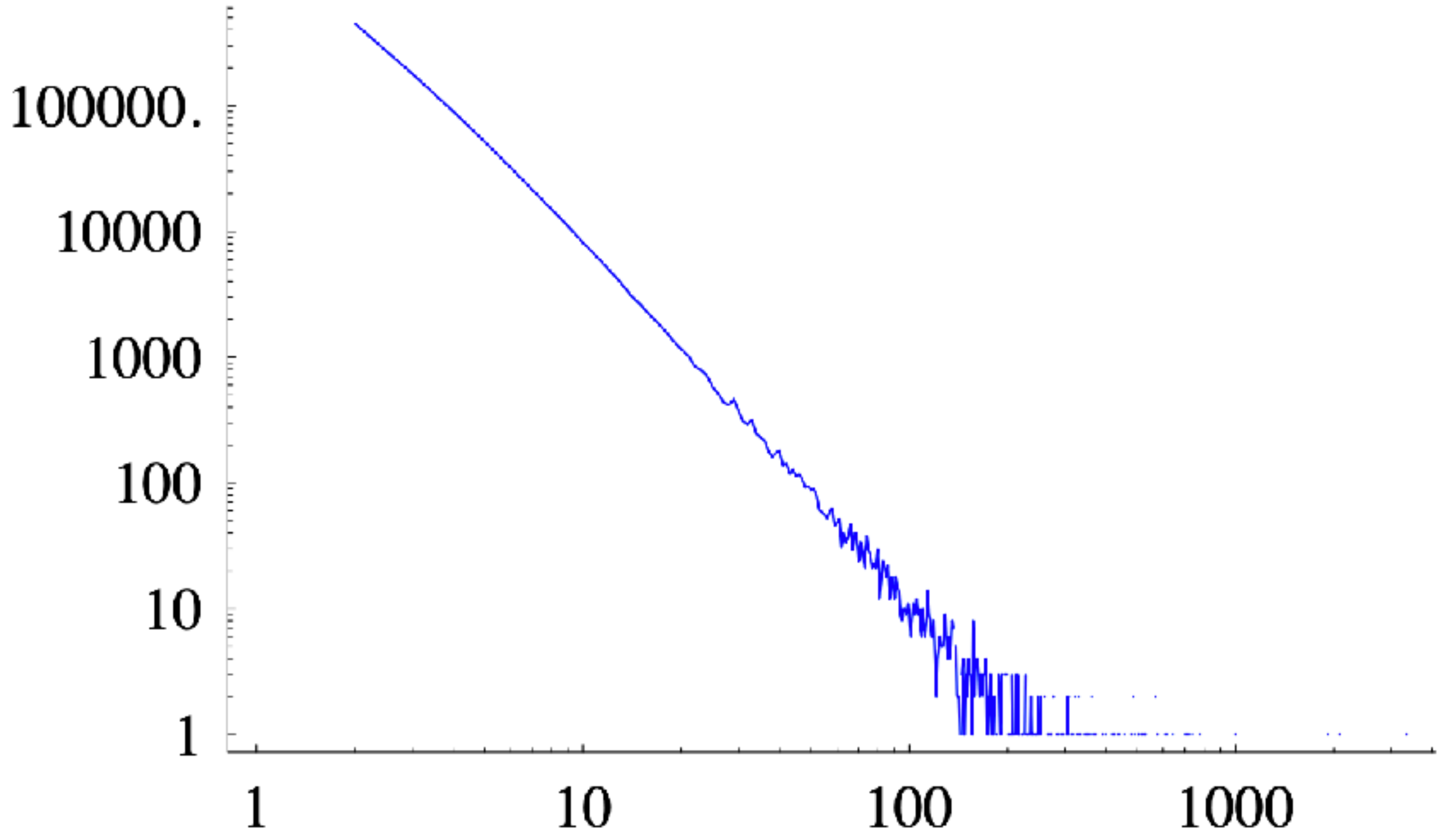


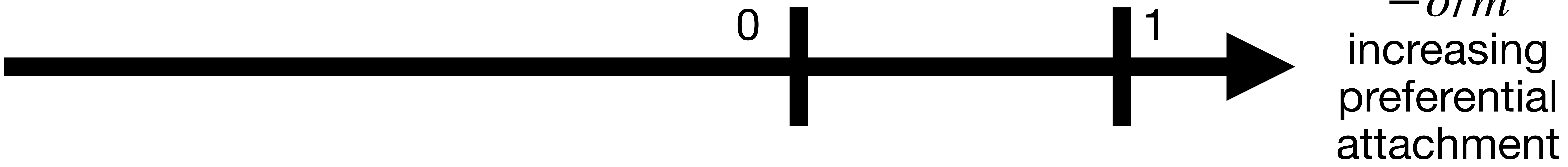
Fig 8.3 of R. Hofstad (2013).
Random Graphs and Complex Networks.
https://doi.org/10.1017/9781316779422

Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



Phase transition

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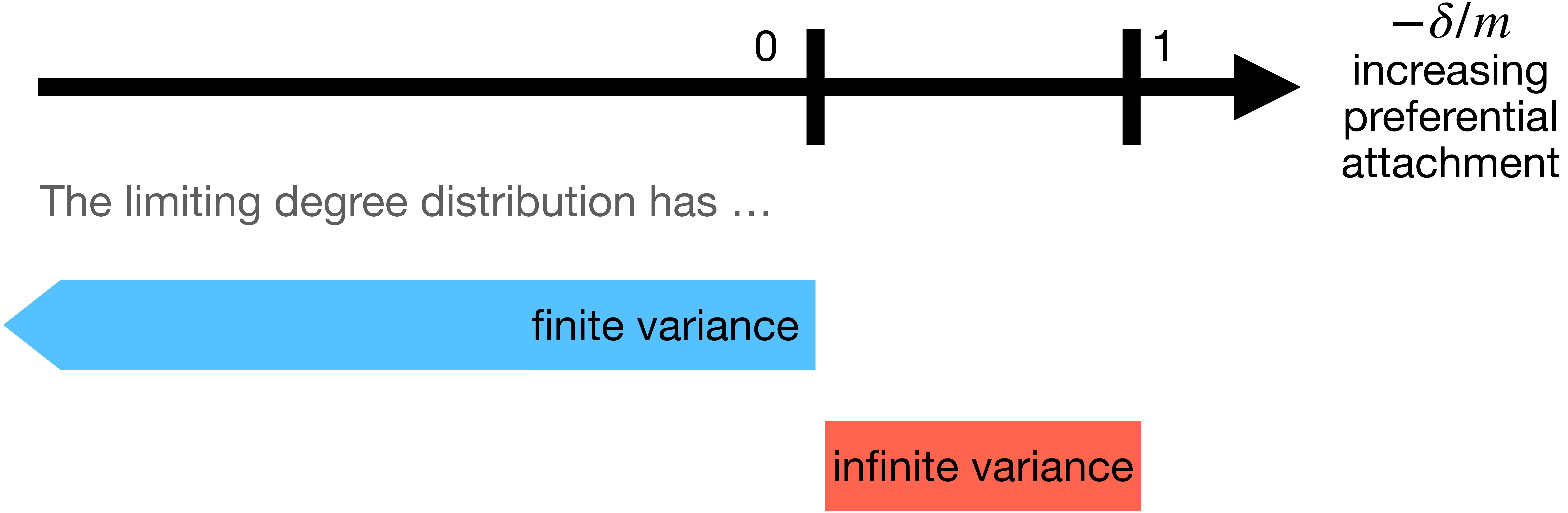
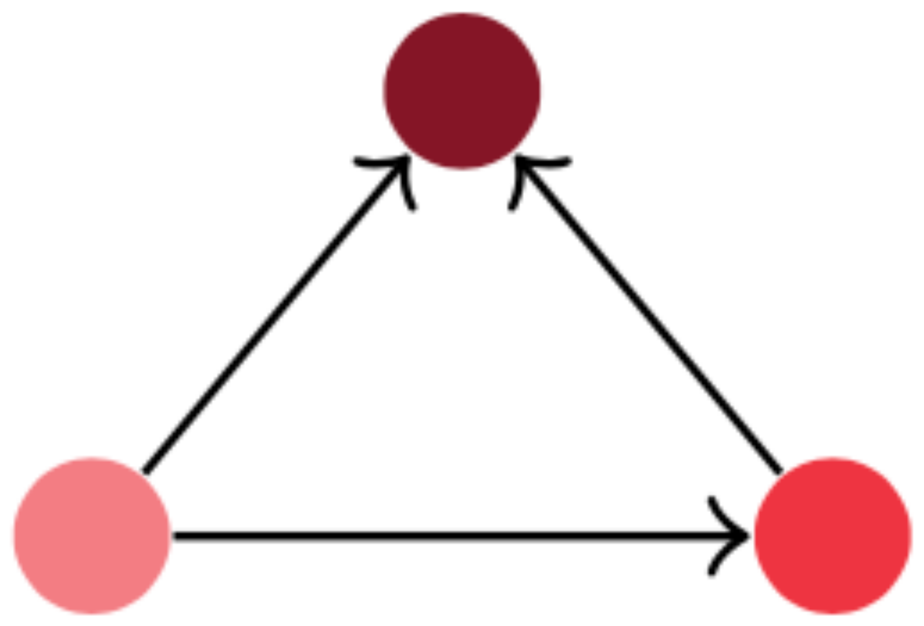


What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013, Garavaglia and Stegehuis 2019]



(a) $t^{(3-\tau)/(\tau-1)}\log(t)$

$\tau = 3 + \delta/m$

Fig 2 of A. Garavaglia and C. Stegehuis (2019). Subgraphs in Preferential Attachment Models. https://doi.org/10.1017/apr.2019.36

What do we know?

- subgraph counts [Garavaglia and Stegehuis 2019]

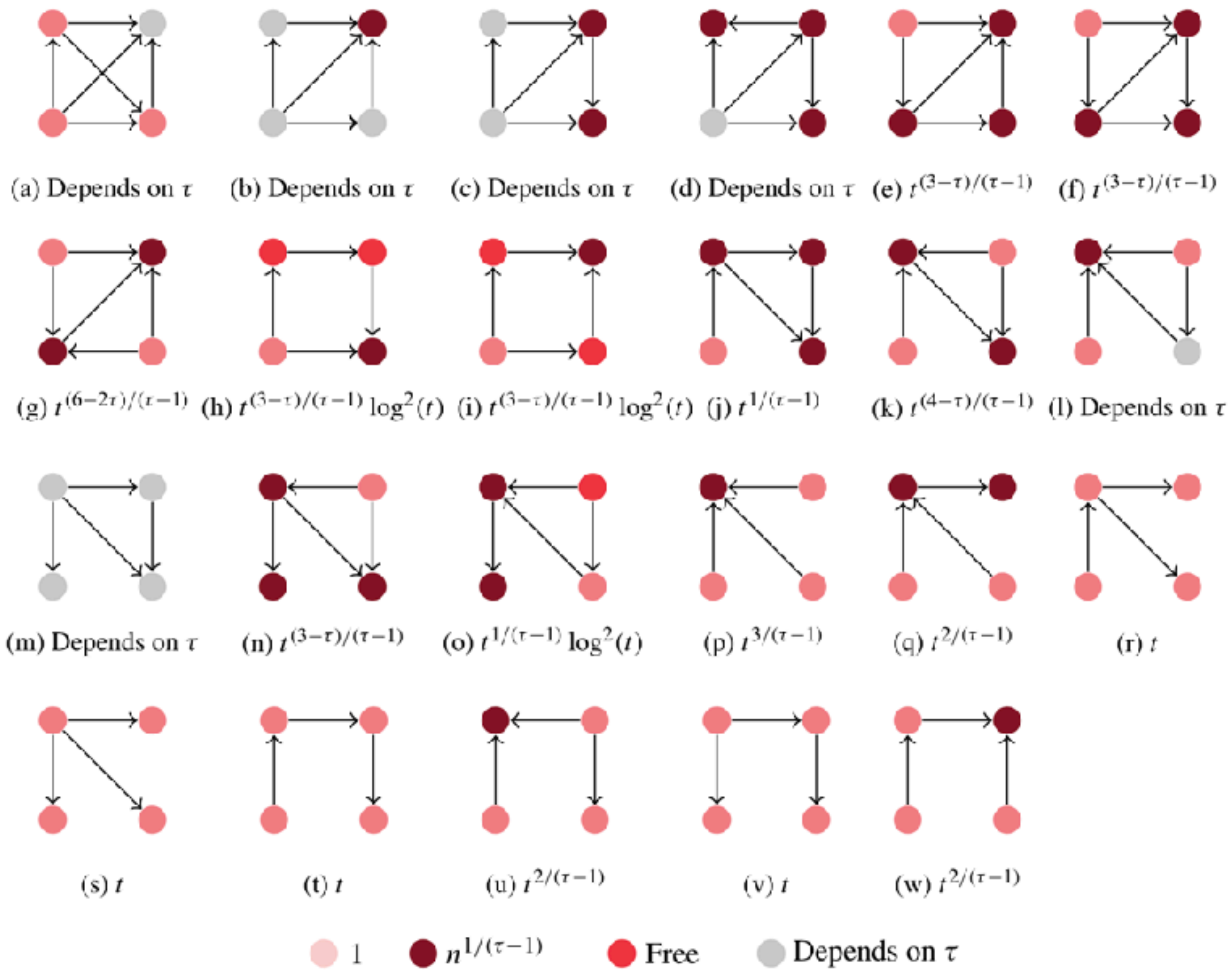
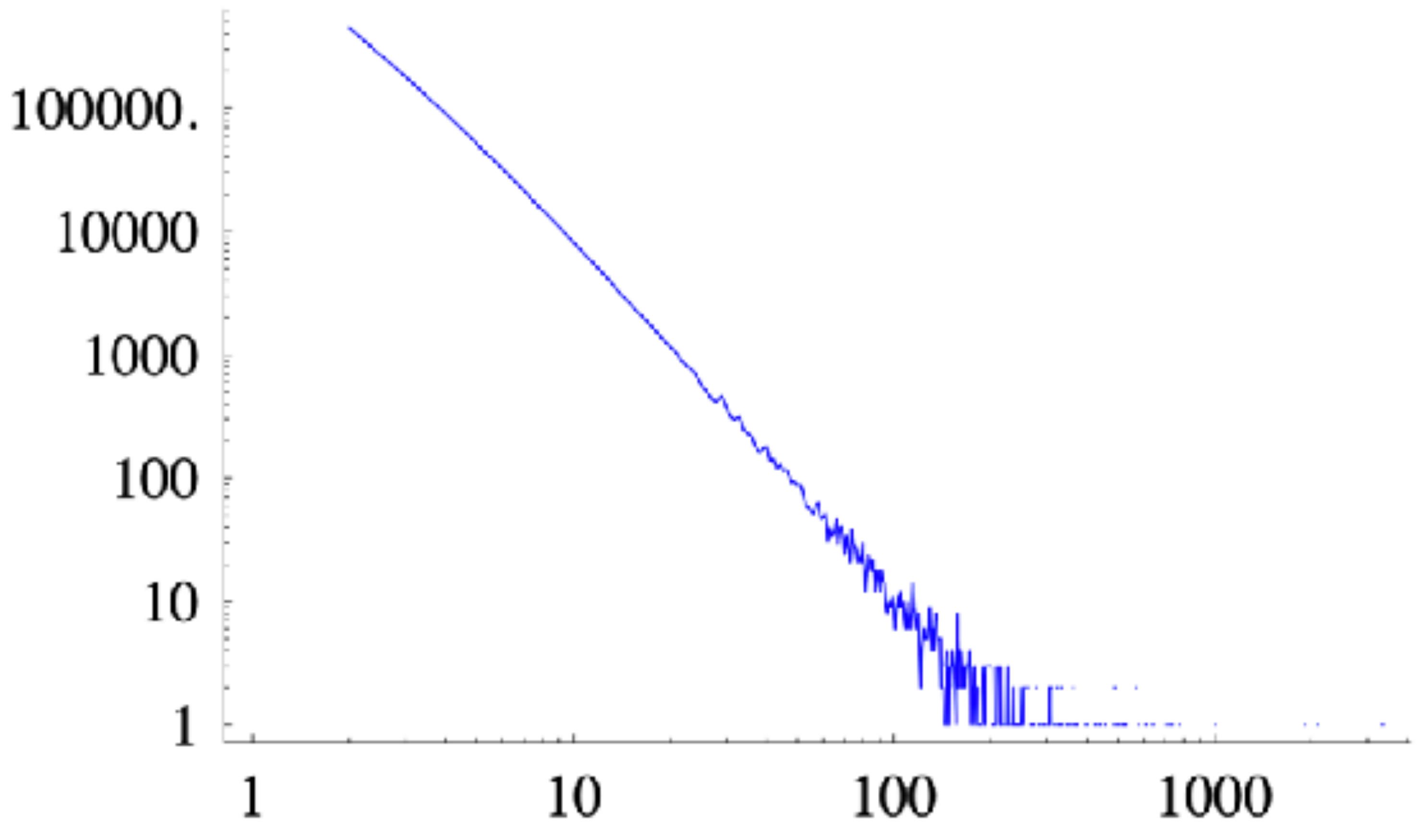
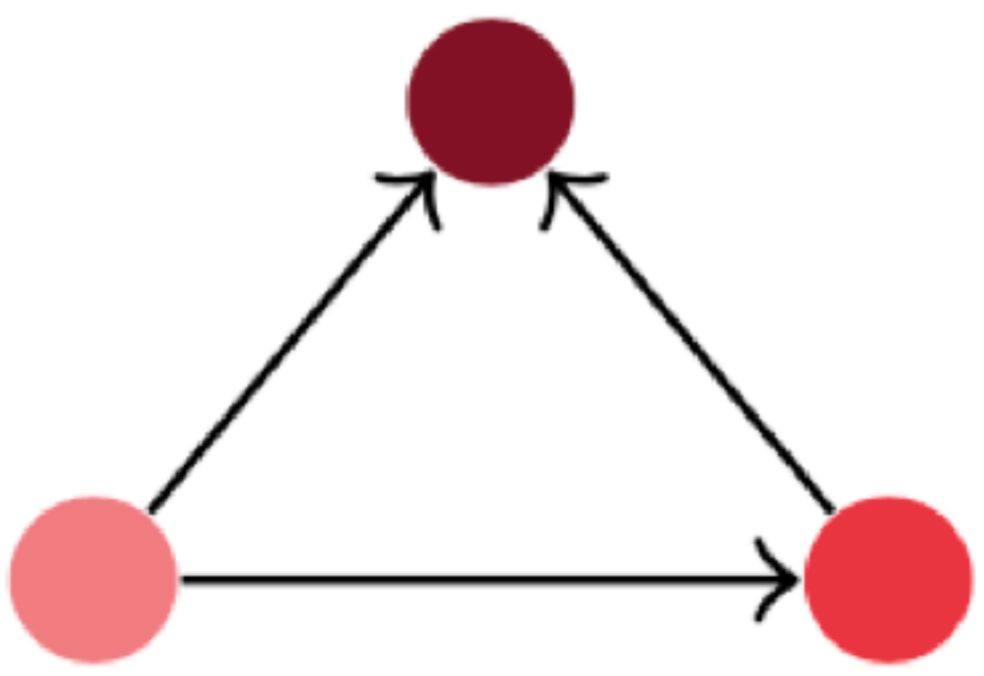


Fig 3 of A. Garavaglia and C. Stegehuis (2019). Subgraphs in Preferential Attachment Models. https://doi.org/10.1017/apr.2019.36

What do we know?

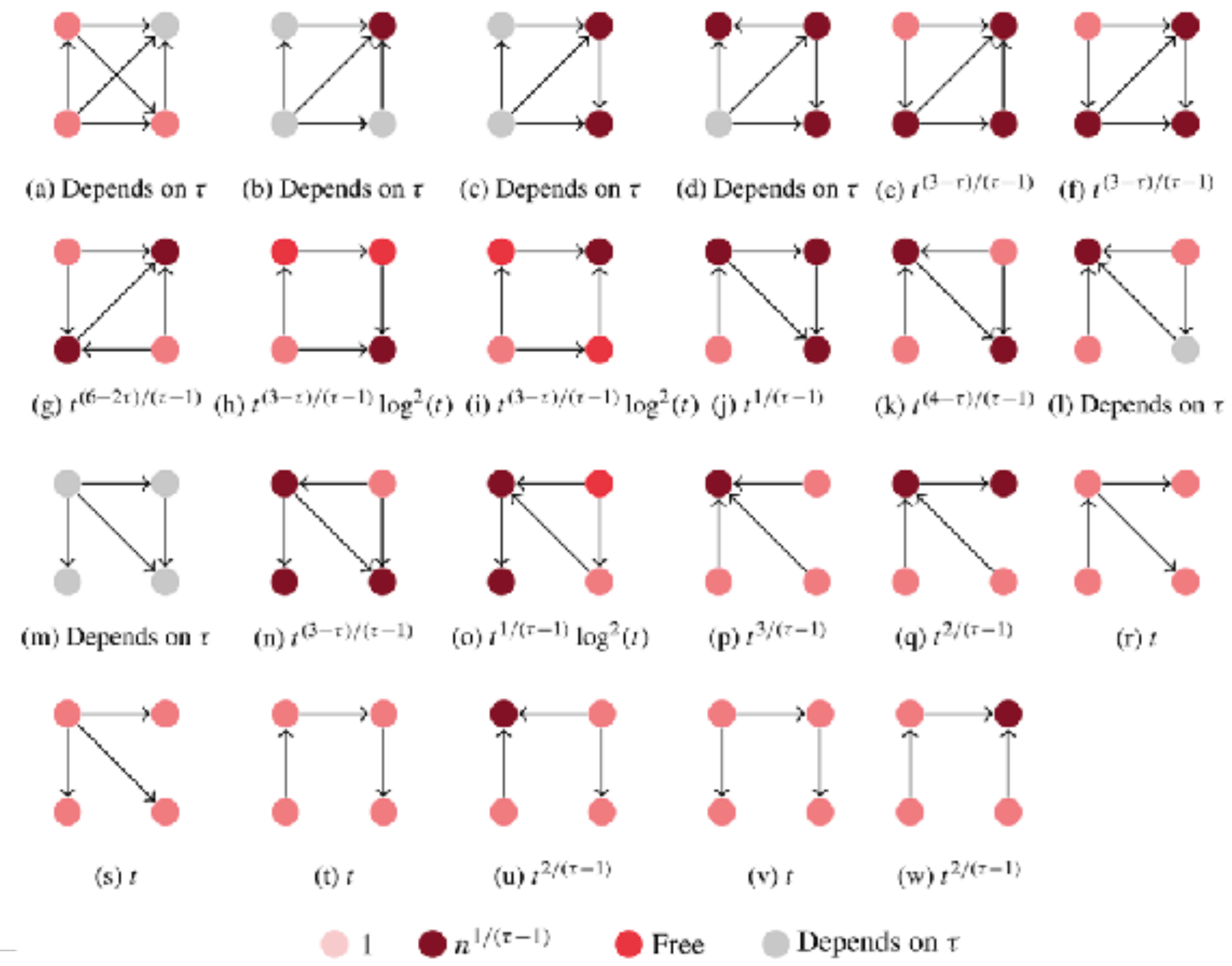


degree distribution



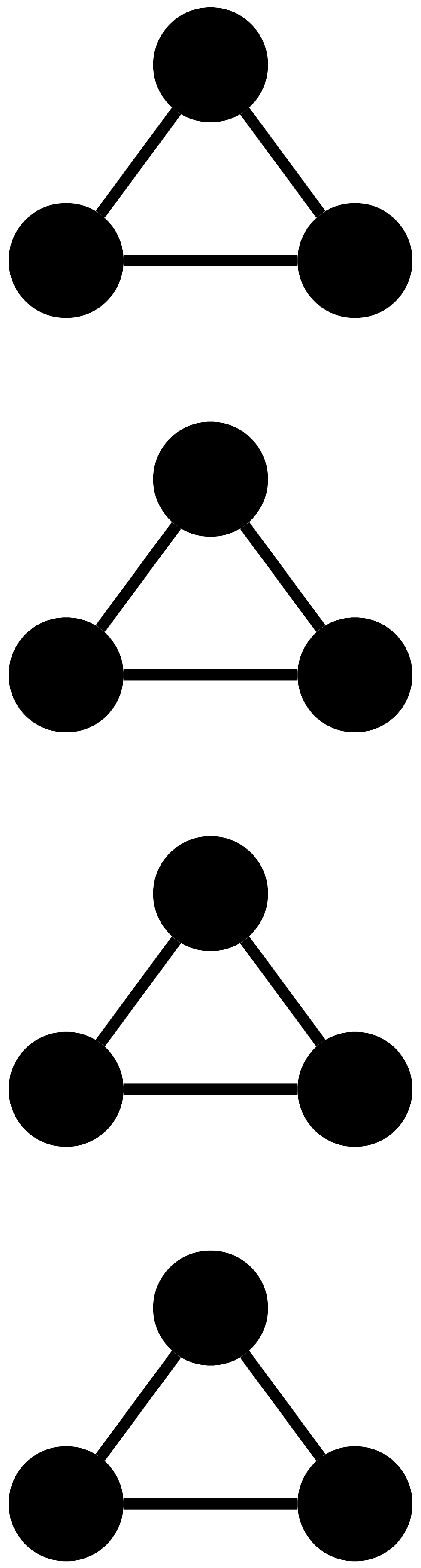
(a) $t^{(3-\tau)/(\tau-1)} \log(t)$

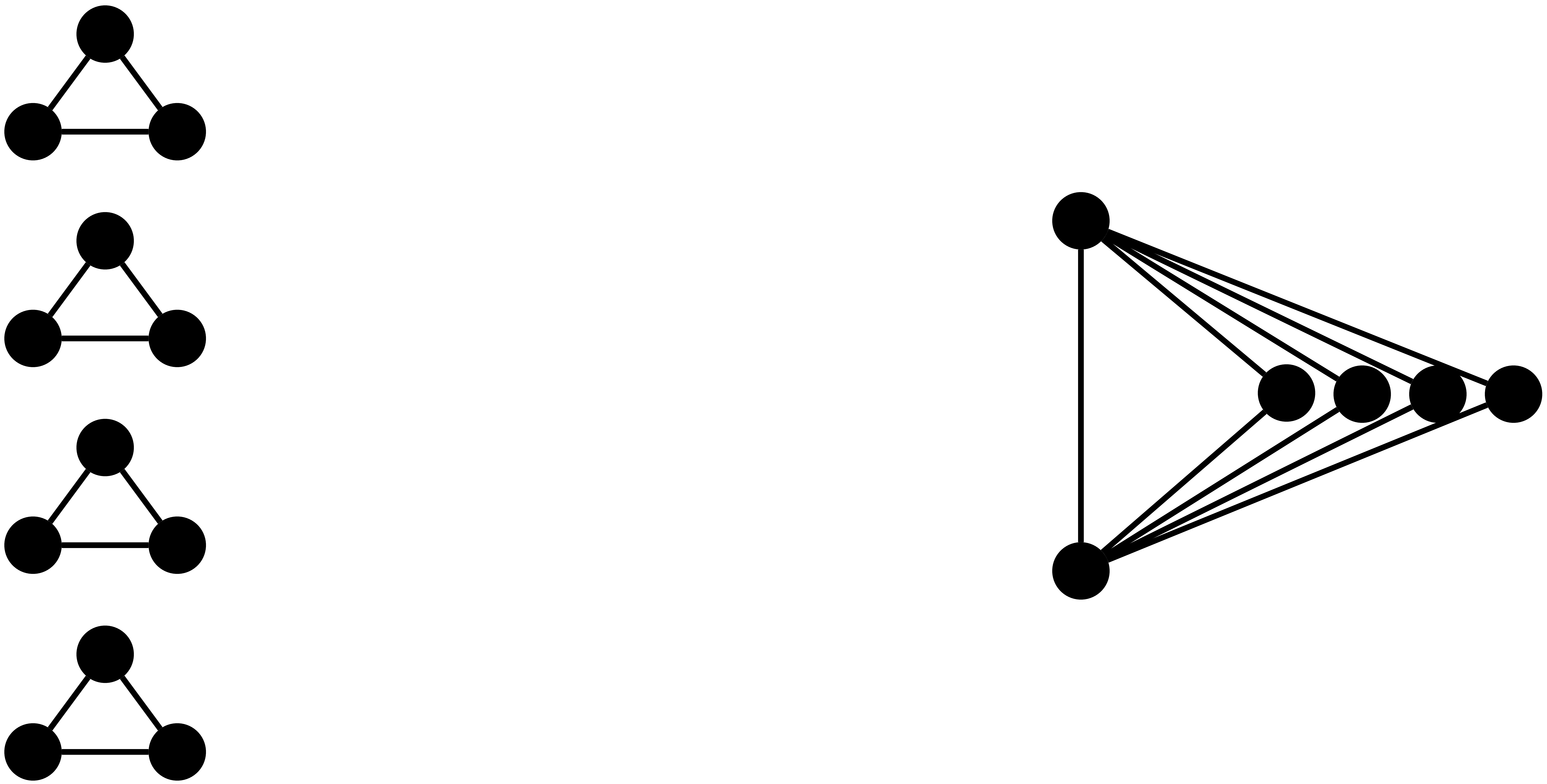
triangle counts



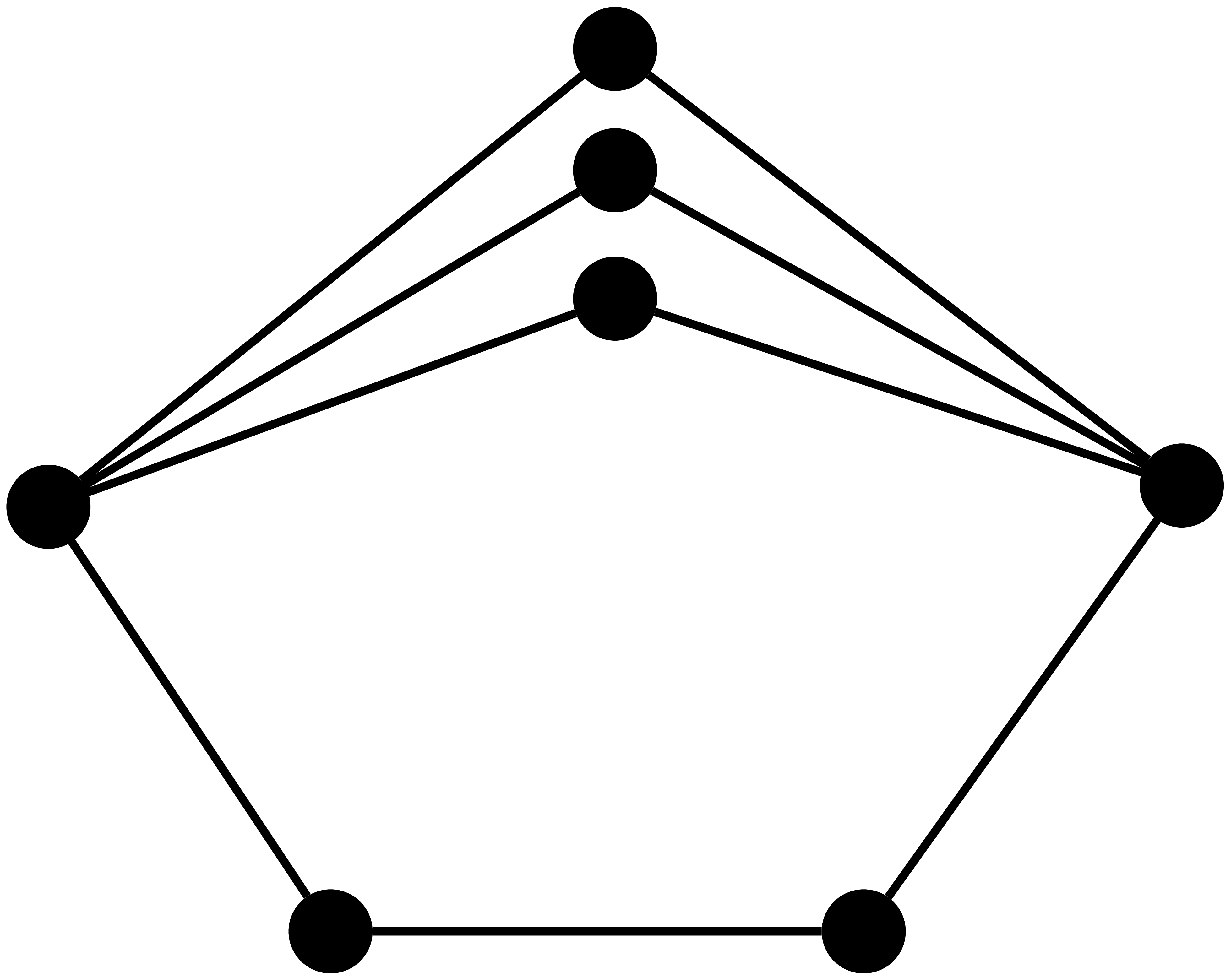
subgraph counts

What should we count?
And how?

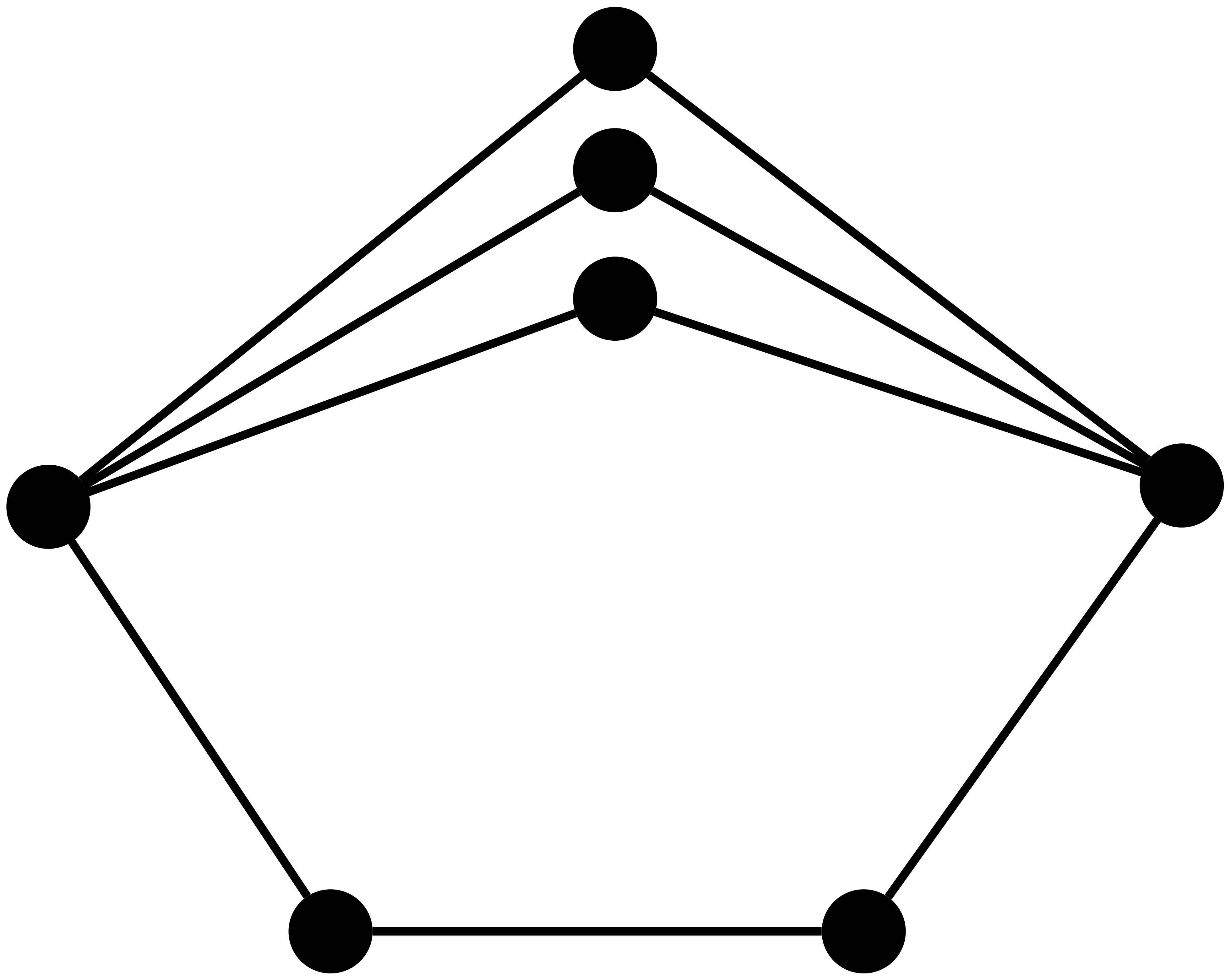




Paths from left to right?

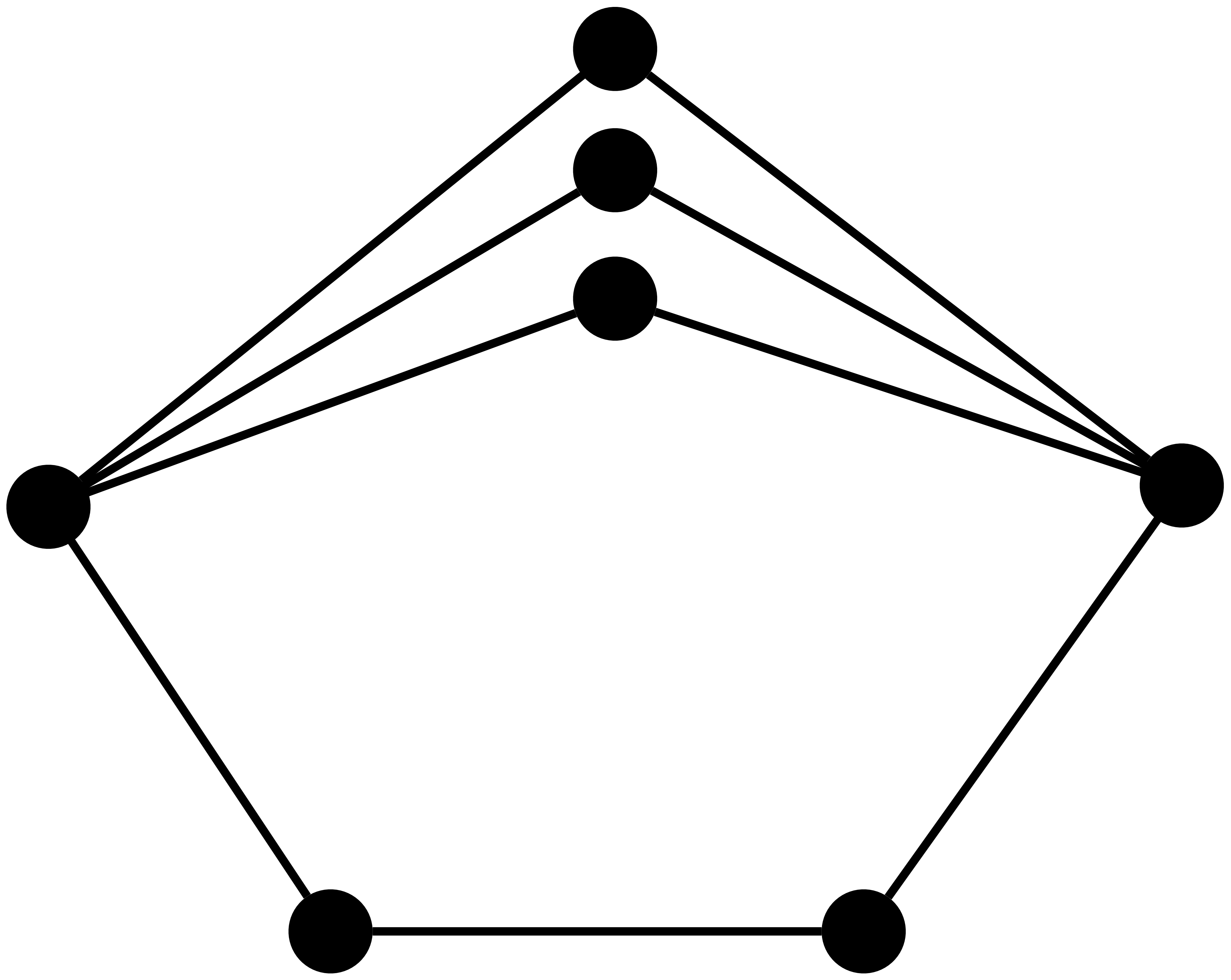


Paths from left to right?



- backtracking?

Paths from left to right?



- backtracking?
- concatenating with loops?

II. Into Topology

Counting everything in every dimension all at once

Betti numbers count repeated connections “in all dimensions”.

Betti numbers count repeated connections “in all dimensions”.

GOOD ✓

“correct” way to count things

Betti numbers count repeated connections “in all dimensions”.

GOOD ✓

“correct” way to count things

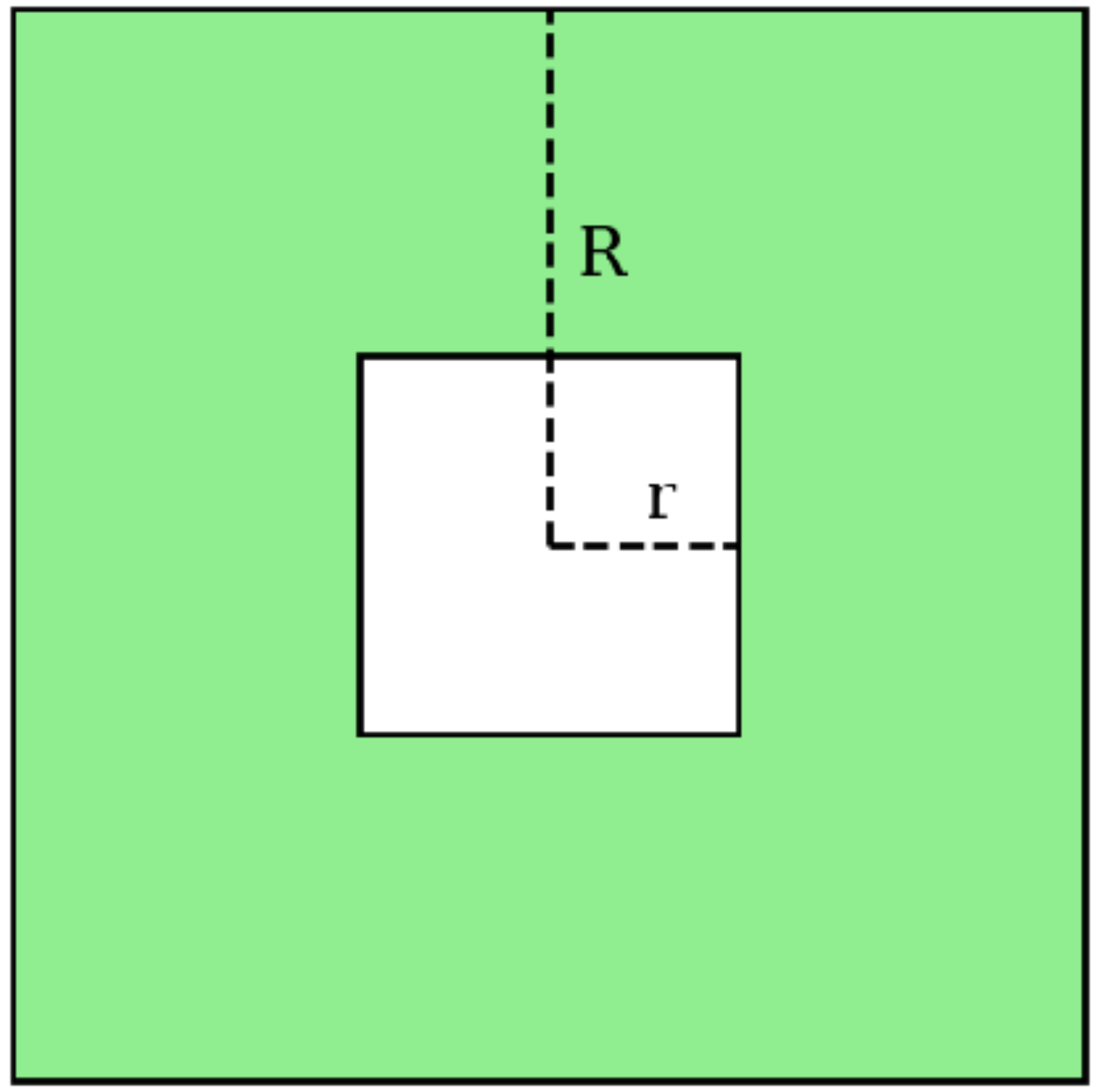
homological algebra

Betti numbers count repeated connections “in all dimensions”.

