The Topology of Preferential Attachment

Higher-Order Connectivity of Random Interactions

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Graph Theorists / Network Scientists

Probabilists / Statisticians

Topologist

Topology of Scale-Free Graphs — Homology and Homotopy

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So, preferential attachment…

- Highly connected hubs

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

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- Highly connected hubs
- Dense core of hubs?

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So, preferential attachment…

- Highly connected hubs
- Dense core of hubs?
- Beyond pairwise connections?

- —> topological properties

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Agenda

preferential attachment

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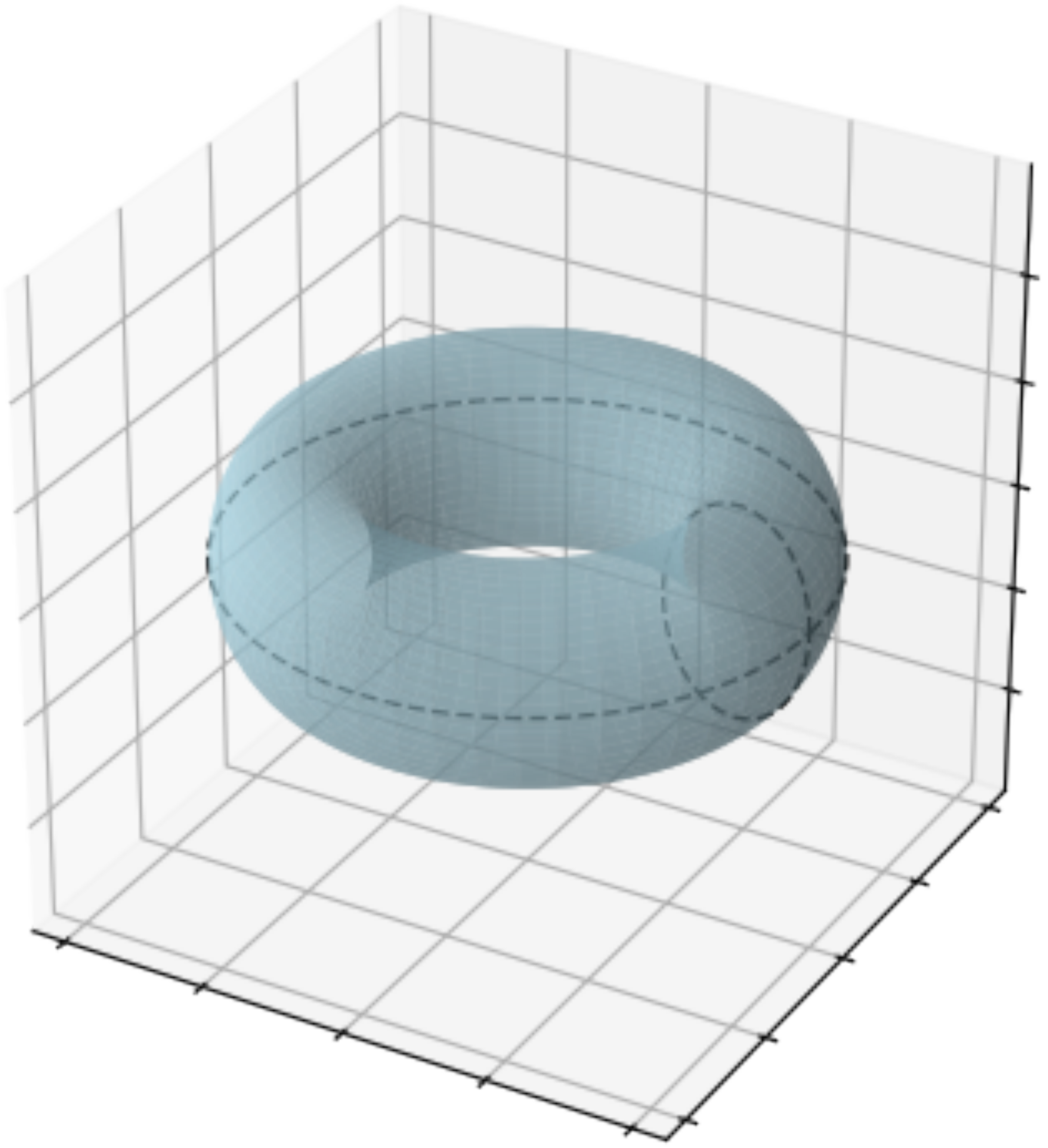
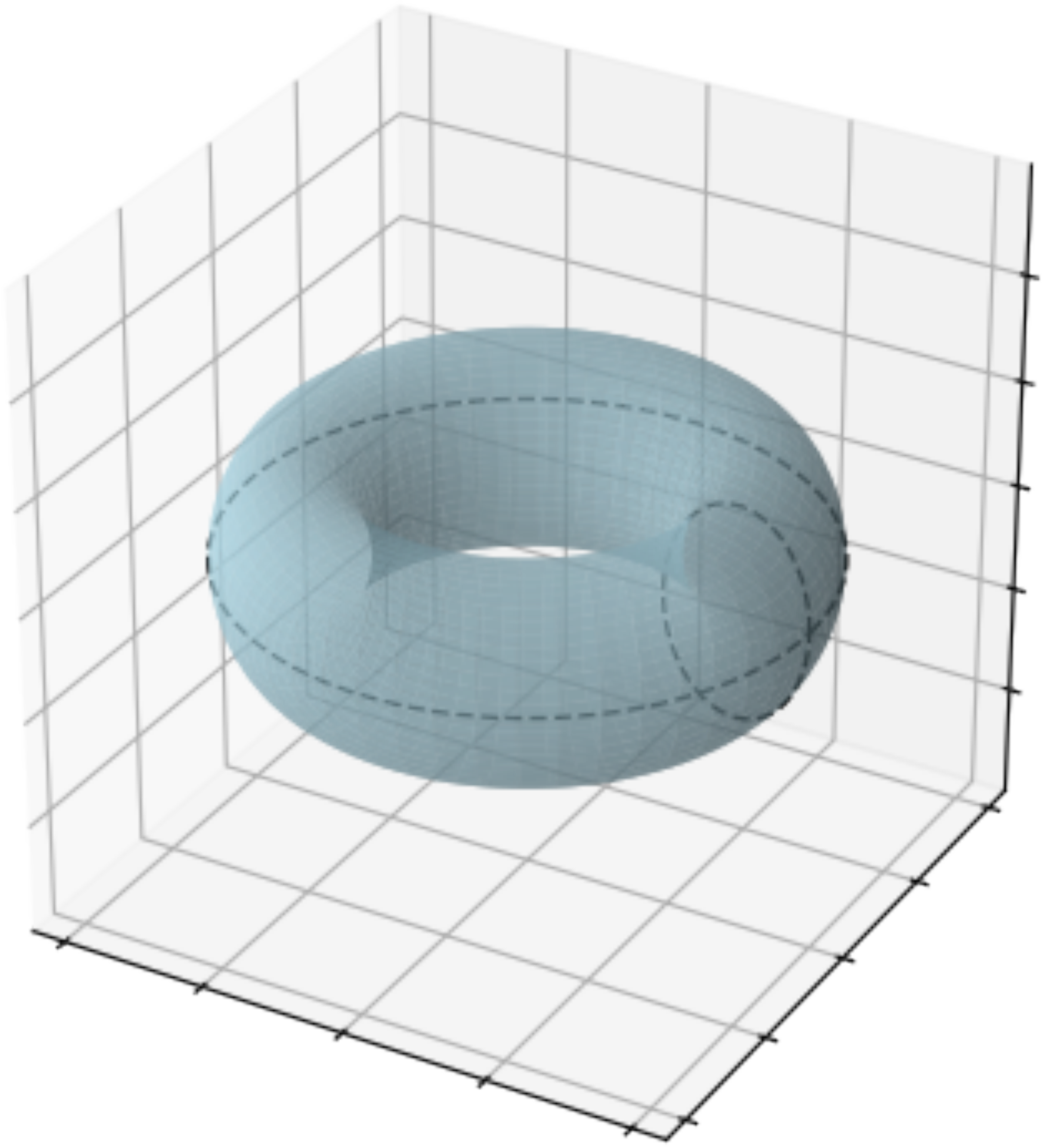
preferential attachment