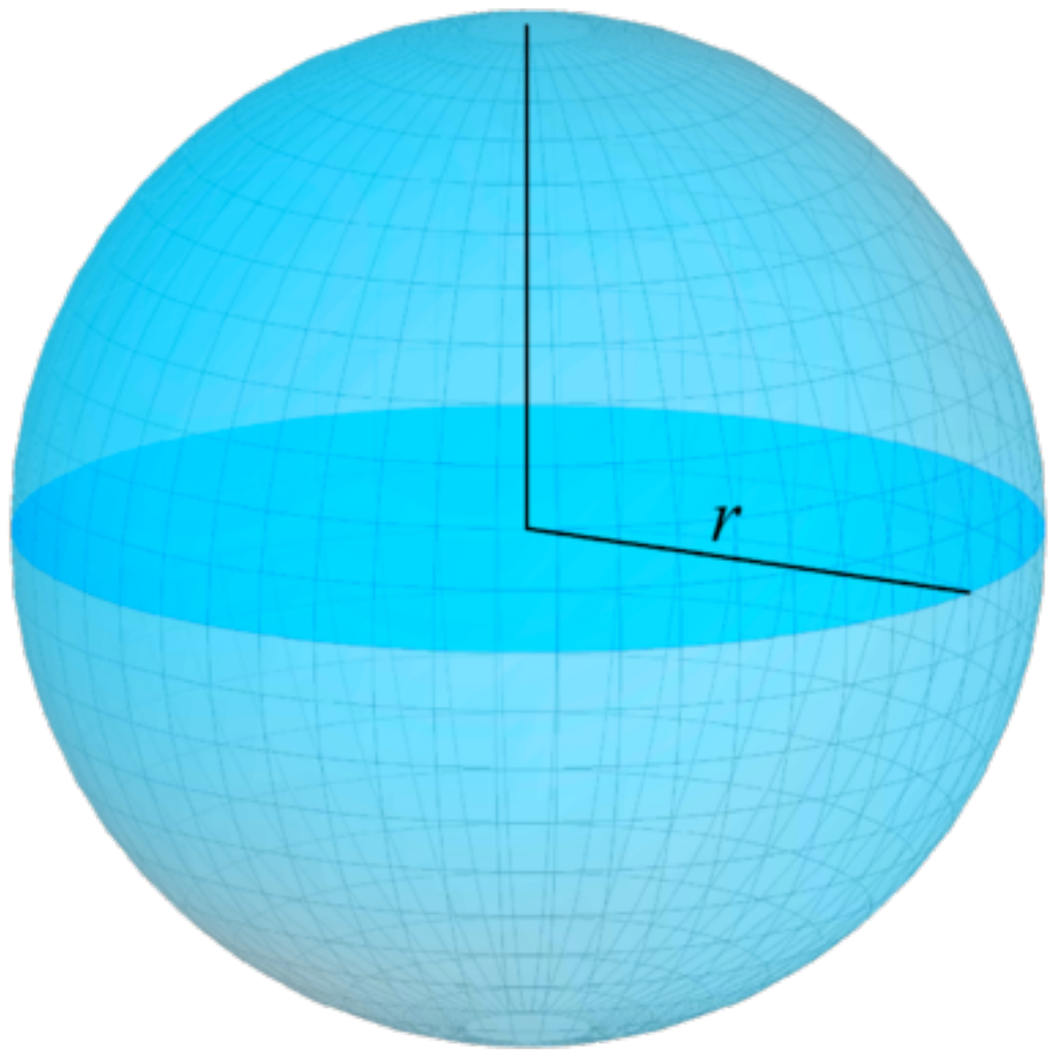
GOOD ✓	**BAD** ✗
“correct” way to count things	hard to write down
homological algebra	hard to do

Betti numbers β_k

Count of Holes



$\beta_1 = 1$: 1 loop

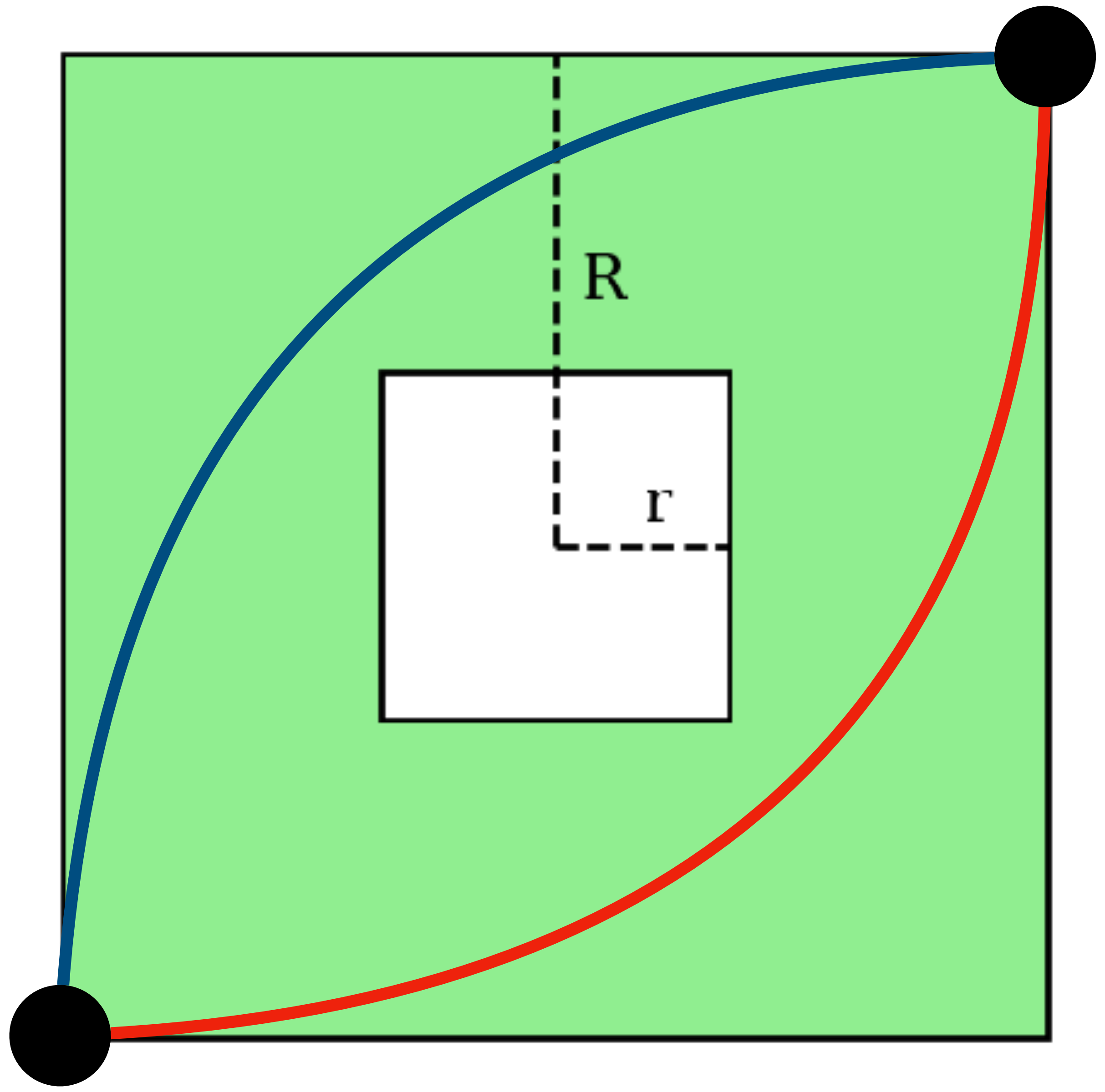


$\beta_1 = 0$: 0 loop

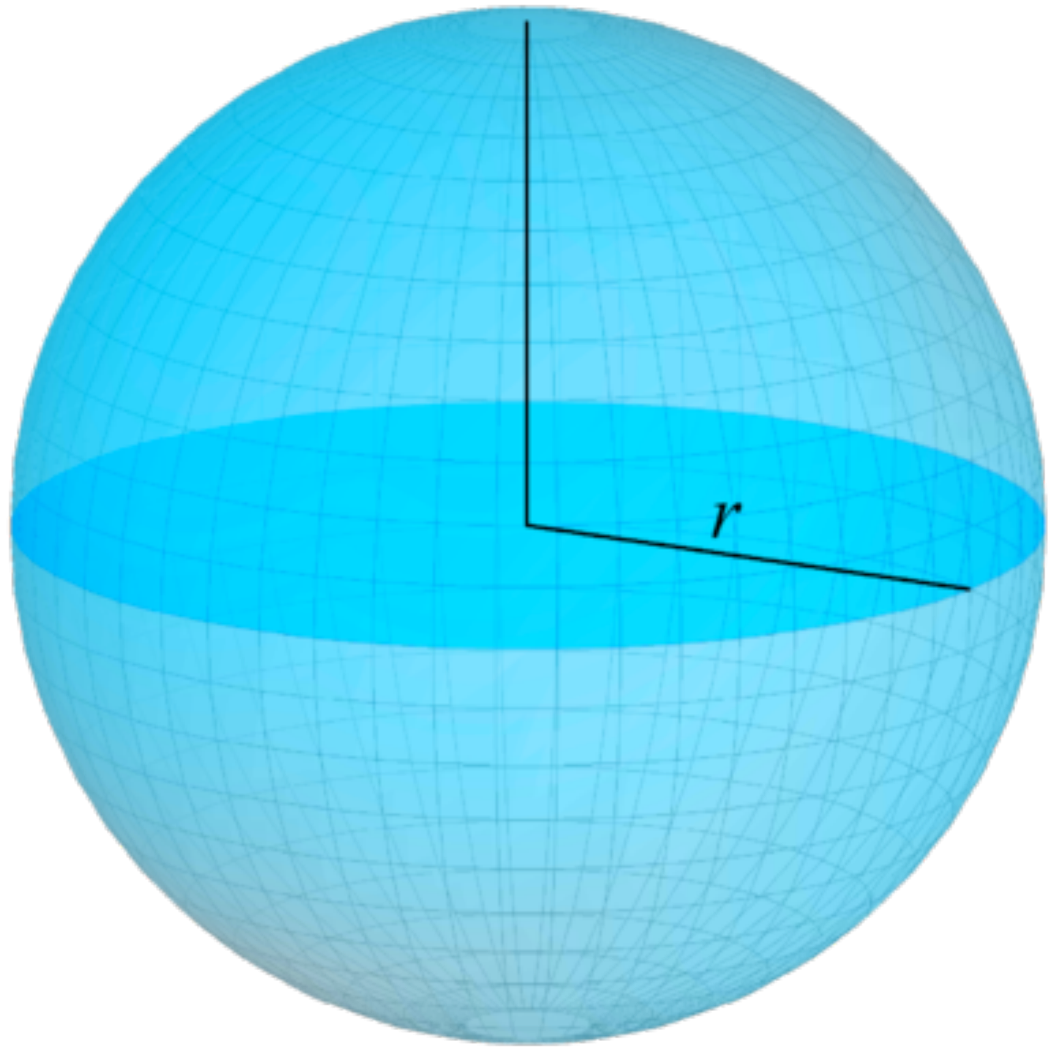
$\beta_2 = 1$: 1 cavity

Betti numbers β_k

Count of Repeated Connections



$\beta_1 = 1$: 1 loop

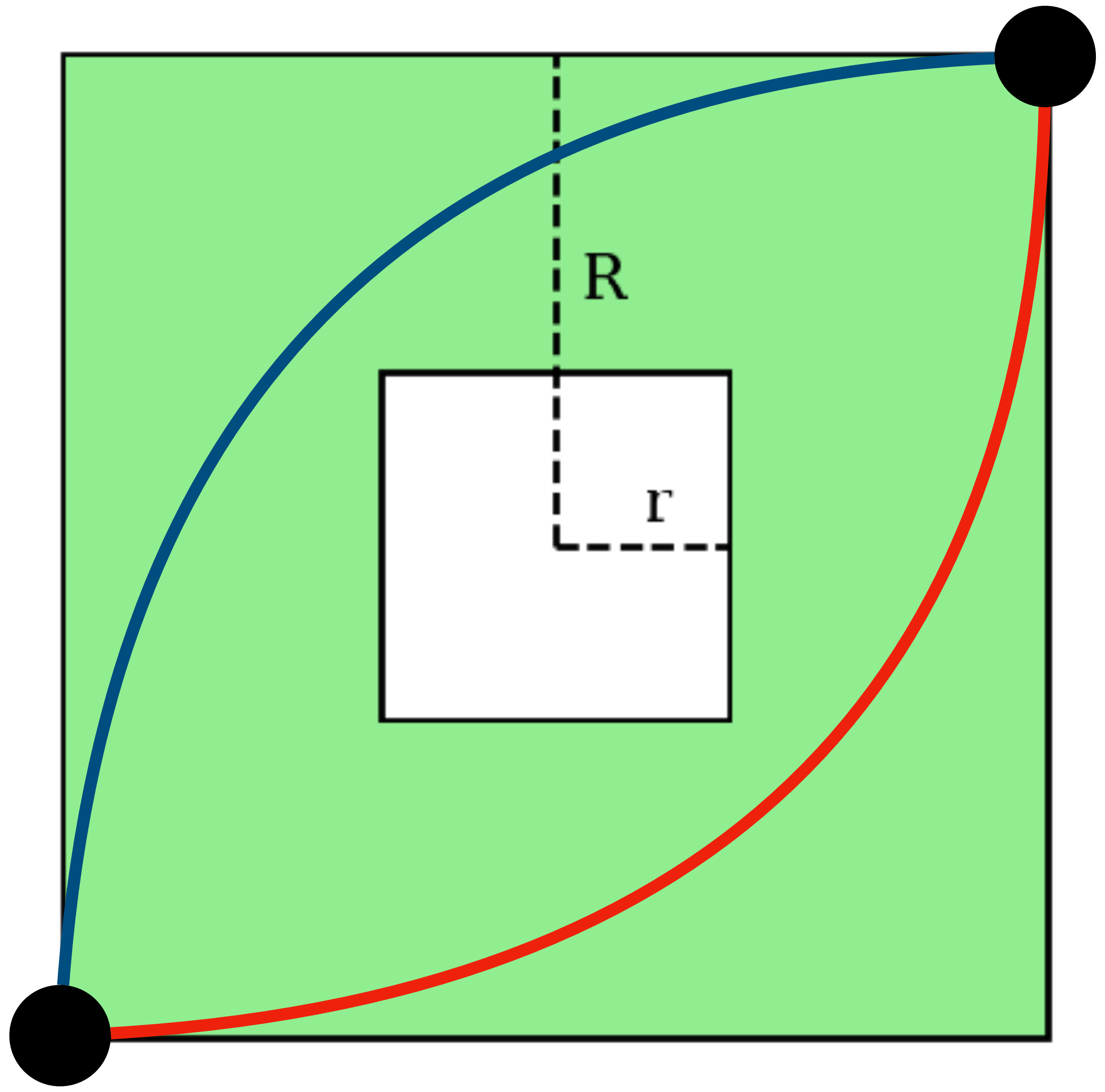


$\beta_1 = 0$: 0 loop

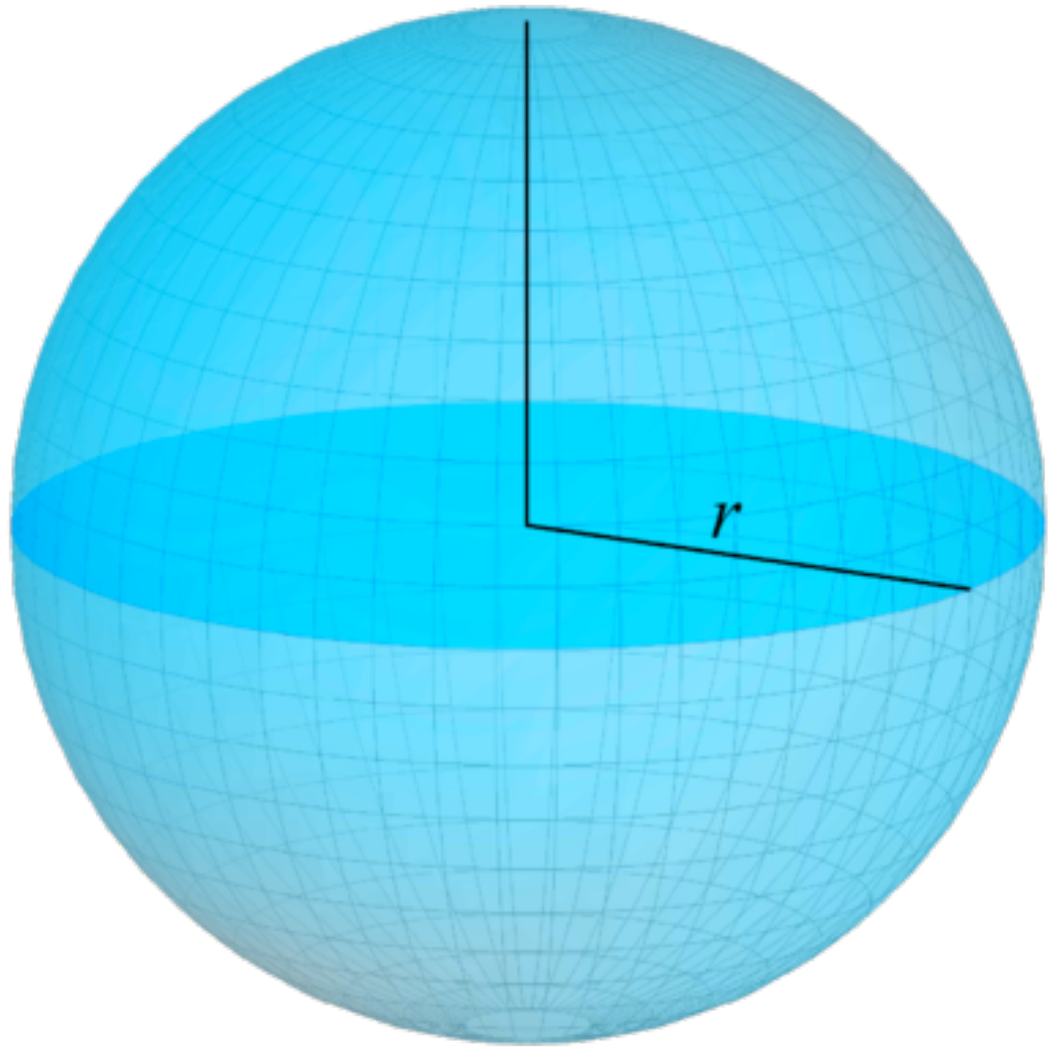
$\beta_2 = 1$: 1 cavity

Betti numbers β_k

Count of Repeated Connections



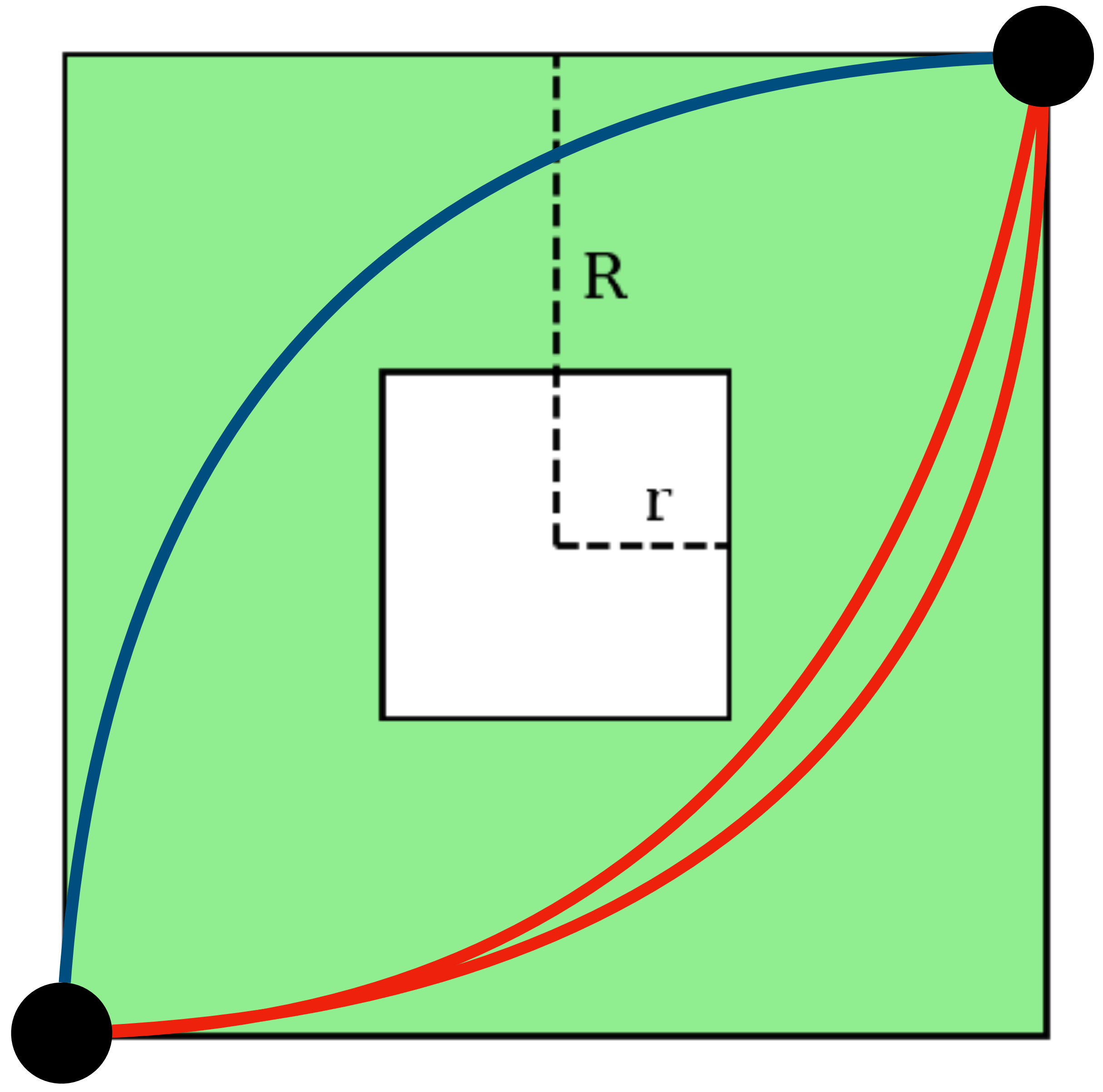
1 alternative path



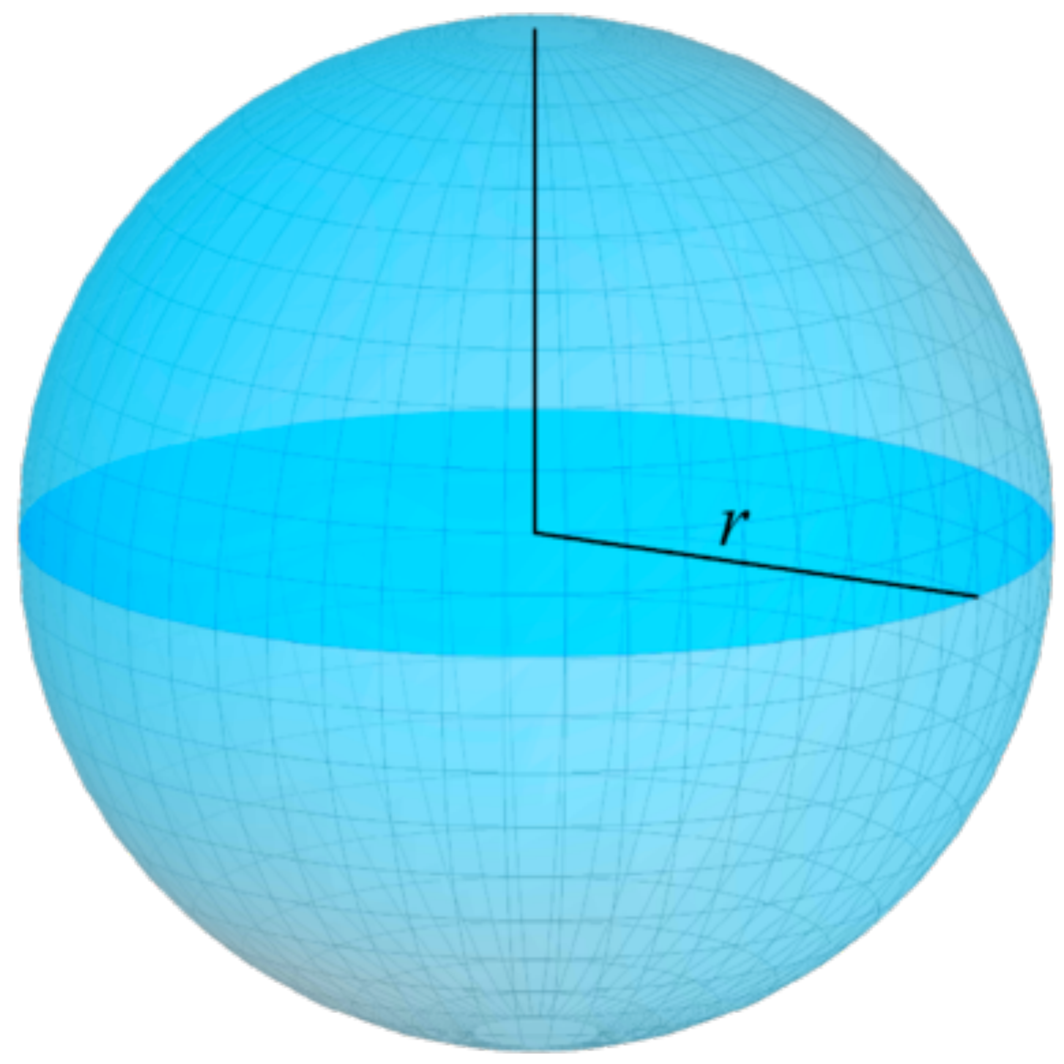
0 loop
1 cavity

Betti numbers β_k

Count of (Independent) Repeated Connections



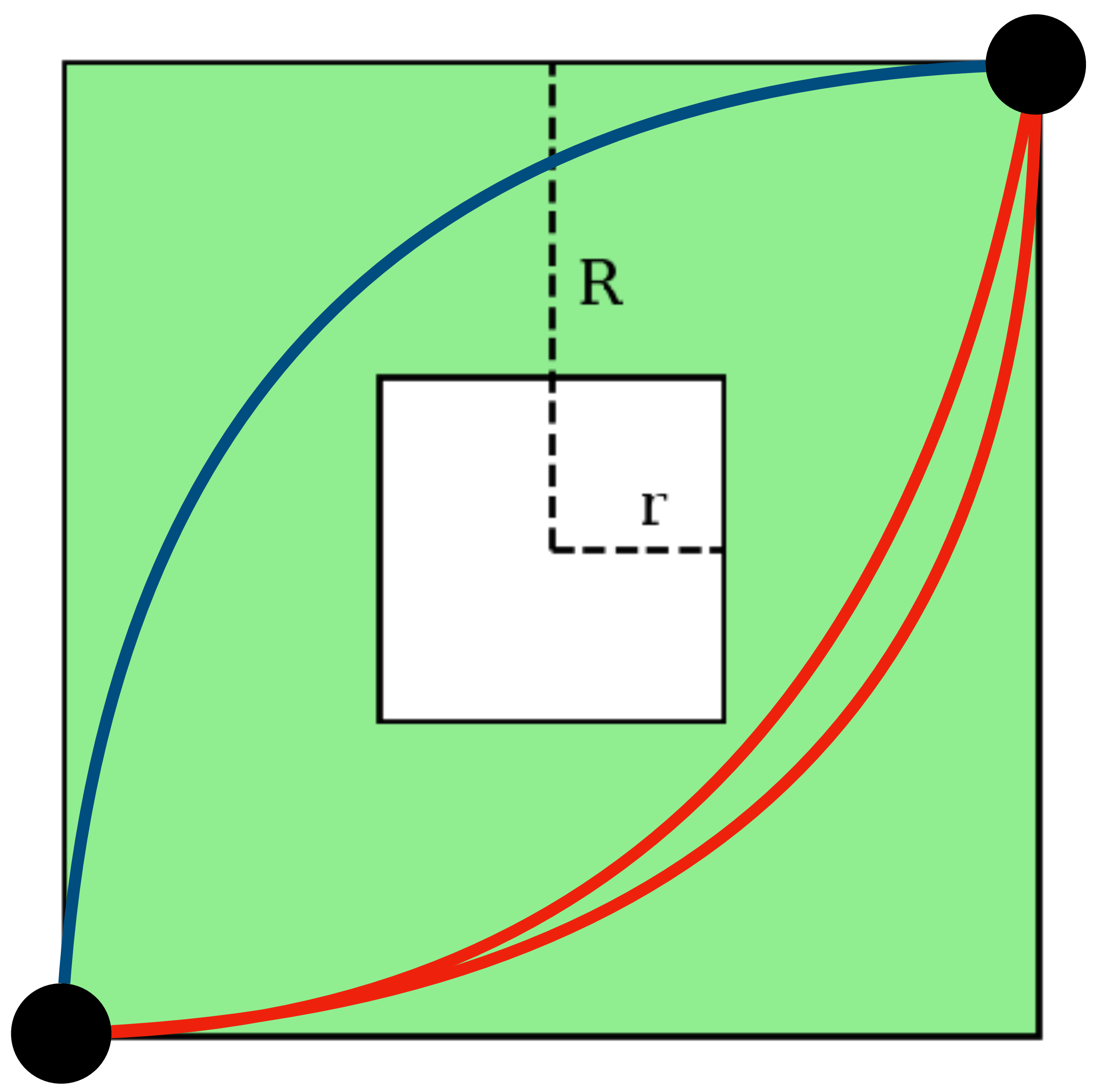
1 alternative path



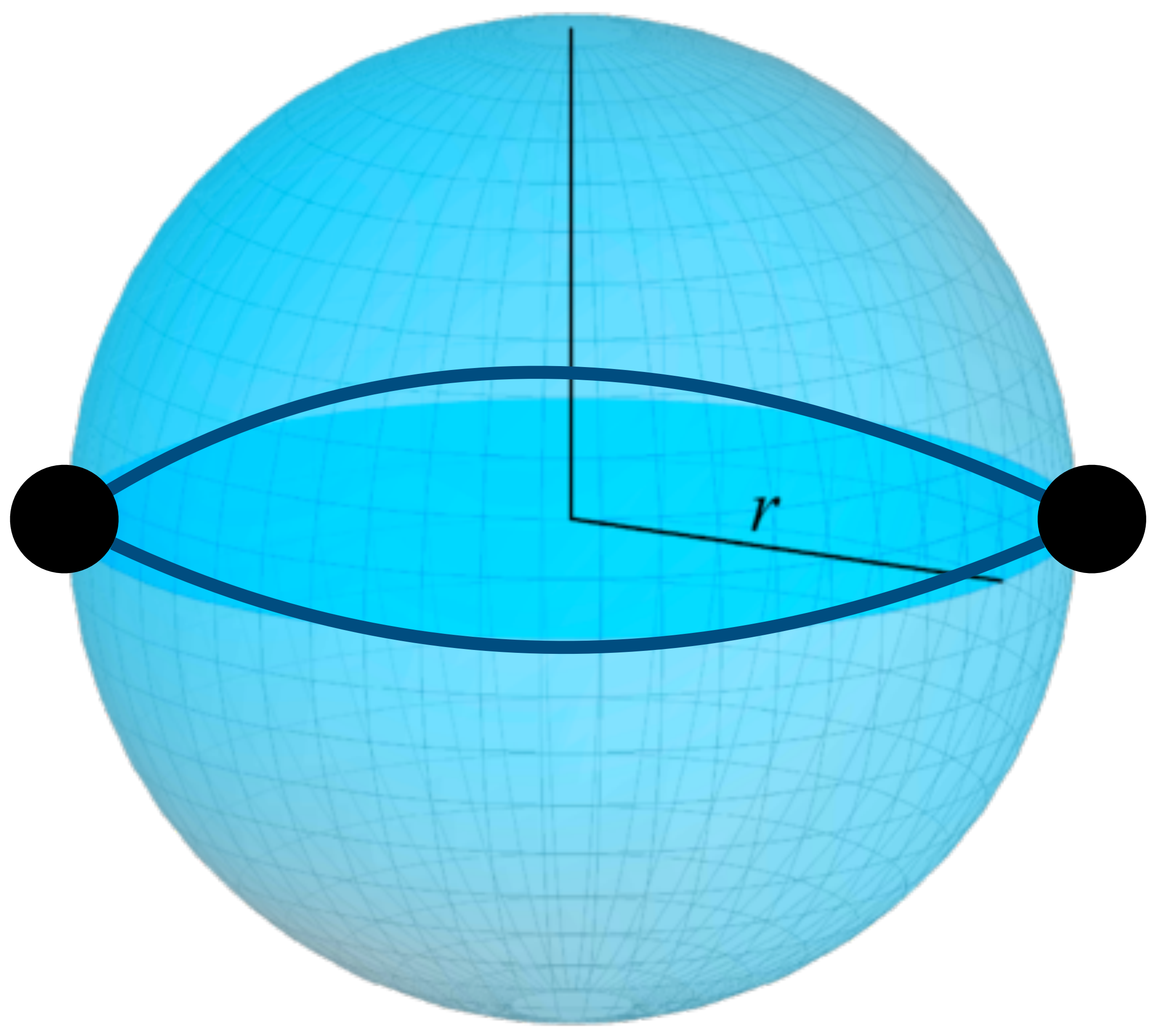
0 loop
1 cavity

Betti numbers β_k

Count of (Independent) Repeated Connections



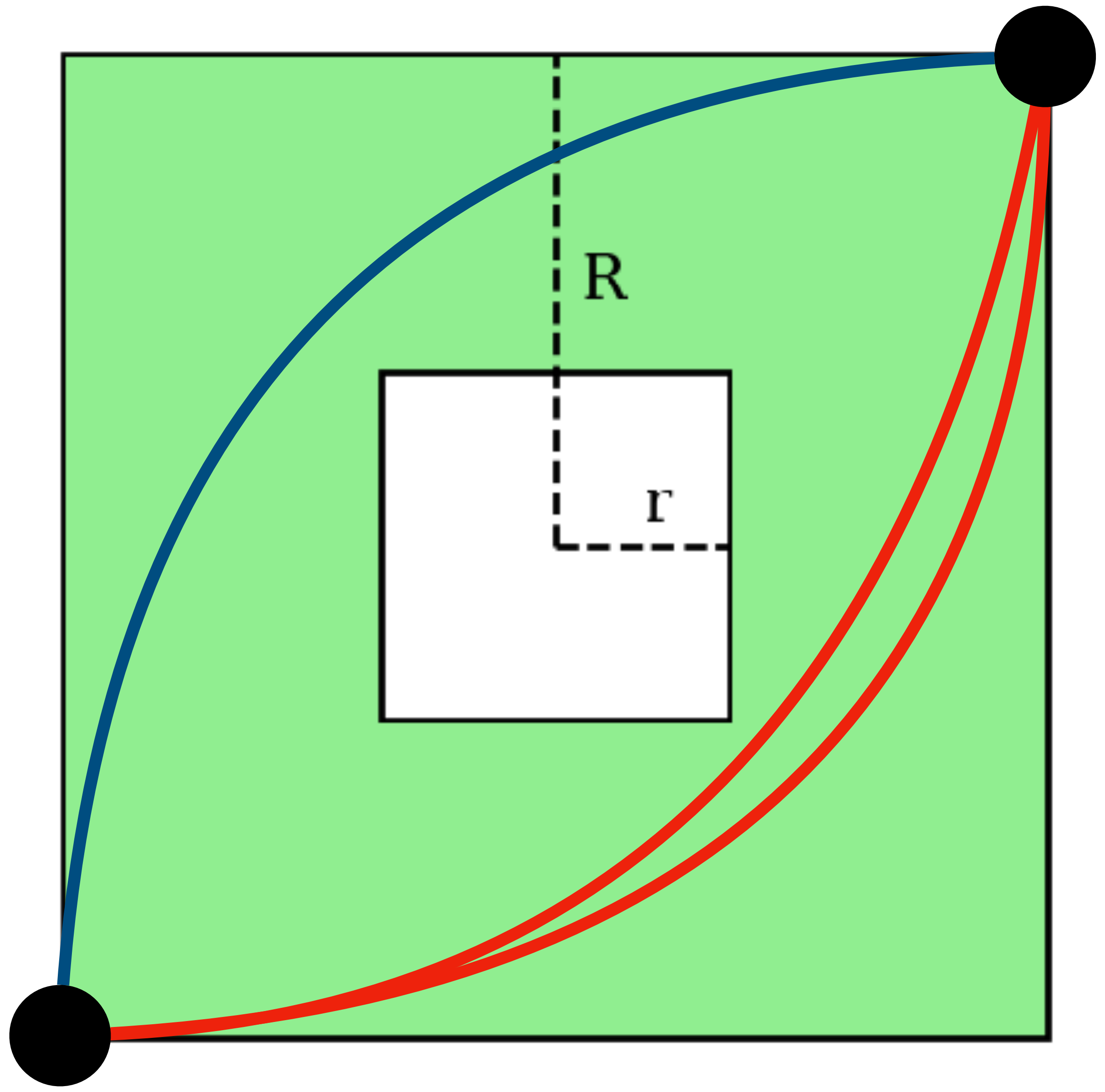
1 alternative path



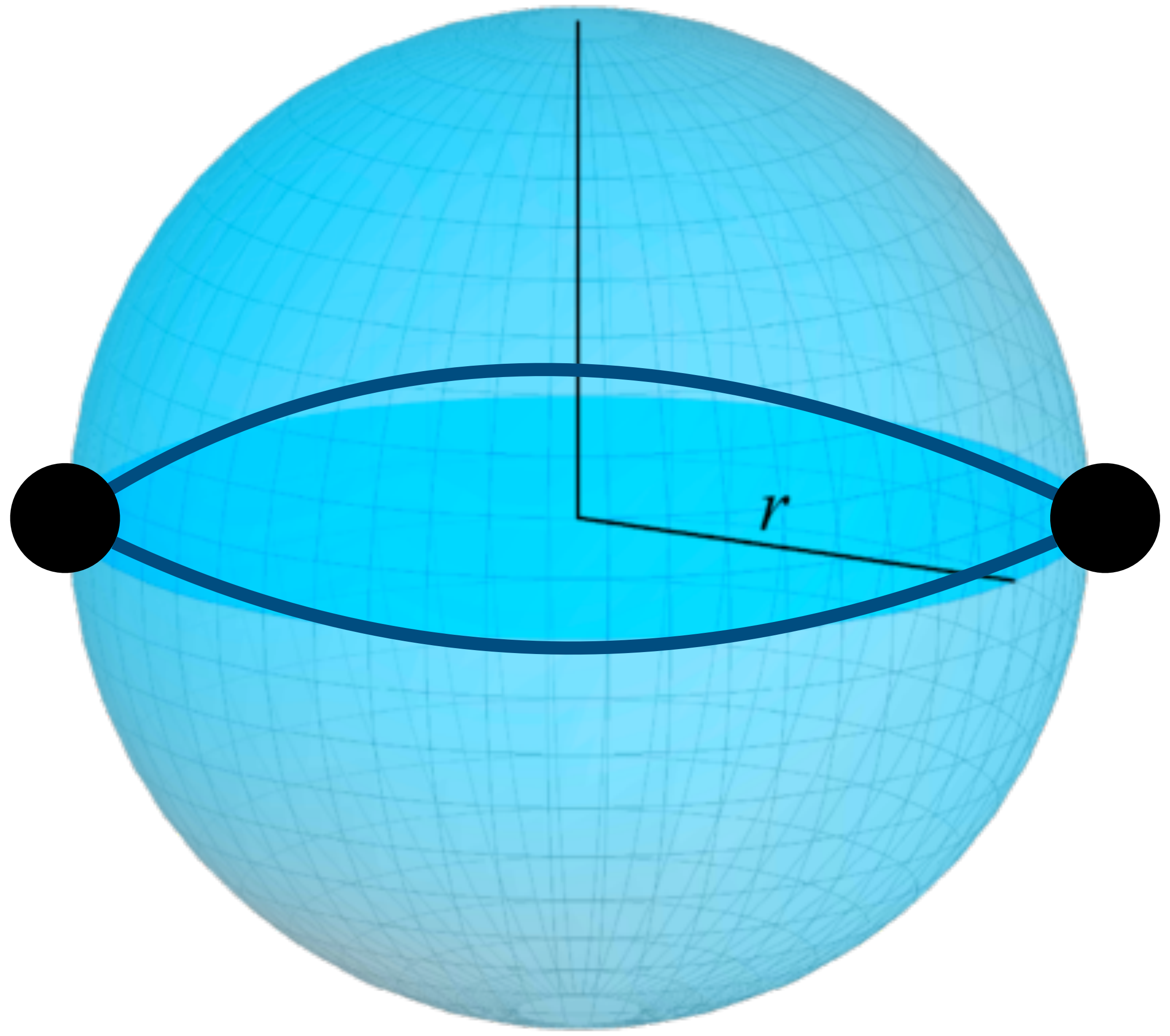
0 alternative path (slide through upper hemisphere)
1 cavity

Betti numbers β_k

Count of (Independent) Repeated Connections



1 alternative path



0 alternative path (slide through upper hemisphere)
1 alternative way to slide a path

**Betti numbers count
repeated connections “in all dimensions”.**

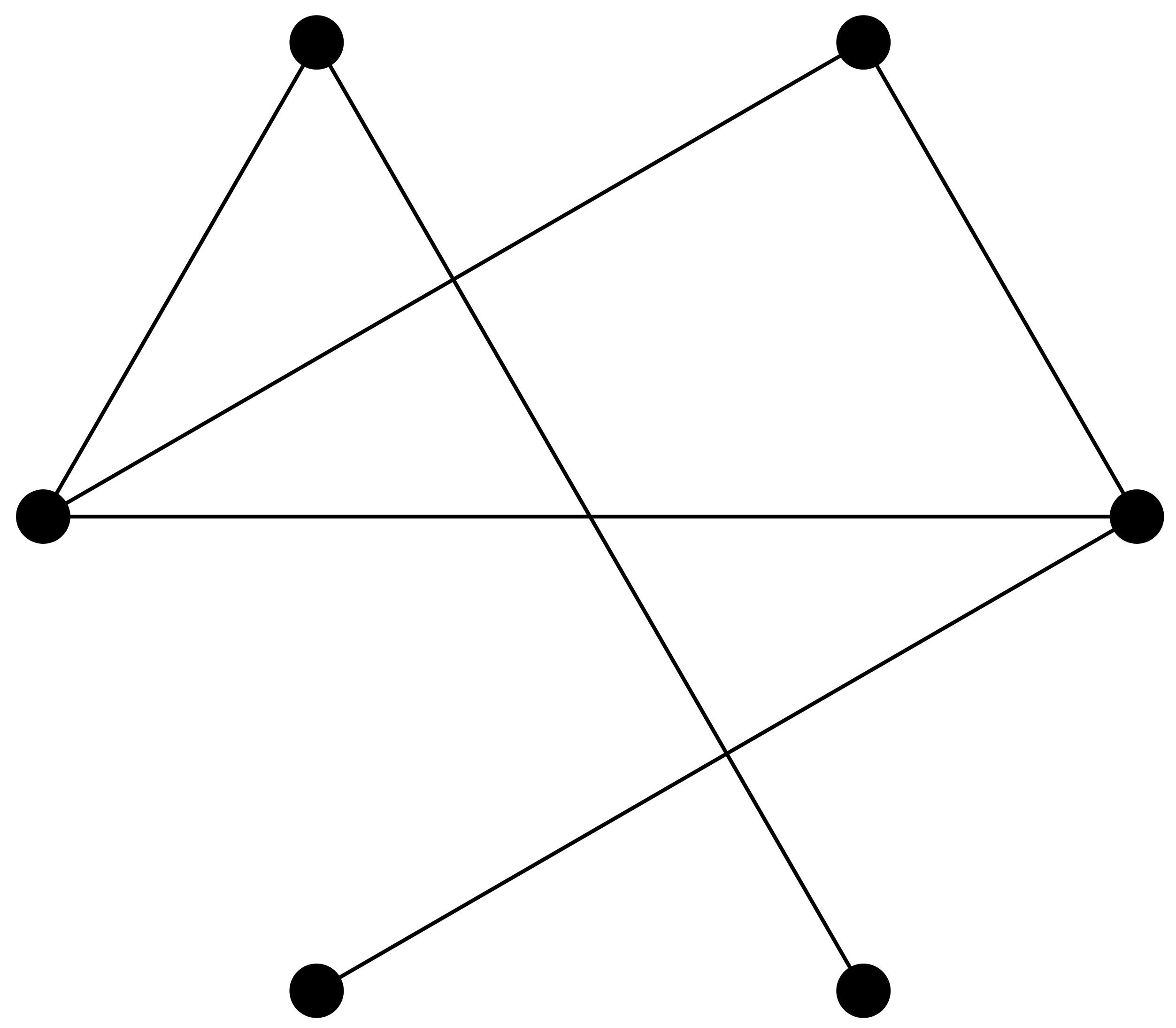
Interlude:

Random Walk in the Literature

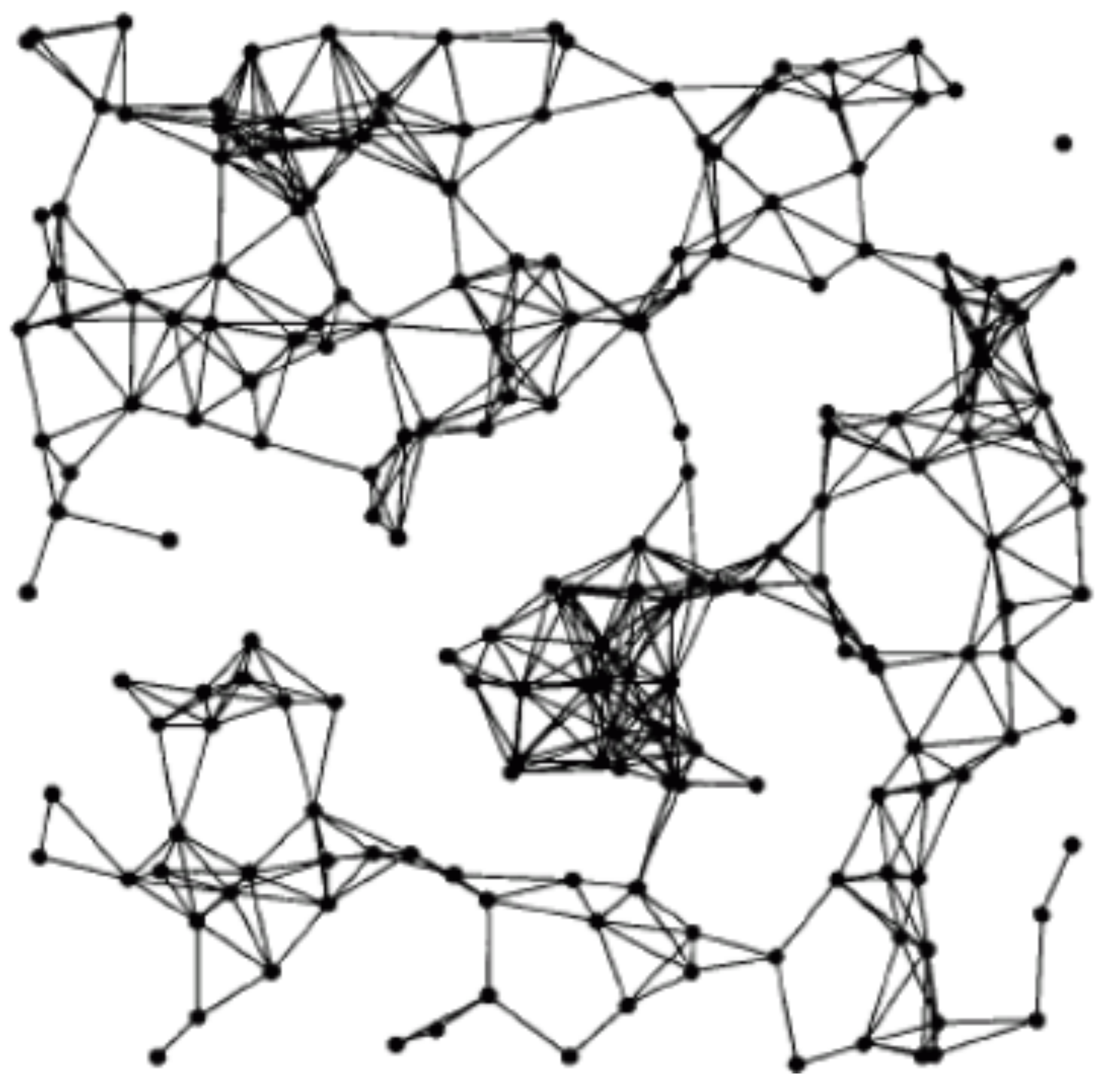
What Random Topologists Already Know

..

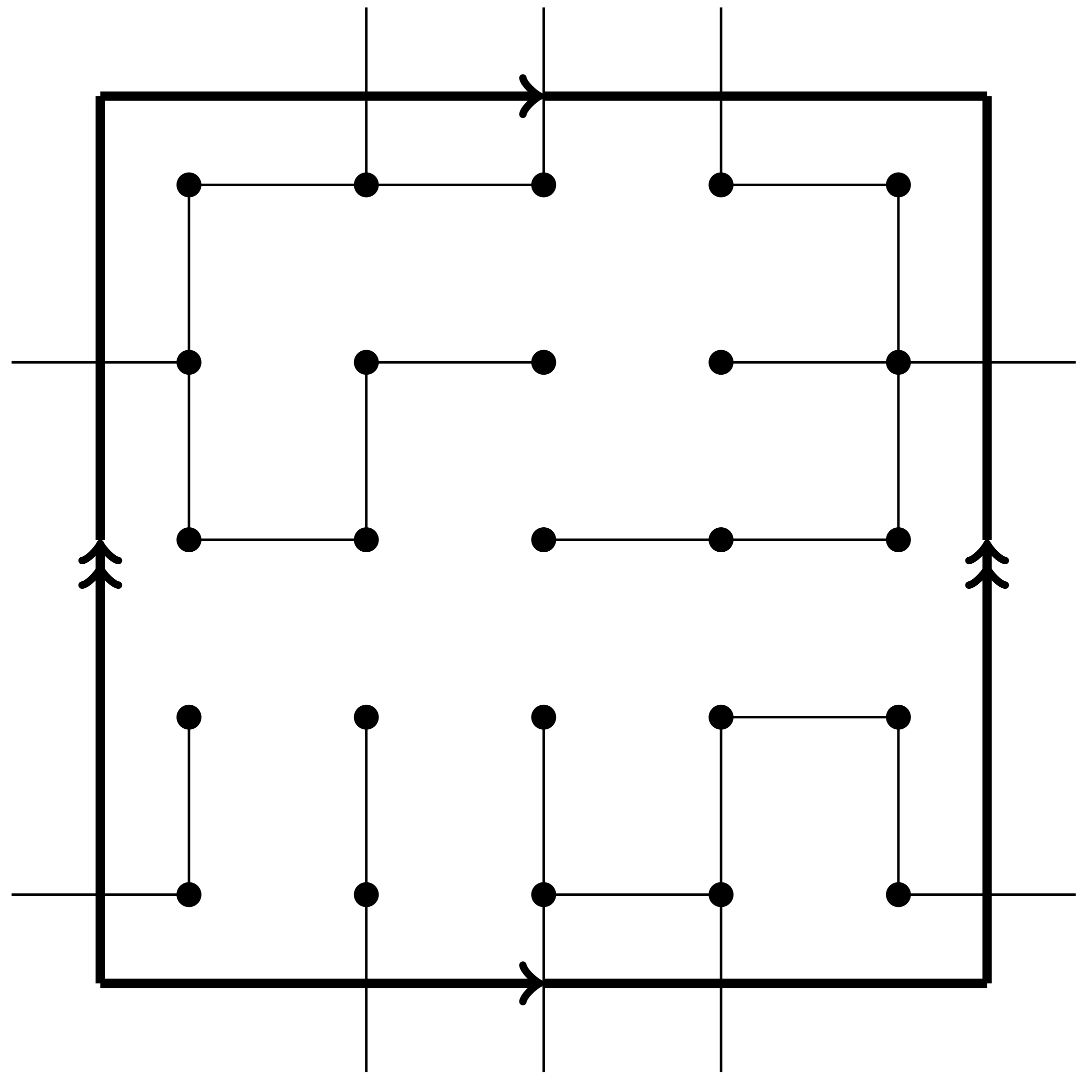
Tapas of Random Topology



Erdo-Renyi Complexes

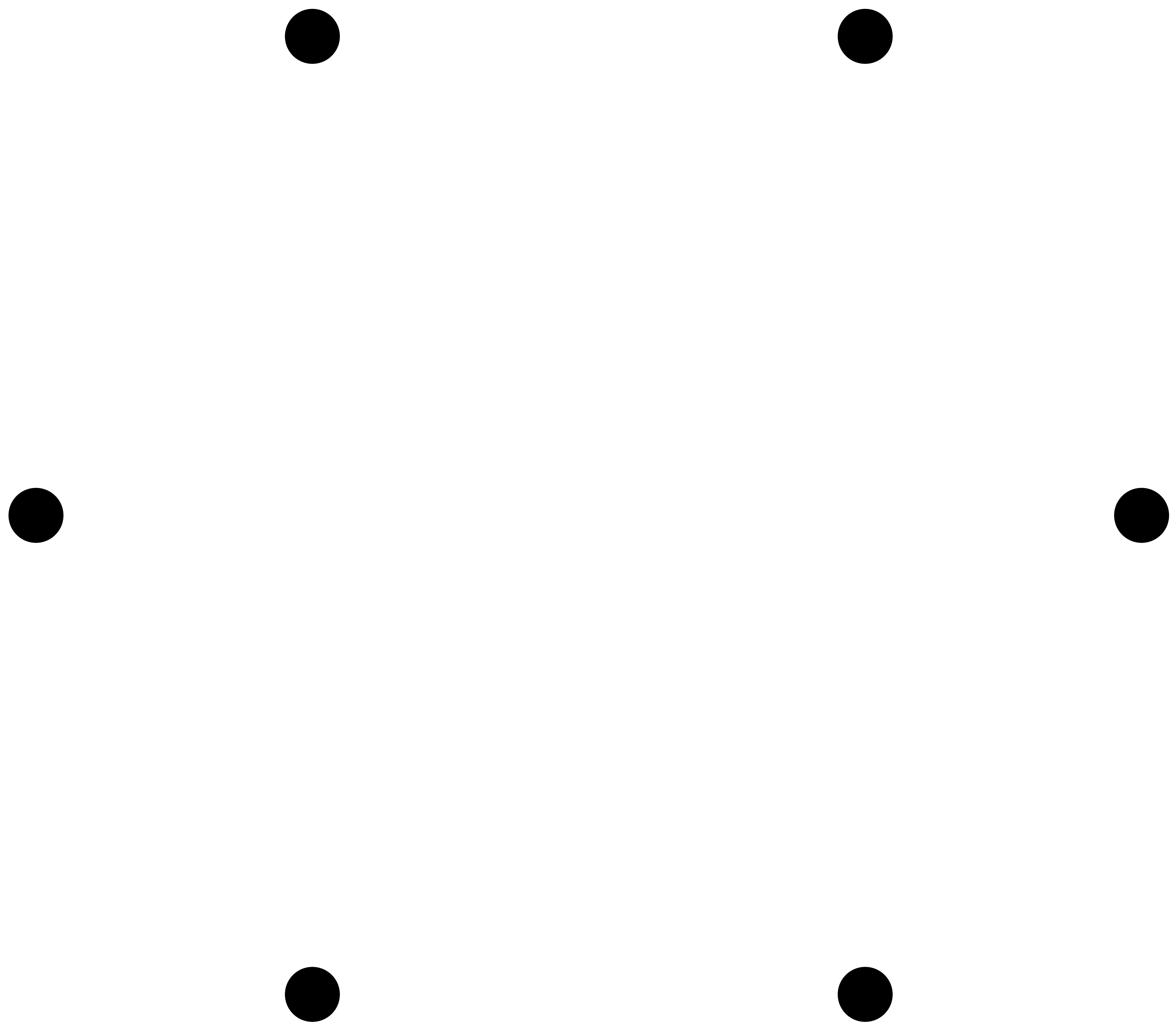


Geometric Complexes

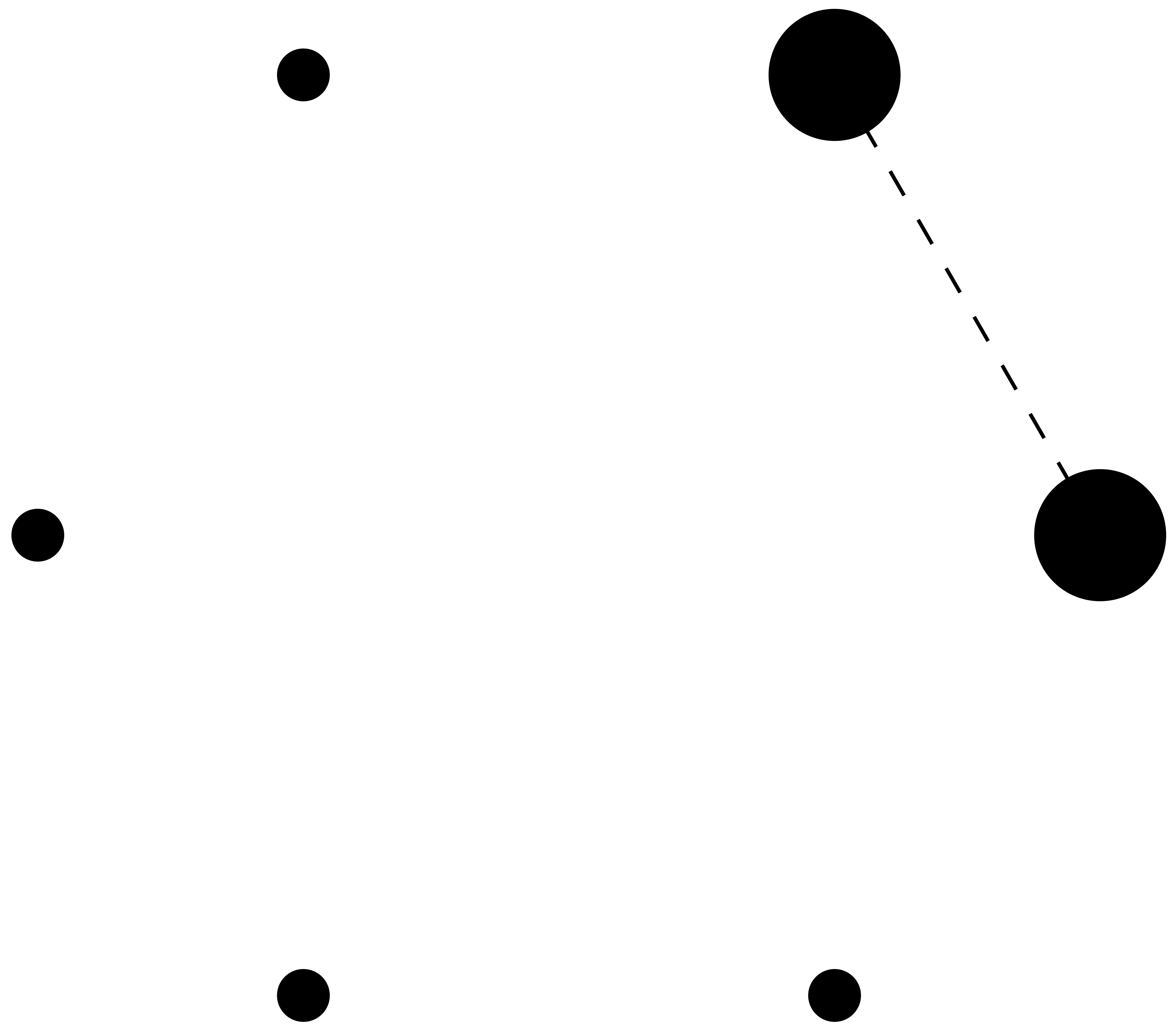


Topological Percolation

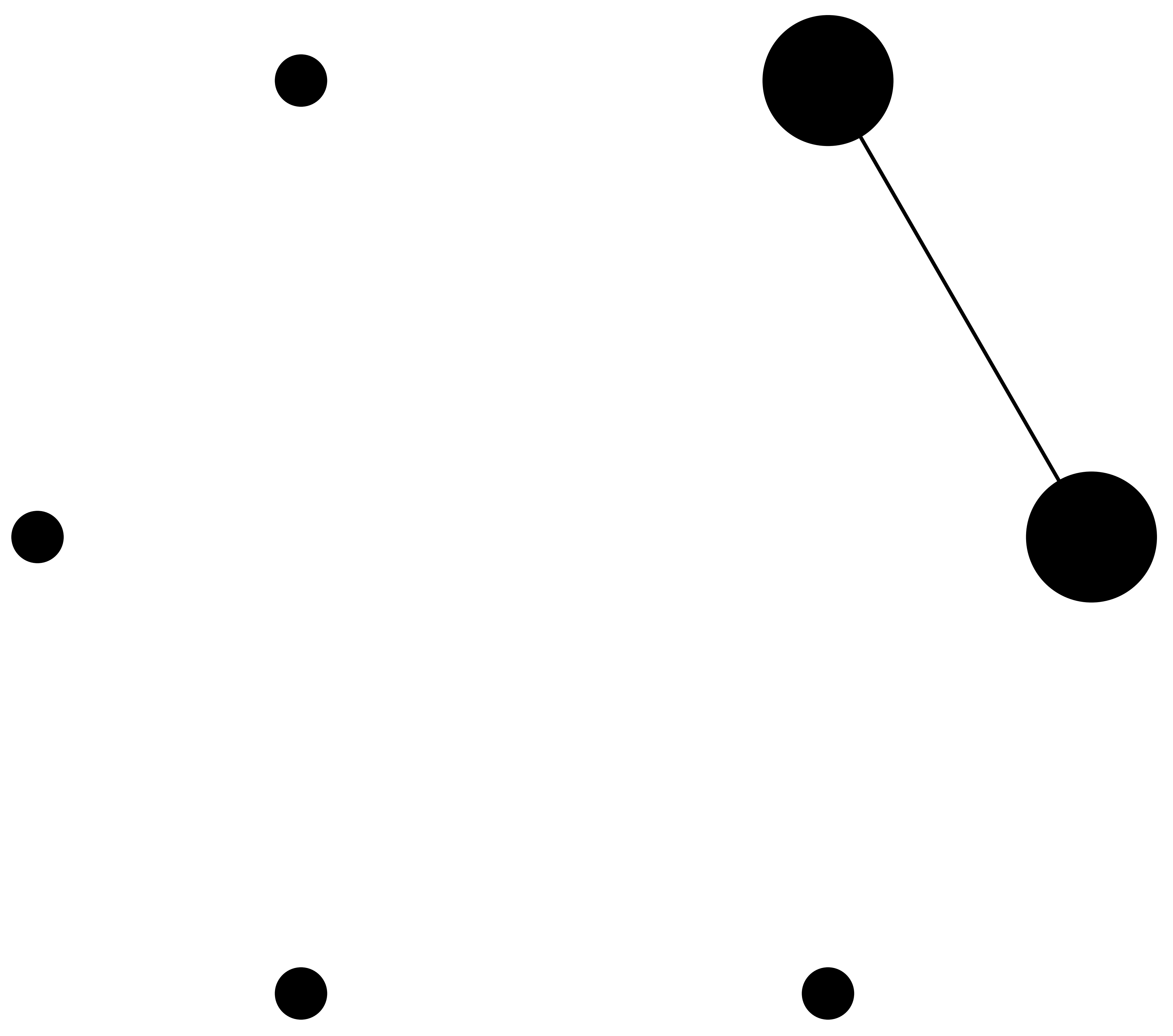
Erdos-Renyi graphs



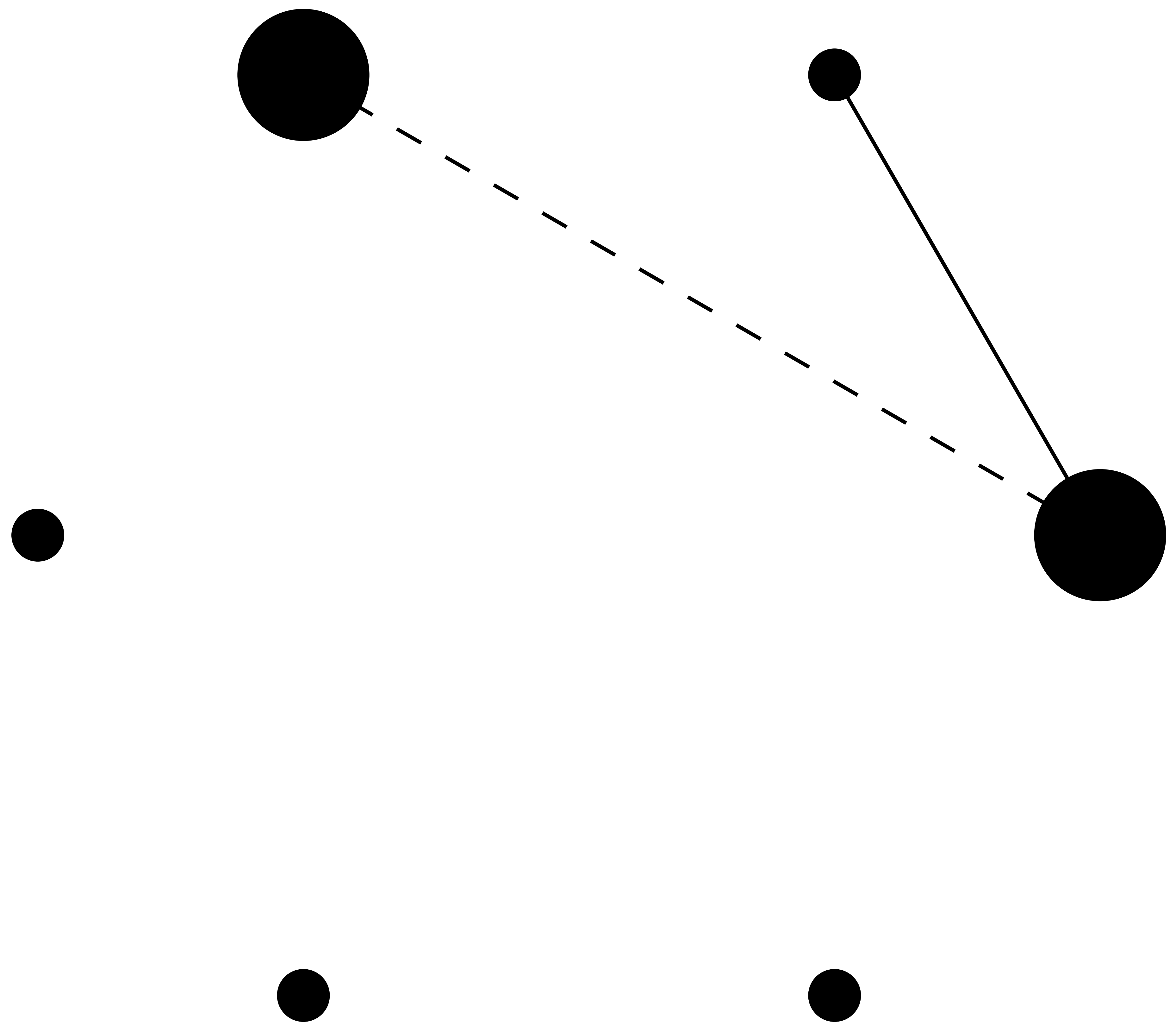
Erdos-Renyi graphs



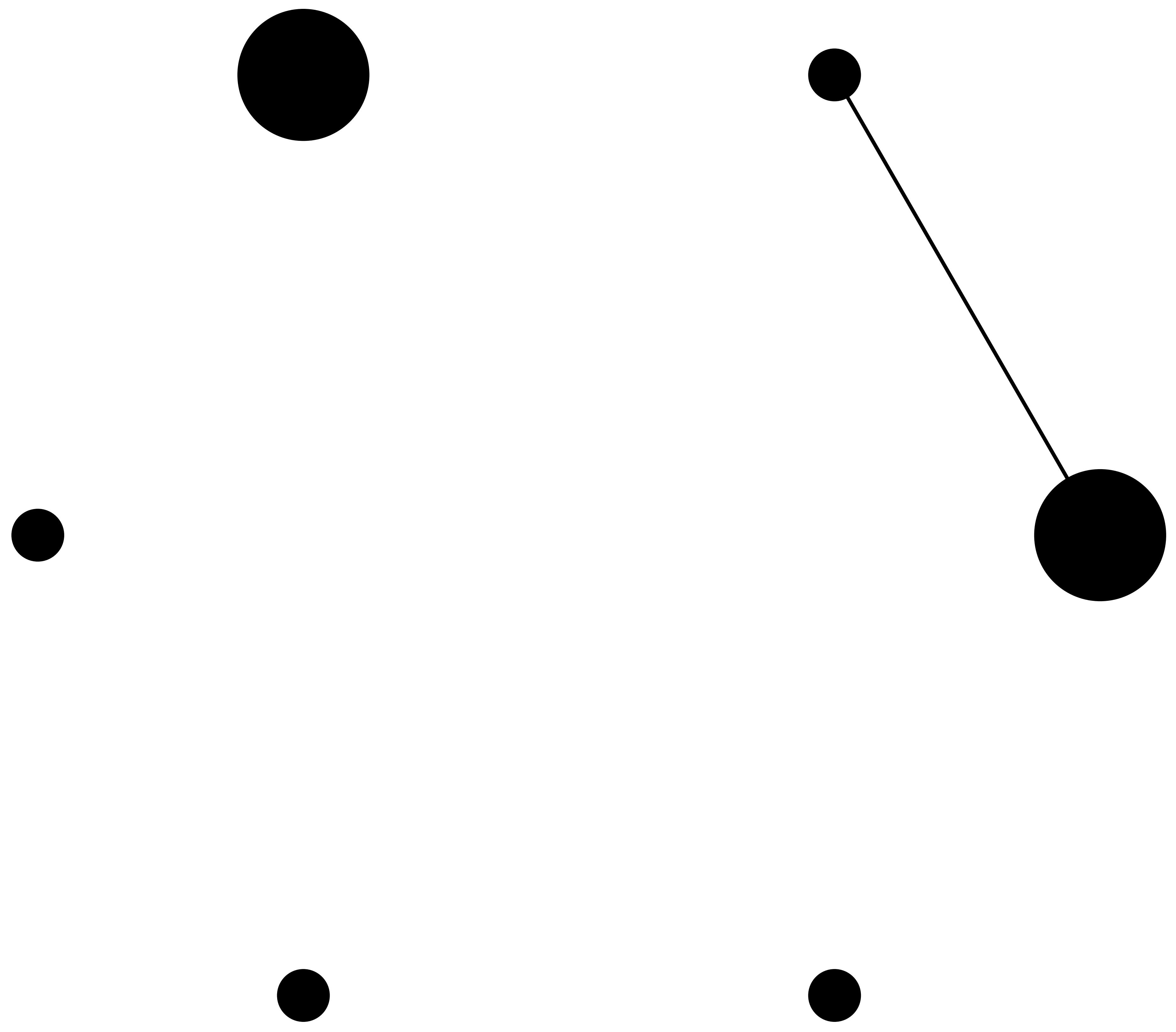
Erdos-Renyi graphs



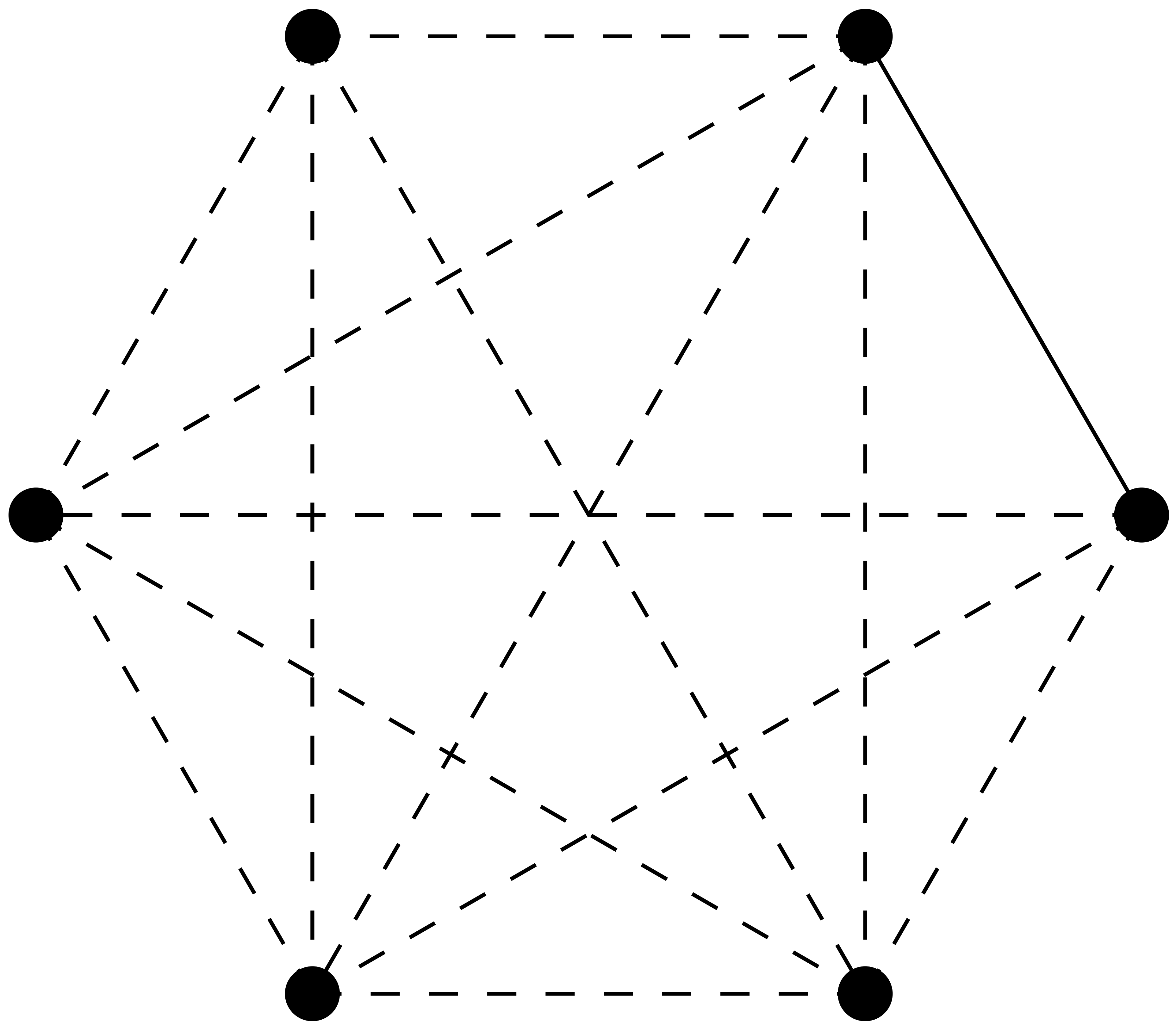
Erdos-Renyi graphs



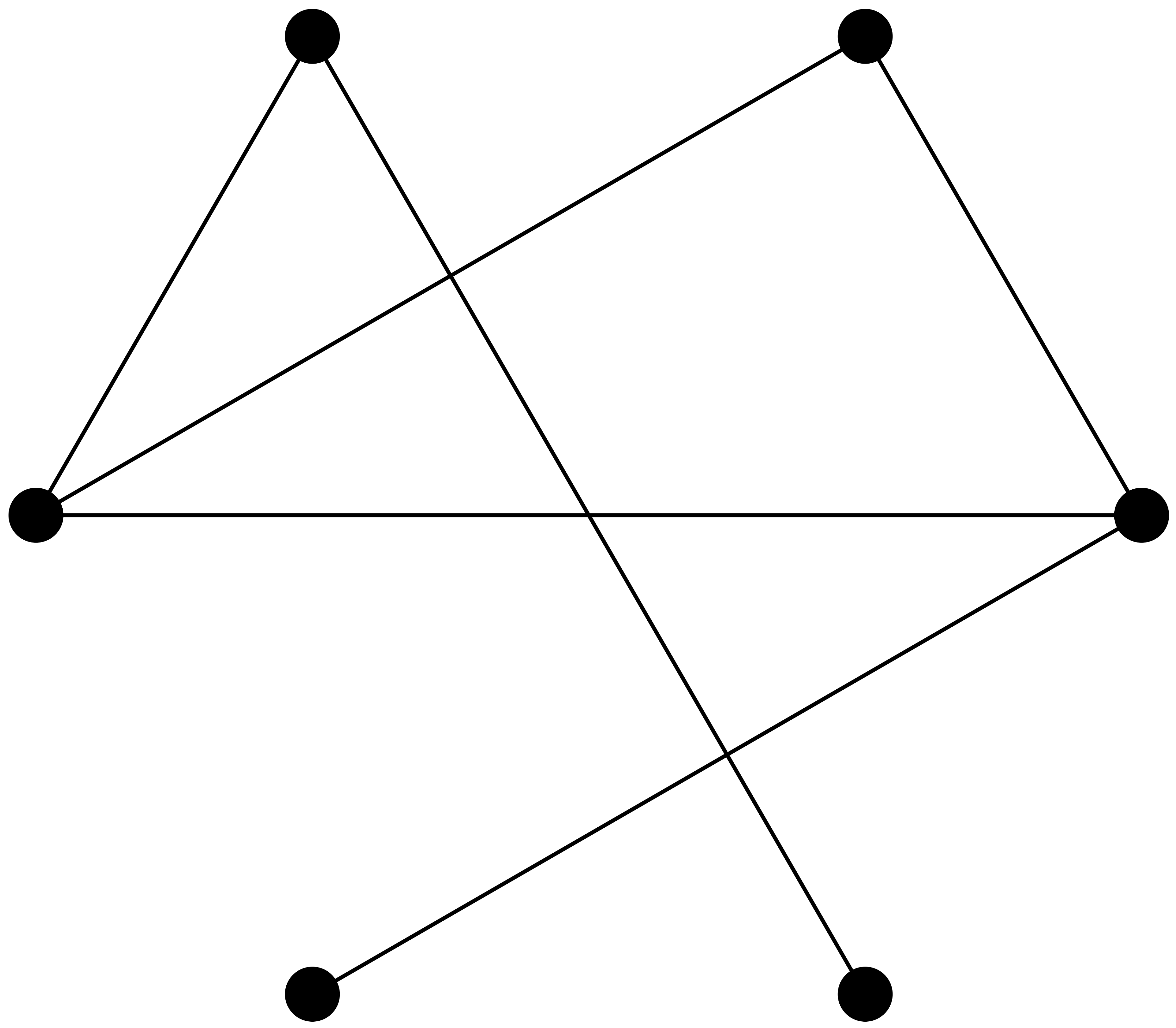
Erdos-Renyi graphs



Erdos-Renyi graphs

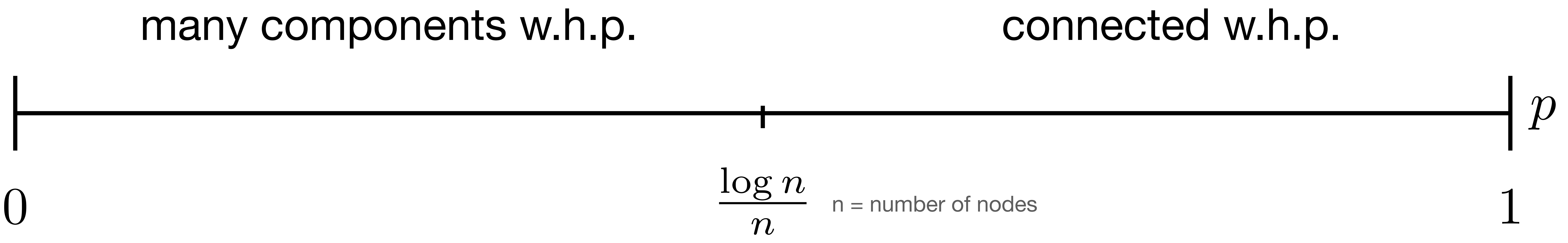


Erdos-Renyi graphs



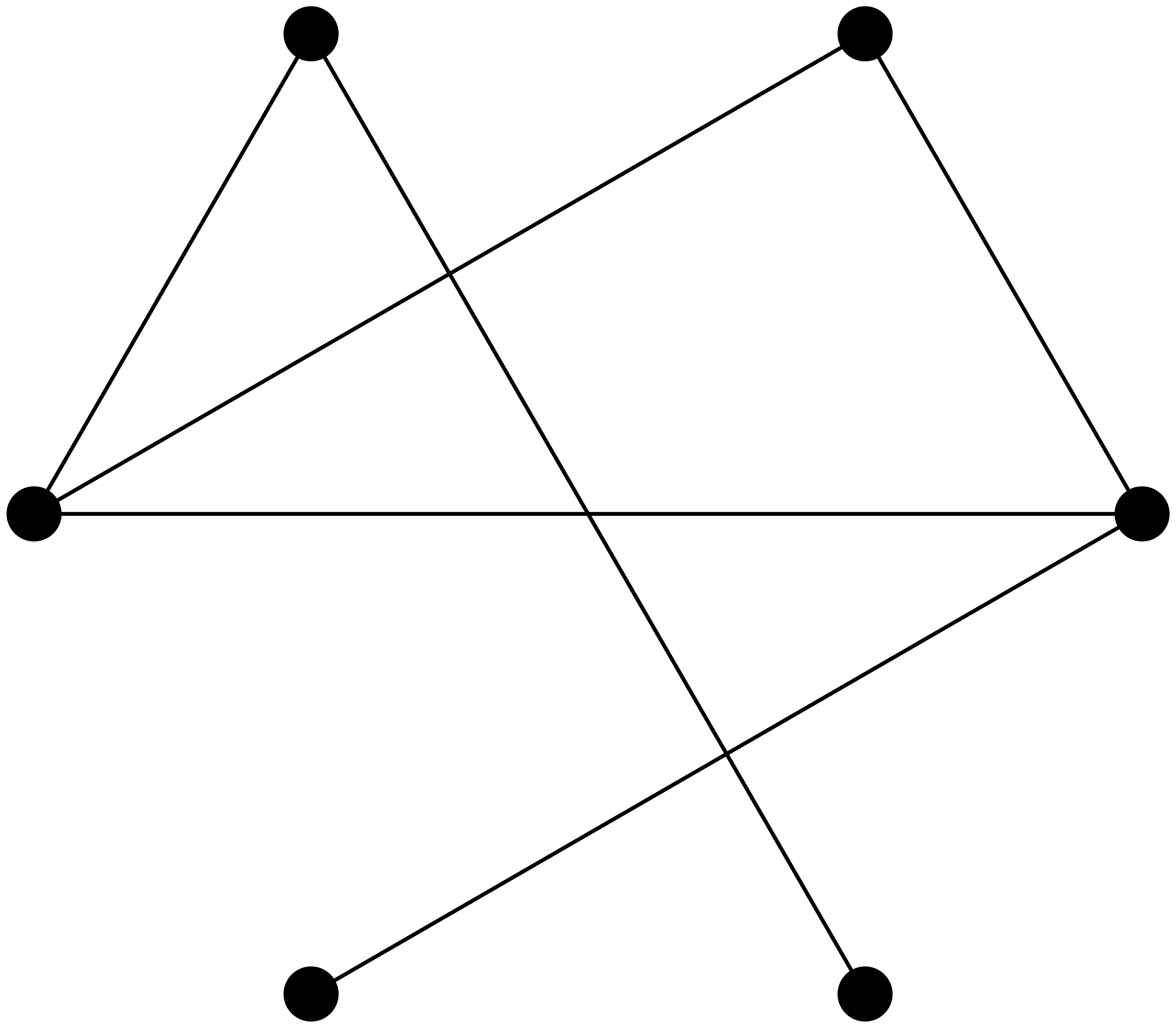
Phase Transition

[Erdos-Renyi 1960]

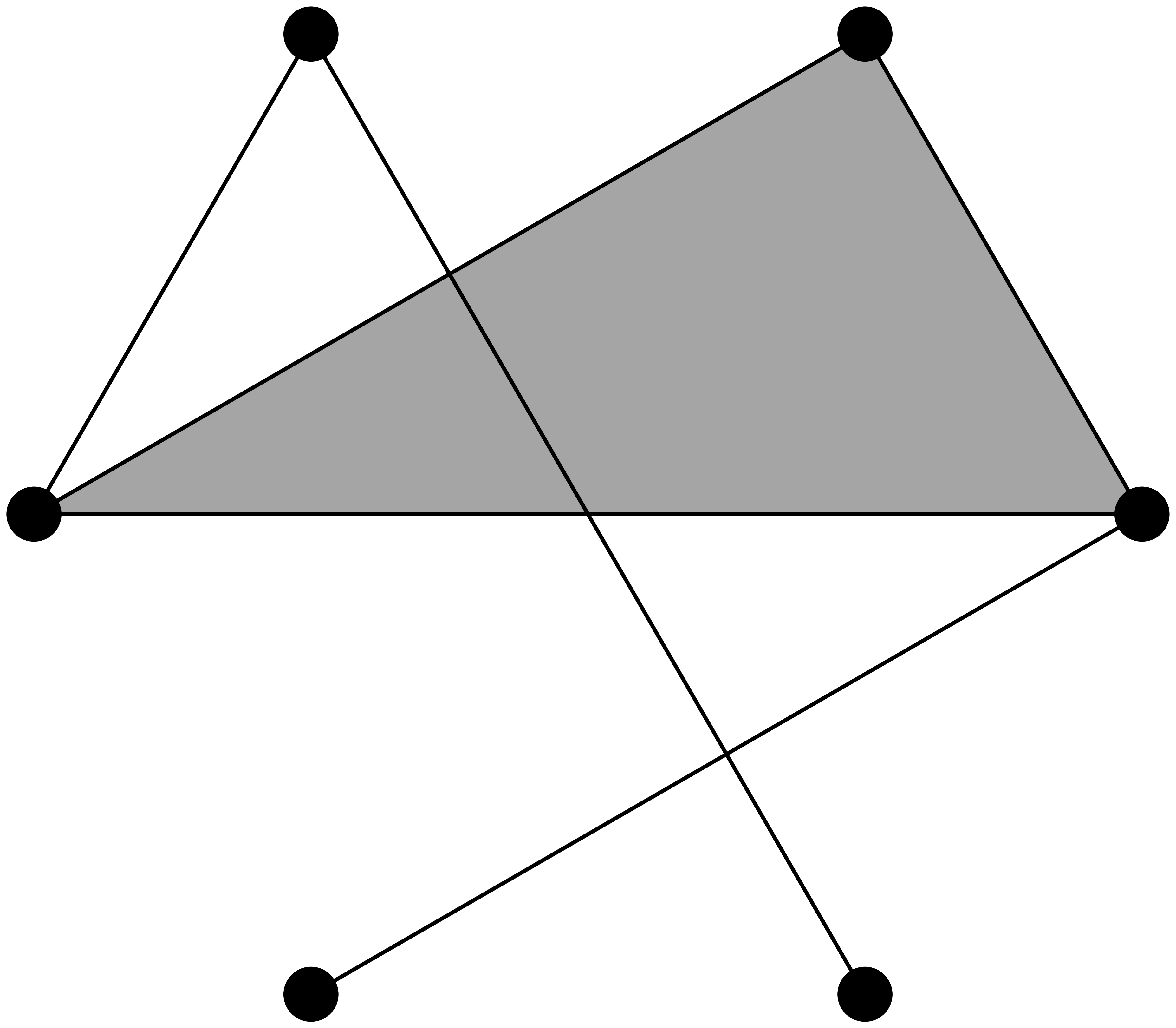


all log terms and constants forgone

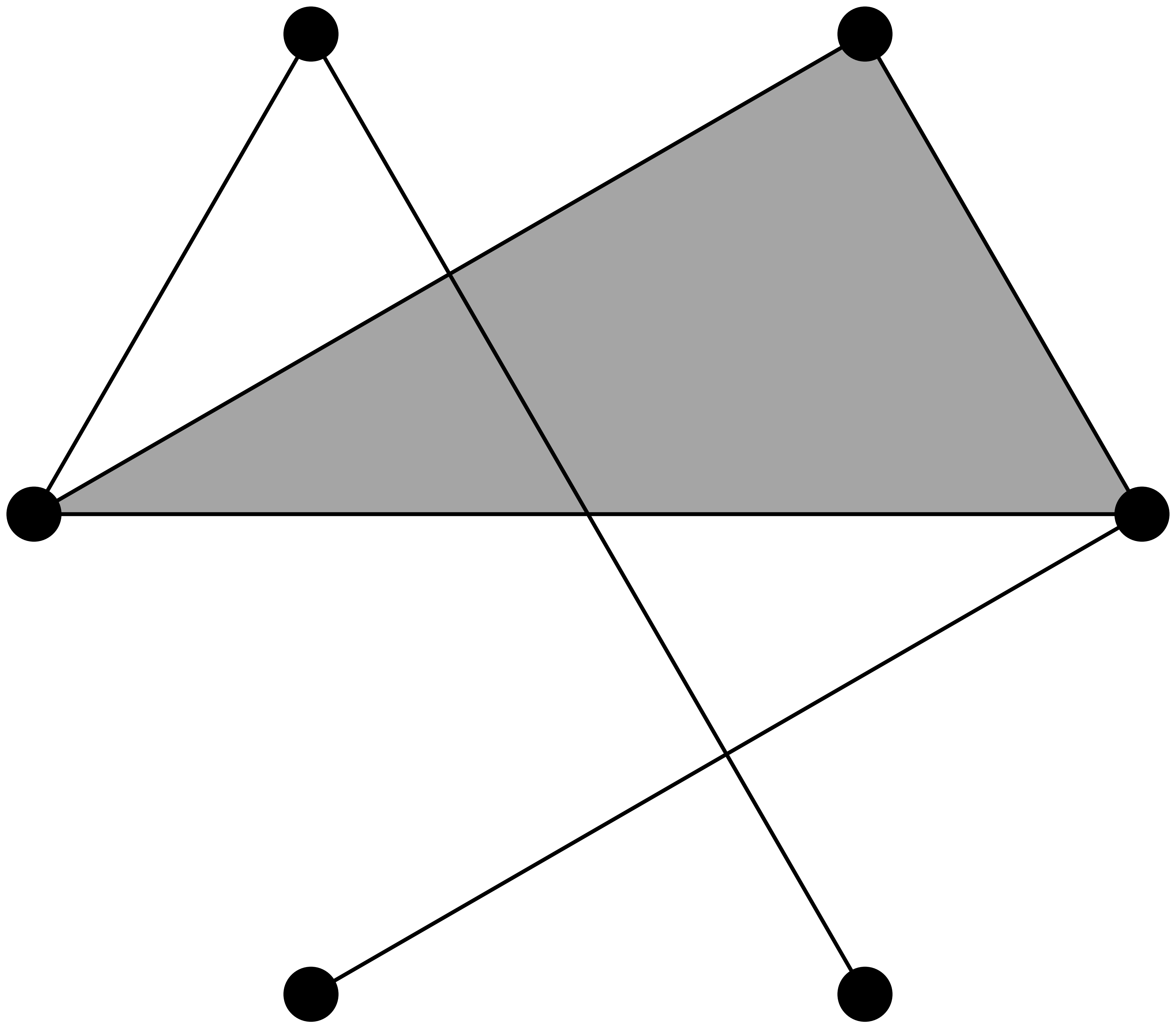
Erdos-Renyi Clique Complex

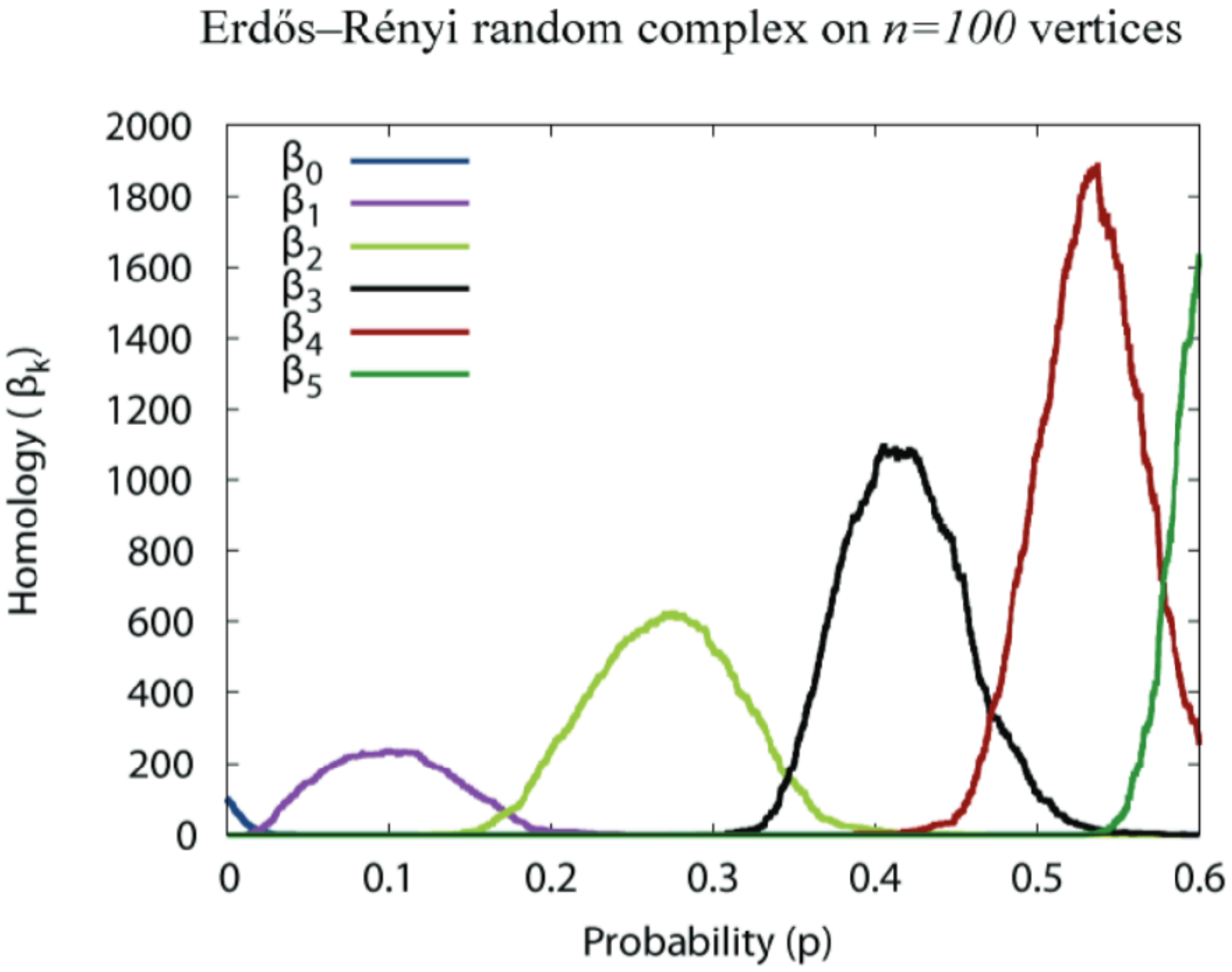


Erdos-Renyi Clique Complex



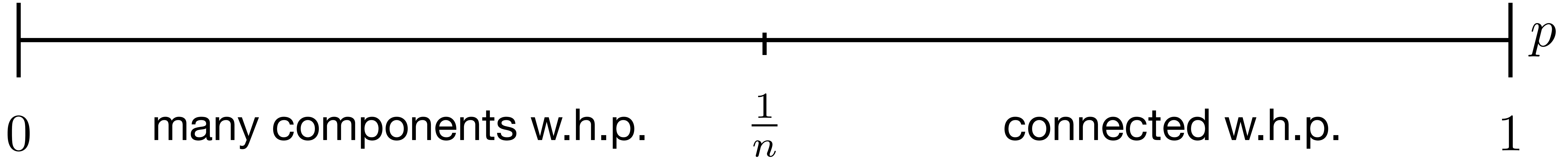
Betti Numbers





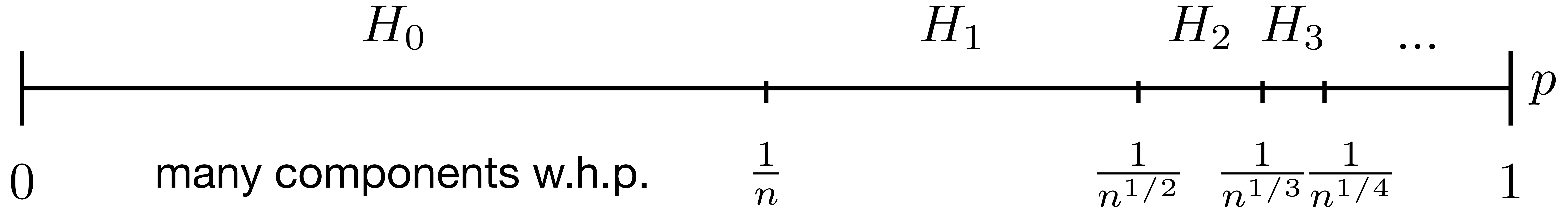
computation and plotting done by Zomorodian

Phase Transition
[Erdos-Renyi 1960]



n = number of nodes
all log terms and constants forgone

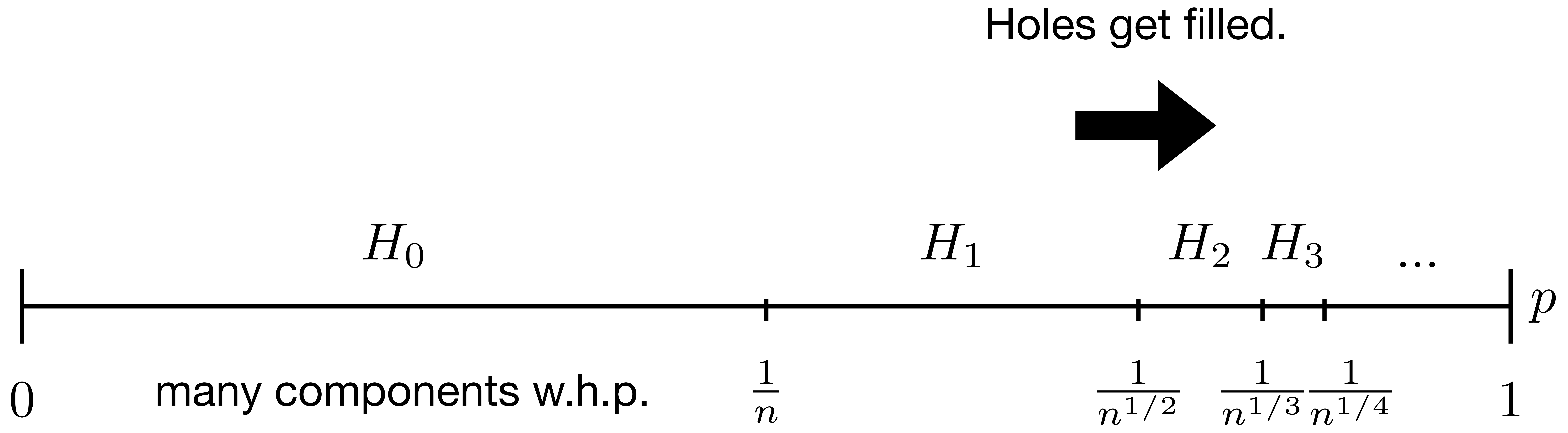
Phase Transition
[Kahle 2009, 2014]



n = number of nodes
all log terms and constants forgone

Phase Transition

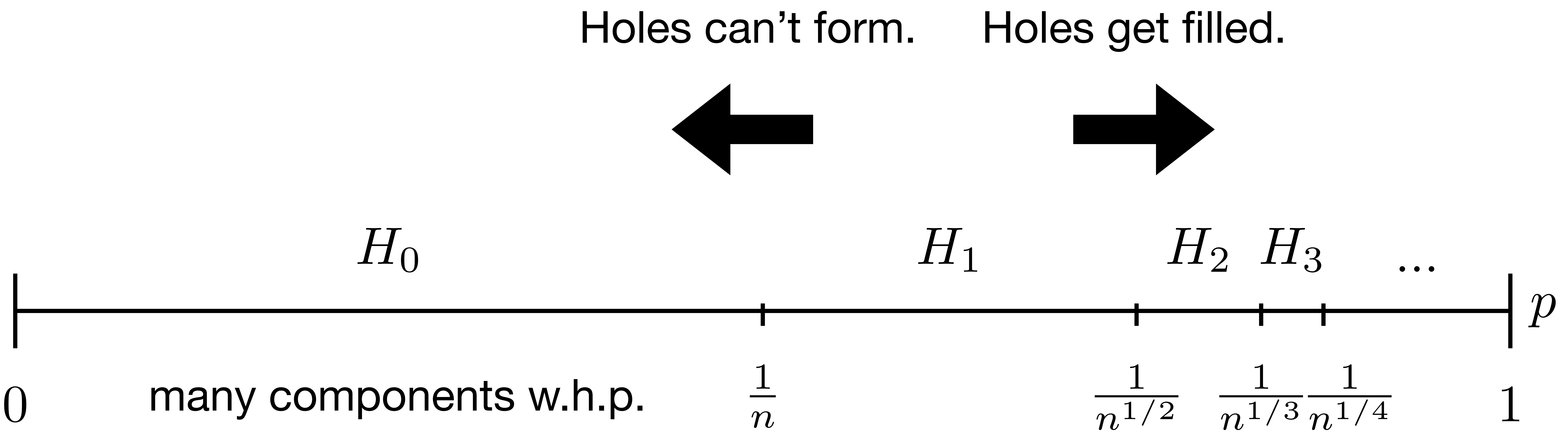
[Kahle 2009, 2014]



n = number of nodes
all log terms and constants forgone

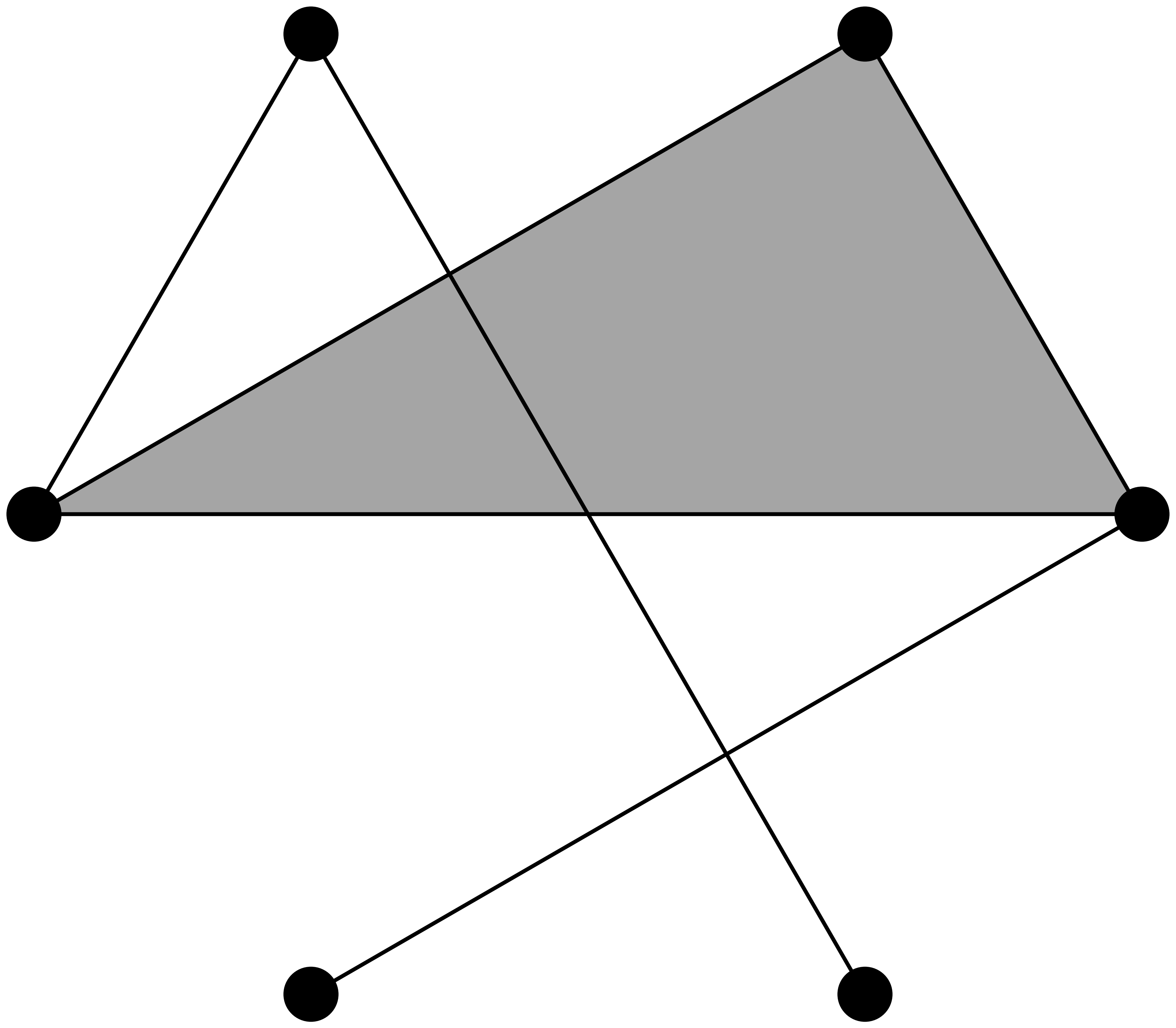
Phase Transition

[Kahle 2009, 2014]



n = number of nodes
all log terms and constants forgone

Erdos-Renyi Clique Complex



Geometric Complexes

- start with Poisson points

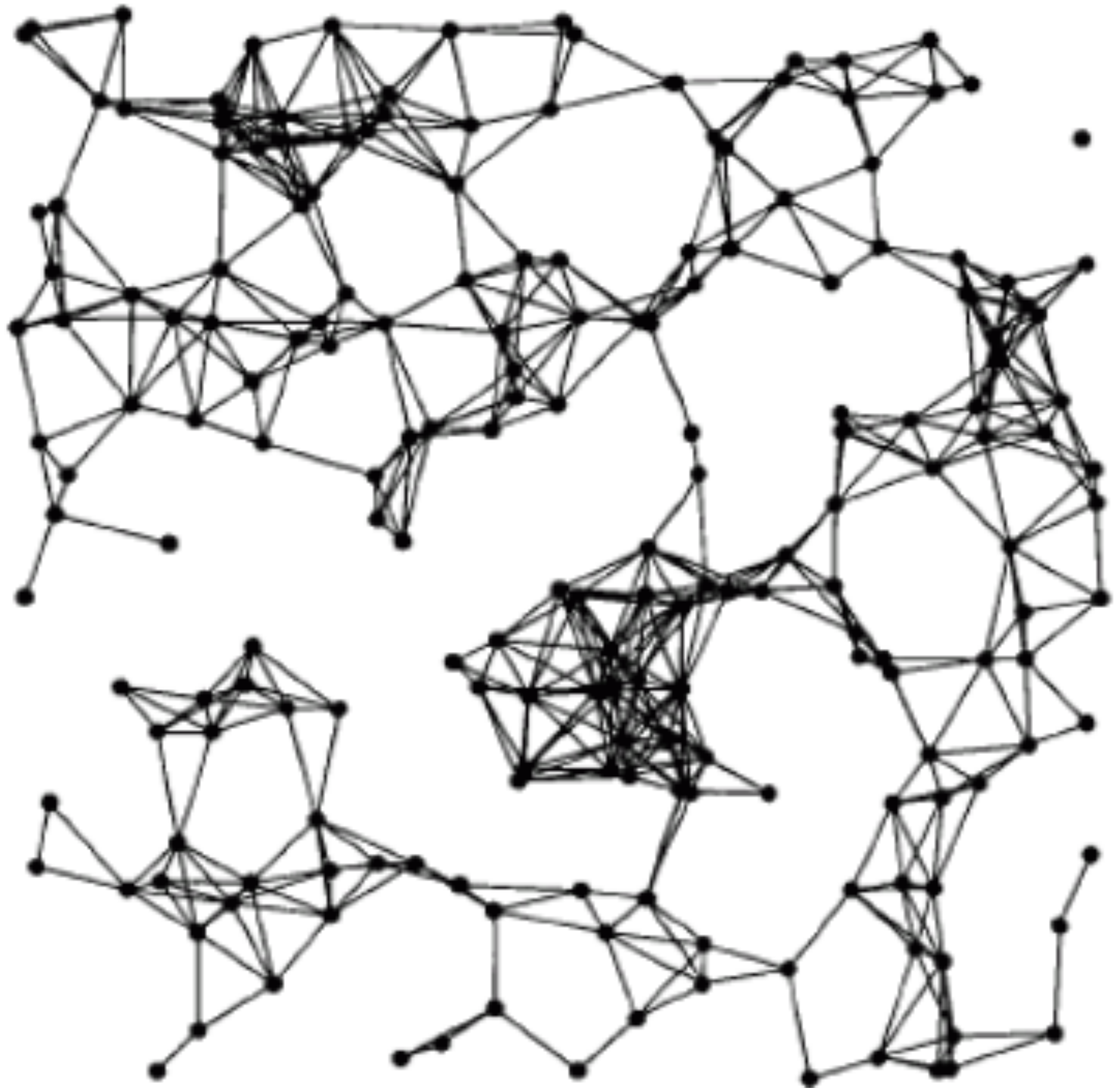


image credit: Penrose

Geometric Complexes

- start with Poisson points
- Rips
- Cech

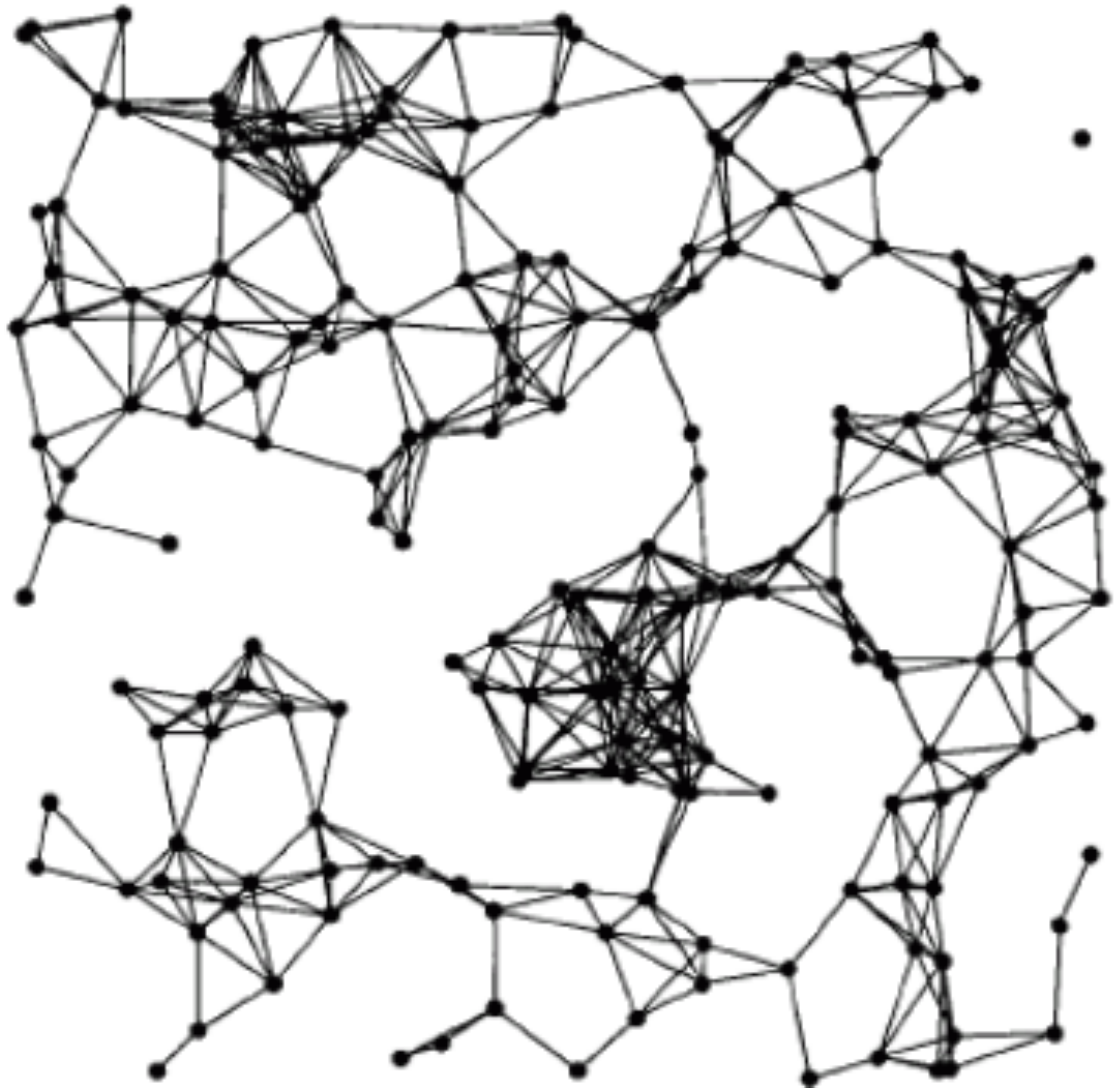


image credit: Penrose

Geometric Complexes

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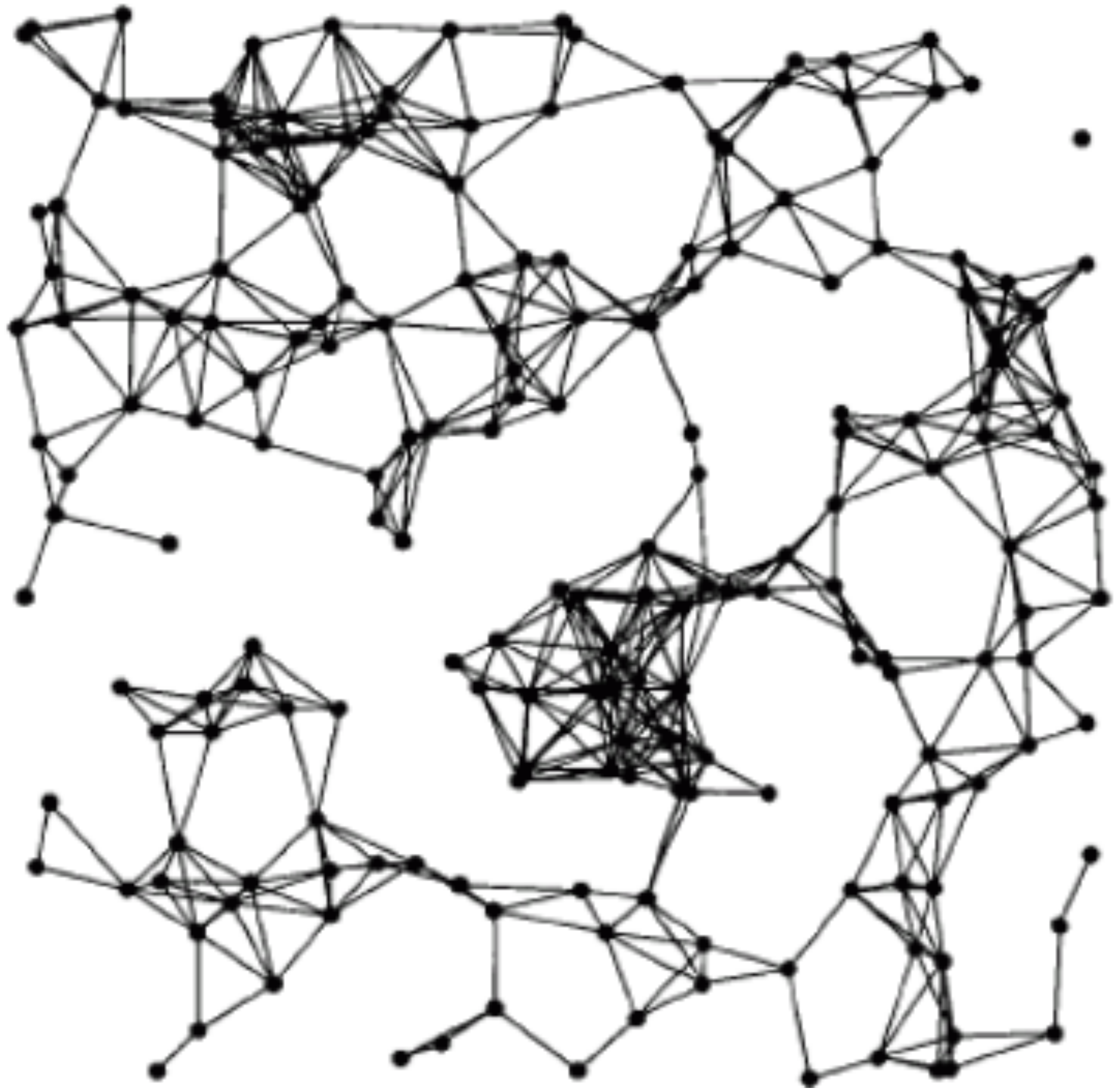

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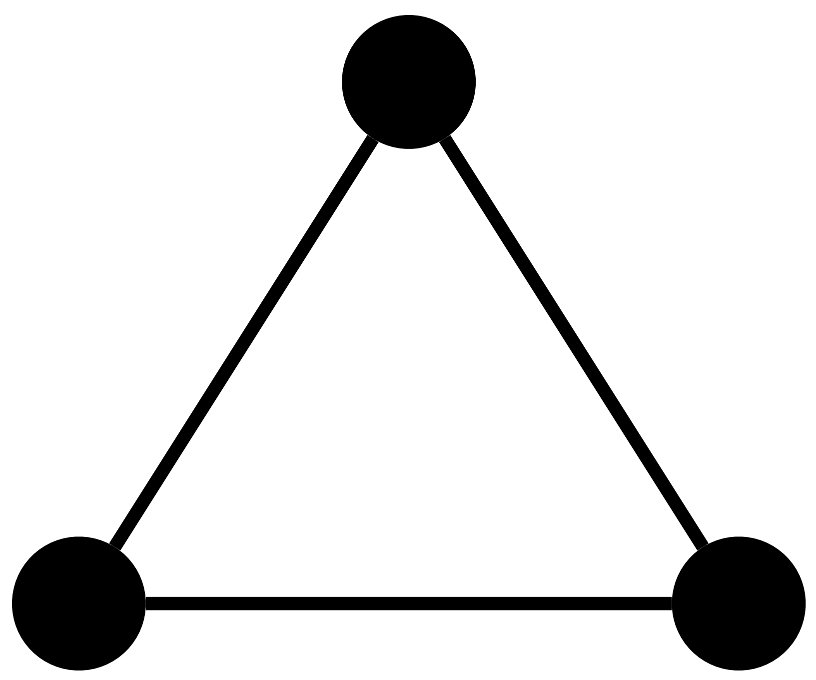

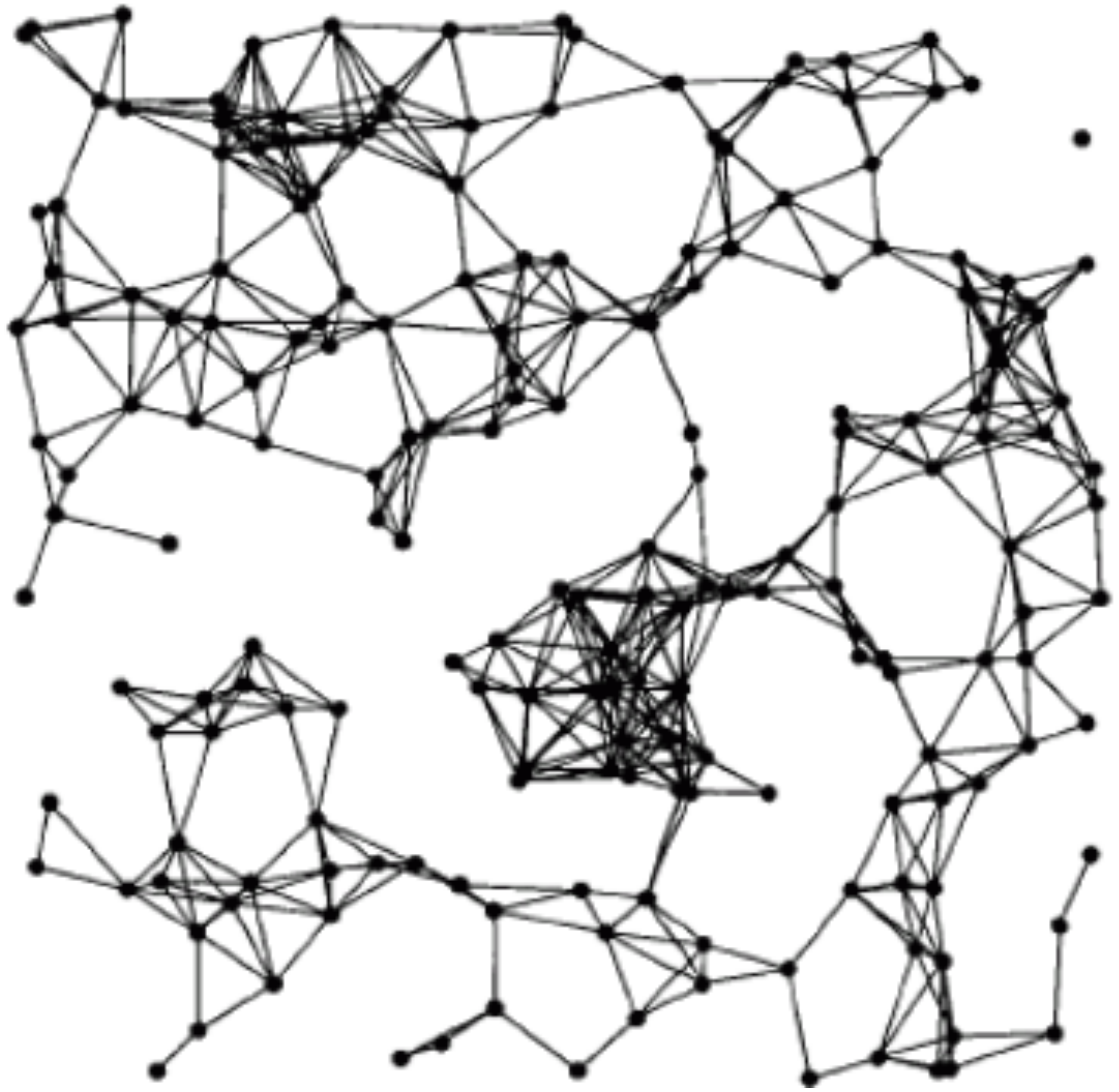

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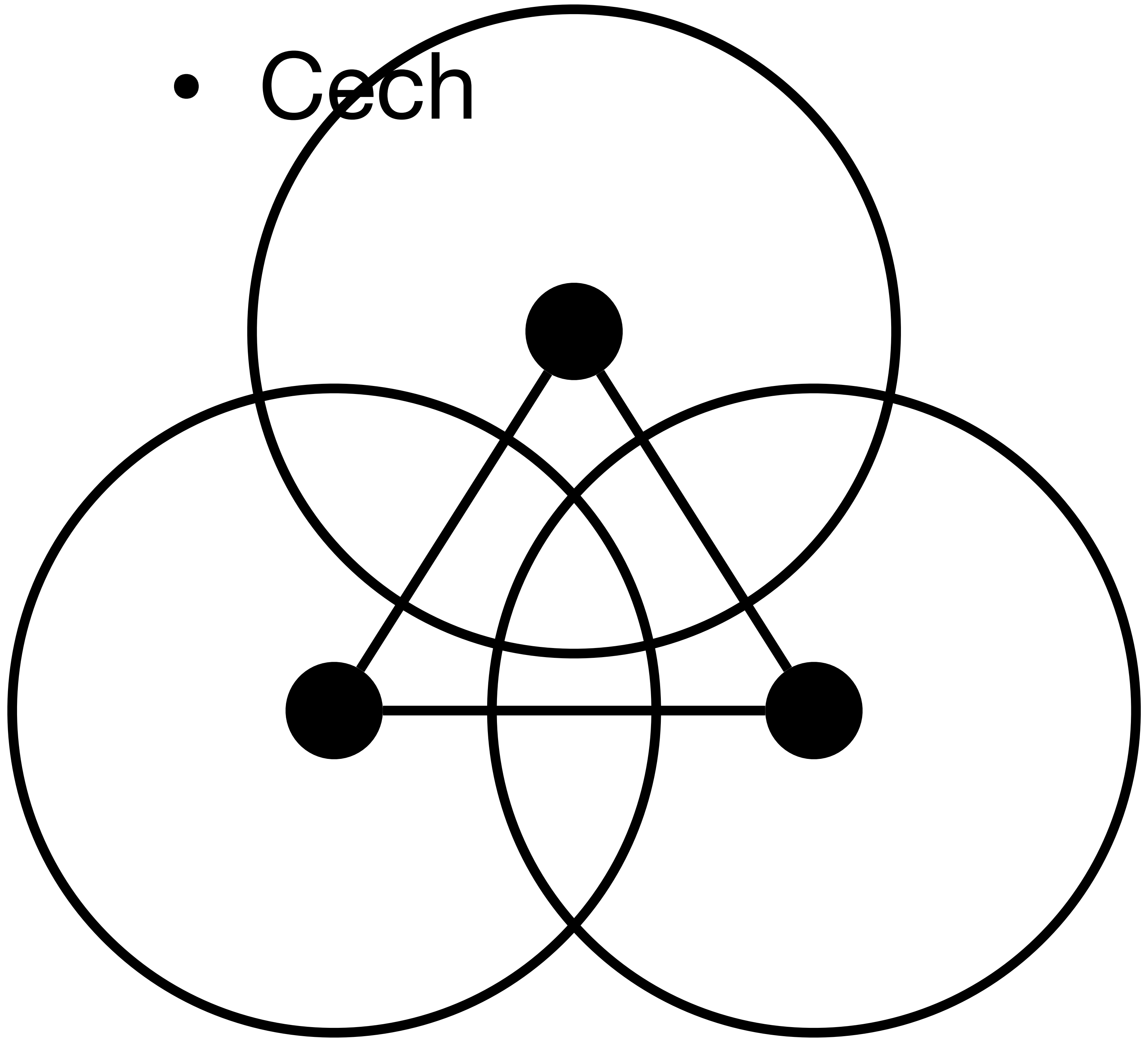

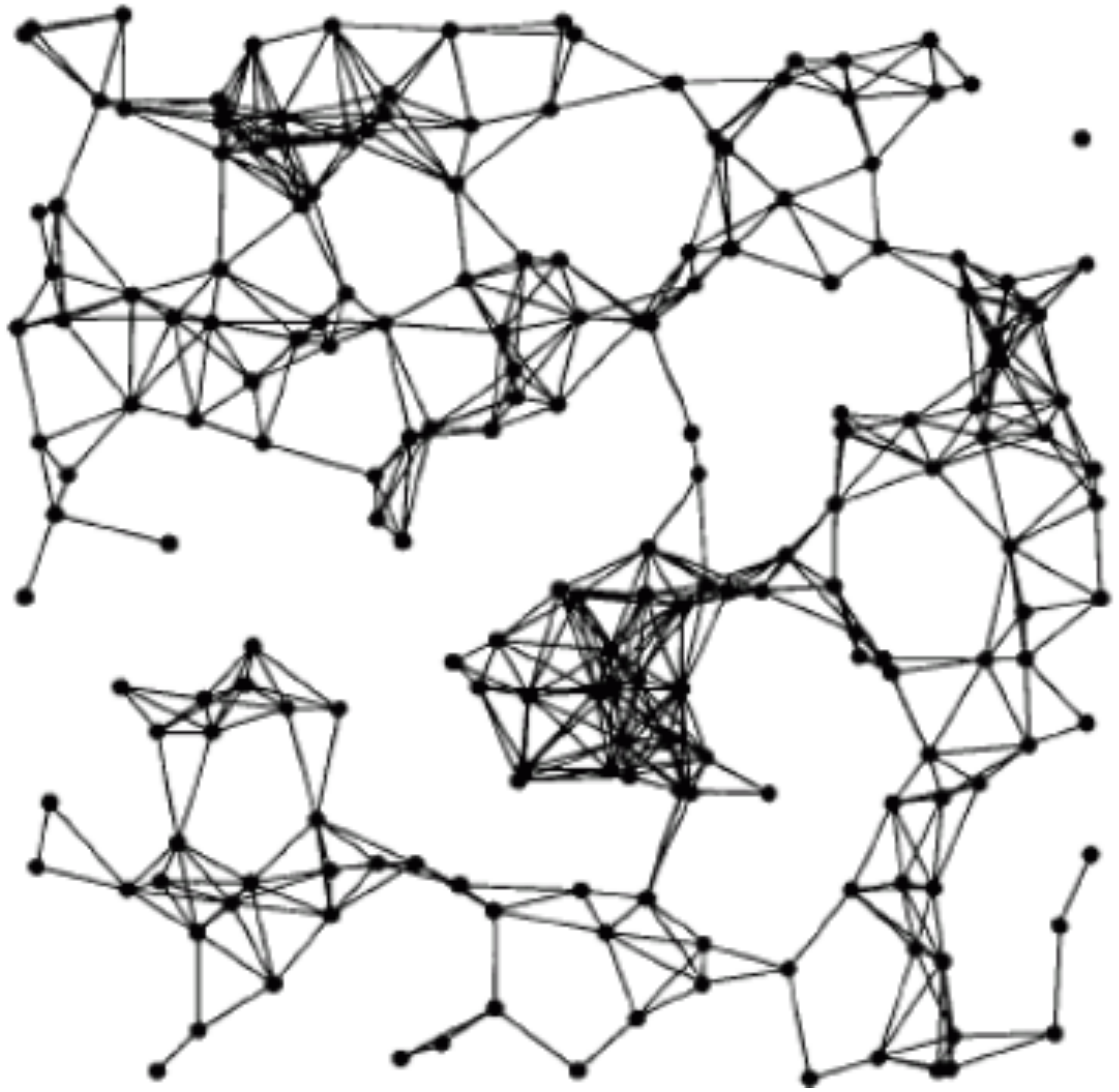

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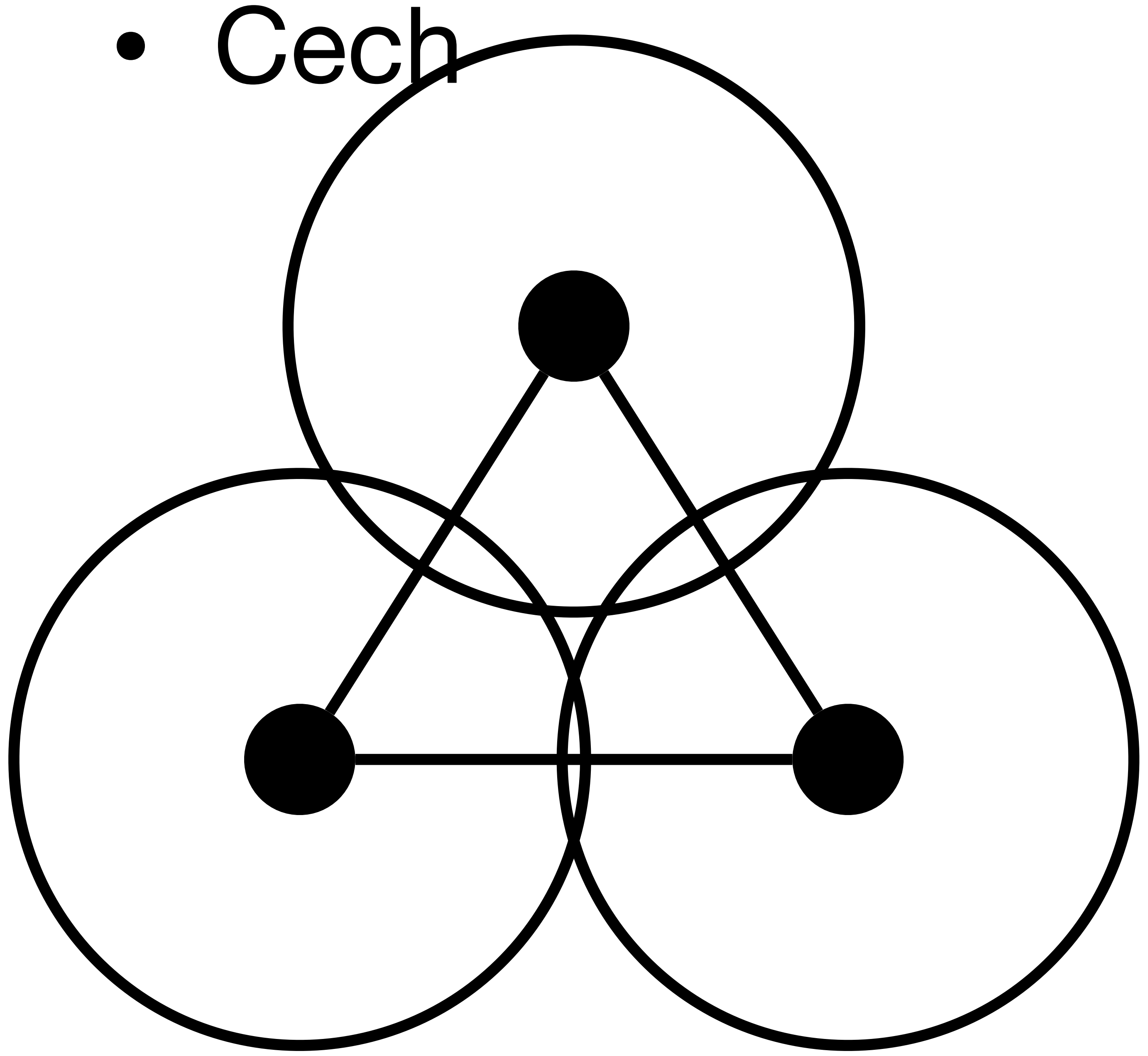

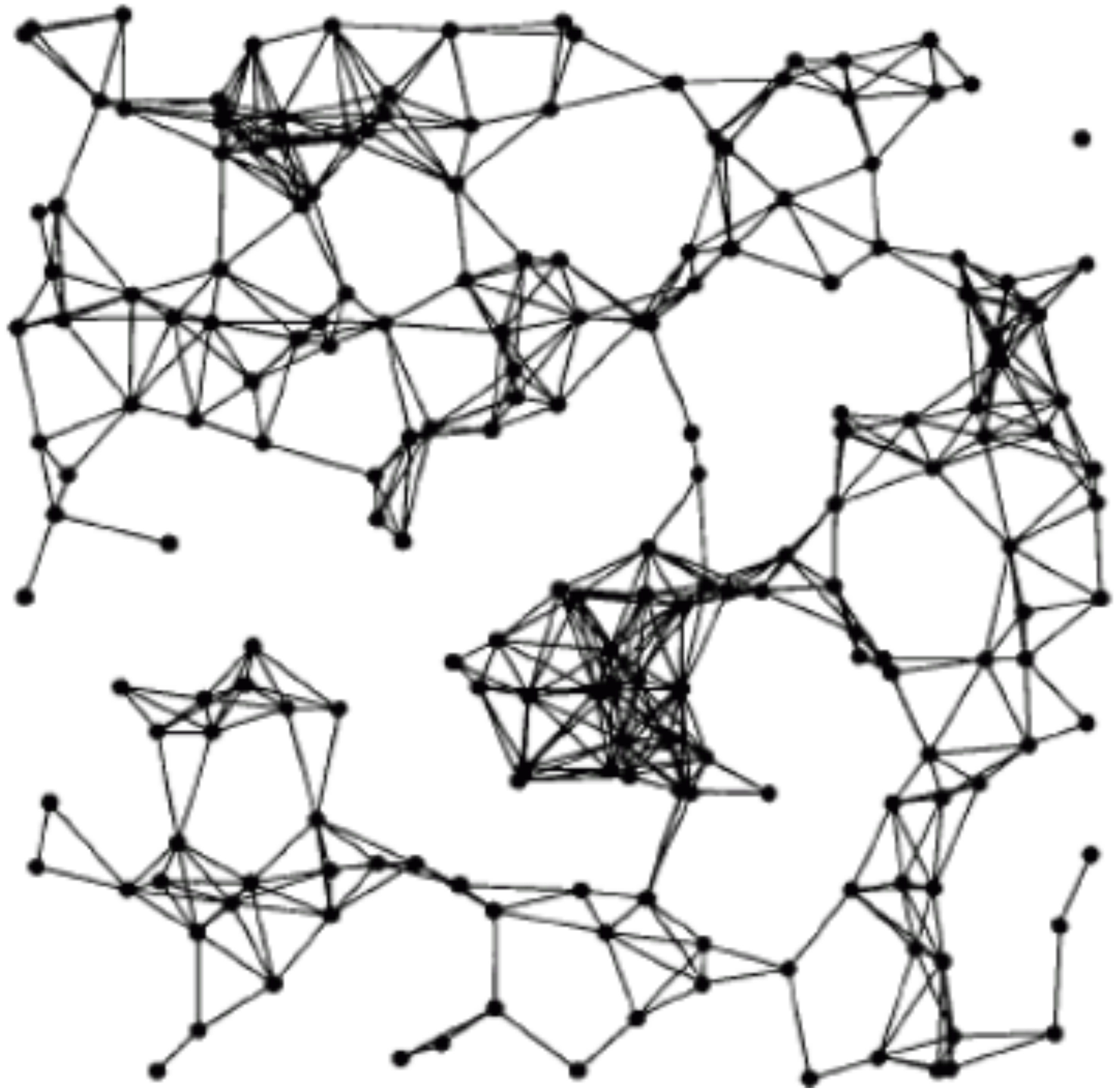

image credit: Penrose

Expected Betti numbers at dimension k

[Kahle 2011]

Expected Betti numbers at dimension k

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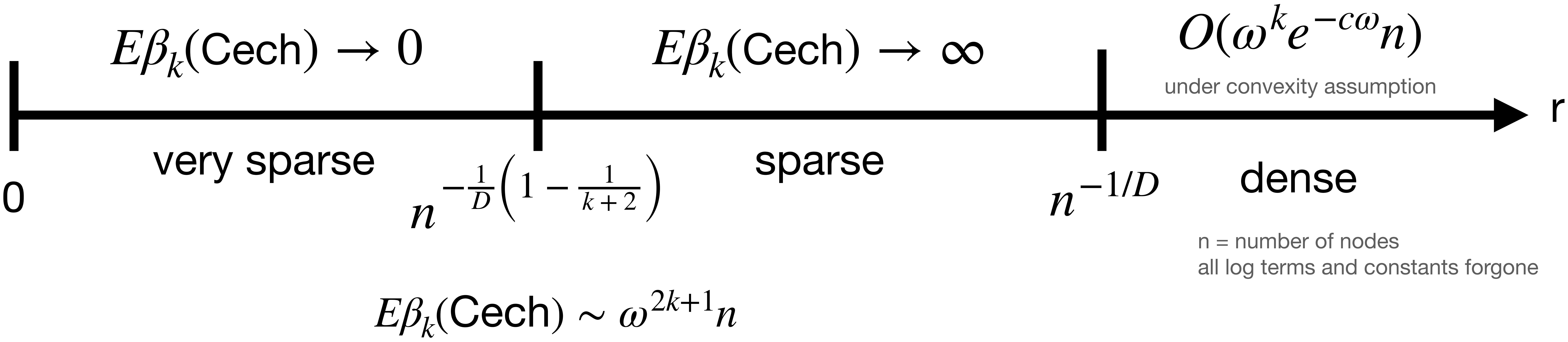
- n, the number of points

Expected Betti numbers at dimension k
[Kahle 2011]

- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension

Expected Betti numbers at dimension k
[Kahle 2011]

- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension



Geometric Complexes

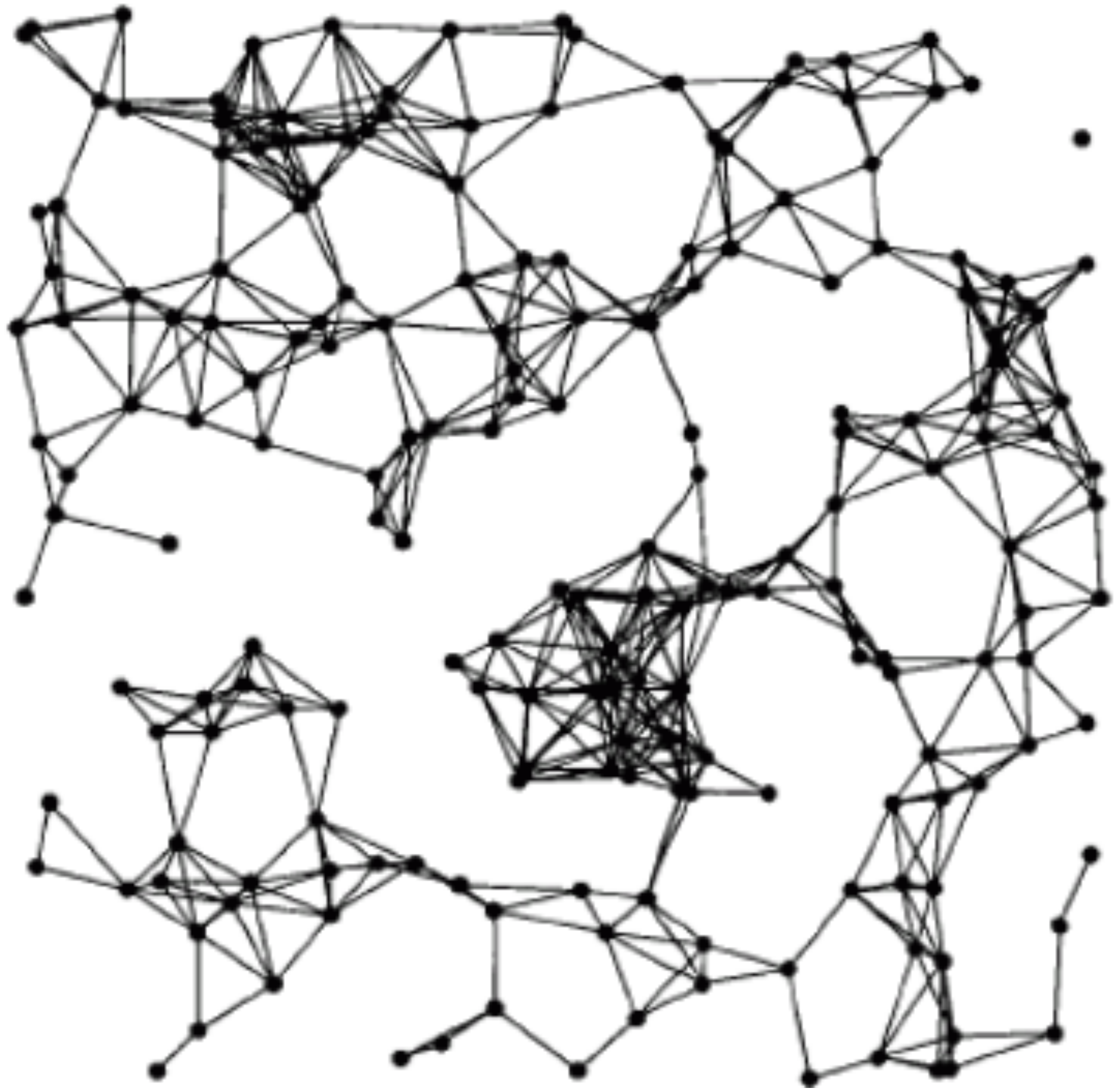
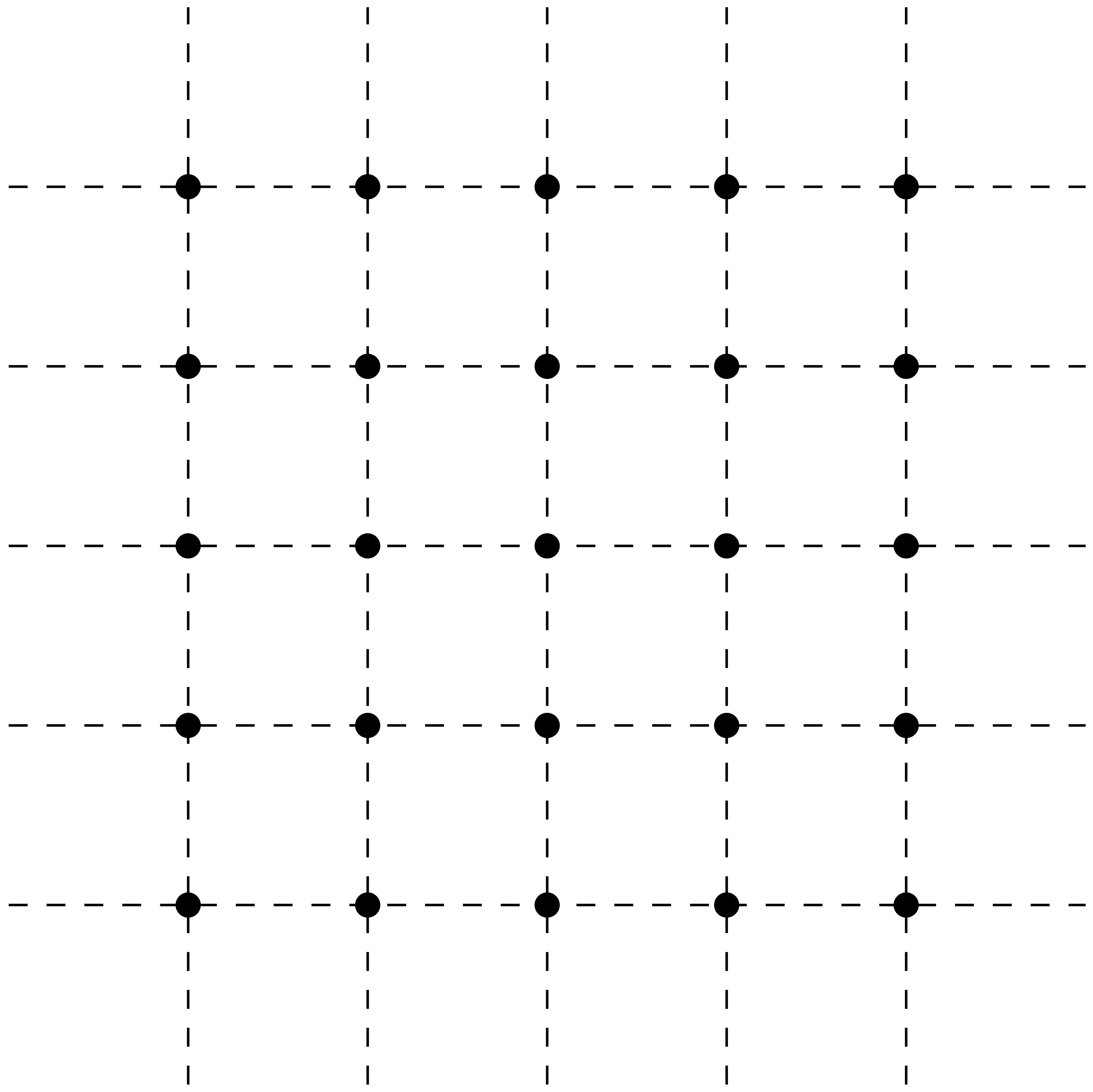
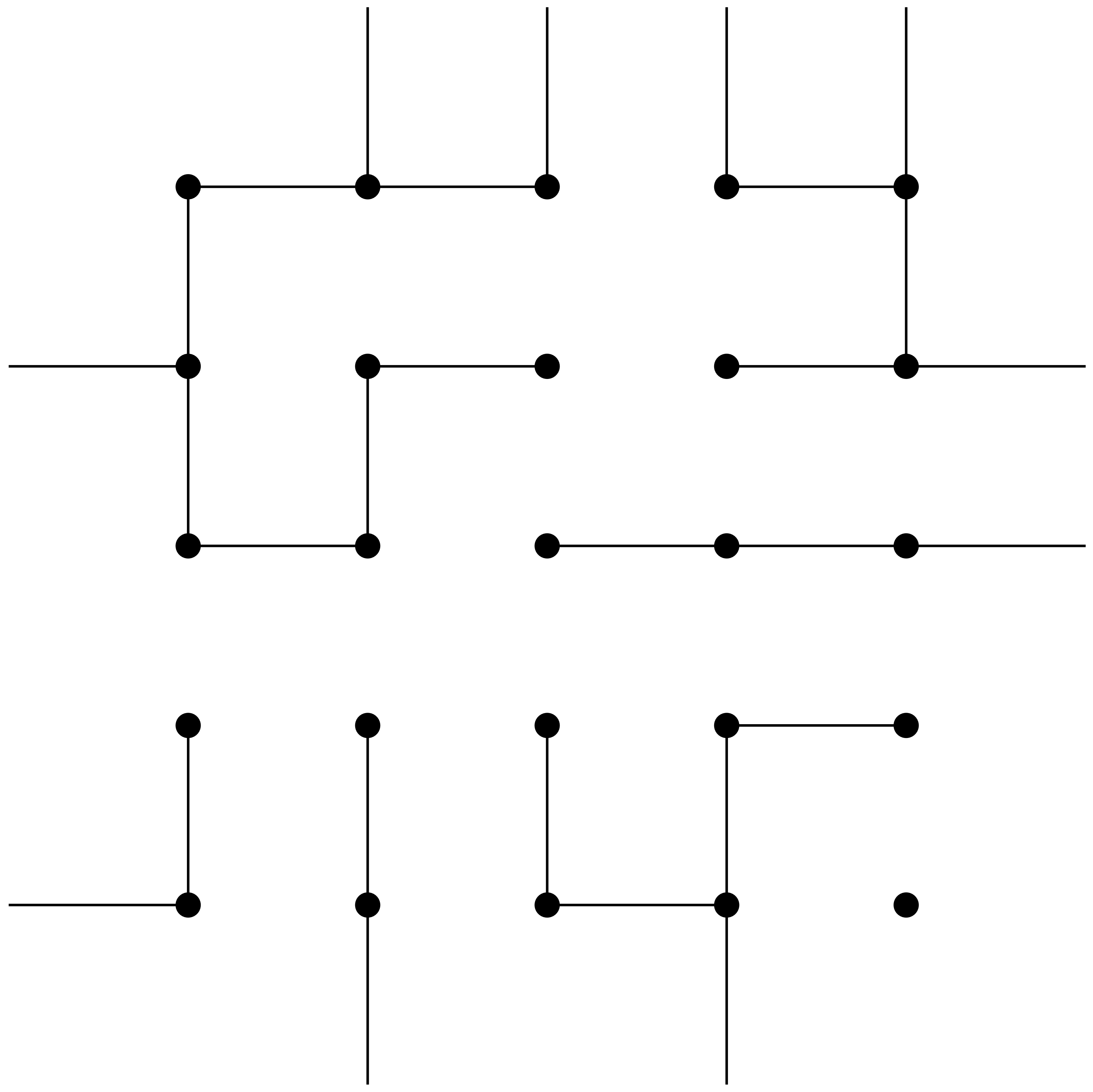


image credit: Penrose

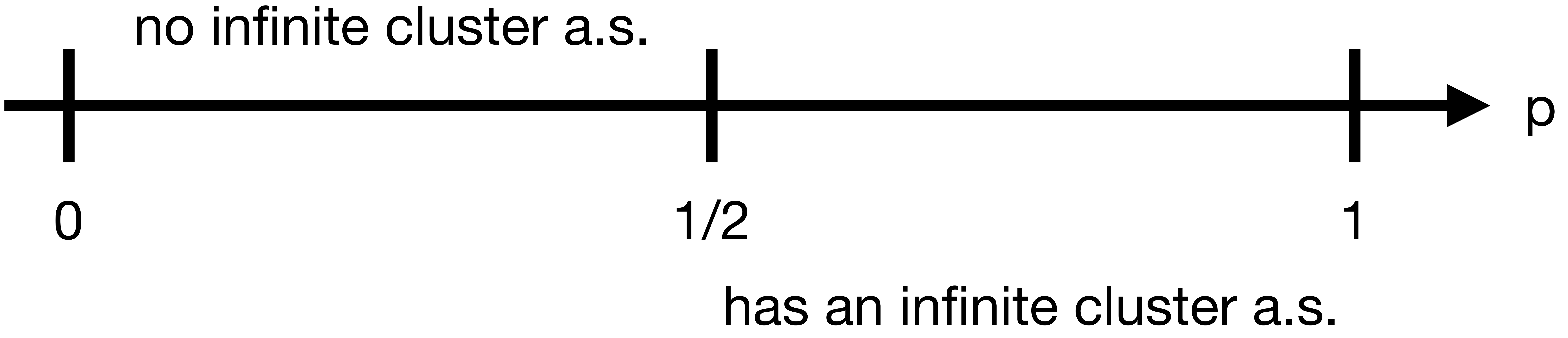
Bernoulli Bond Percolation



Bernoulli Bond Percolation

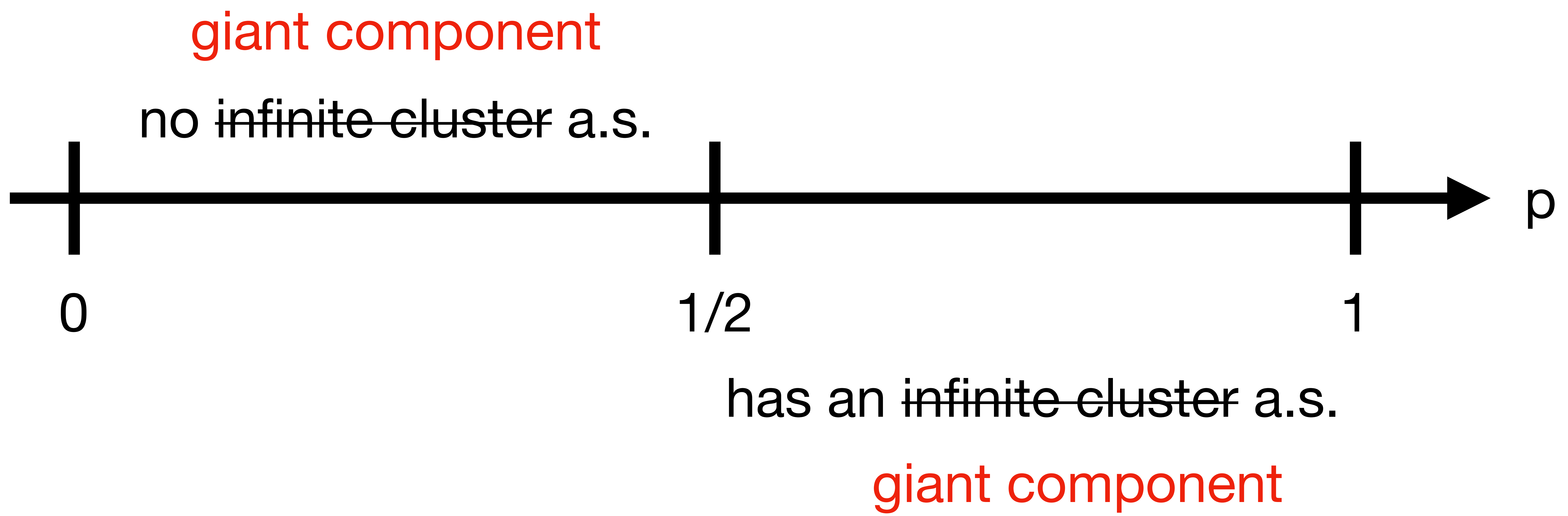


Phase Transition
[Harris 1960, Kesten 1980]



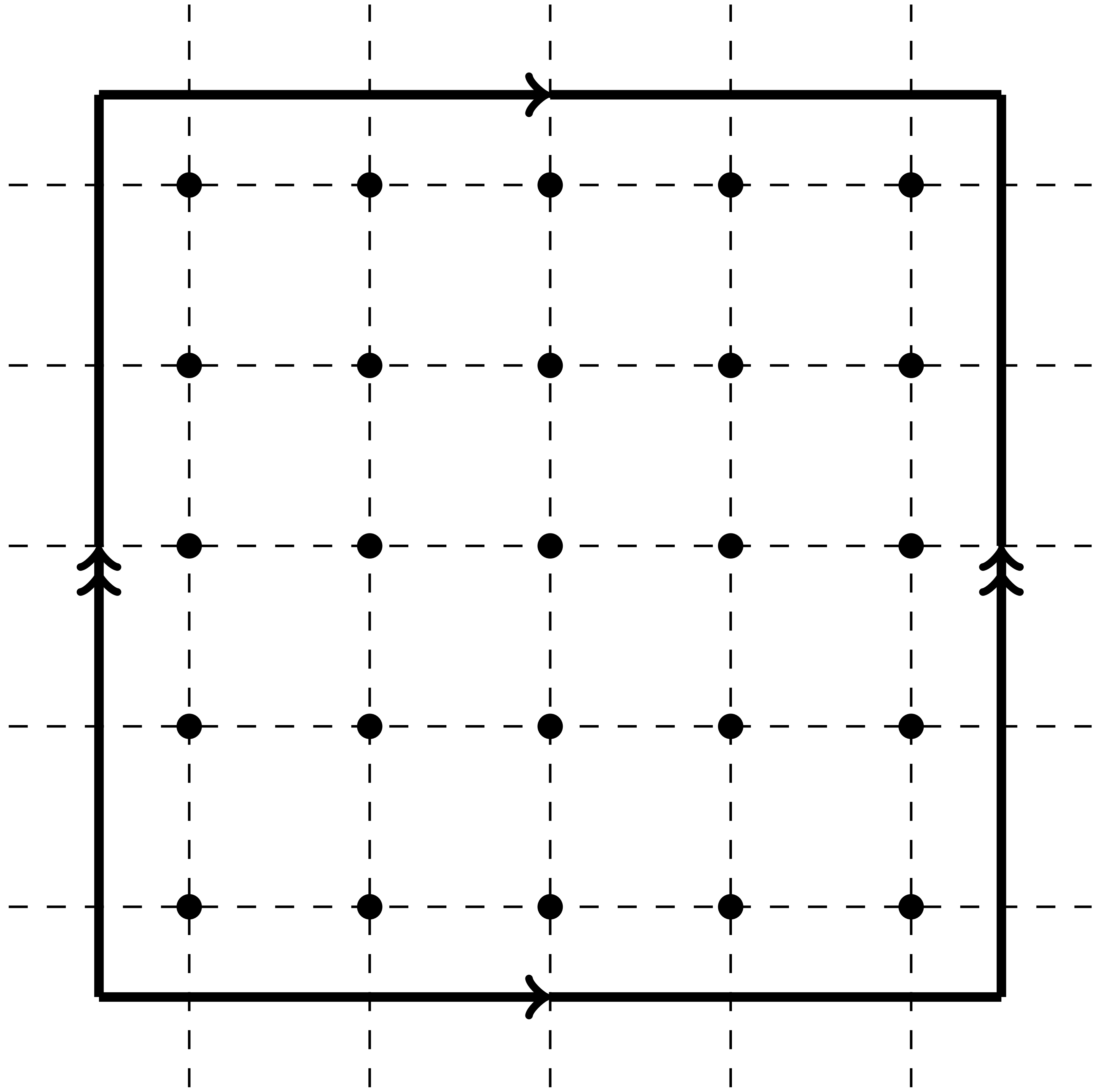
Phase Transition

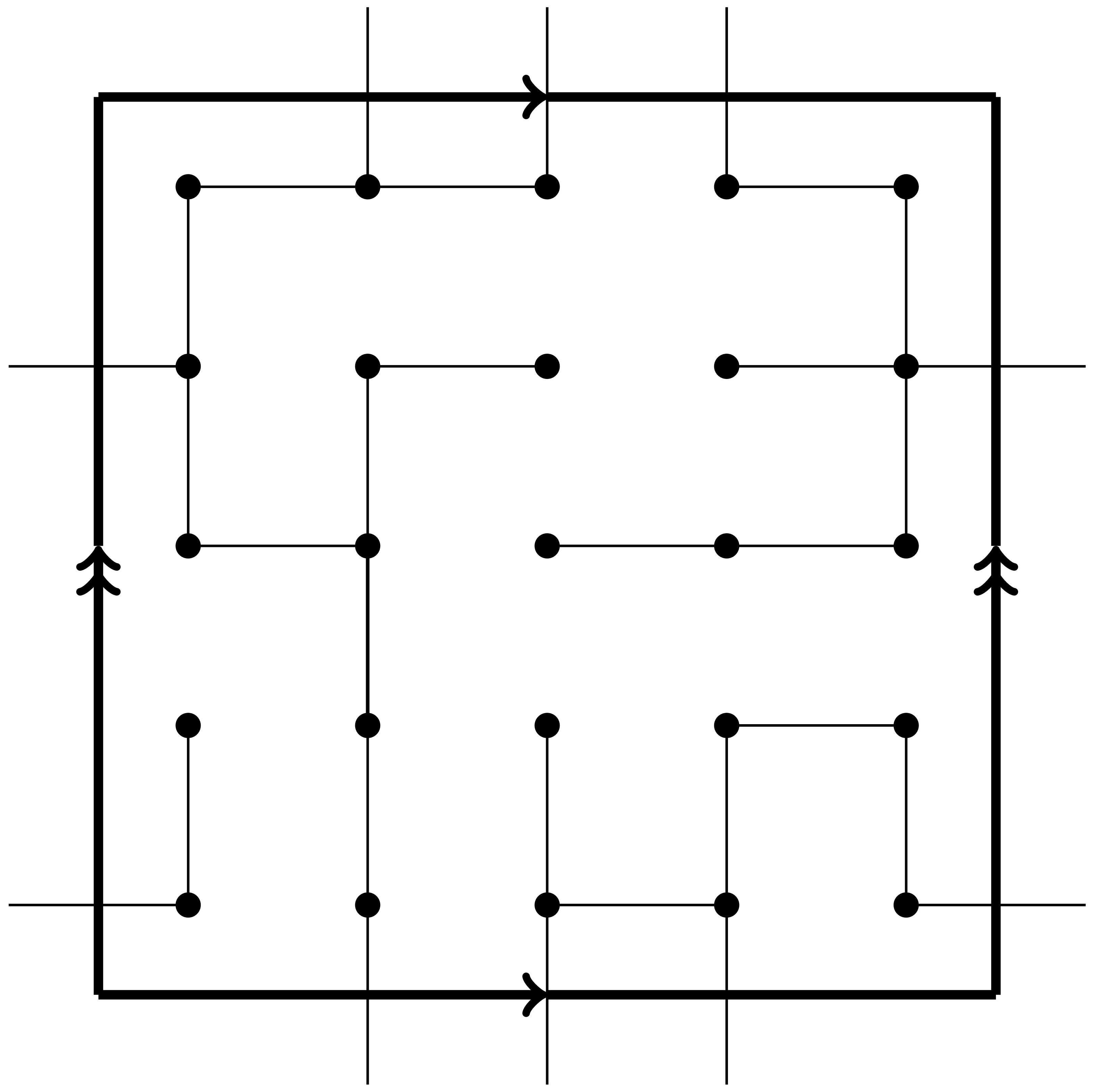
[Harris 1960, Kesten 1980]

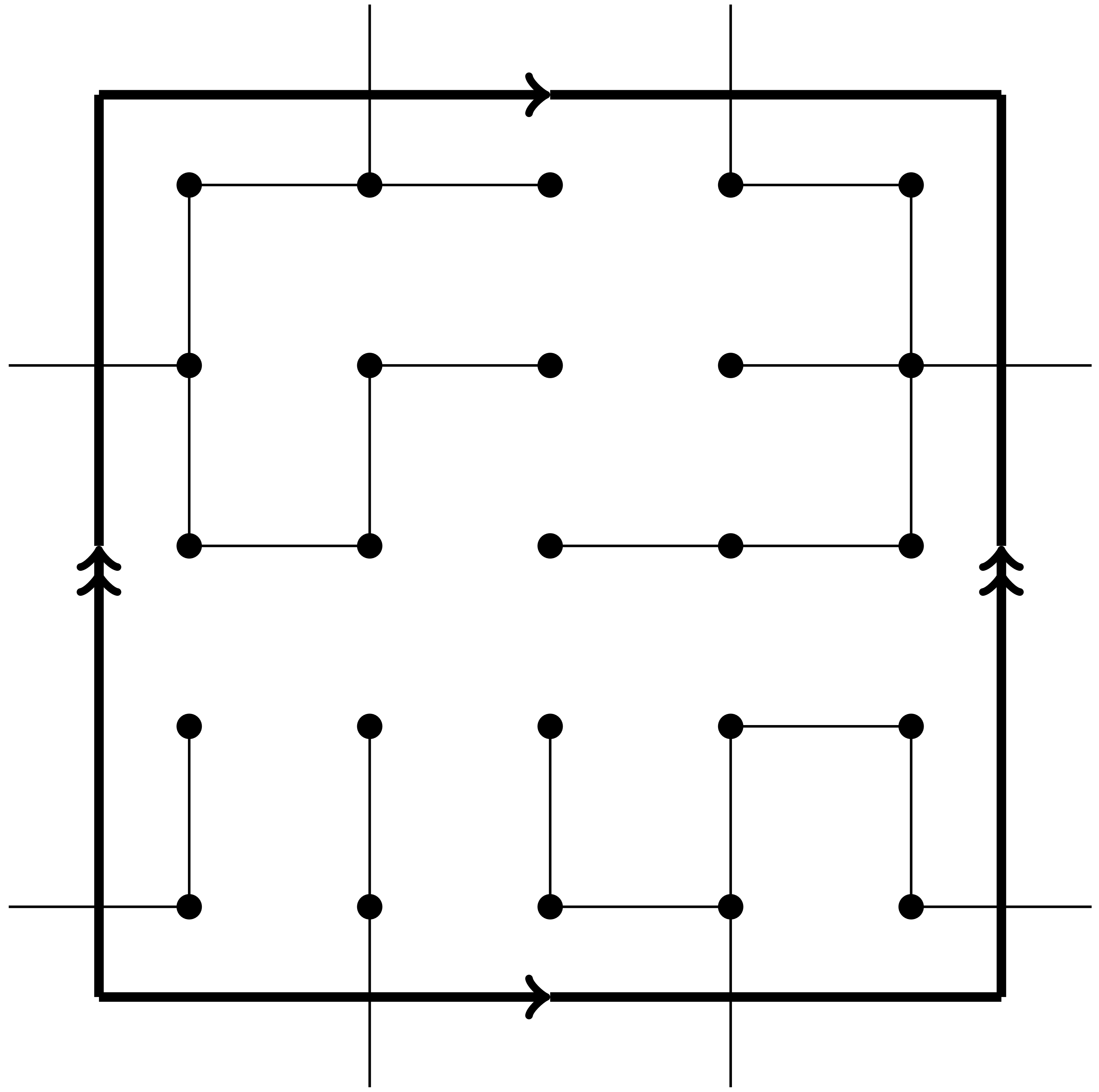


Giant Cycles?

Bernoulli Bond Percolation

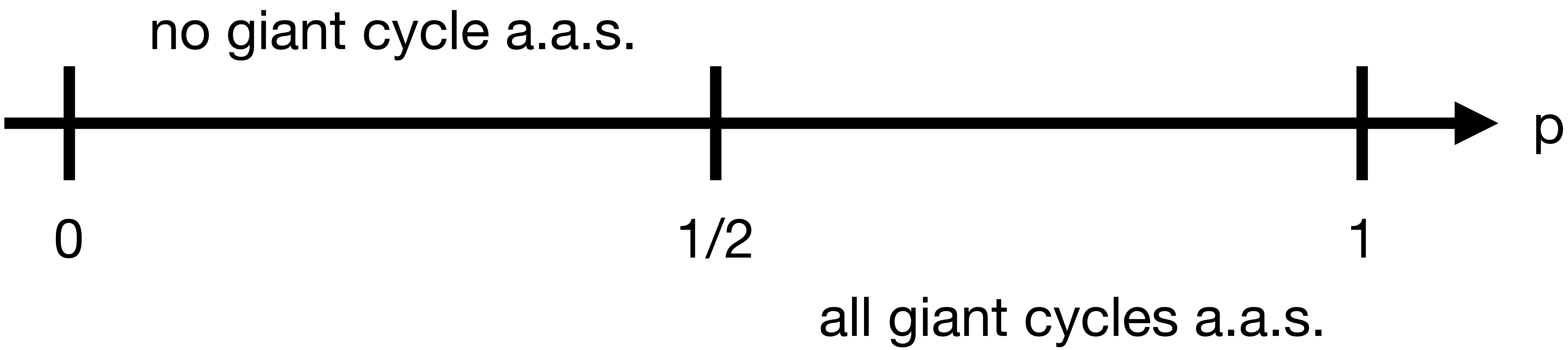




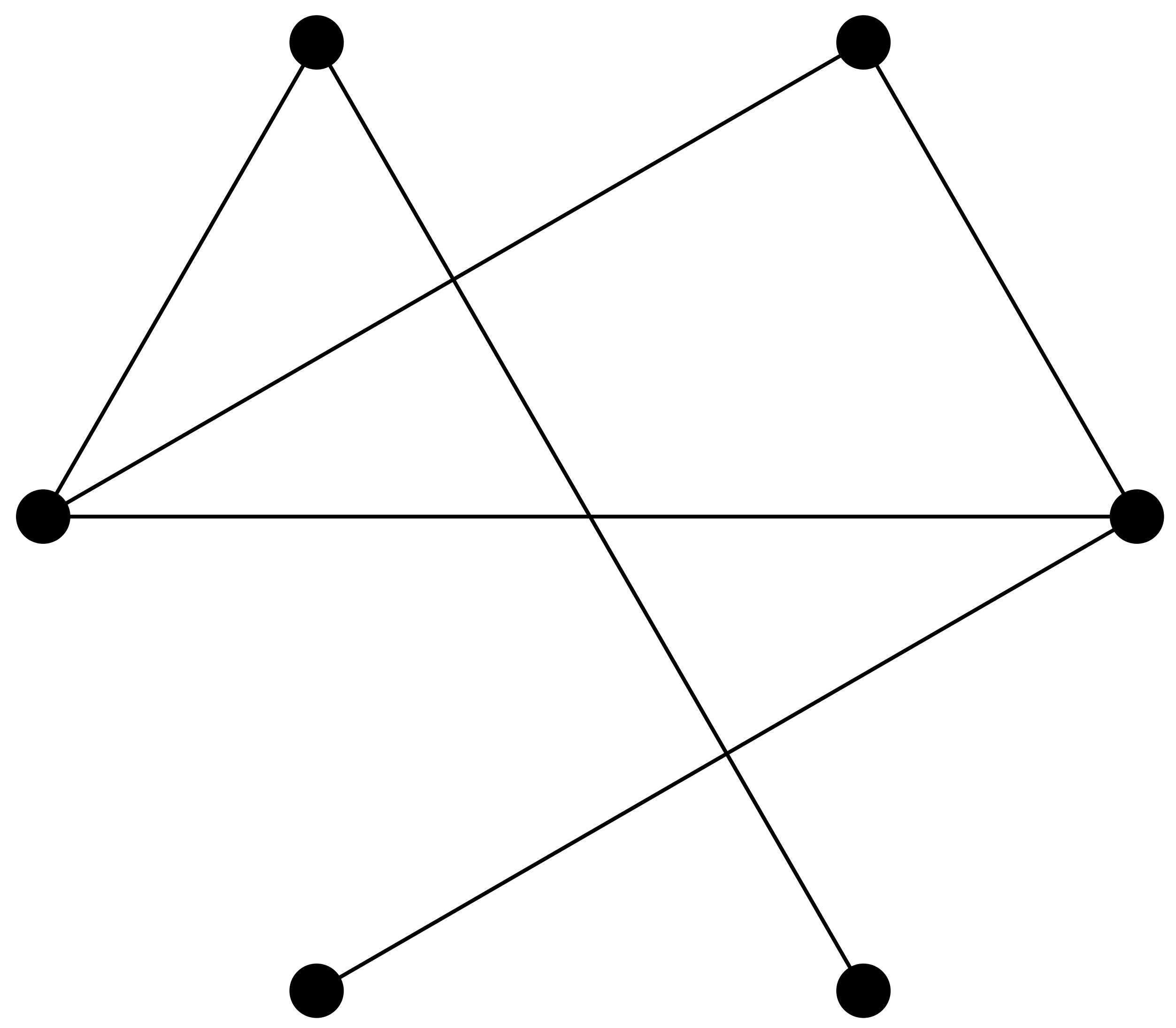


Phase Transition

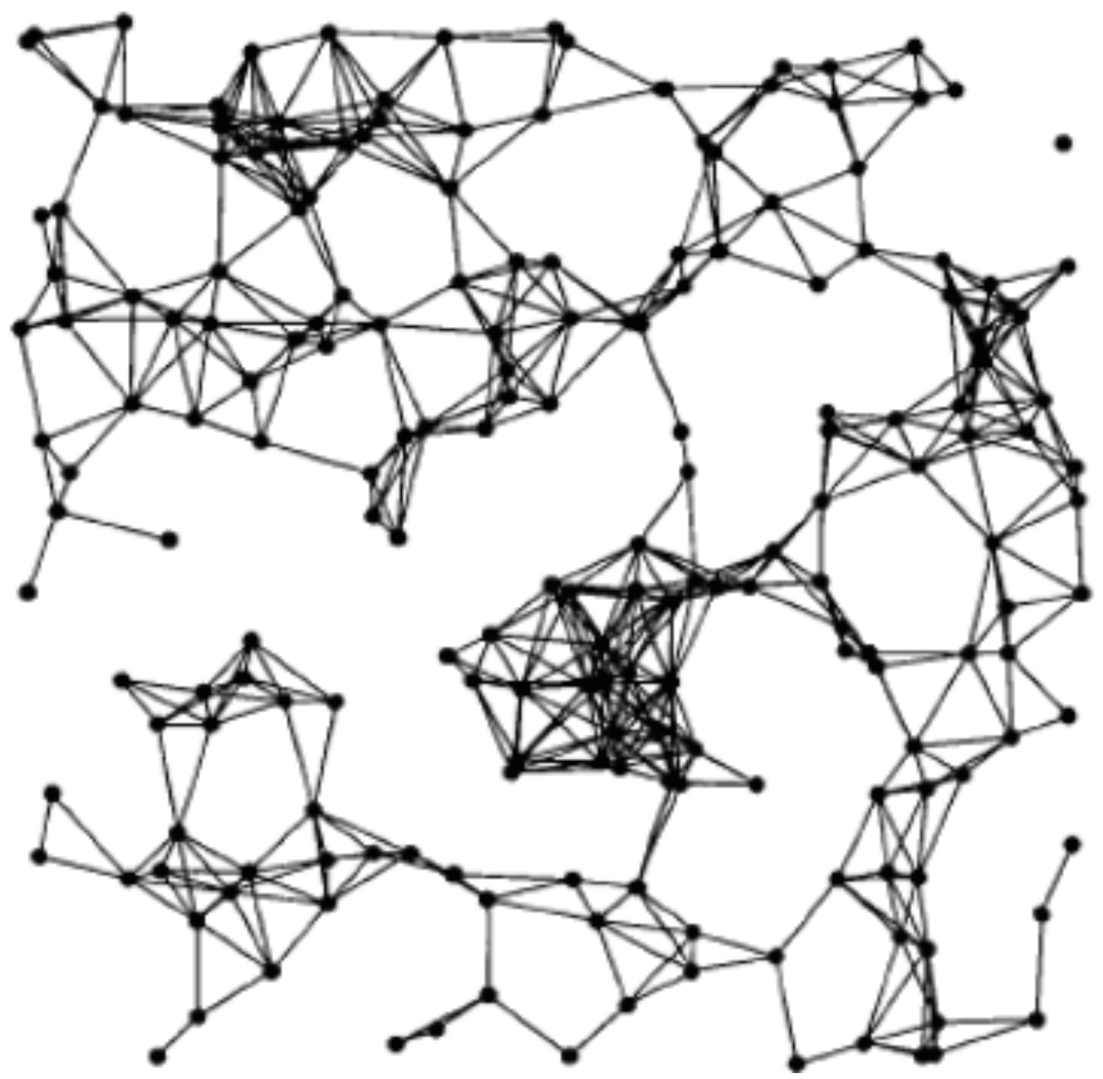
[Duncan-Kahle-Schweinhart, 2021]



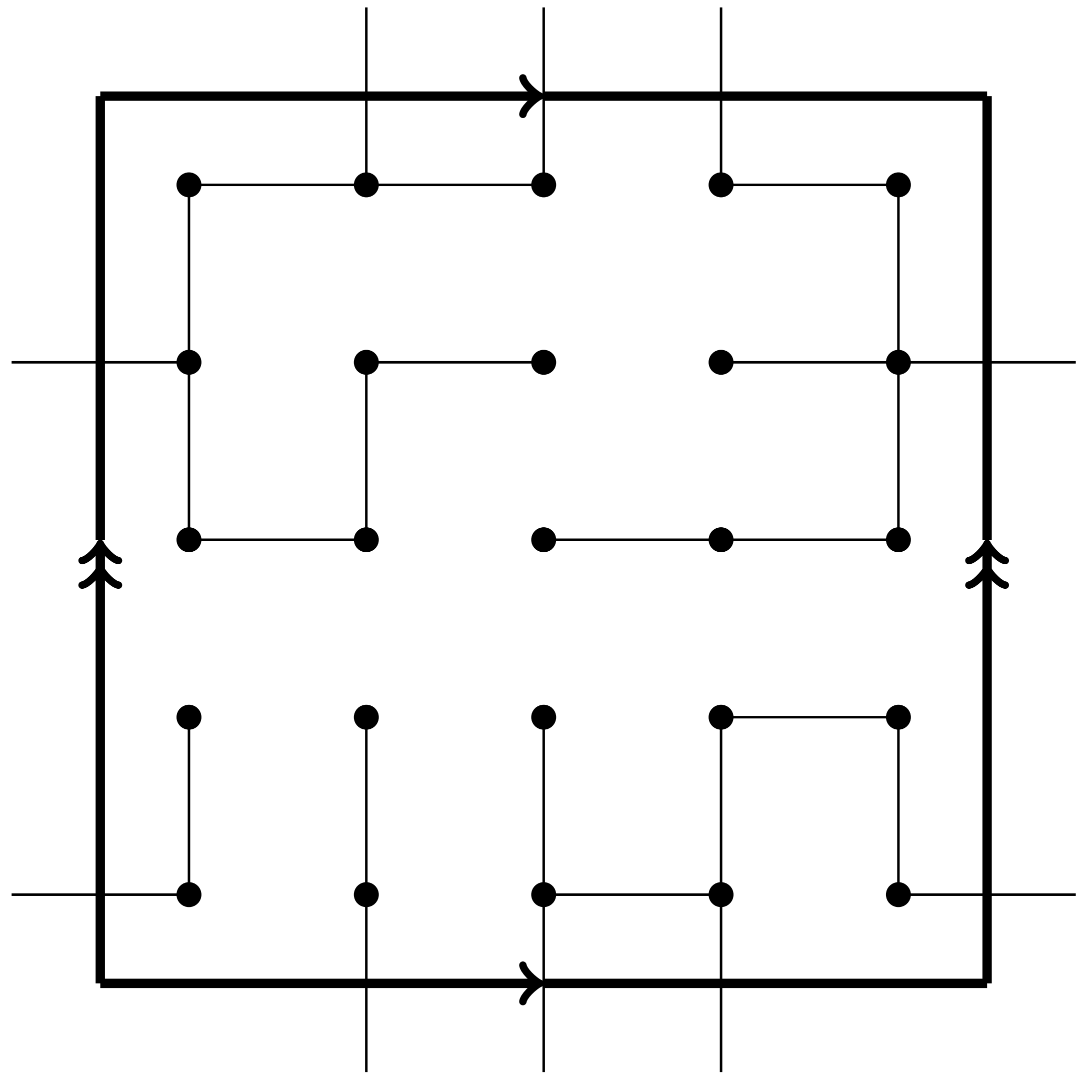
Tapas at Random Topology



Erdo-Renyi Complexes

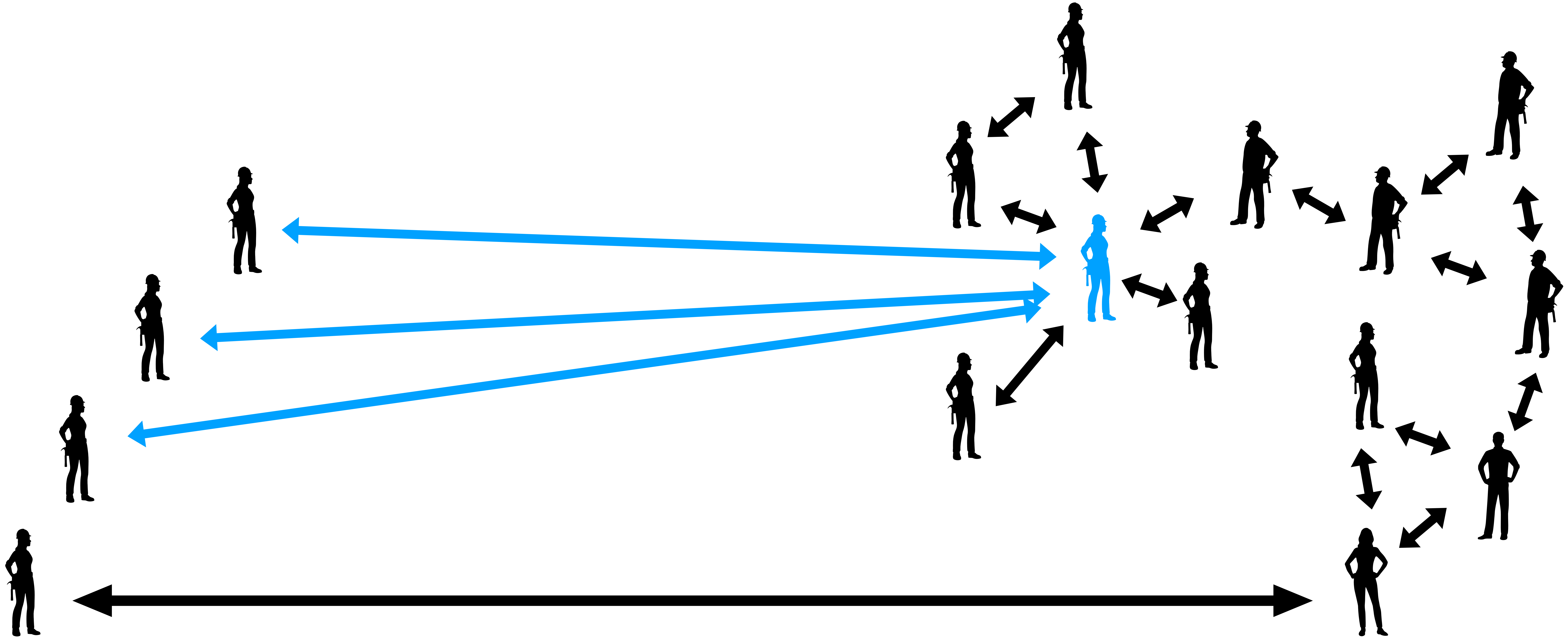


Geometric Complexes



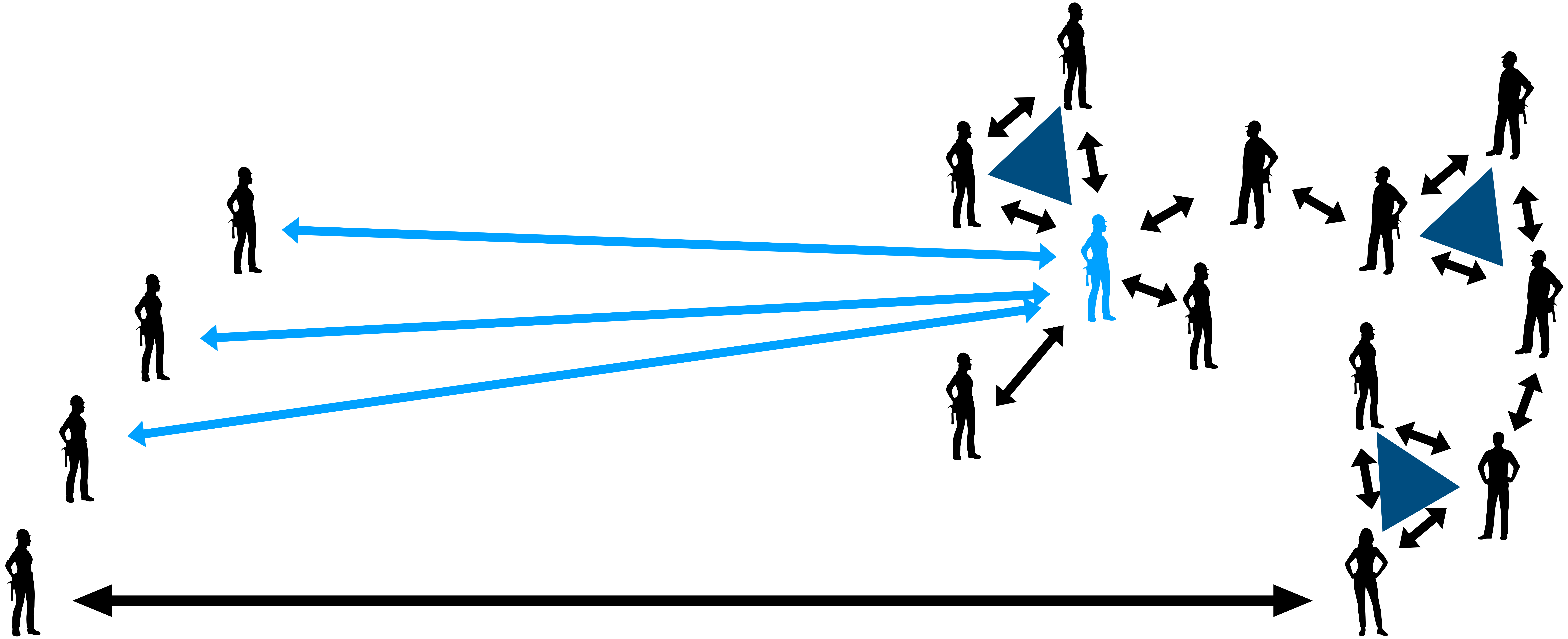
Topological Percolation

**Betti numbers count
repeated connections “in all dimensions”.**



Clique Complex

aka Flag Complex

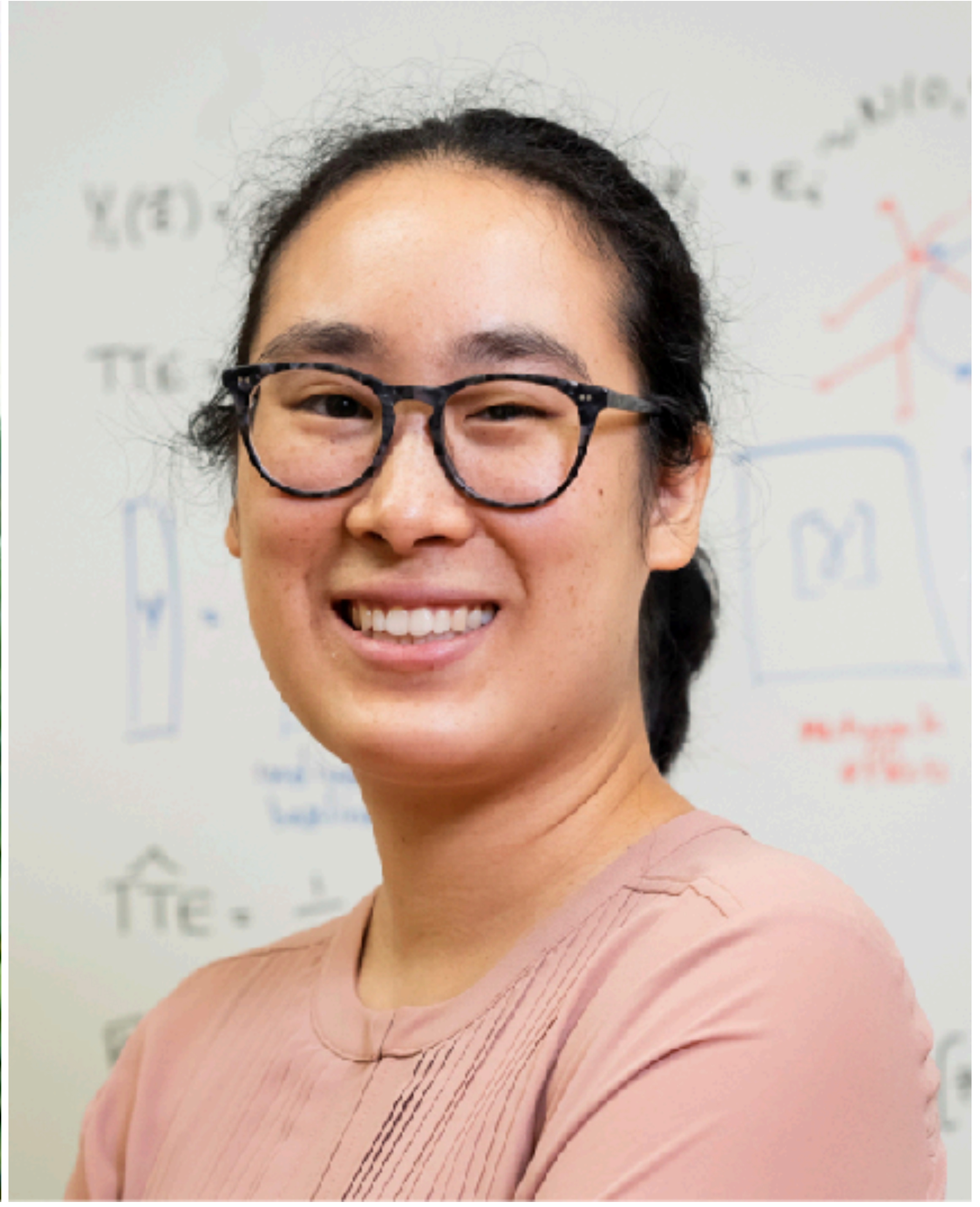


III Topology of Preferential Attachment

… …

My Lovely Collaborators

Avhan Misra



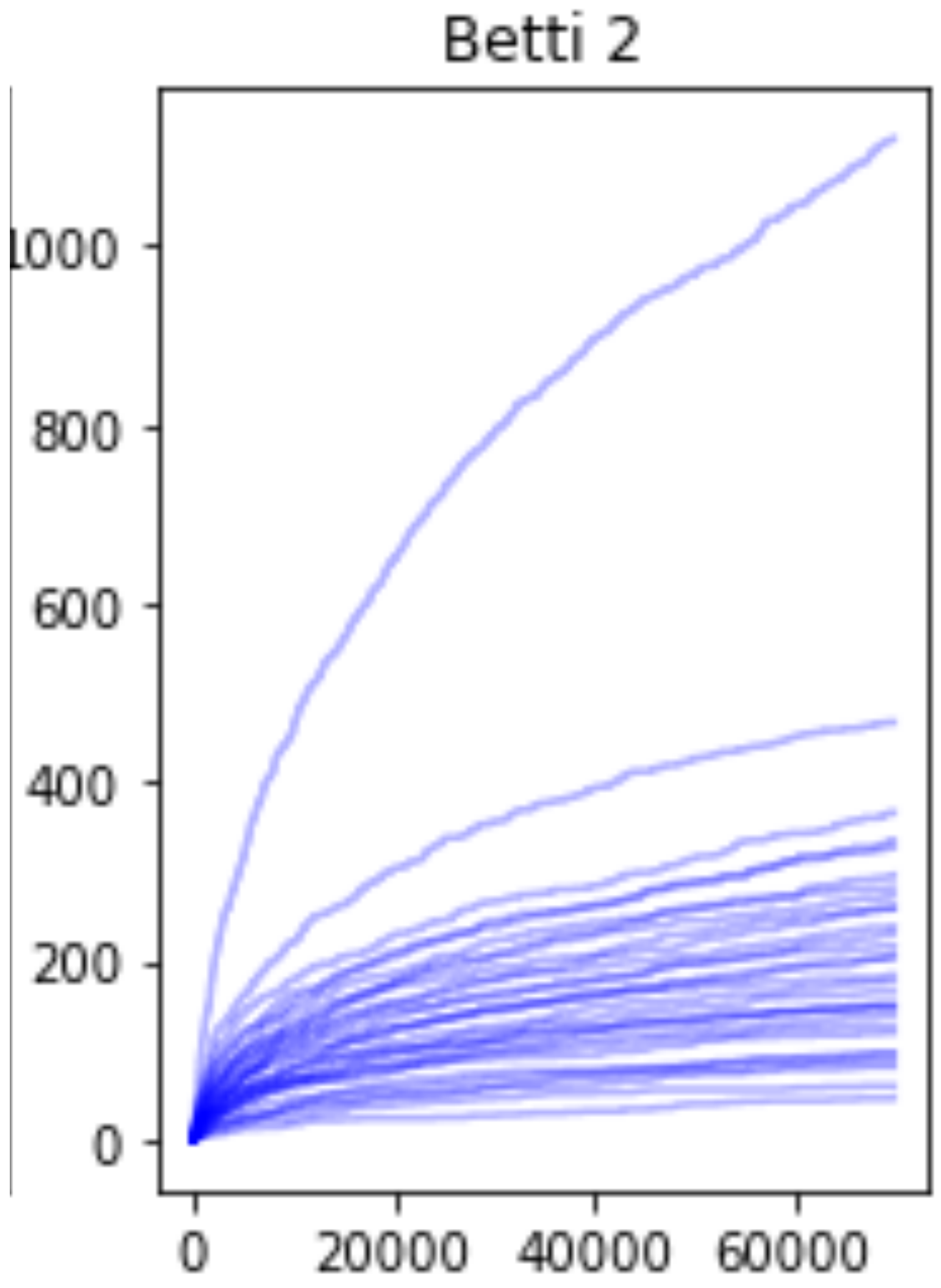
Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

Betti Number β_q

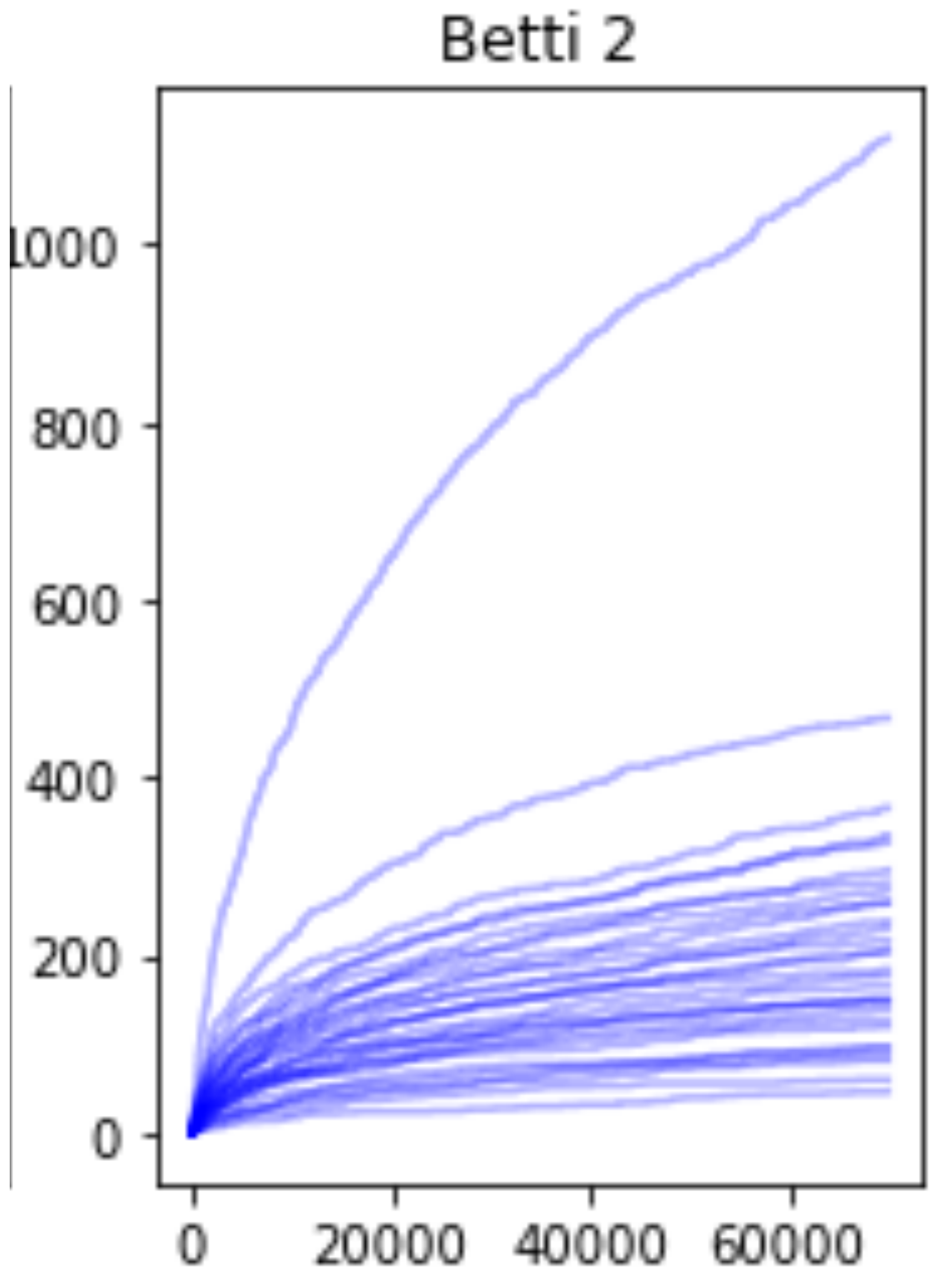
Betti Number β_q



Different curves, different random seeds.
All curves have the same model parameters.

Betti Number β_q

- increasing trend



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Betti Number β_q

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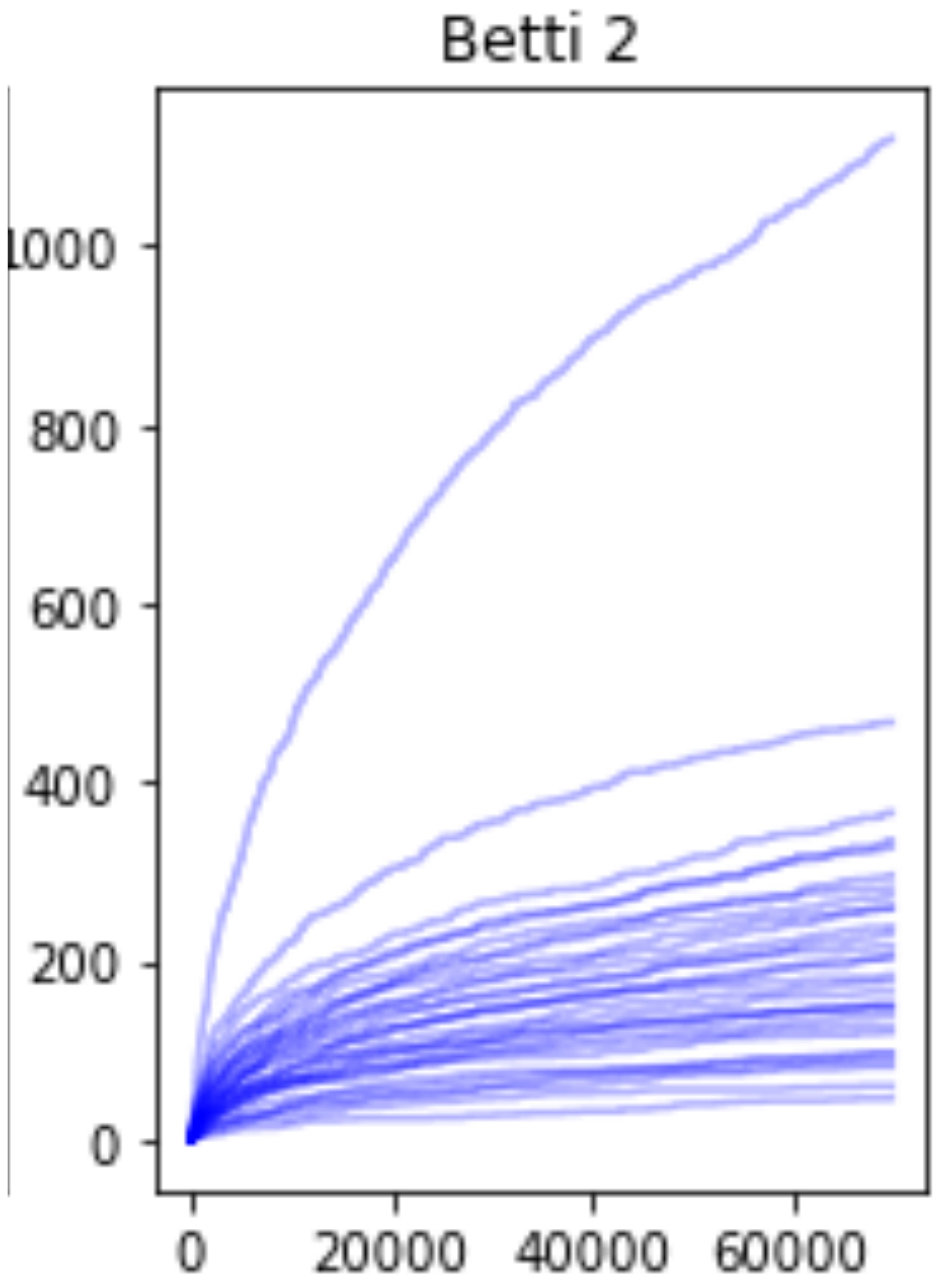

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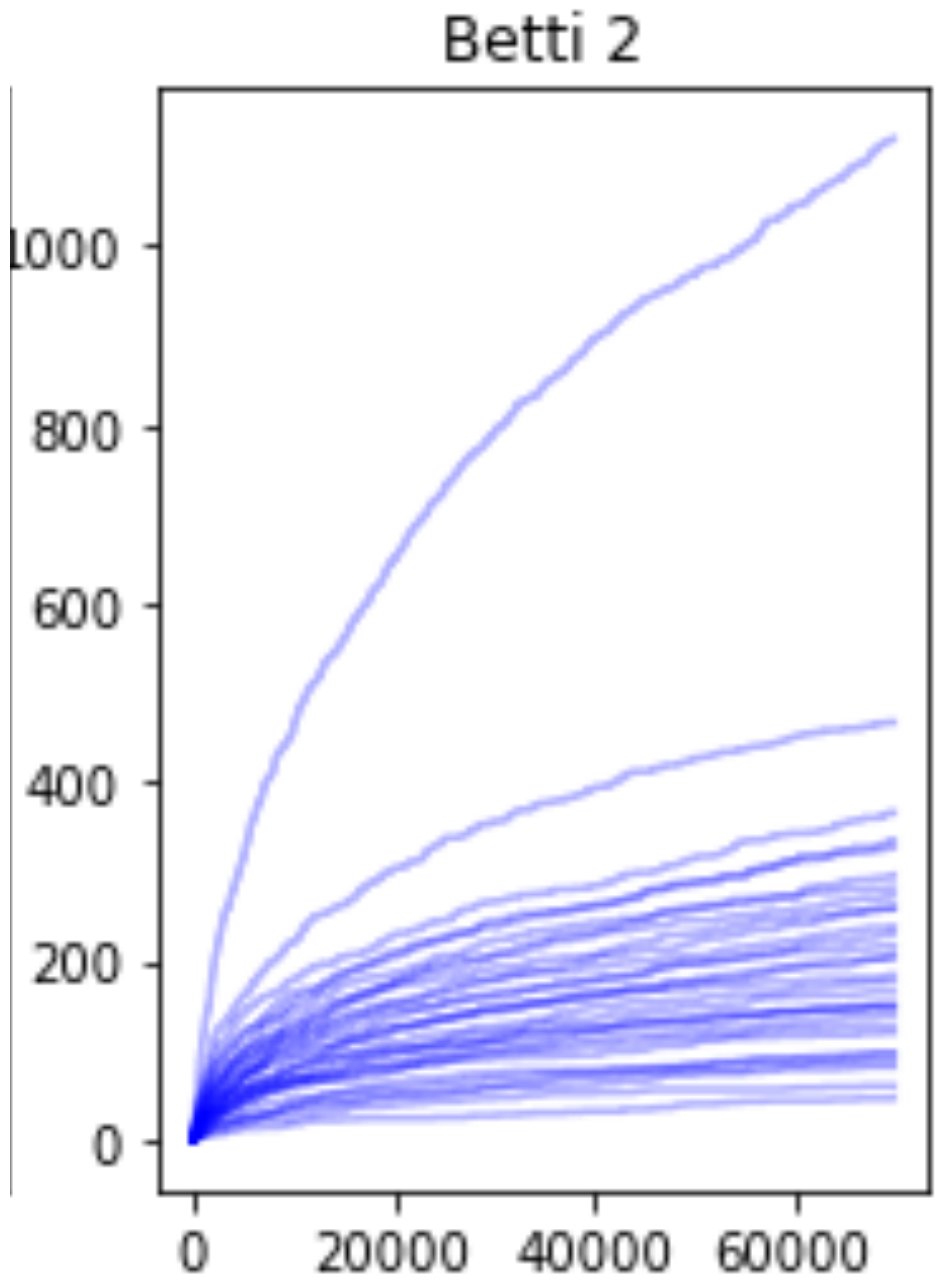
Betti Number β_q

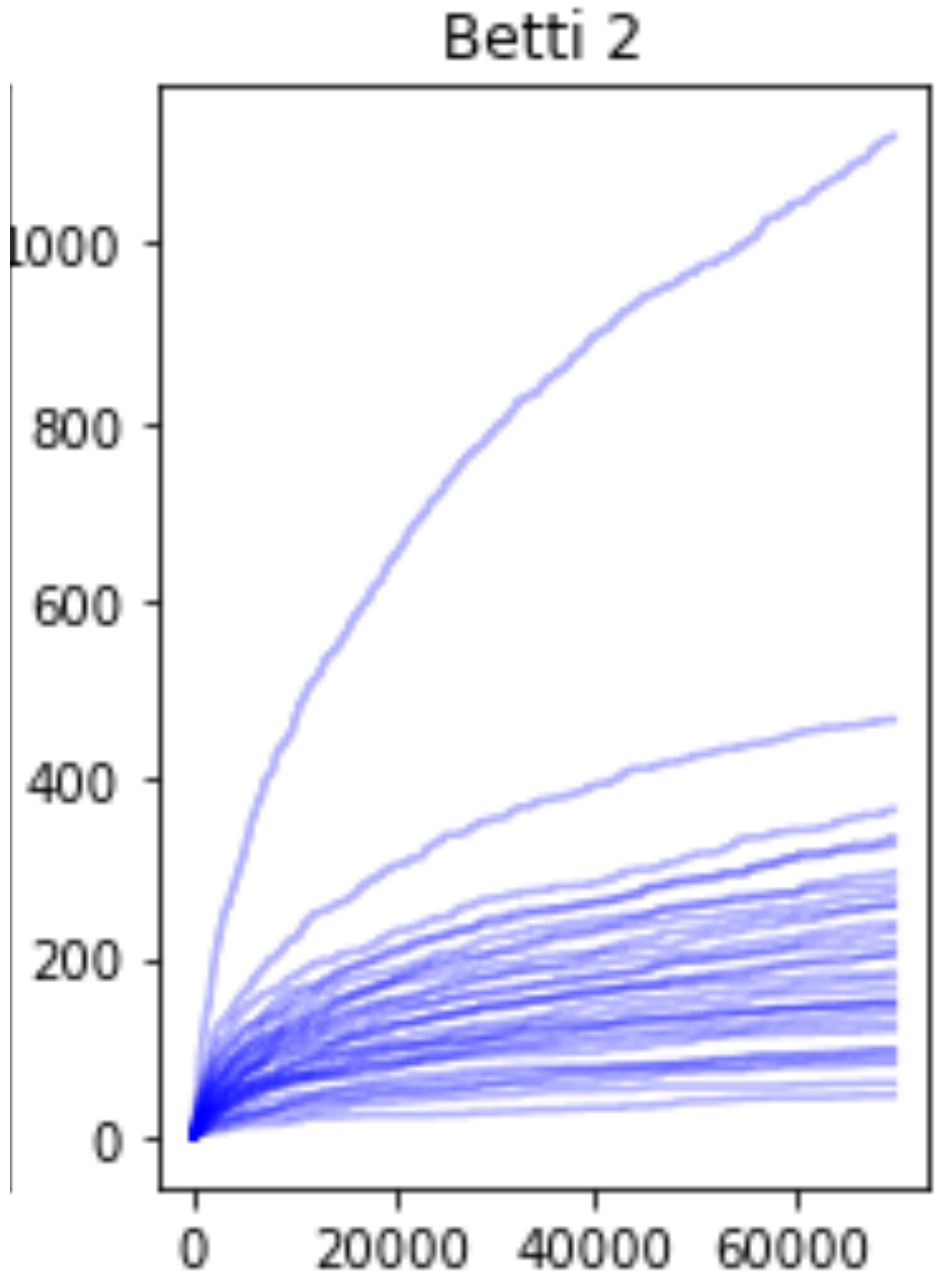
- increasing trend
- concave growth
- outlier



Different curves, different random seeds.
All curves have the same model parameters.

Betti Number β_q

- With probability at least $1 - \varepsilon$,
- $c_\varepsilon(\text{num of nodes}^{1-4x}) \leq \beta_2 \leq C_\varepsilon(\text{num of nodes}^{1-4x})$
  - $x = 1 - \dfrac{1}{2 + \delta/m} \in (0,1)$ decreases with the preferential attachment strength
    - $P[T \text{ attaches to } i] \propto T^{-x}$



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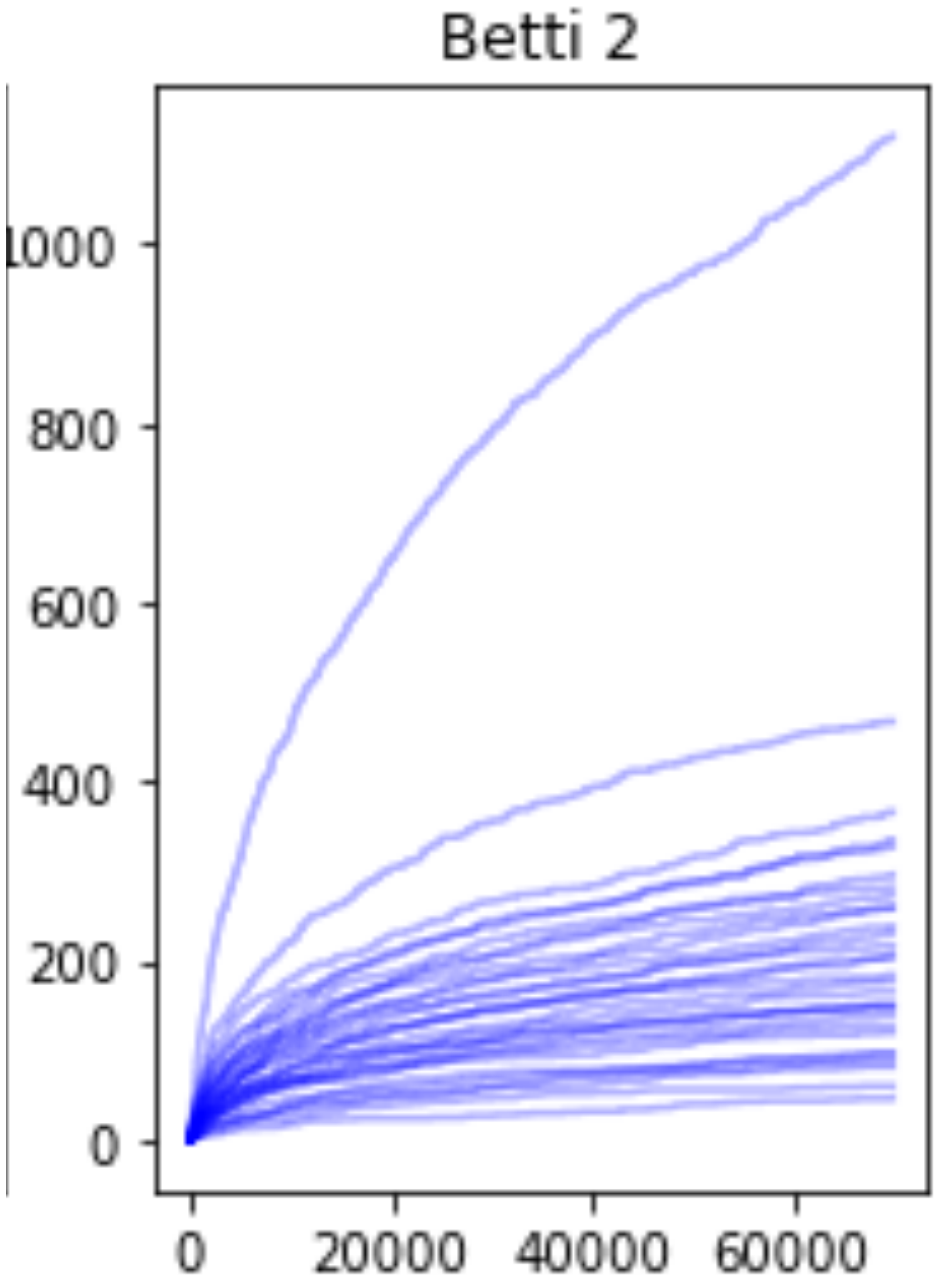
  - If $\boxed{1 - 4x} < 0$, then $\beta_2 \leq C_\varepsilon$.



Betti Number β_q

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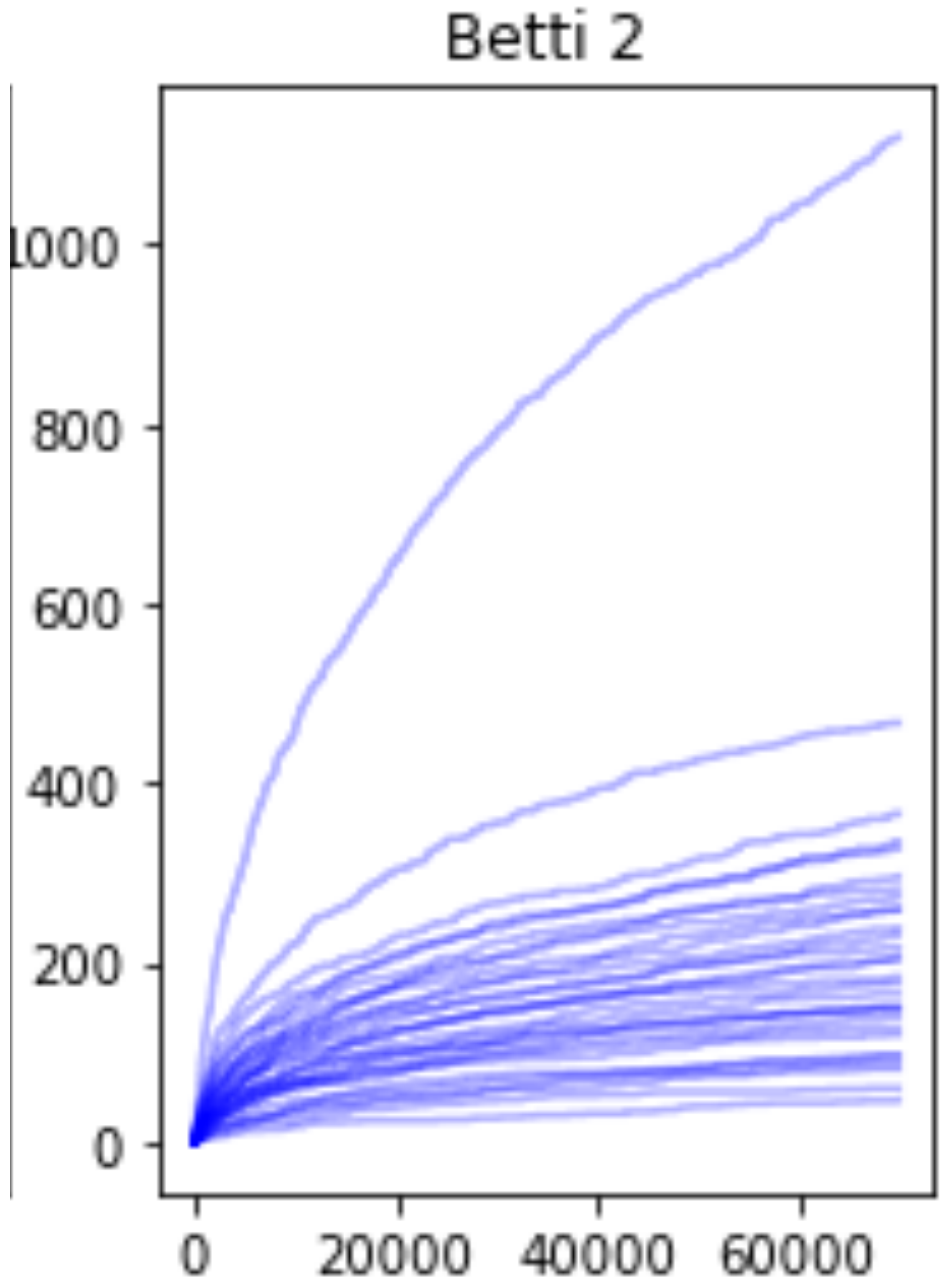
  - If $1 - 4x < 0$, then $\beta_2 \leq C_\varepsilon$.
- c_ε(num of nodes$^{1-\boxed{2q}x}$) $\leq \beta_q \leq C_\varepsilon$(num of nodes$^{1-\boxed{2q}x}$) for $q \geq 2$.

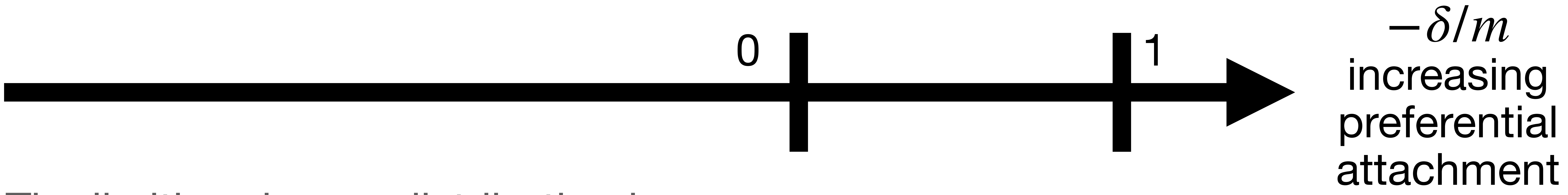


Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



The limiting degree distribution has …

Phase transition

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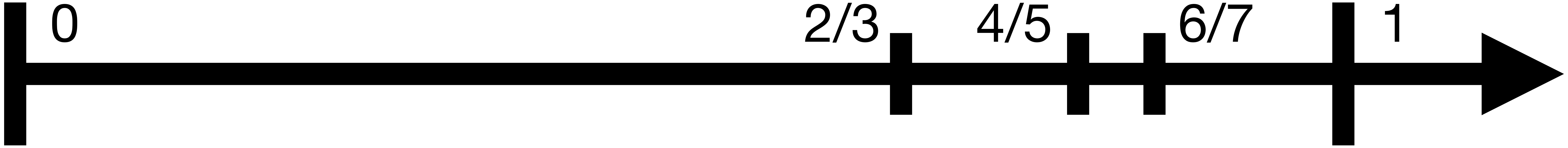

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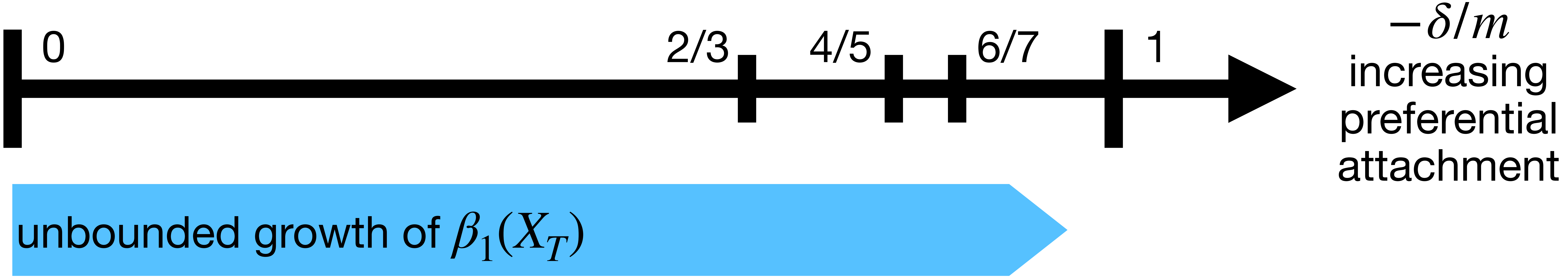

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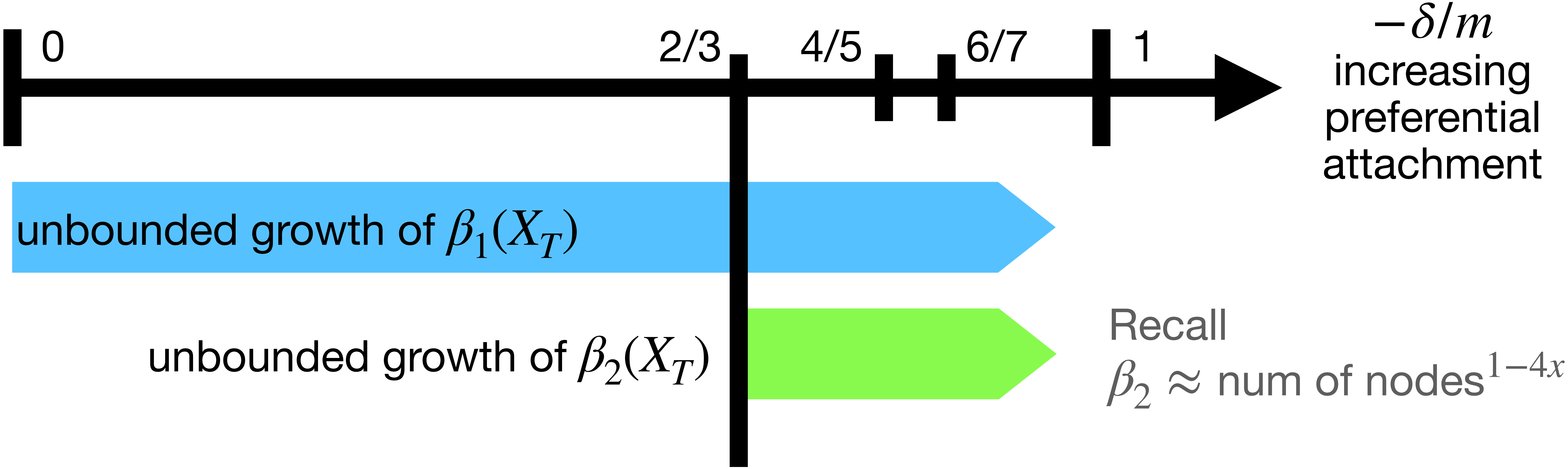

Recall

$\beta_2 \approx$ num of nodes^{1-4x}

Phase transition

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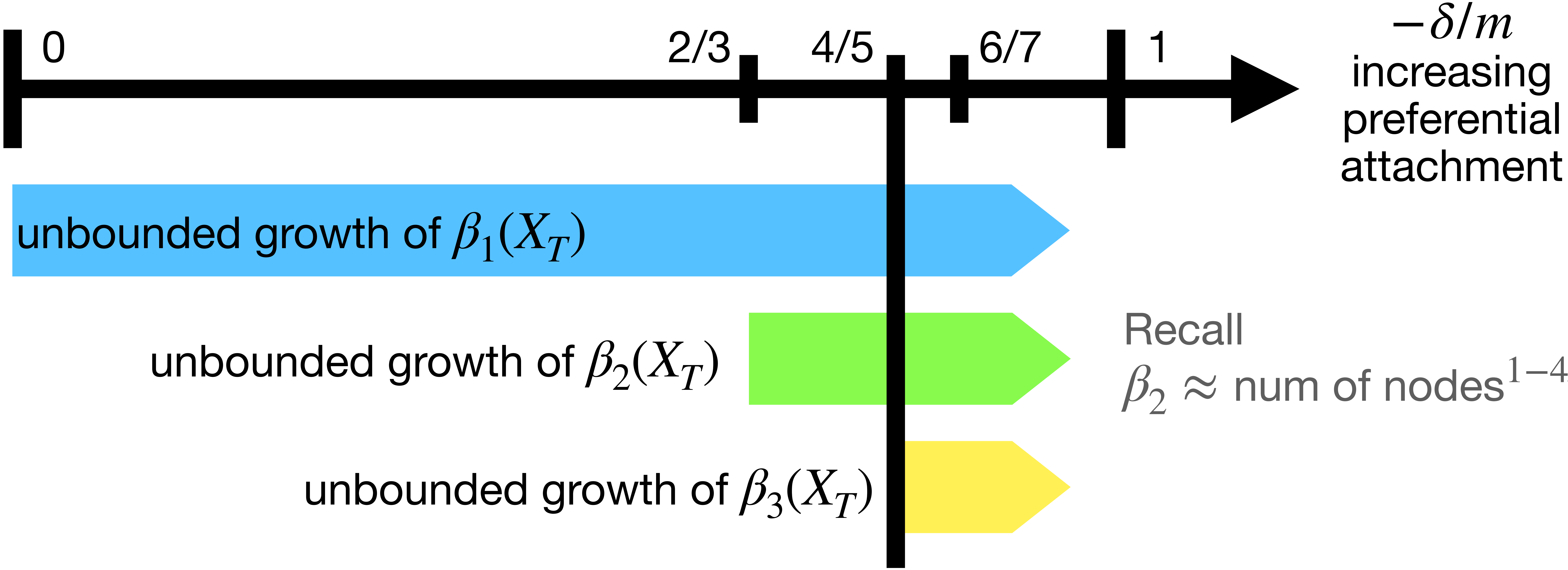

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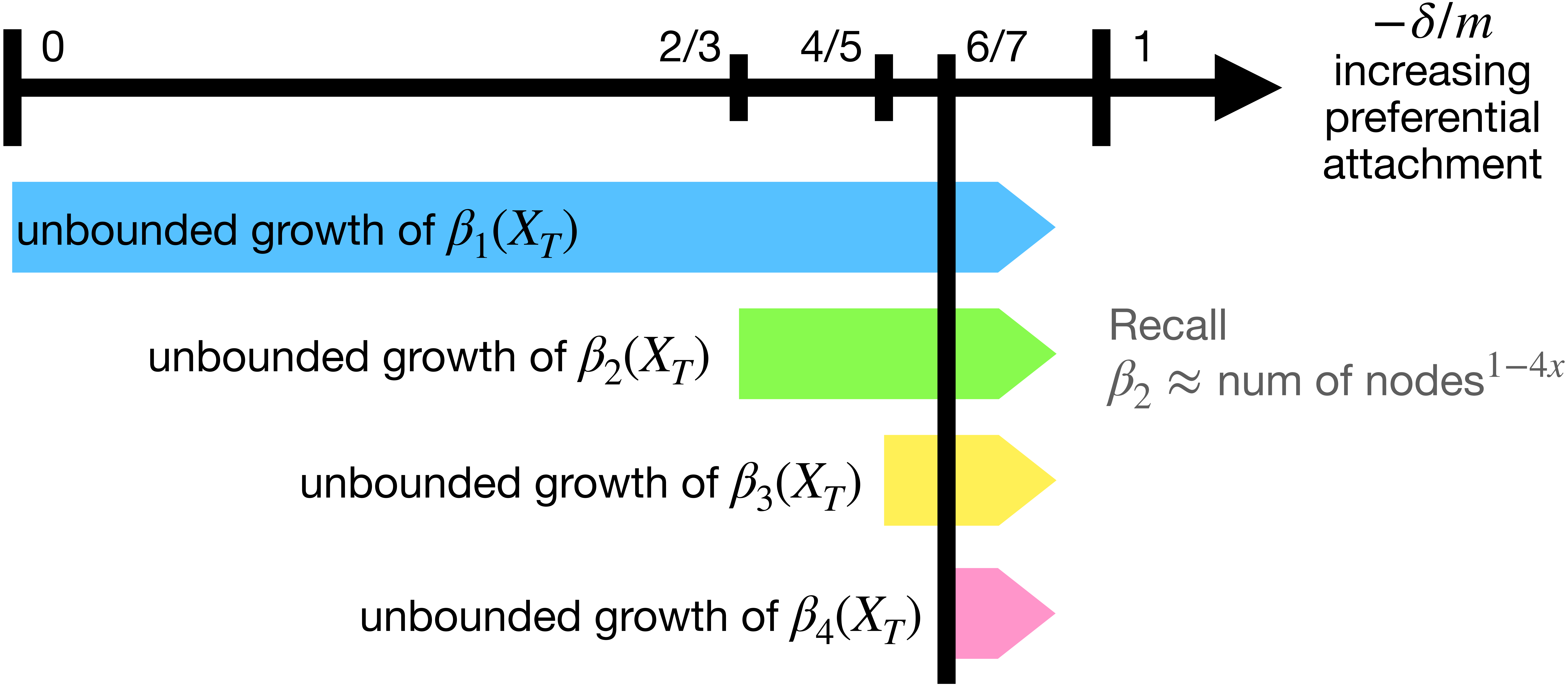

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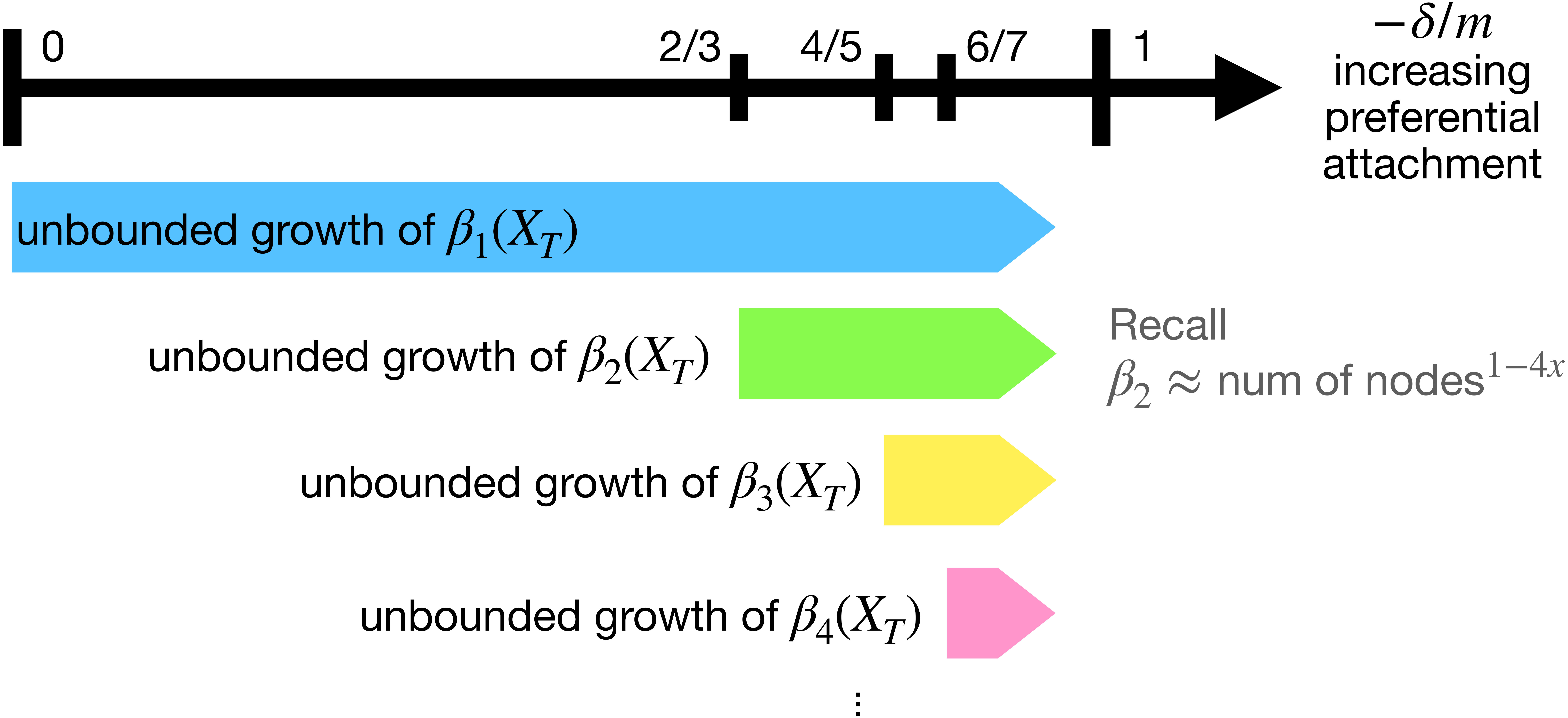


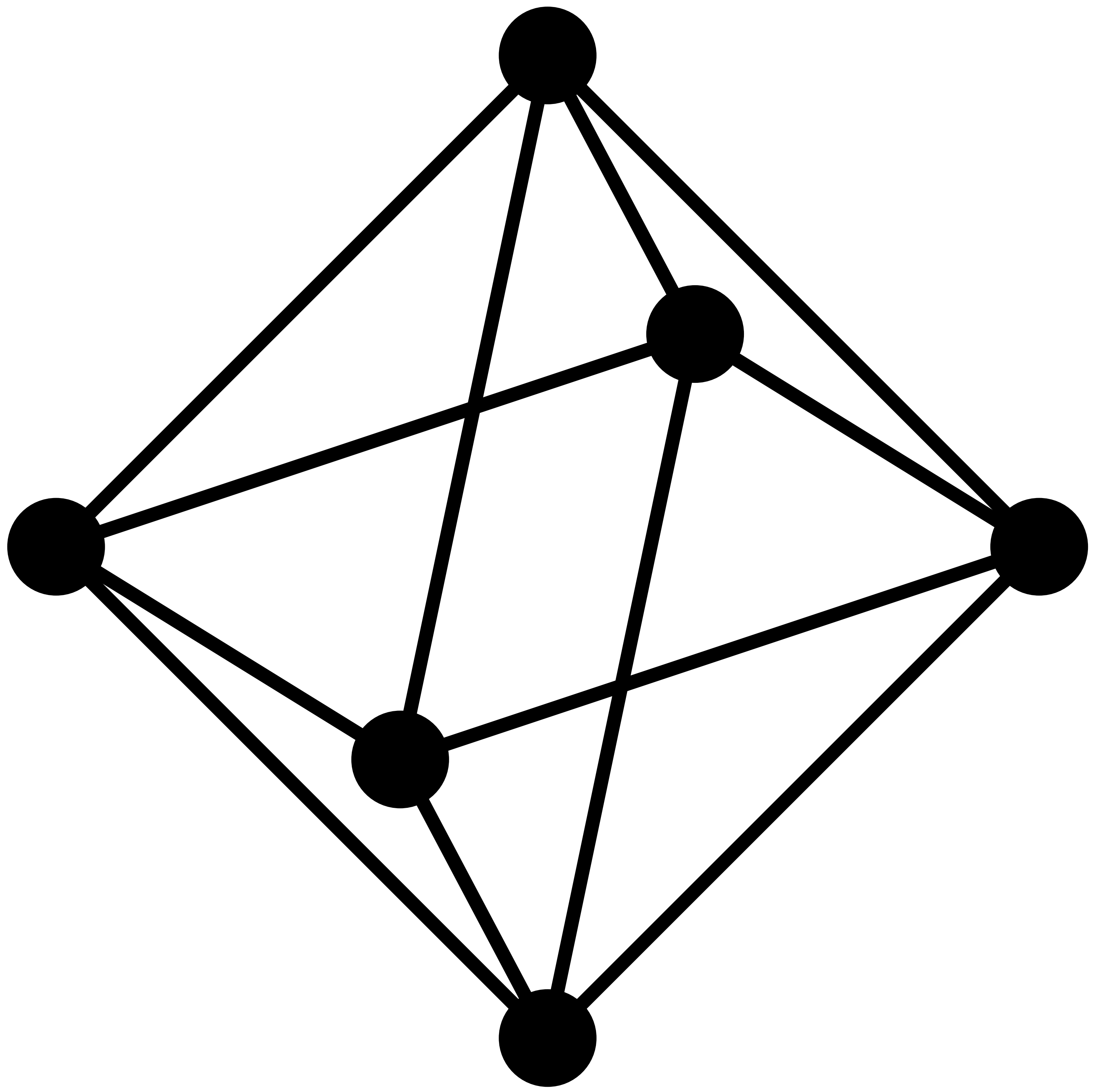
Recall

$\beta_2 \approx$ num of nodes^{1-4x}

Theorem: $\beta_2 \approx$ num of nodes^{1-4x}

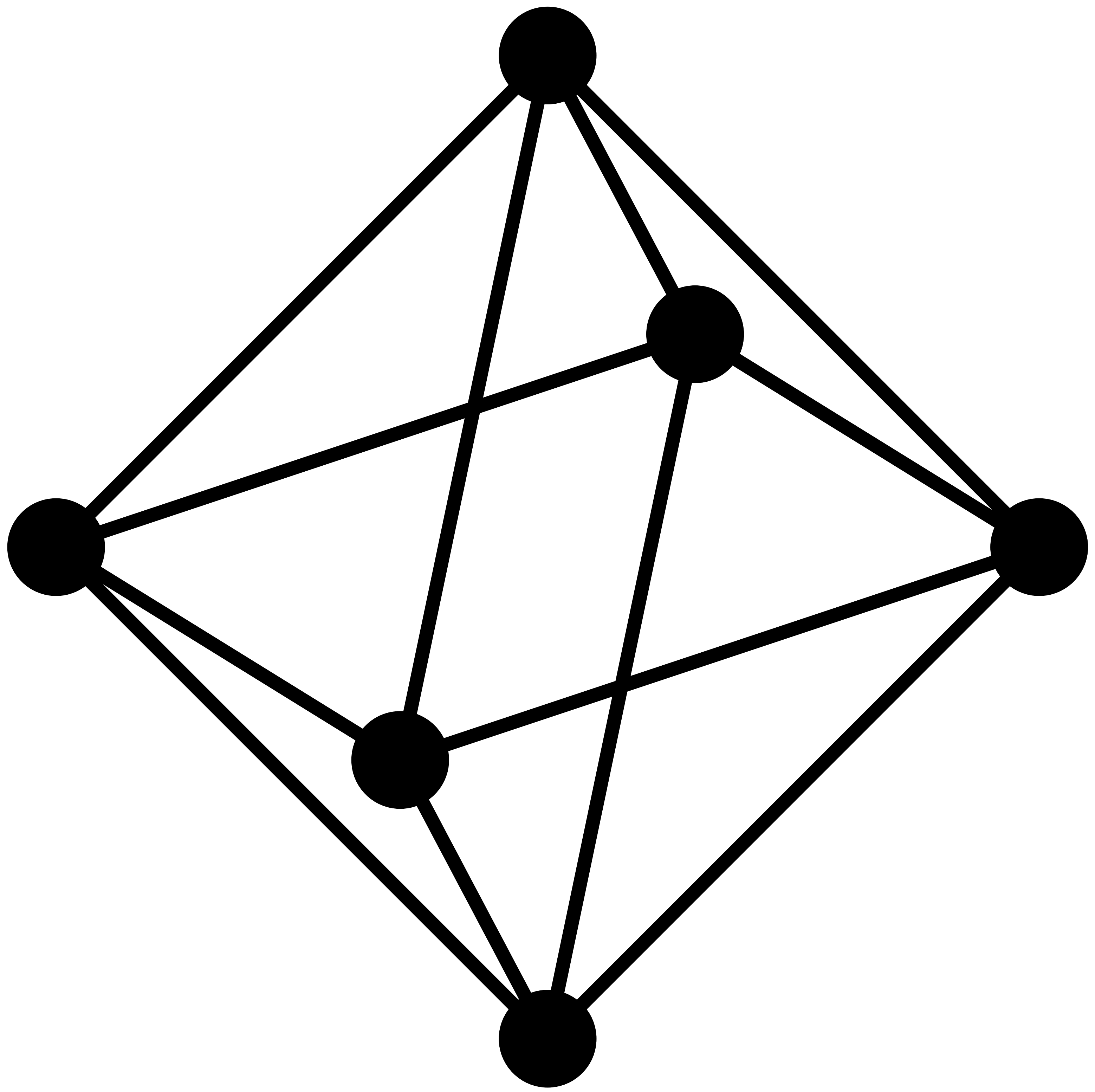
Proof?

Proof of $\beta_2 \approx$ num of nodes^{1-4x}



$\beta_2 = 1$

Proof of $\beta_2 \approx$ num of nodes^{1-4x}



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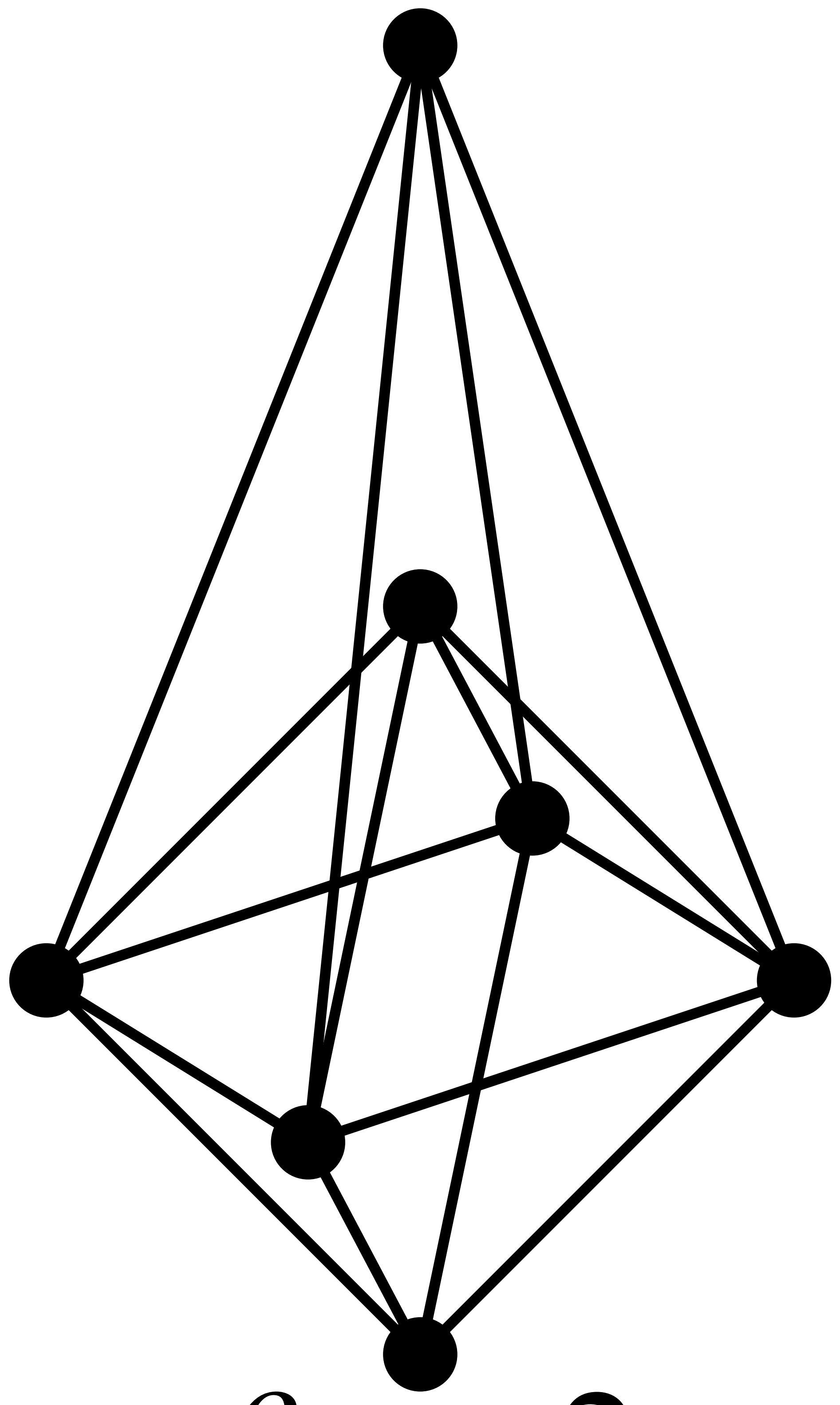
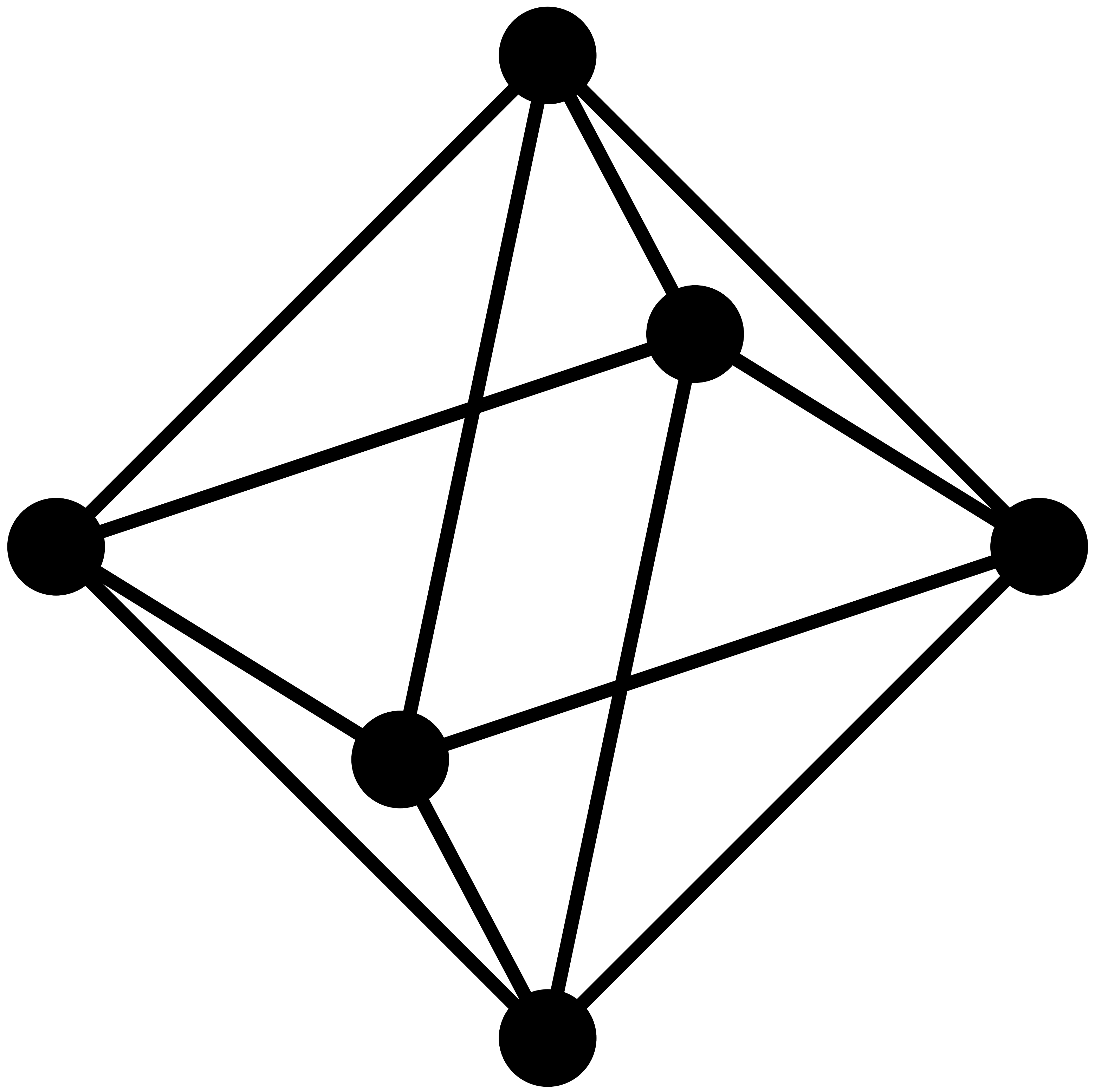
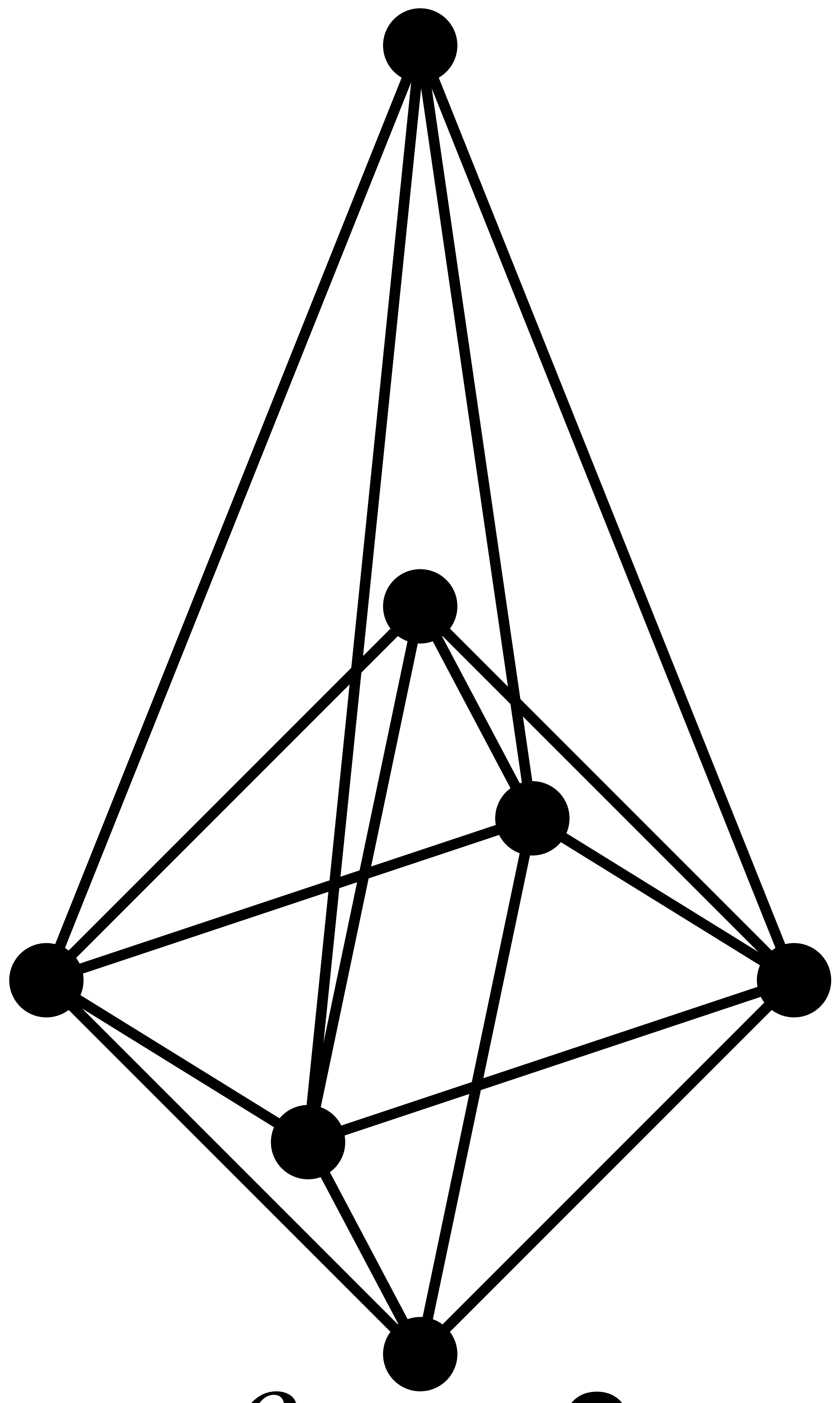


$\beta_2 = 2$

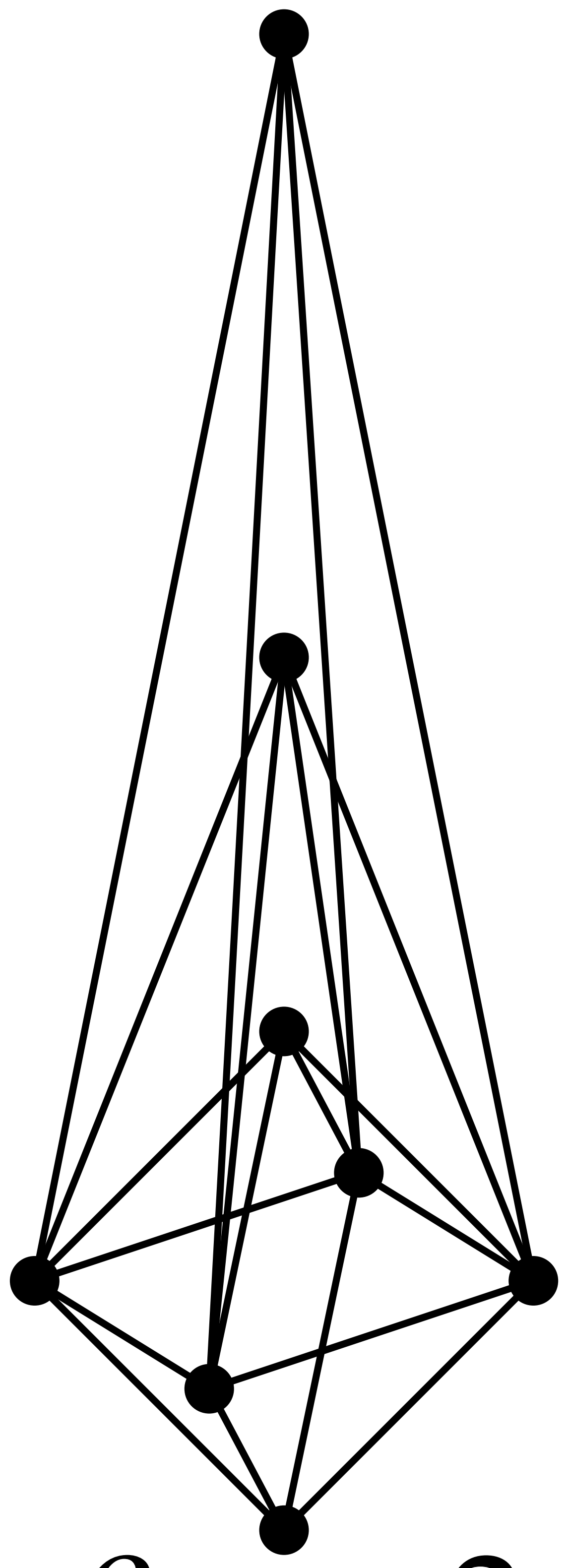
Proof of $\beta_2 \approx$ num of nodes^{1-4x}



$\beta_2 = 1$



$\beta_2 = 2$



$\beta_2 = 3$

Subtleties

- Need homological algebra to relate Betti numbers with counts

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- Need homological algebra to relate Betti numbers with counts
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results in the language of homological algebra
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs

Theorem: $\beta_2 \approx$ num of nodes^{1-4x}

IV. What lies ahead

order of magnitude of
Betti numbers

more precise estimates?

other inhomogeneous complexes?

order of magnitude of Betti numbers

more precise estimates?

other inhomogeneous complexes?

order of magnitude of Betti numbers

breakthroughs for graphs????

The threshold for percolation in $\mathbb{Z}^3$?

Gwynne, 2023. private conversation

Some wishful thinking…

The threshold for percolation in $\mathbb{Z}^3$?

Gwynne, 2024. private conversation

Some wishful thinking…

- Maybe $\mathbb{Z}^3$ and $(\mathbb{Z}/n)^3$ are similar enough…

The threshold for percolation in $\mathbb{Z}^3$?

Gwynne, 2024. private conversation

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The threshold for percolation in $\mathbb{Z}^3$?

Gwynne, 2024. private conversation

Some wishful thinking…

- Maybe $\mathbb{Z}^3$ and $(\mathbb{Z}/n)^3$ are similar enough…
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- They are equal! [Duncan, Kahle, Schweihart, 2024, Theorem 2]

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Gwynne, 2024. private conversation

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The threshold for percolation in $\mathbb{Z}^3$?

Gwynne, 2024. private conversation

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For a site percolation model

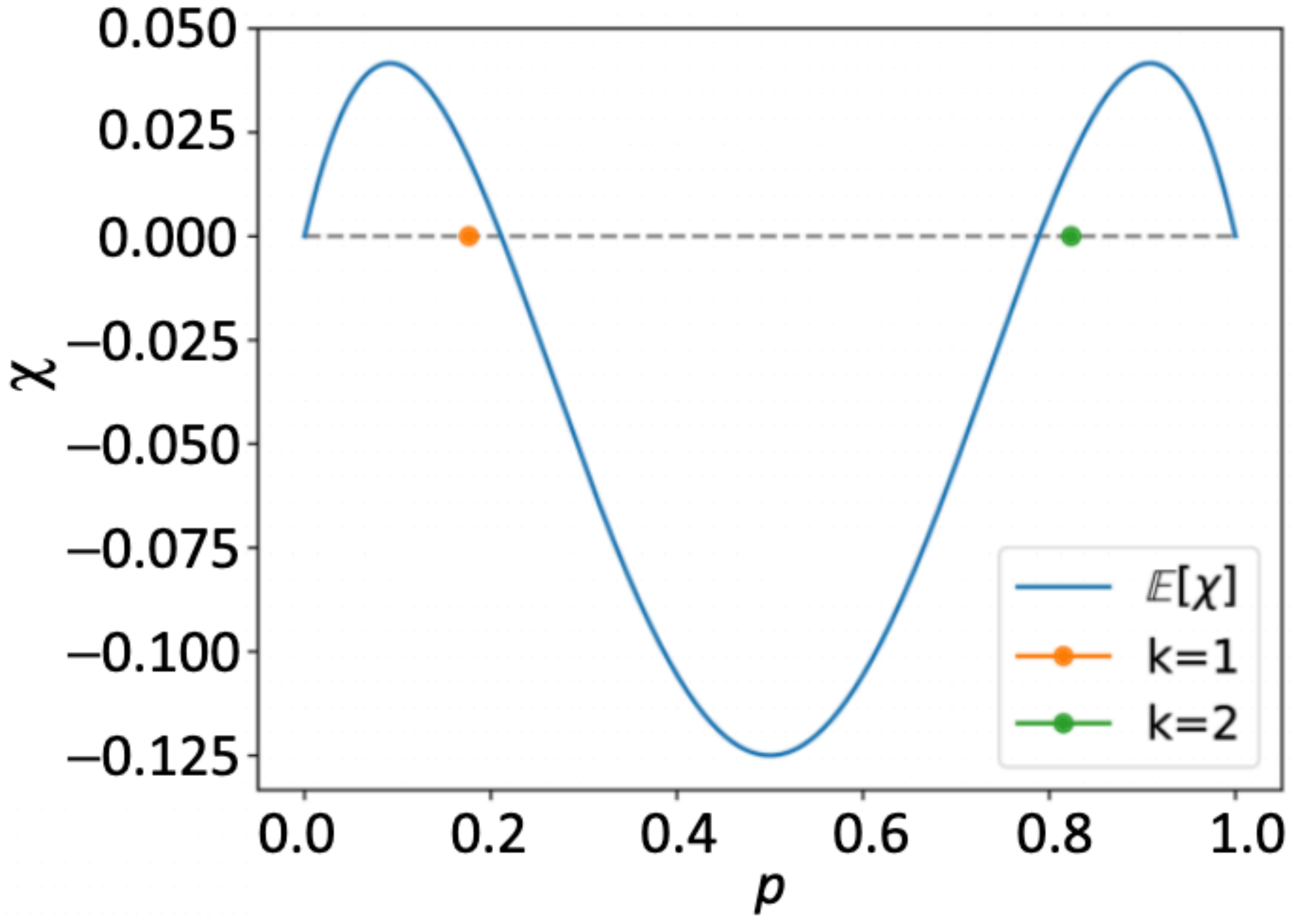

Fig 7e of [Bobrowski and Skraba, 2020]

more precise estimates?

other inhomogeneous complexes?

order of magnitude of Betti numbers

breakthroughs for graphs????

more precise estimates?

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order of magnitude of Betti numbers

random chain complexes (extension of random matrices)?

breakthroughs for graphs????

more precise estimates?

other inhomogeneous complexes?

order of magnitude of Betti numbers

random chain complexes (extension of random matrices)?

breakthroughs for graphs????

local minima and homology of random high-dimensional functionals

What did we learn today?

- Random topology is cool.

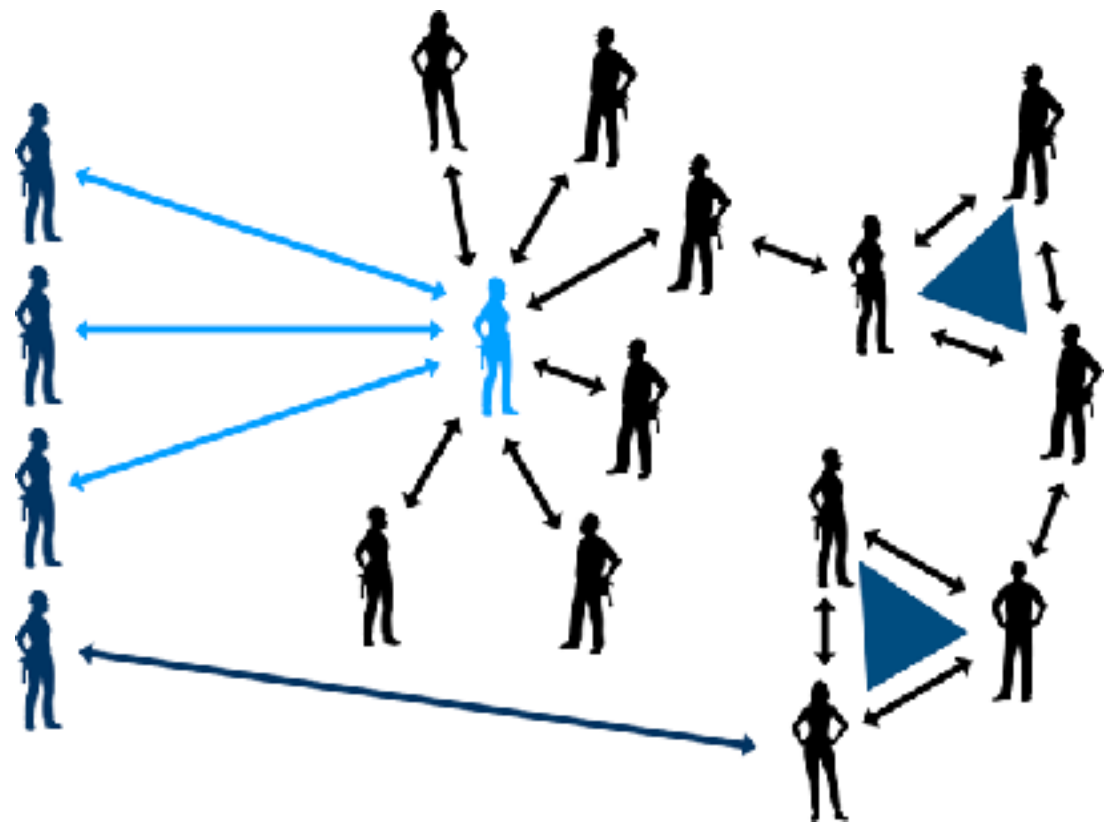
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.

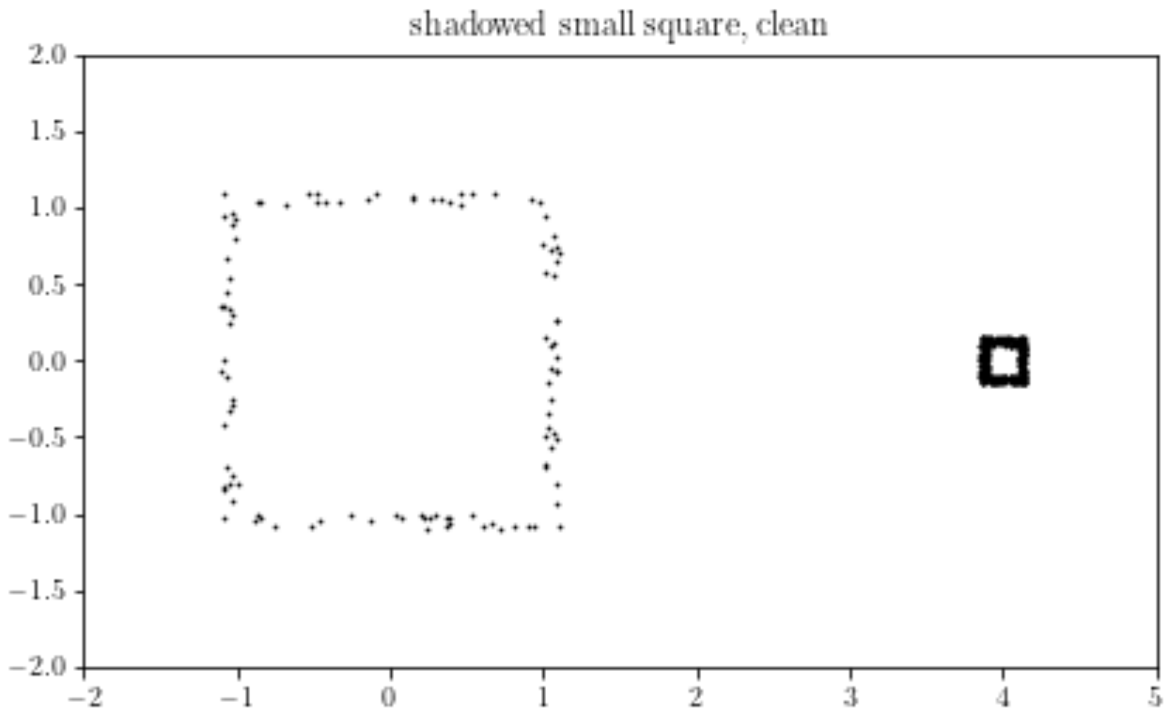
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

Chunyin Siu **Stanford University, School of Medicine**
siuc@stanford.edu



paper

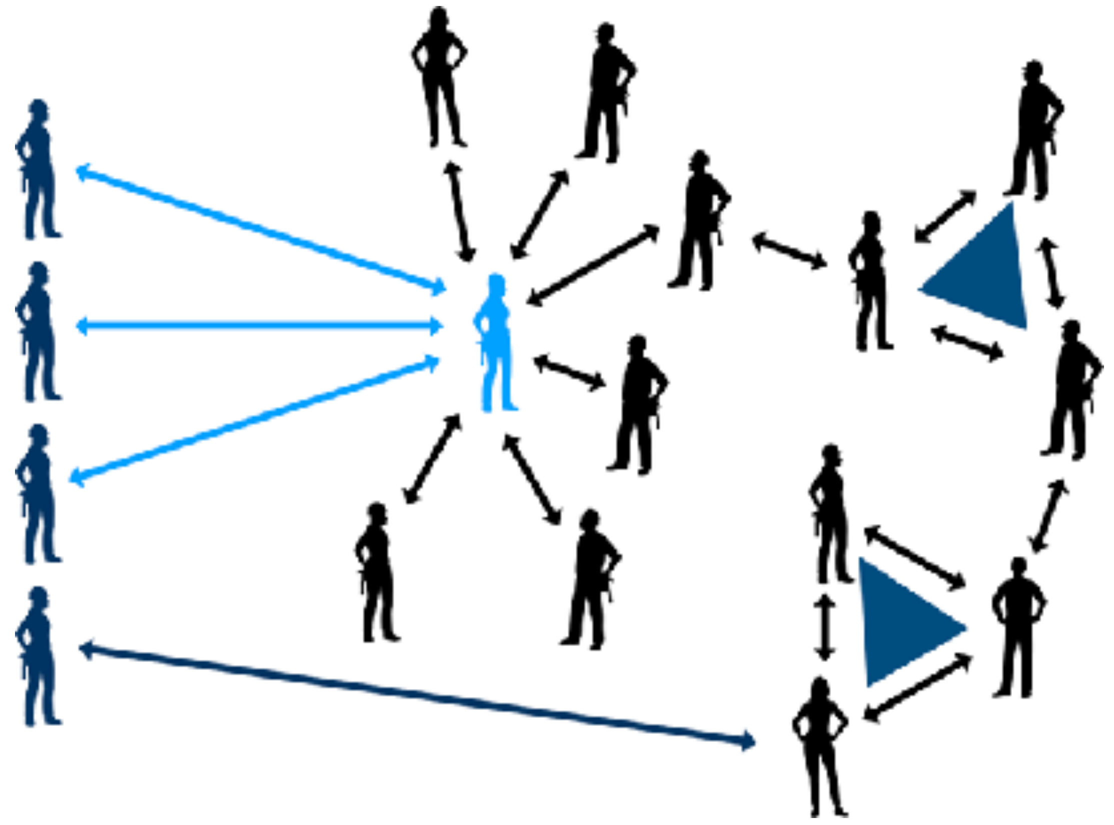


my video about small holes

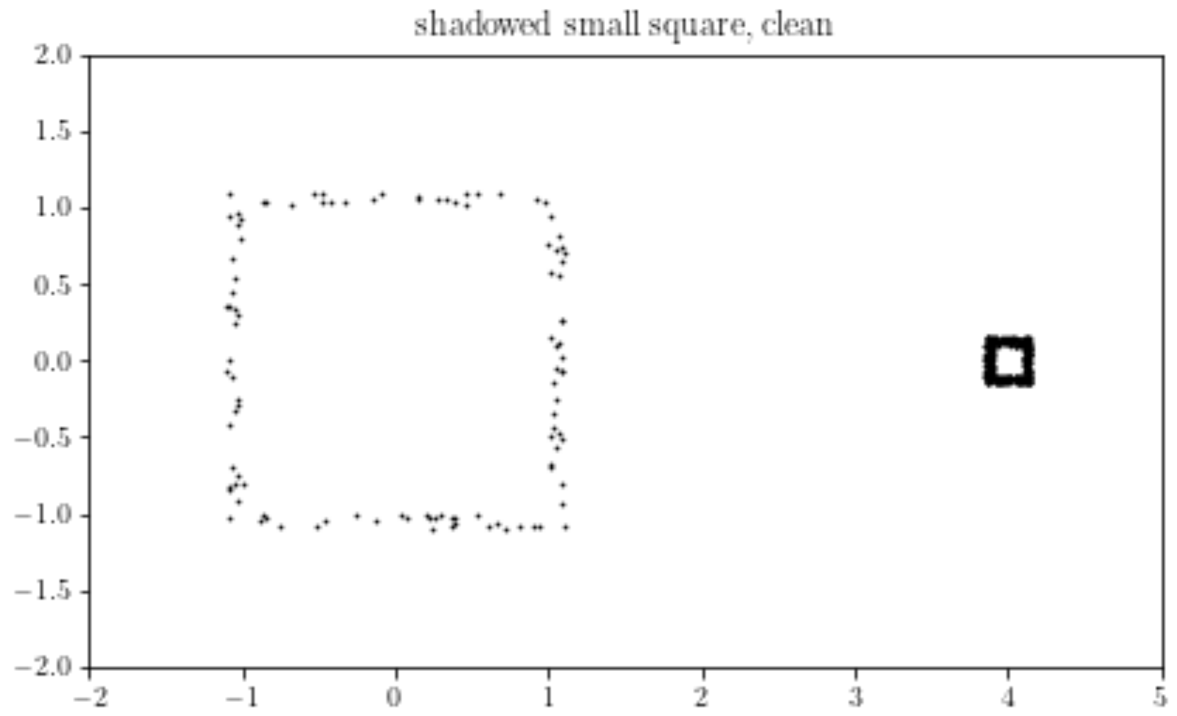
Thank you!

Chunyin Siu **Stanford University, School of Medicine**

siuc@stanford.edu



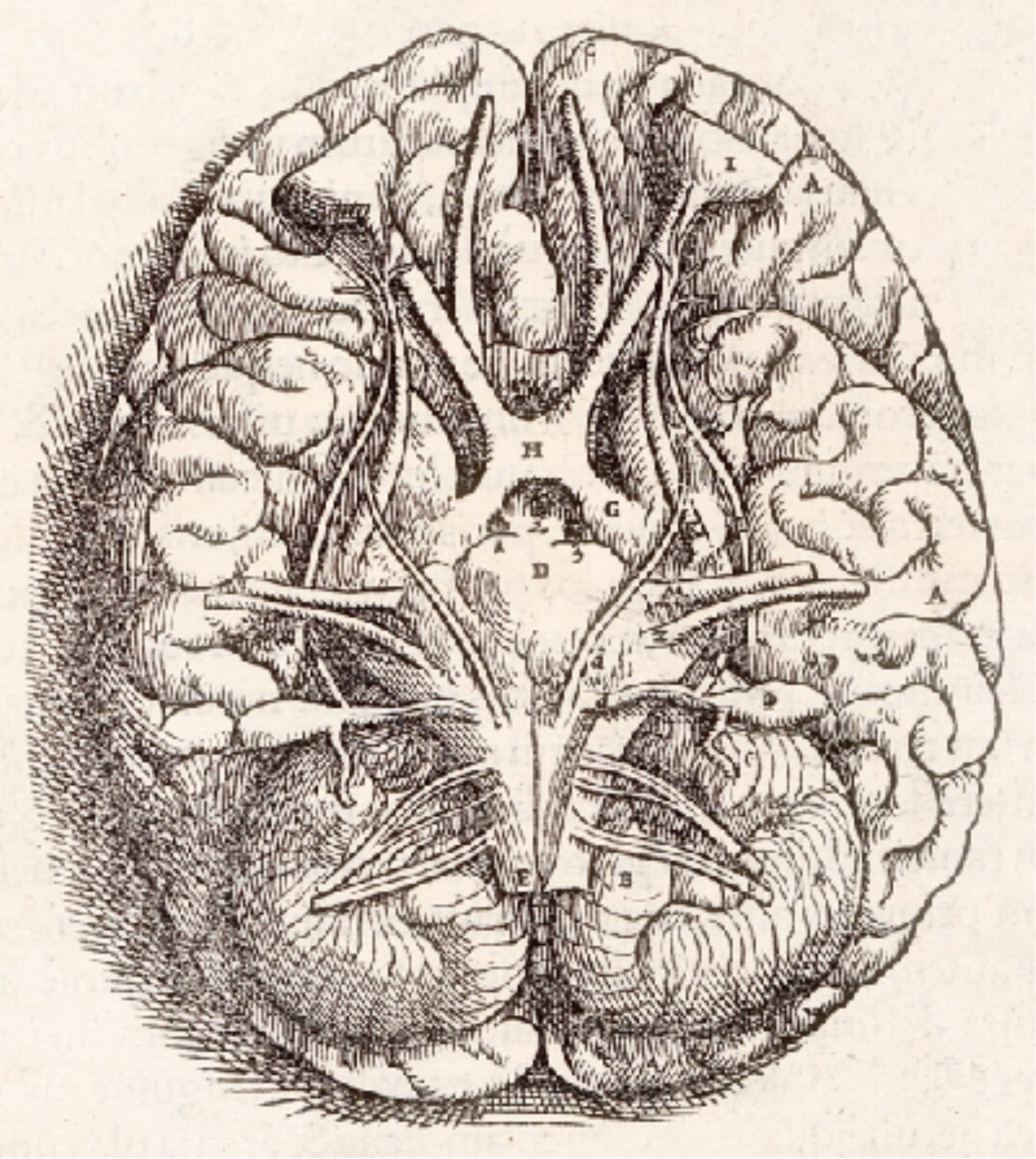
arxiv paper



my video about small holes

Applications of Topology

Topology and Networks



Neuroscience

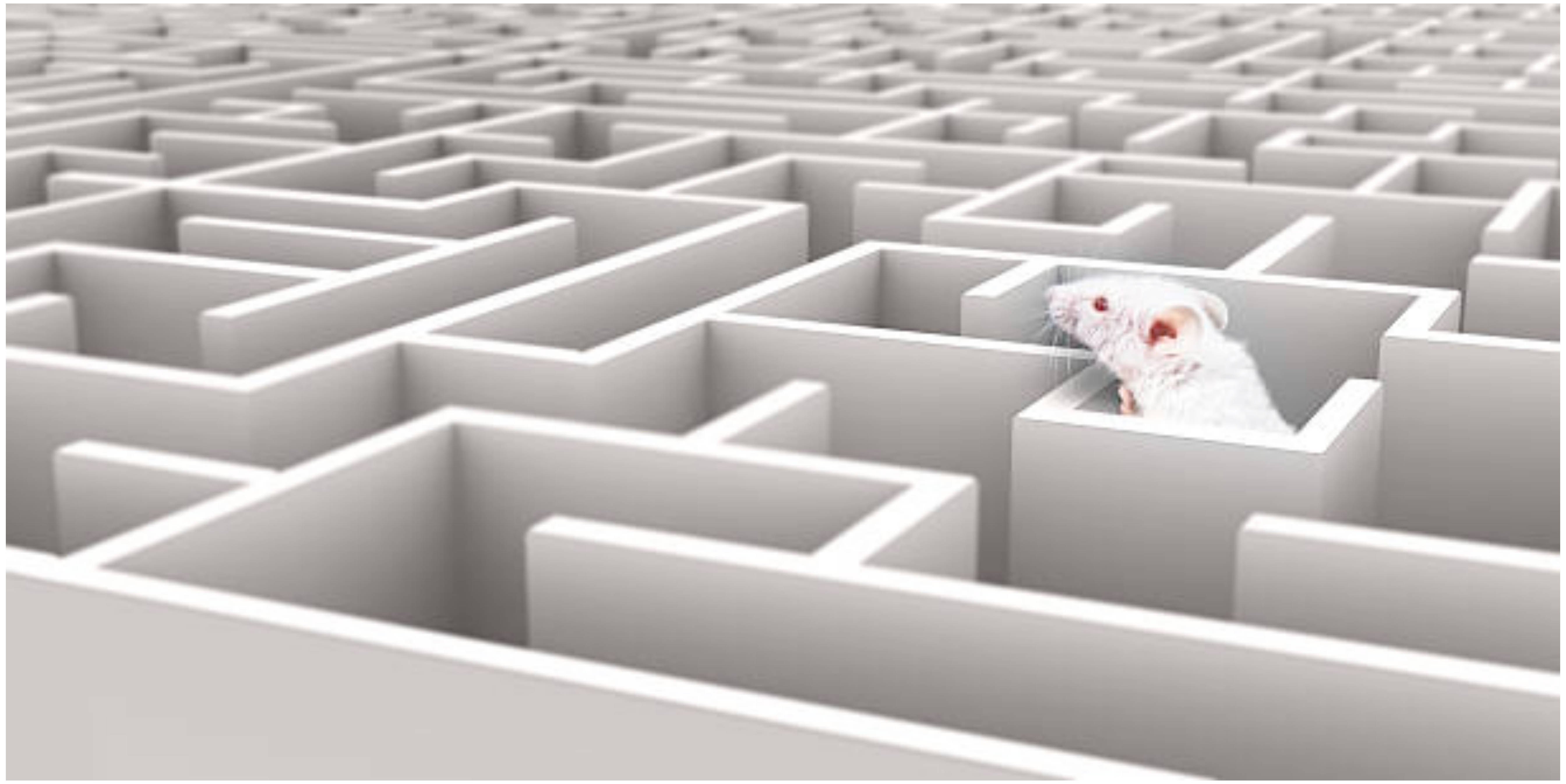
from Andreas Vesalius' Fabrica

https://en.wikipedia.org/wiki/
File:1543,_Andreas_Vesalius%27_Fabrica,_Base_Of_The_Brain.jpg

arXiv

Research Network

Neuroscience



by NeilLockhart, extracted from https://media.istockphoto.com/photos/white-mouse-in-white-maze-looking-over-walls-picture-id636373812?
k=6&m=636373812&s=612x612&w=0&h=iyeo-EXXgBKjfLR6sCb3K-RHotTYKqJHoxDYqP-sllg=

Place Cell Model
[Curto and Itskov, 2008; Dabaghian et al, 2012]

- “Place cells” fire when the rat is near their locations.
- The rat remembers which cells fire together.

Connectivity of the co-firing network

v.s. connectivity of the navigation space

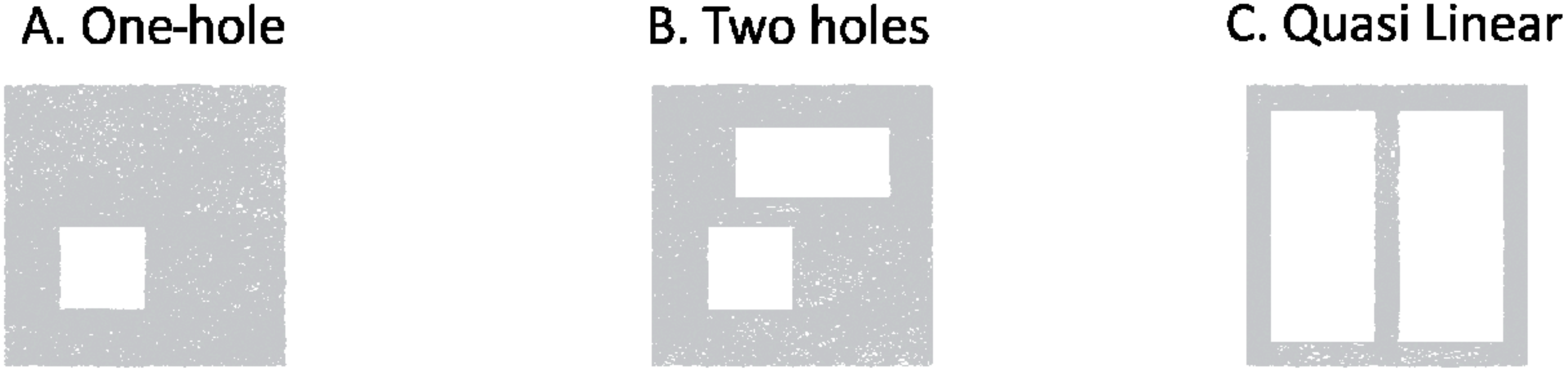


Figure 4 of [Dabaghian et al, 2012]

Connectivity of Real Data

[Dabaghian et al, 2014]

Connectivity of Real Data

[Dabaghian et al, 2014]

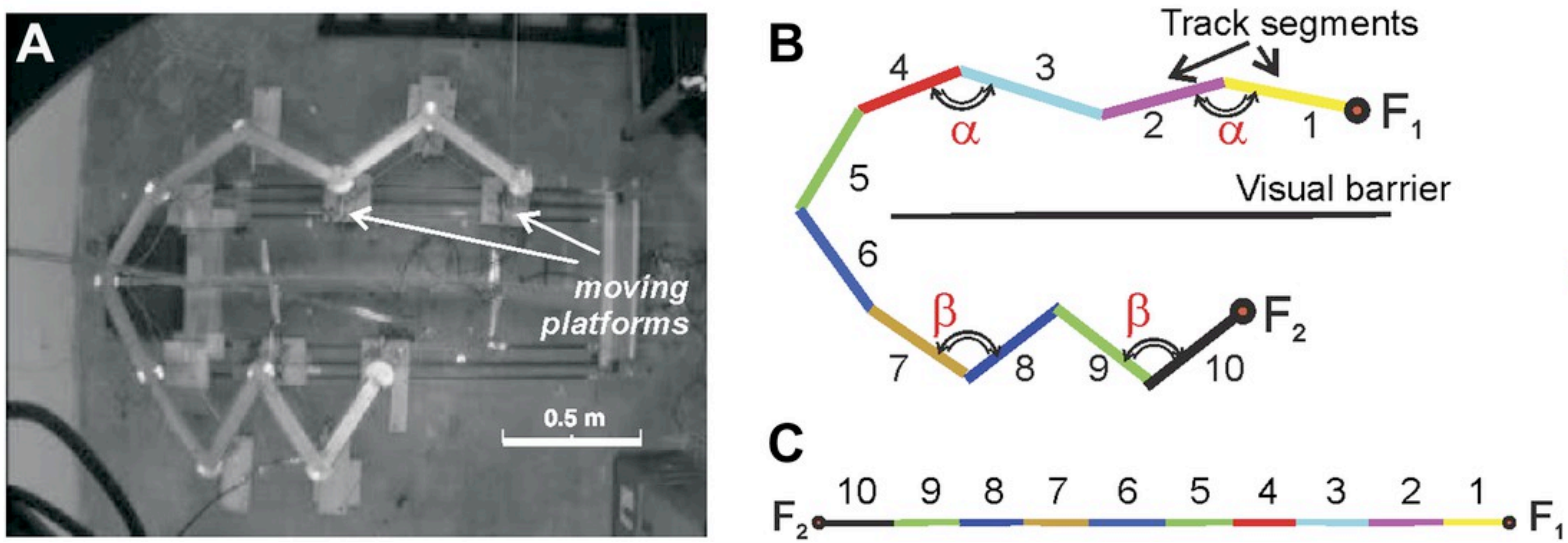
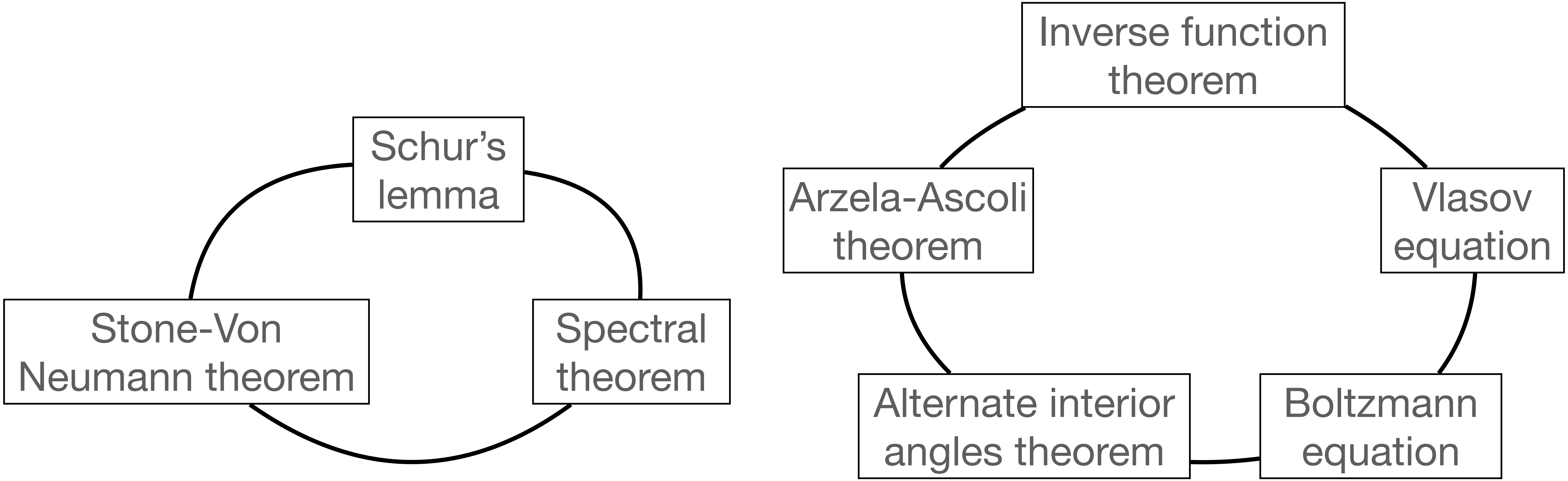


Fig 1 of [Dabaghian et al, 2014]

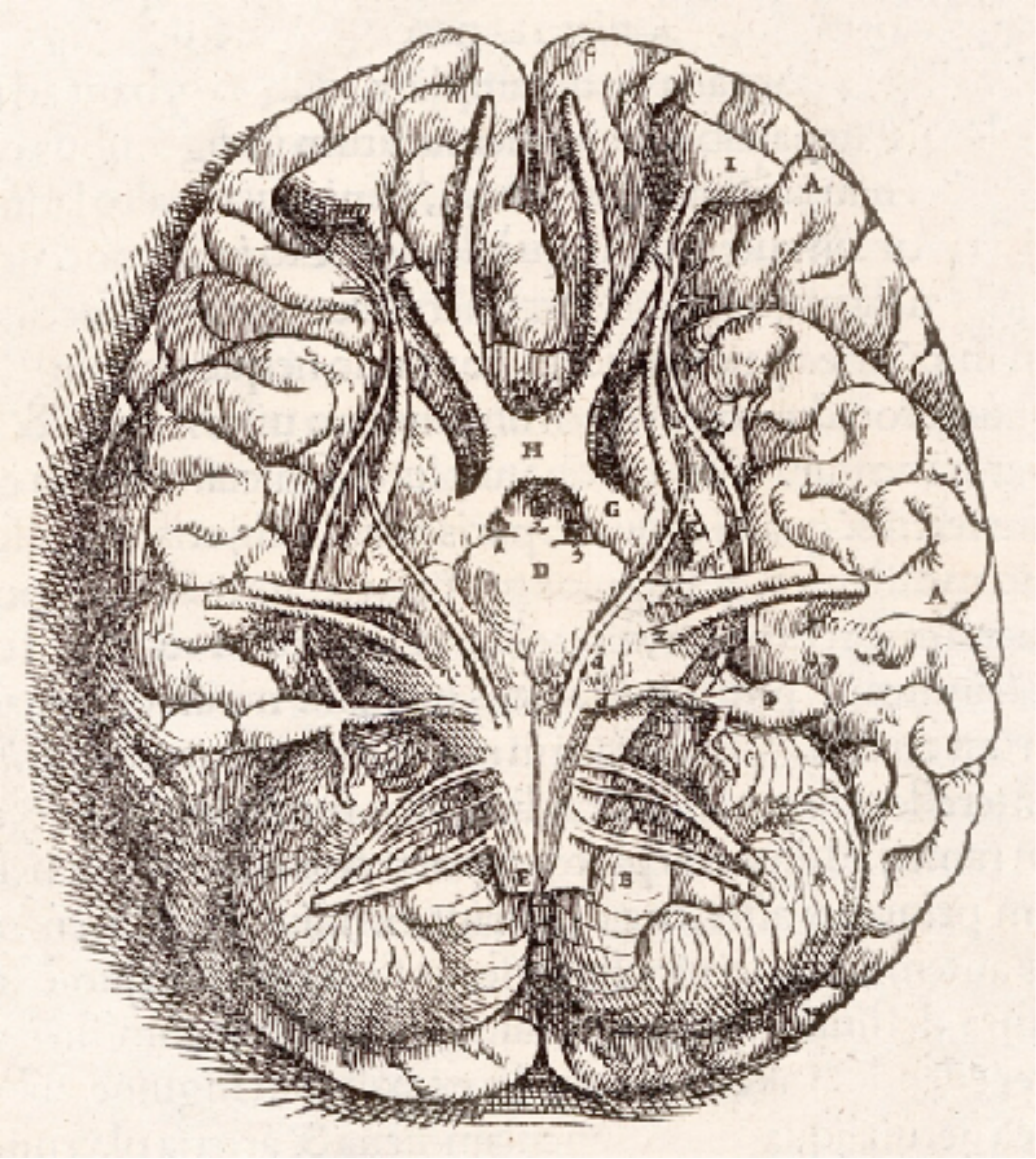
Research Network
[Salikov et al, 2018]

- Co-occurence complex in Math research paper

Gap in Understanding



Topology and Networks



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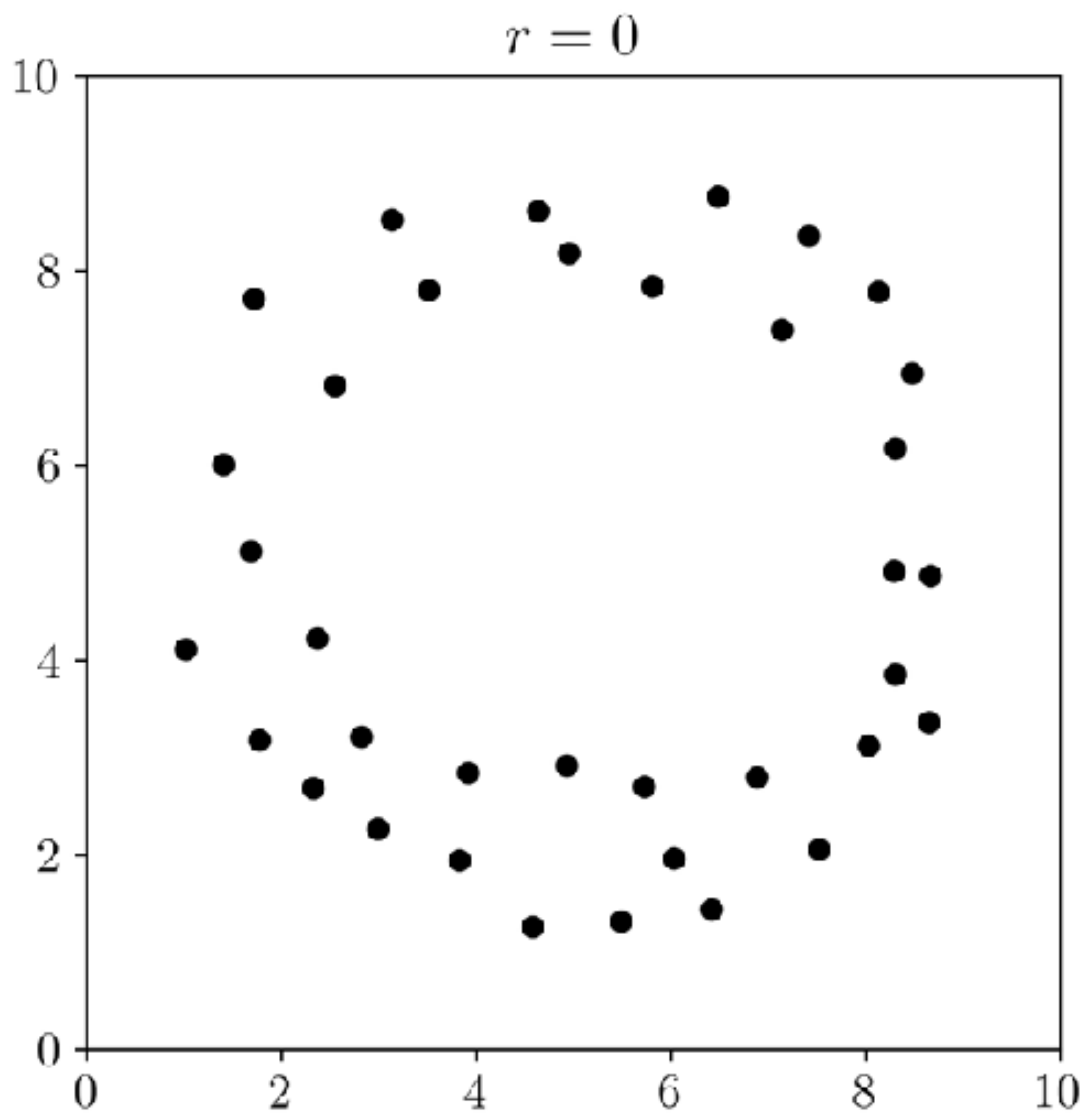
arXiv

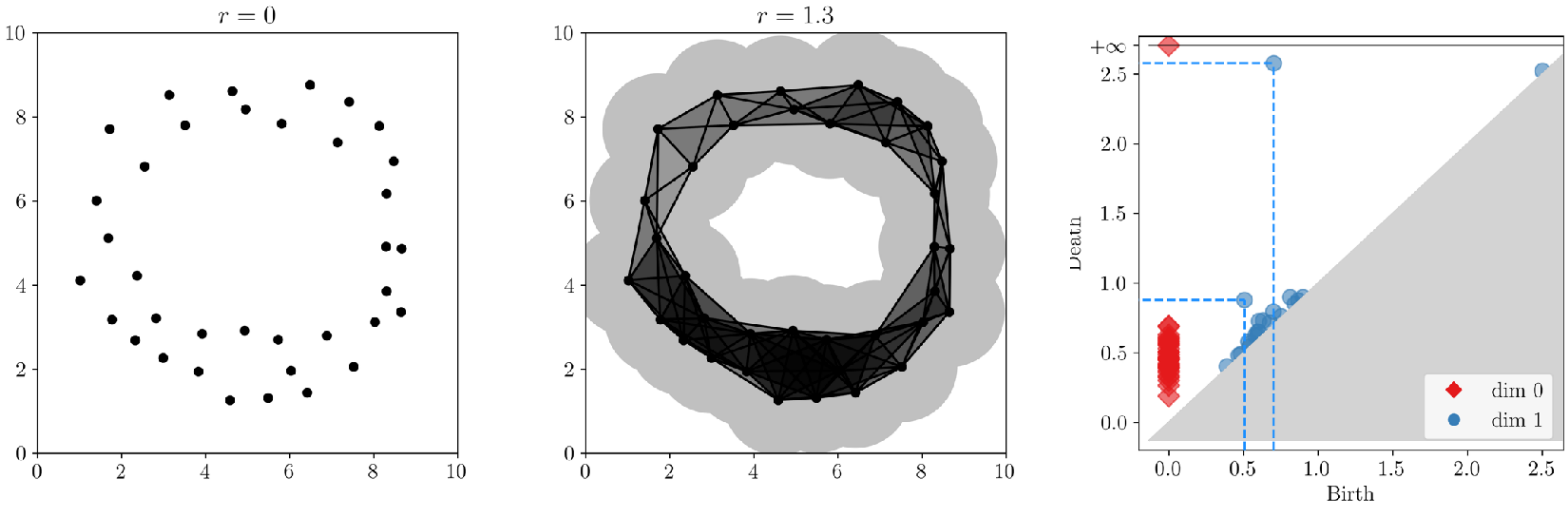
Research Network

I. A Probabilist’s Apology

Why Random Topology

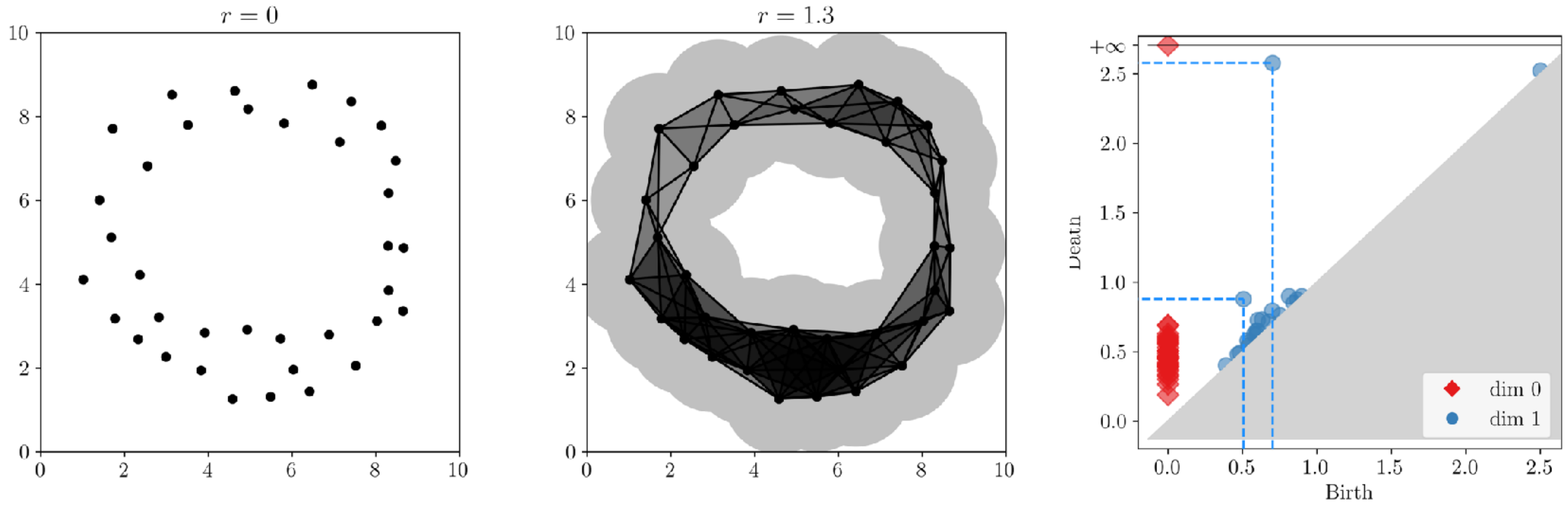


$r = 0$
0
2
4
6
8
10
0
2
4
6
8
10

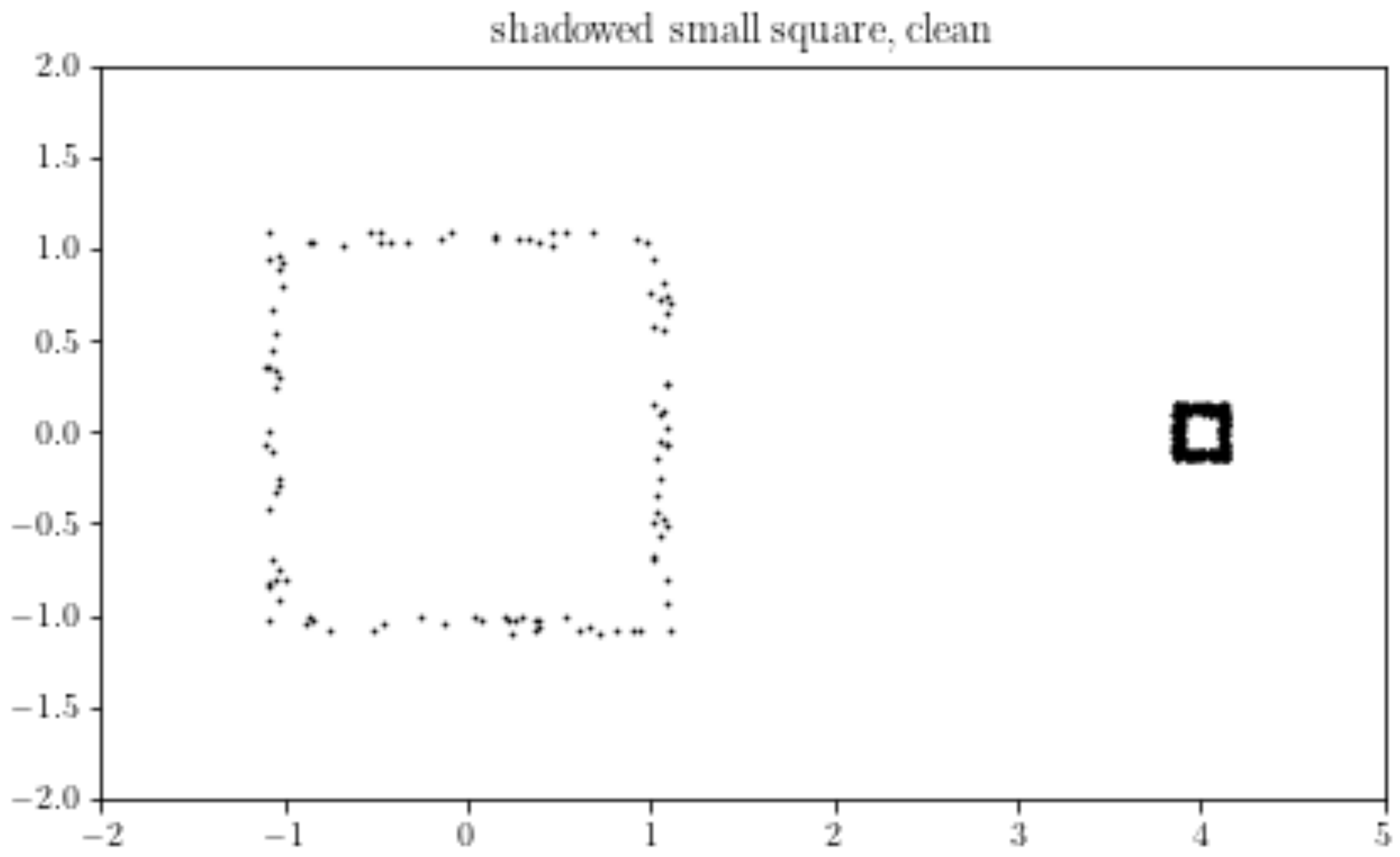


plots generated by Andrey Yao

Size is Signal

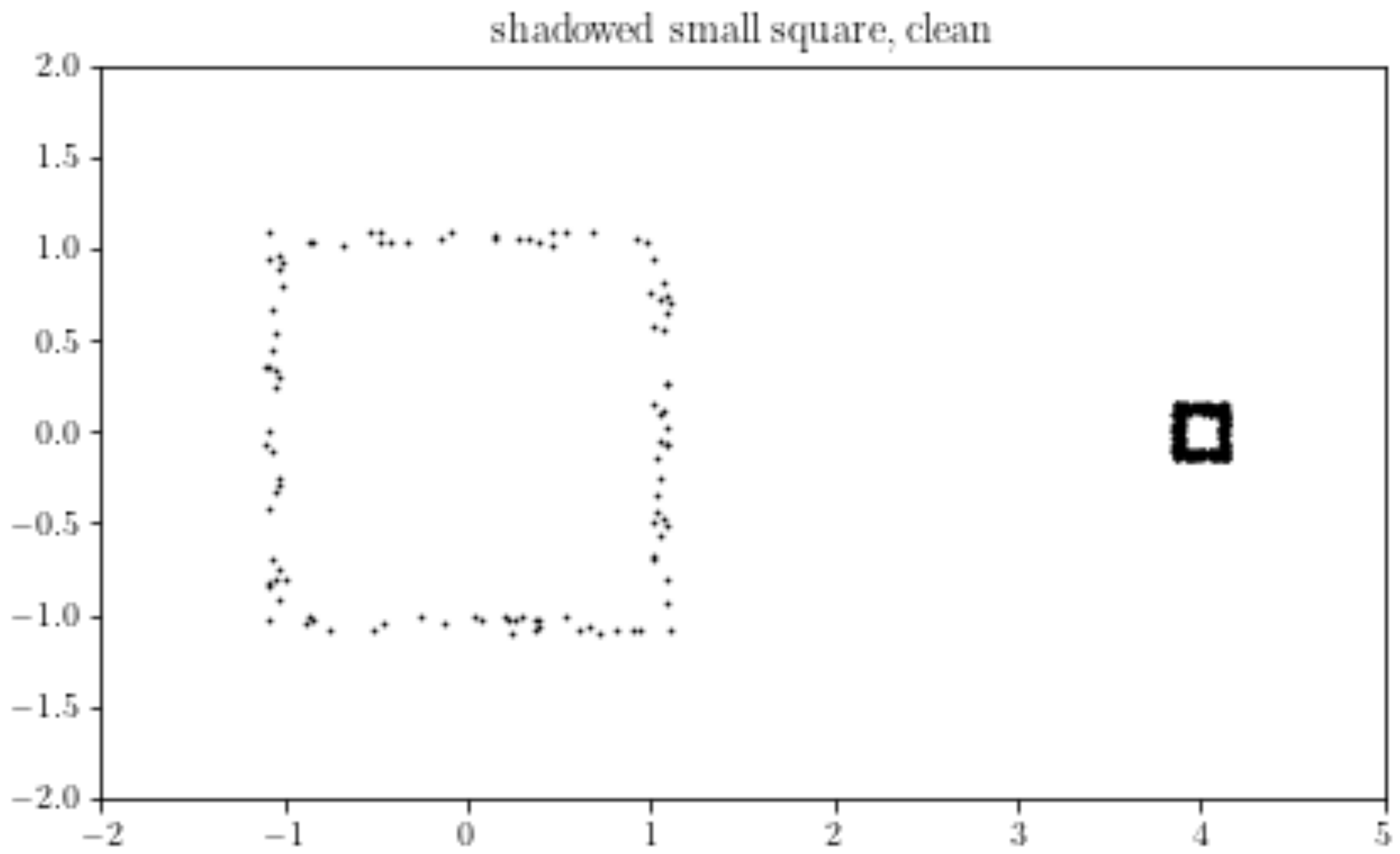


Or is it?

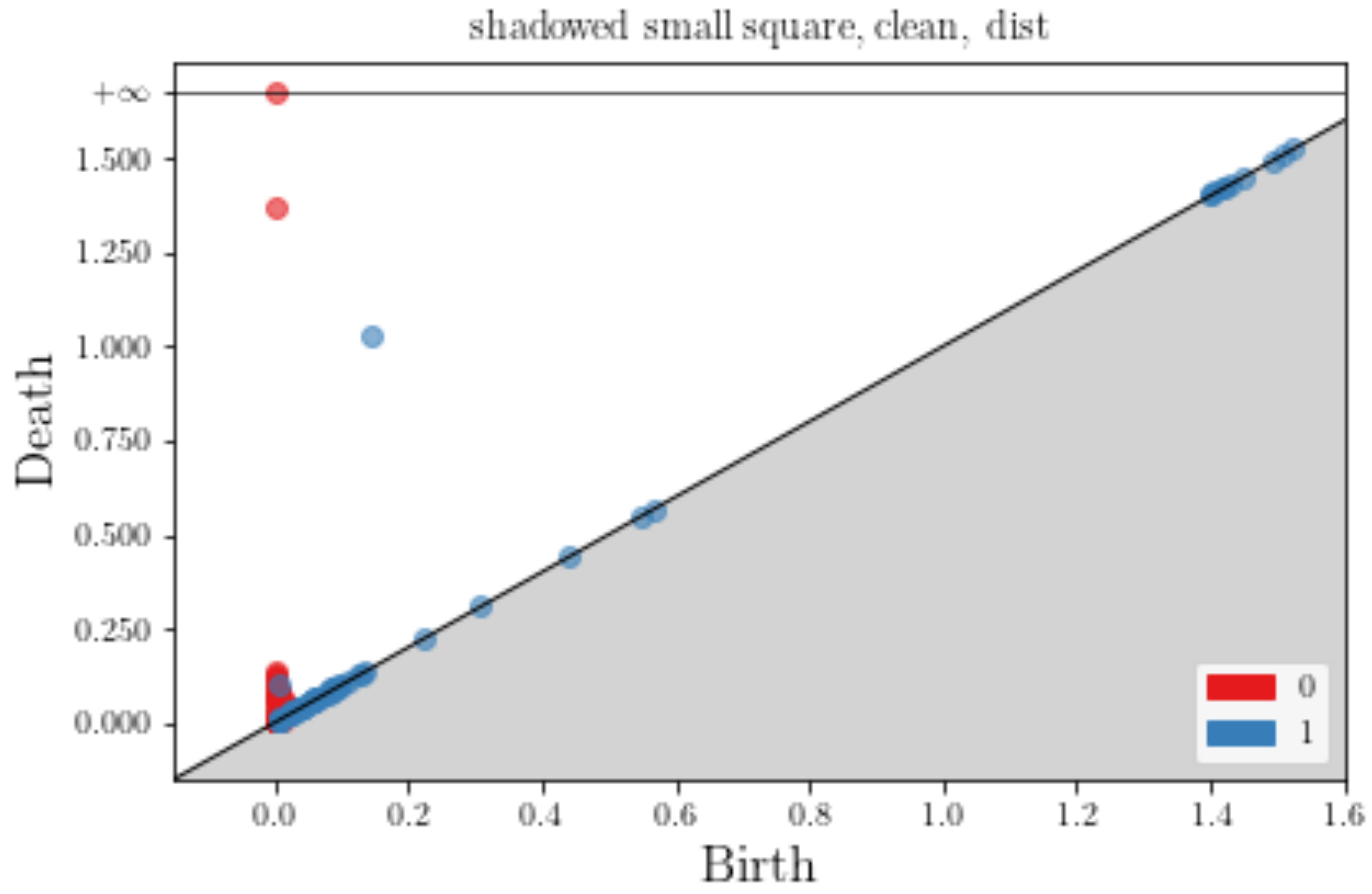


shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5

Or is it?



shadowed small square, clean



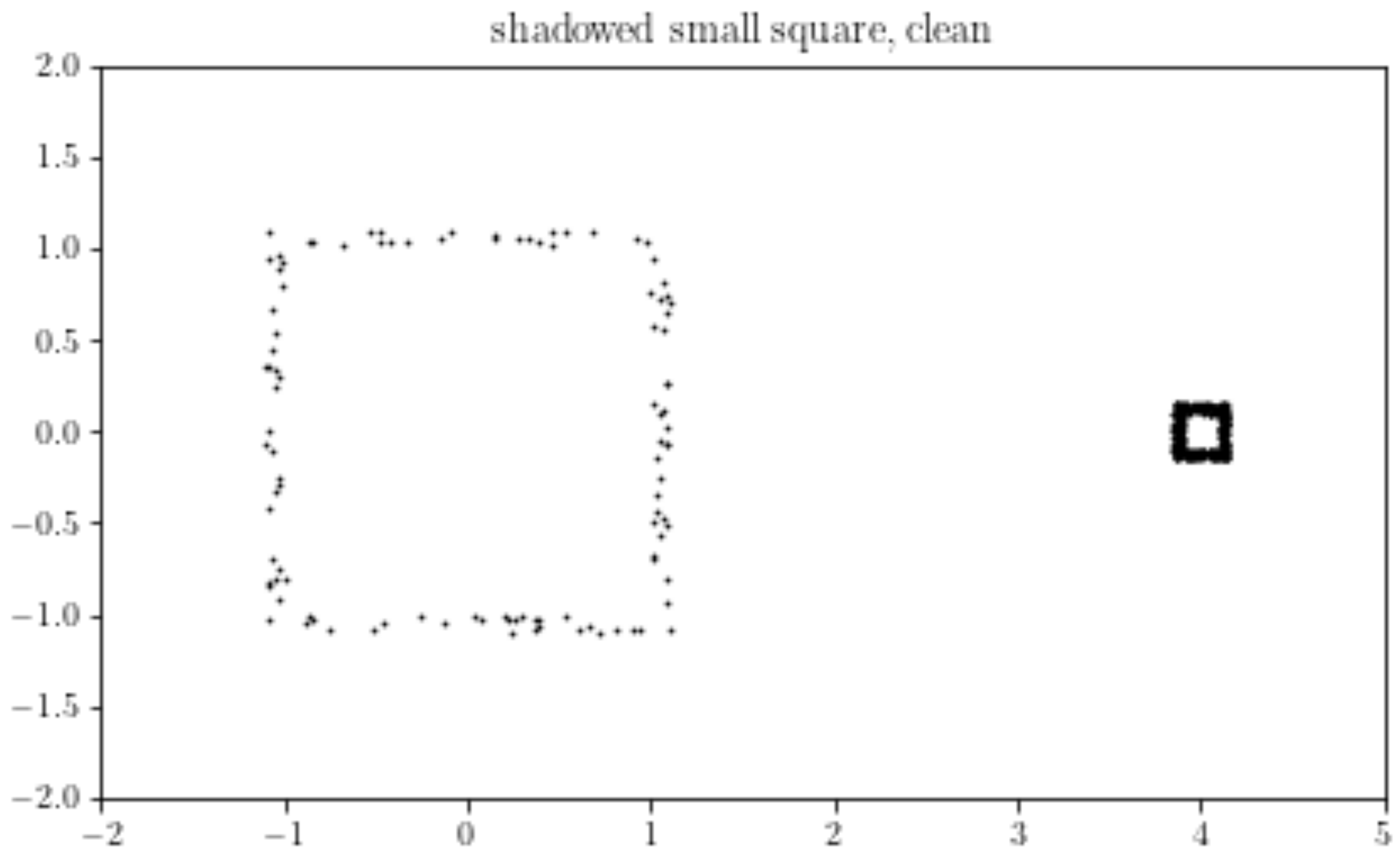
shadowed small square, clean, dist
Death
Birth
0
1

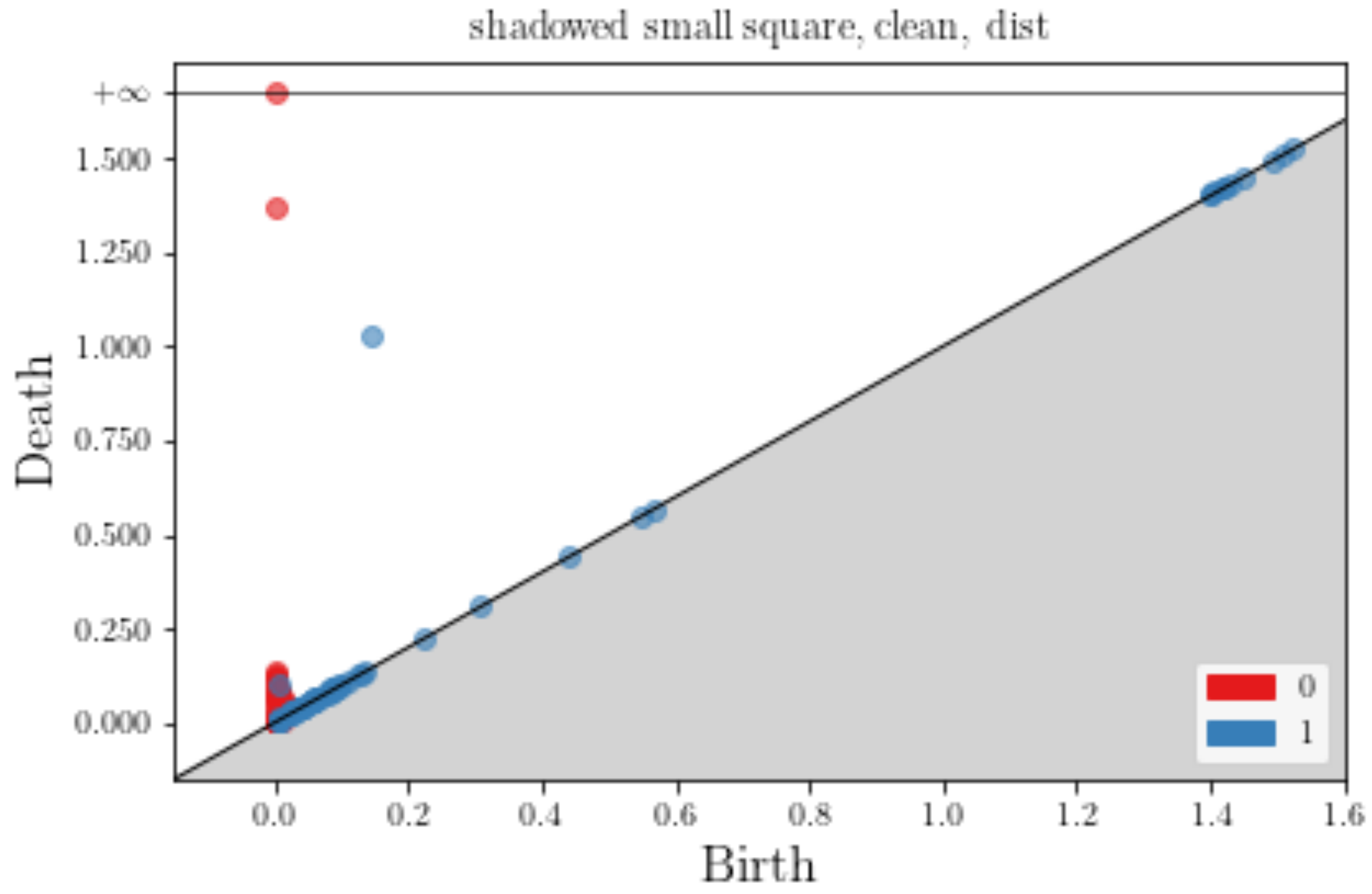
Size is Signal?

Surprise

~~Size~~ is Signal.

Random points don’t do that.





Signal is what is not random.

Signal is what is not **random.**
So what is **random?**

Functional Convergence at dimension k?

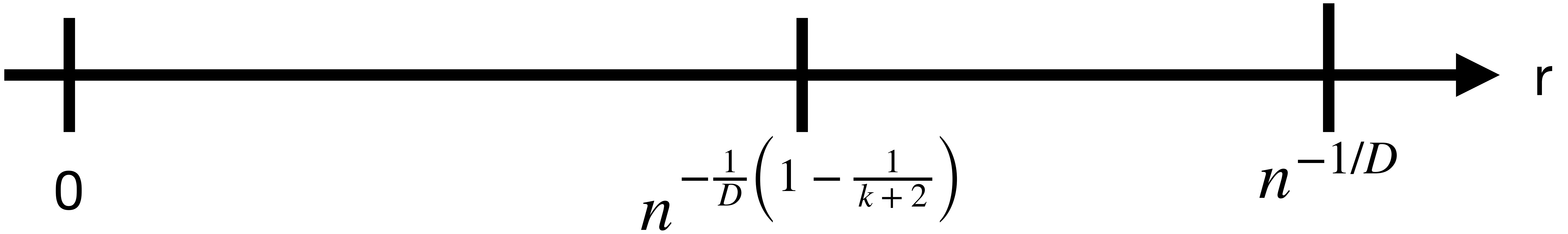
[Thomas and Owada 2020]

Functional Convergence at dimension k?
[Thomas and Owada 2020]

- Cech: weak convergence in finite-dimensional sense

Functional Convergence at dimension k?
[Thomas and Owada 2020]

- Cech: weak convergence in finite-dimensional sense

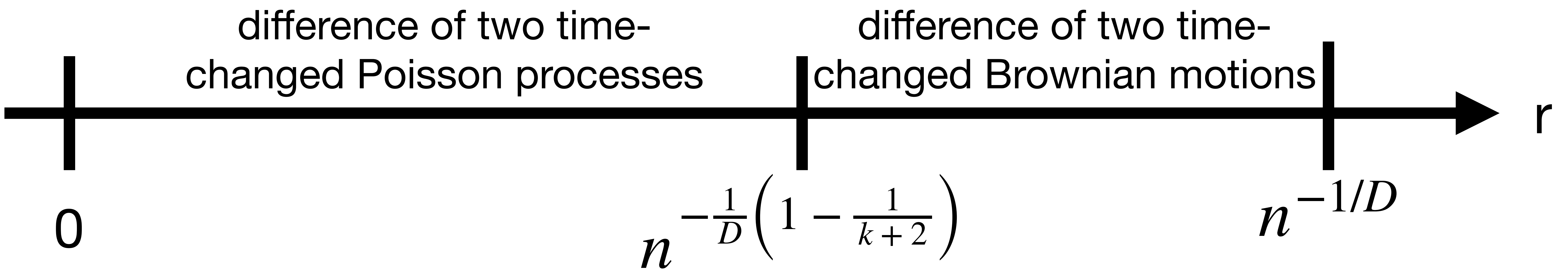


n = number of nodes
all log terms and constants forgone

Functional Convergence at dimension k?

[Thomas and Owada 2020]

- Cech: weak convergence in finite-dimensional sense



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Functional Convergence at dimension k?

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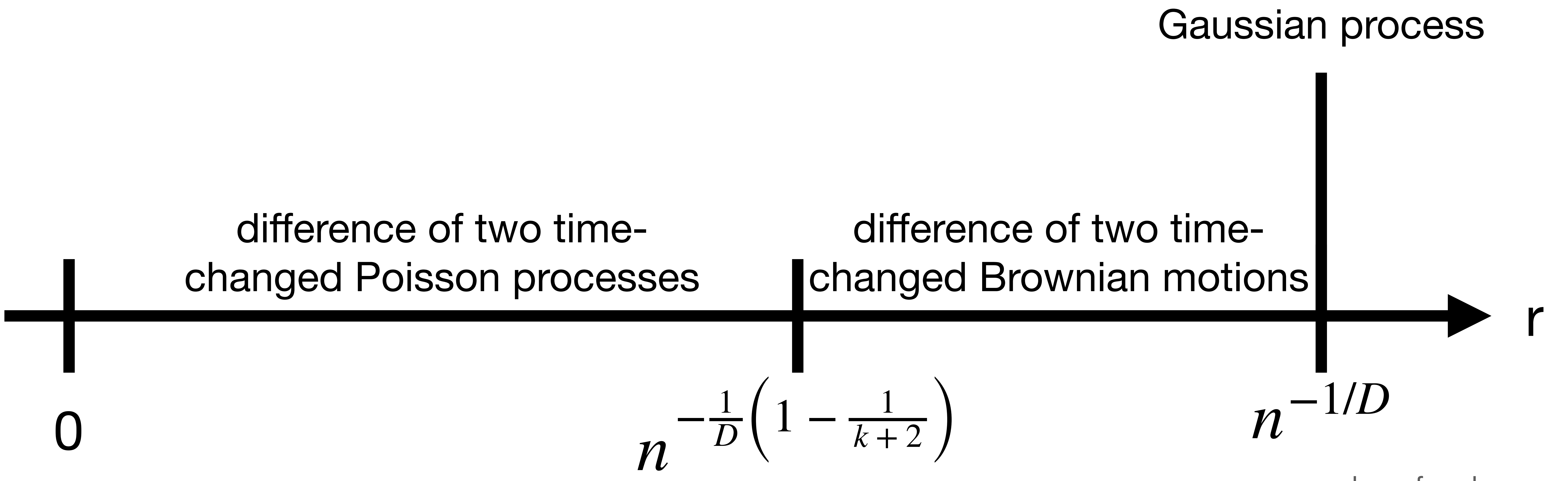
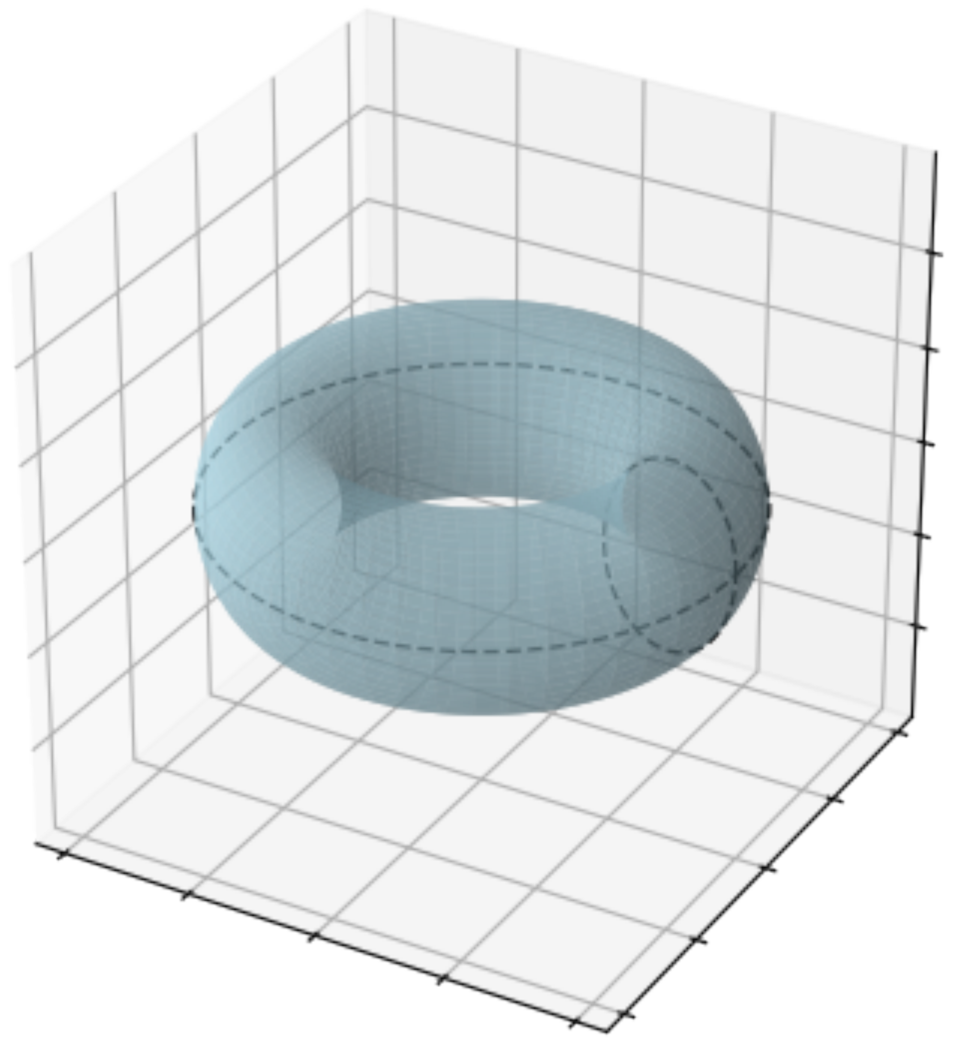
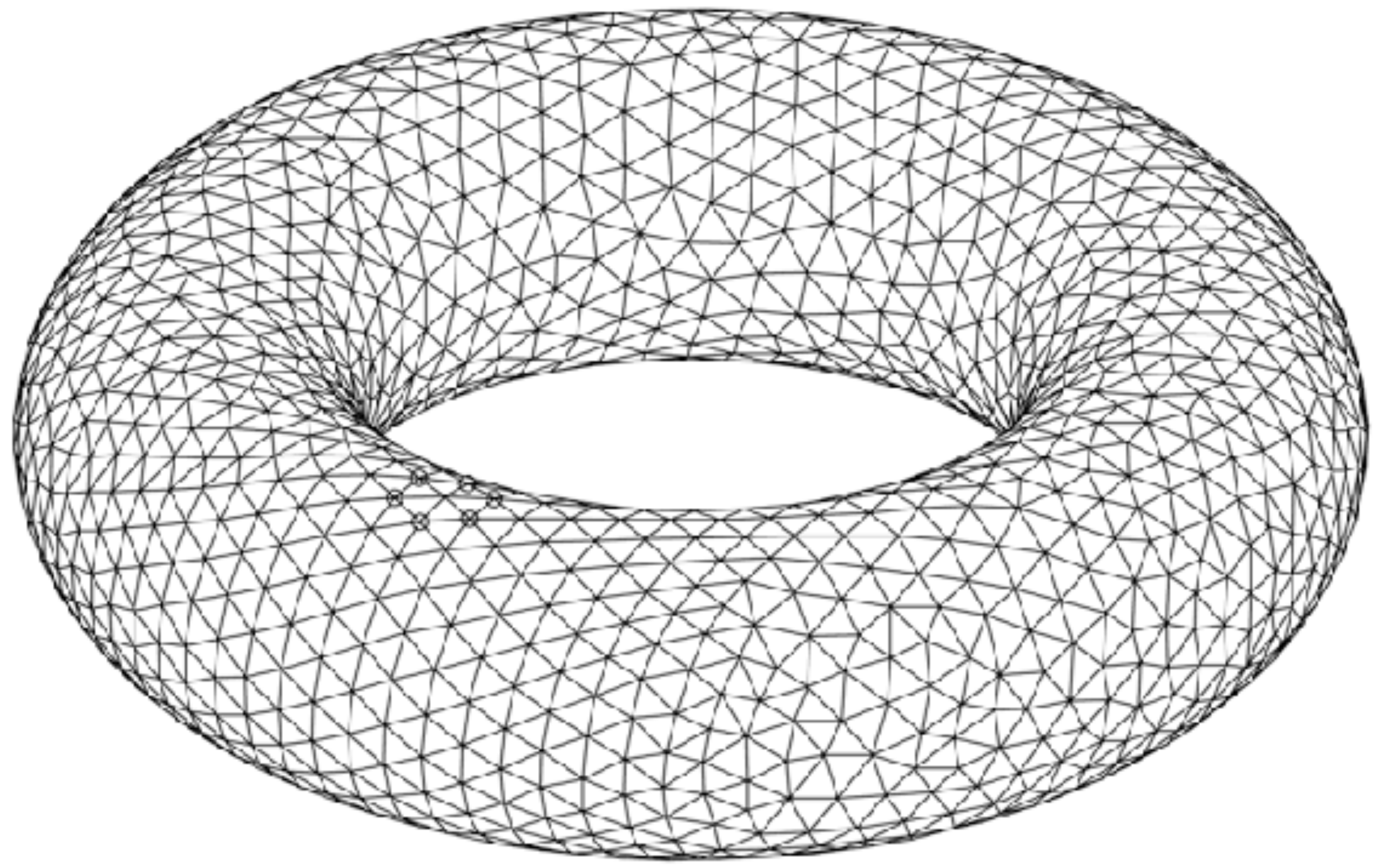


n = number of nodes
all log terms and constants forgone

Topology



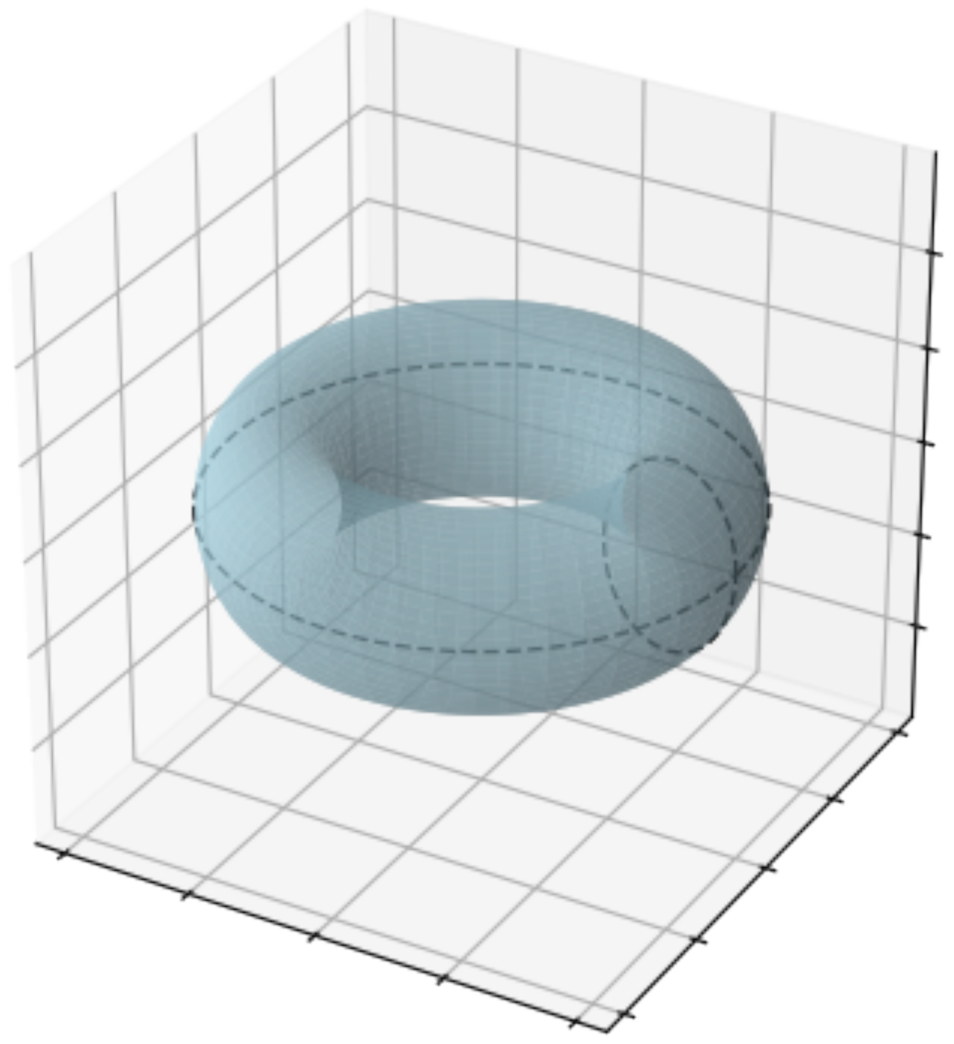
Andrey Yao



Maintenance Script (?)
https://handwiki.org/wiki/File:Torus-triang.png

Topology

New Models



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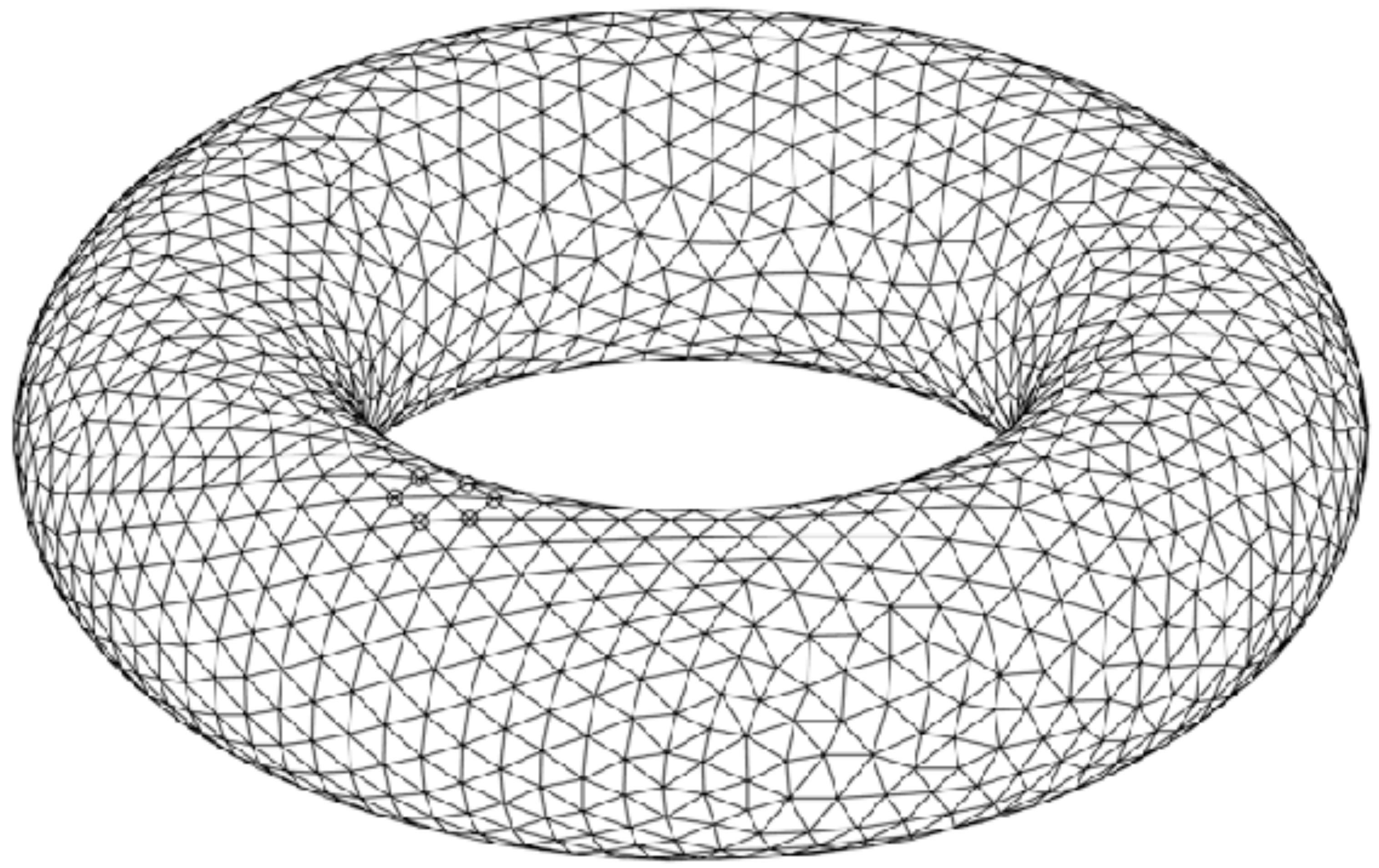

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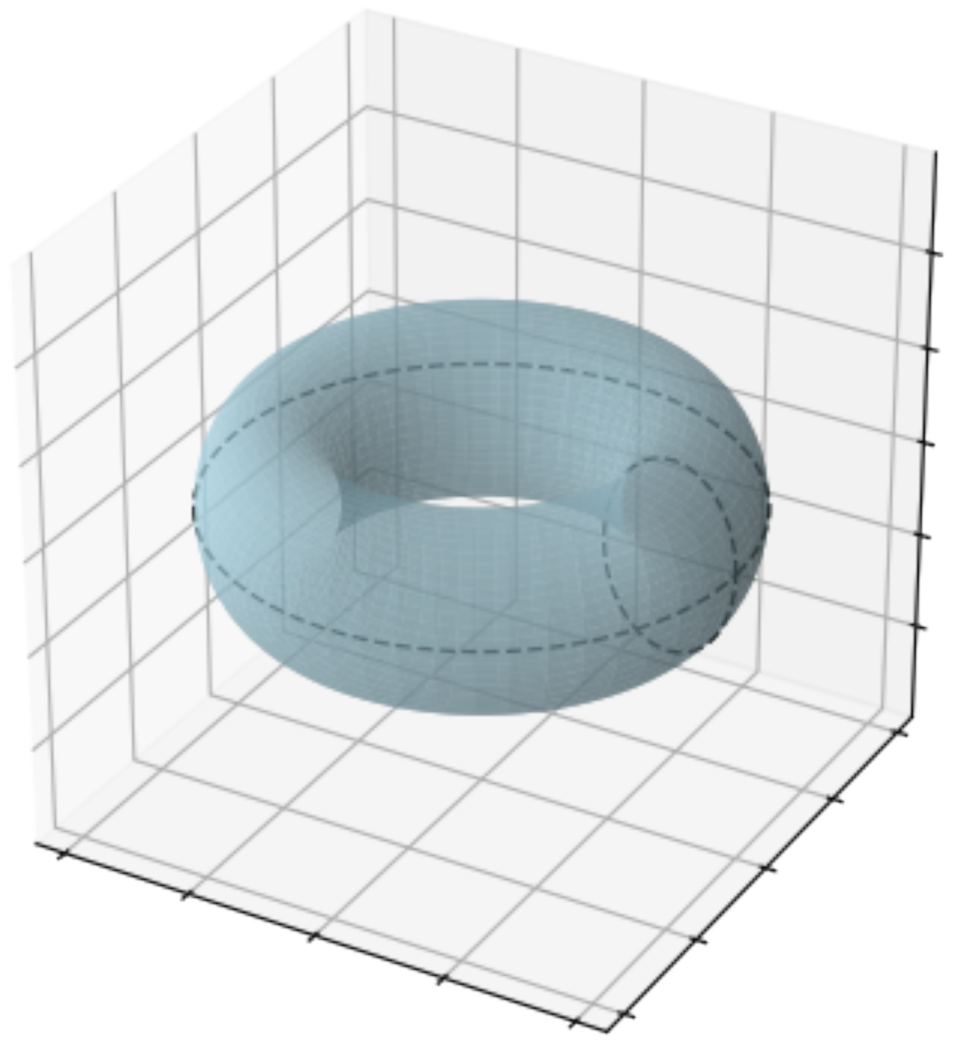

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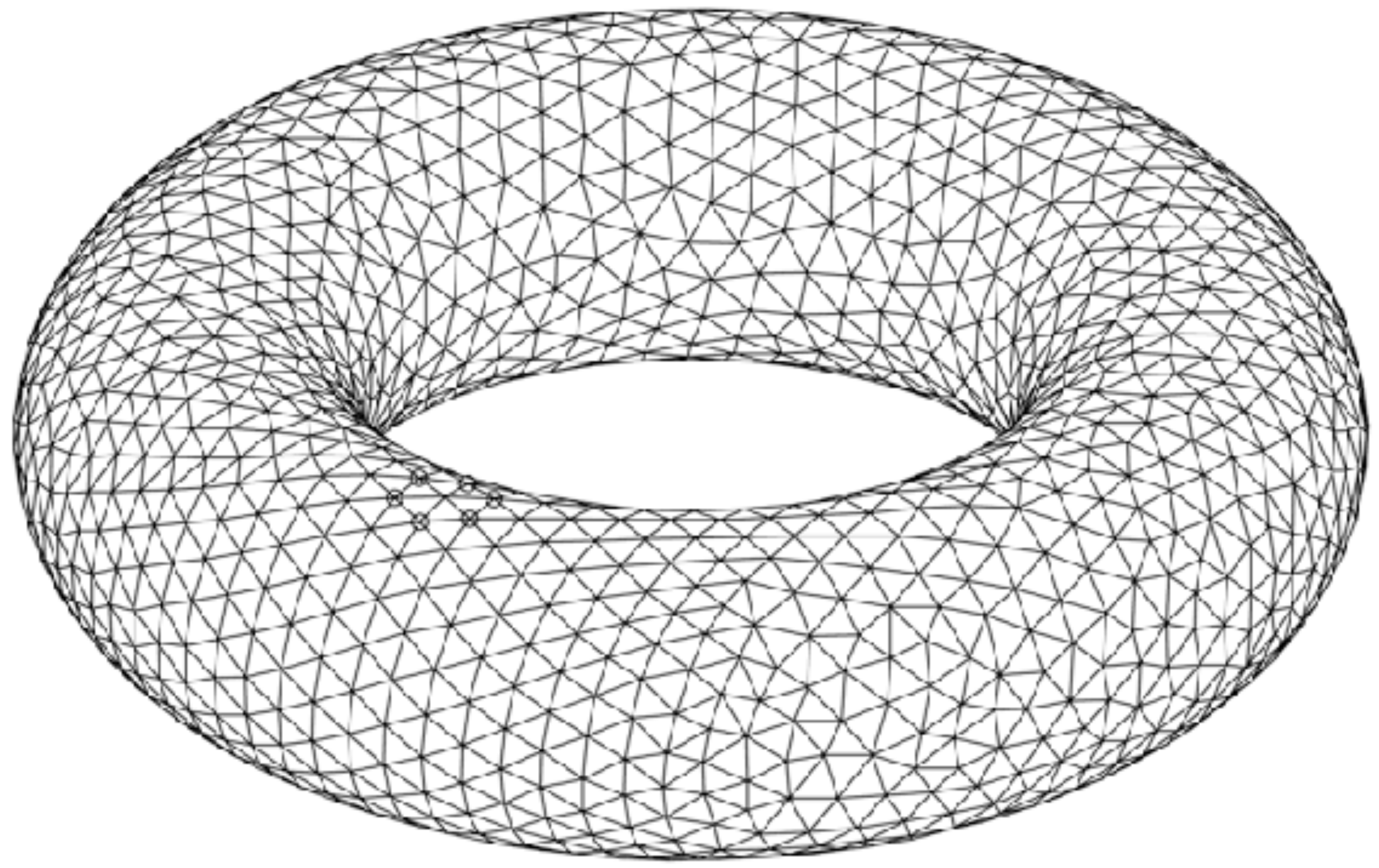
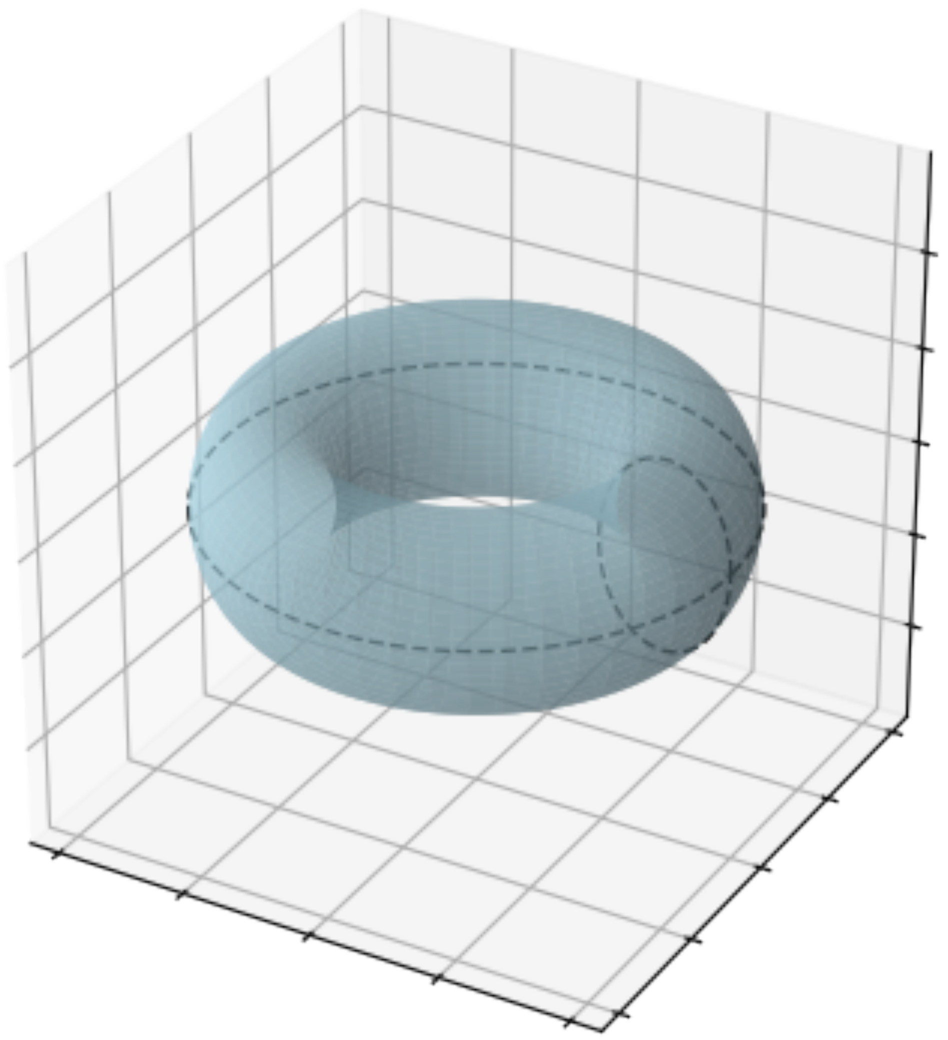


Maintenance Script (?)
https://handwiki.org/wiki/File:Torus-triang.png

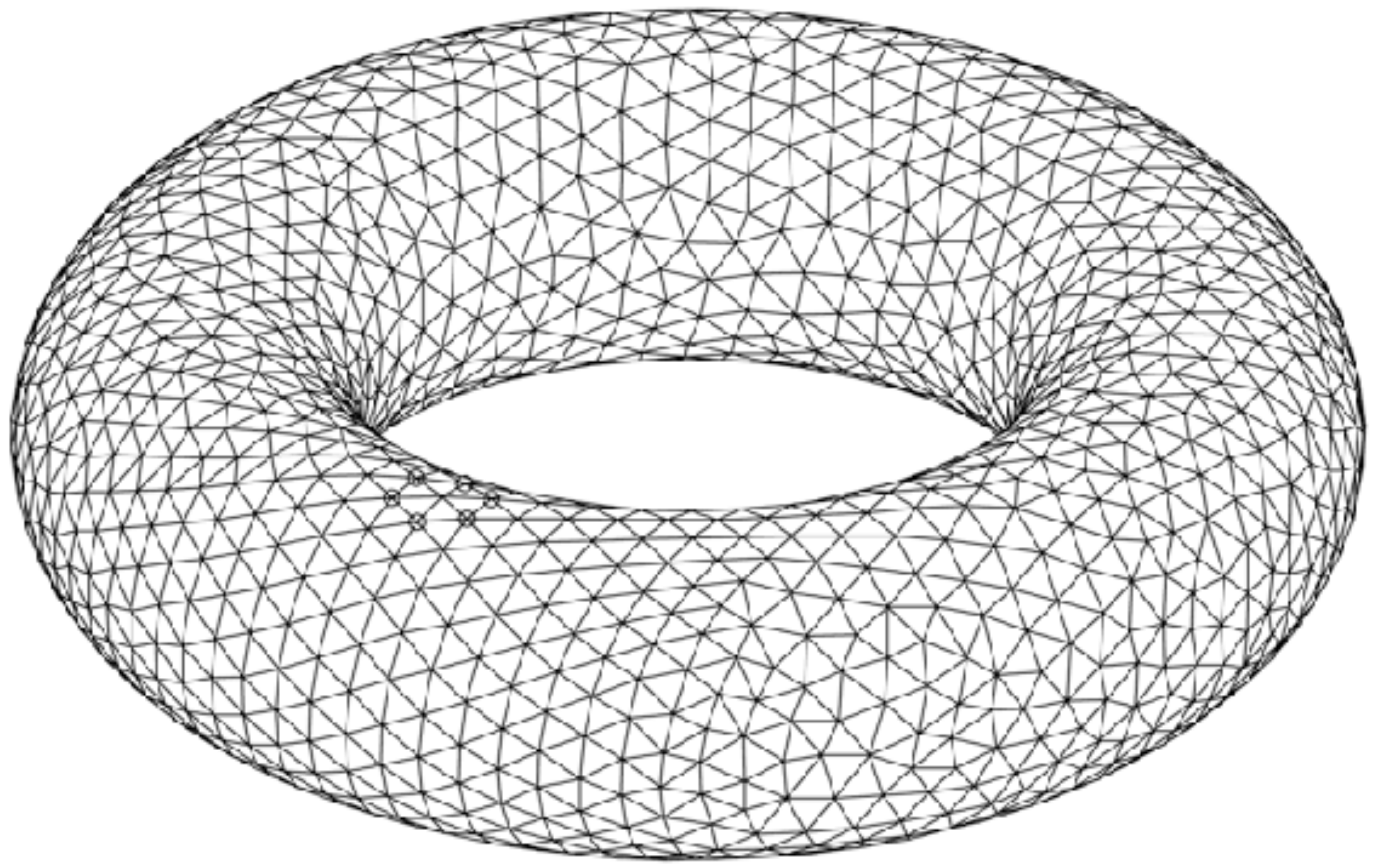
New Models

New Questions

Topology



Andrey Yao



Maintenance Script (?)
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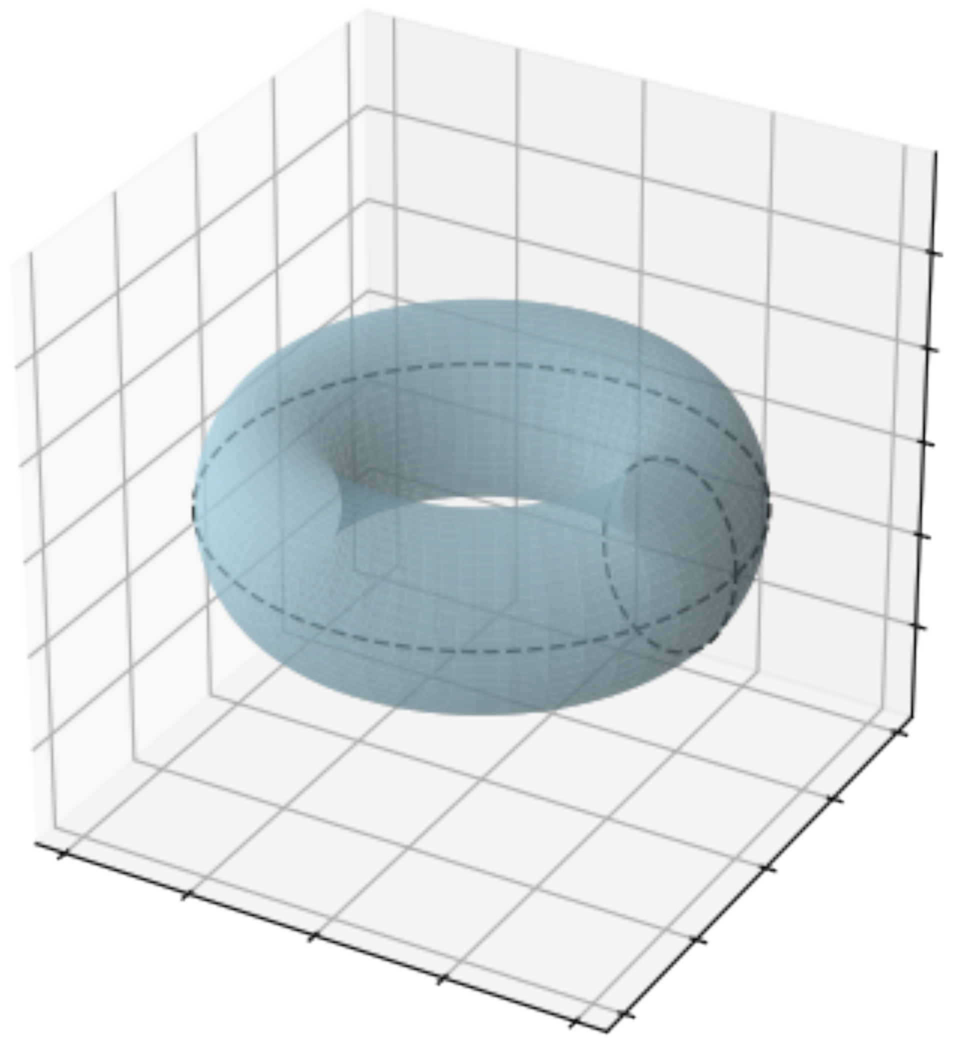
New Models

New Questions

New Techniques

Topology



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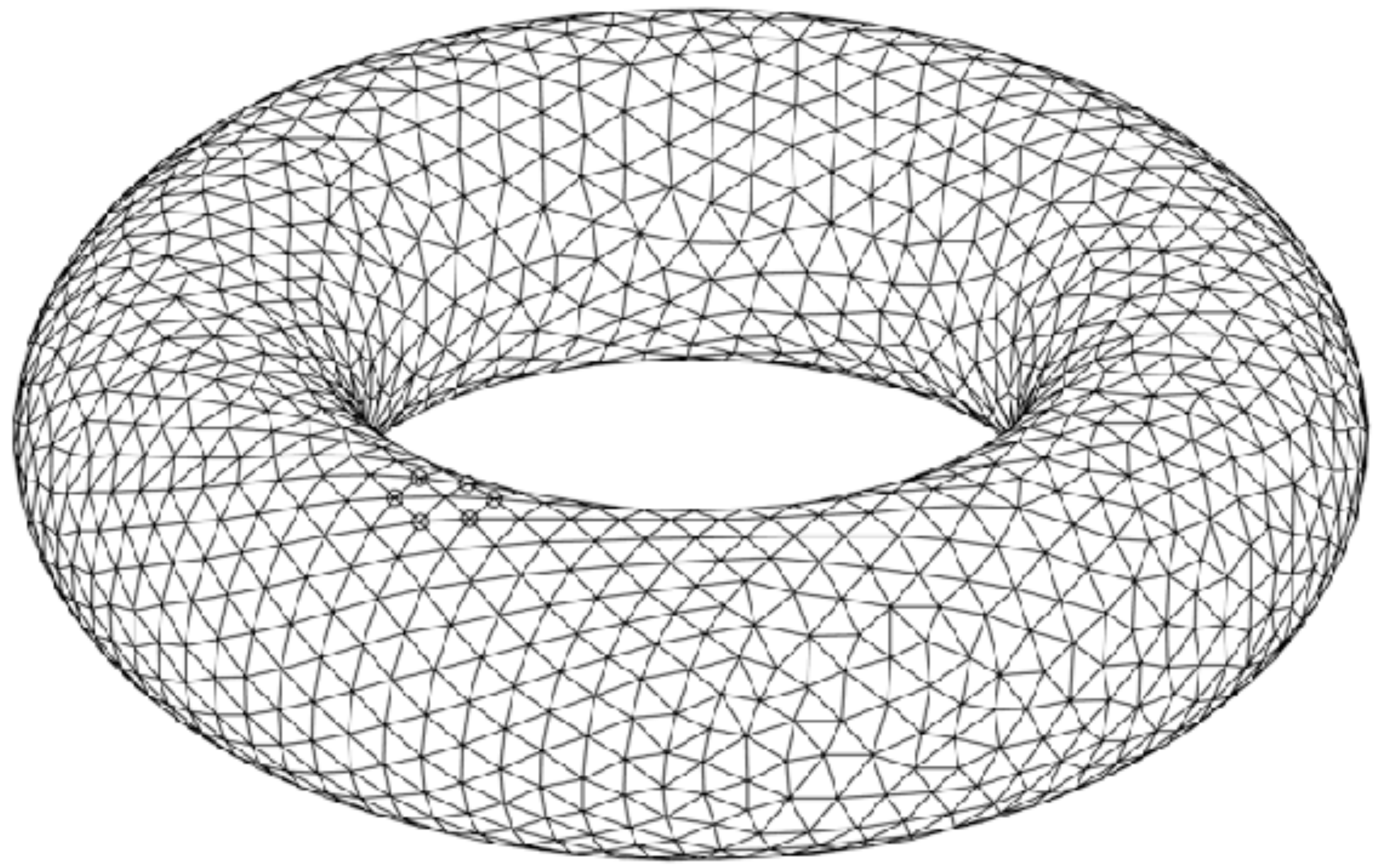

Maintenance Script (?)
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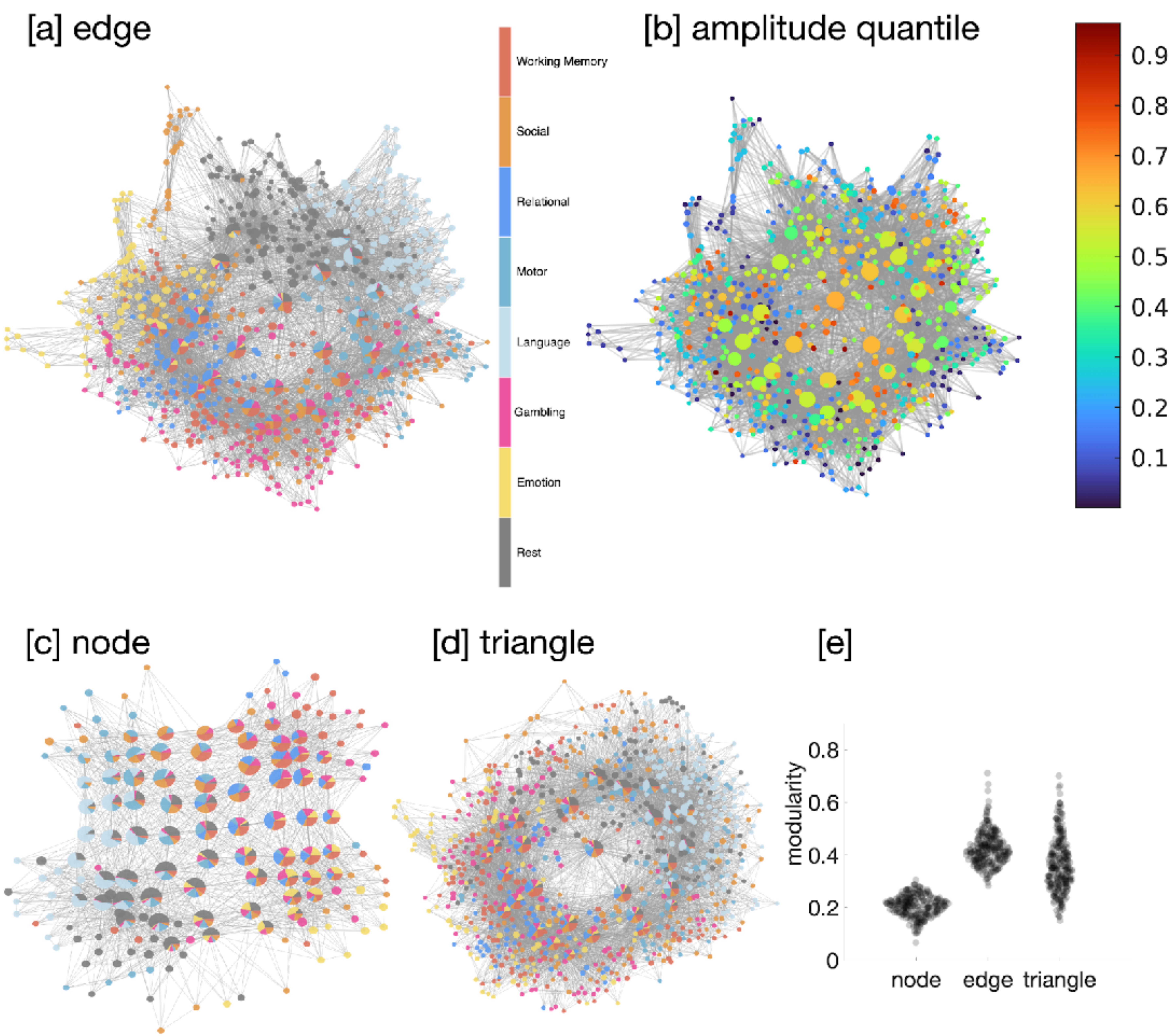
New Models

New Questions

New Techniques

New Answers


[a] edge
Working Memory
Social
Relational
Motor
Language
Gambling
Emotion
Rest
[b] amplitude quantile
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
[c] node
[d] triangle
[e]
modularity
0.8
0.6
0.4
0.2
0
node
edge
triangle

cheatsheet

- Duncan, Kahle, Schweihart, 2024, Theorem 2
  For arbitrary dimensional torus, do bond percolation (start with vertices and add edges), the critical threshold for the first homology is the same as the one for the critical threshold for connectivity in the infinite lattice of the same dimension.
  https://mason.gmu.edu/~bschwei/PlaquettePercolation.pdf

- ``Hexagonal” site percolation
  https://arxiv.org/pdf/1910.10146