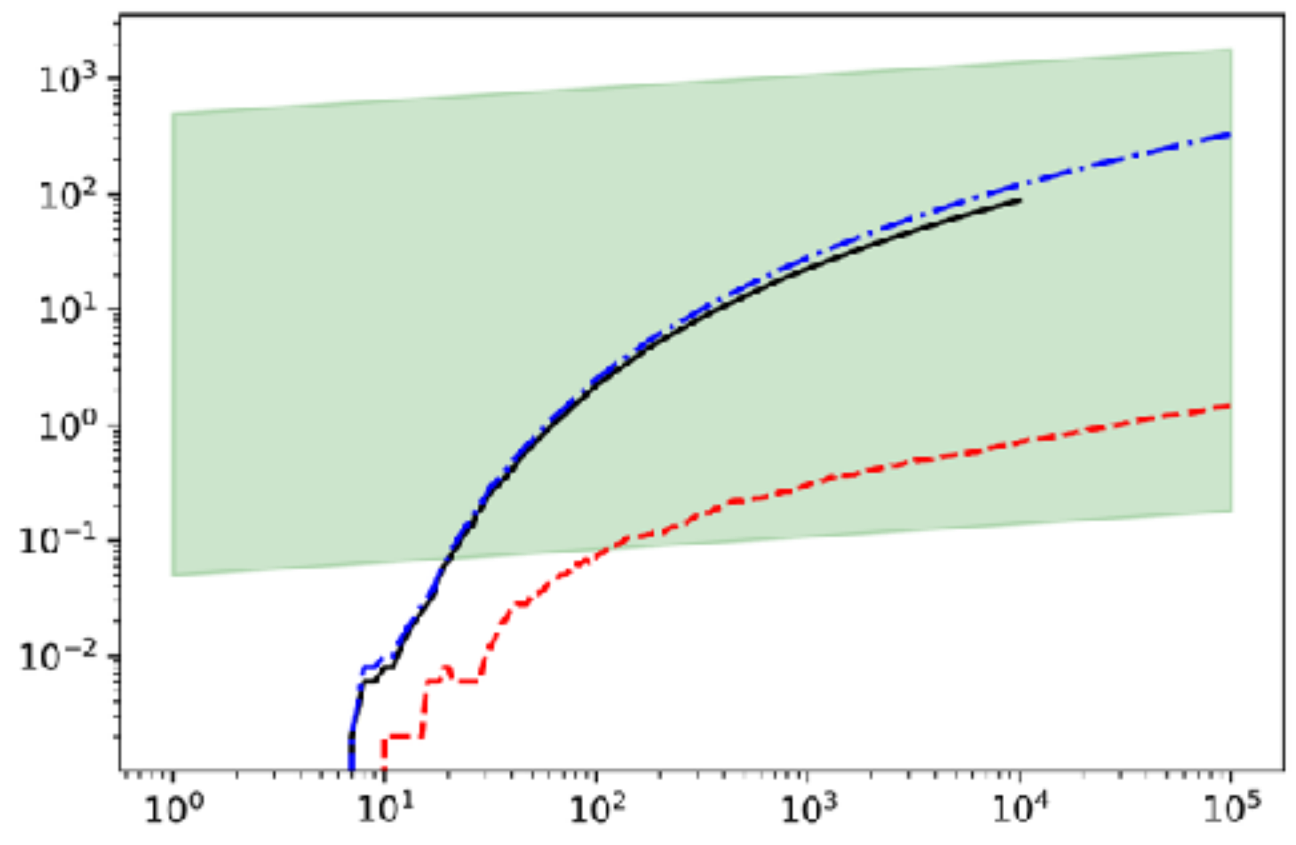
topology

Agenda

preferential attachment



topology



our result

I. Preferential Attachment

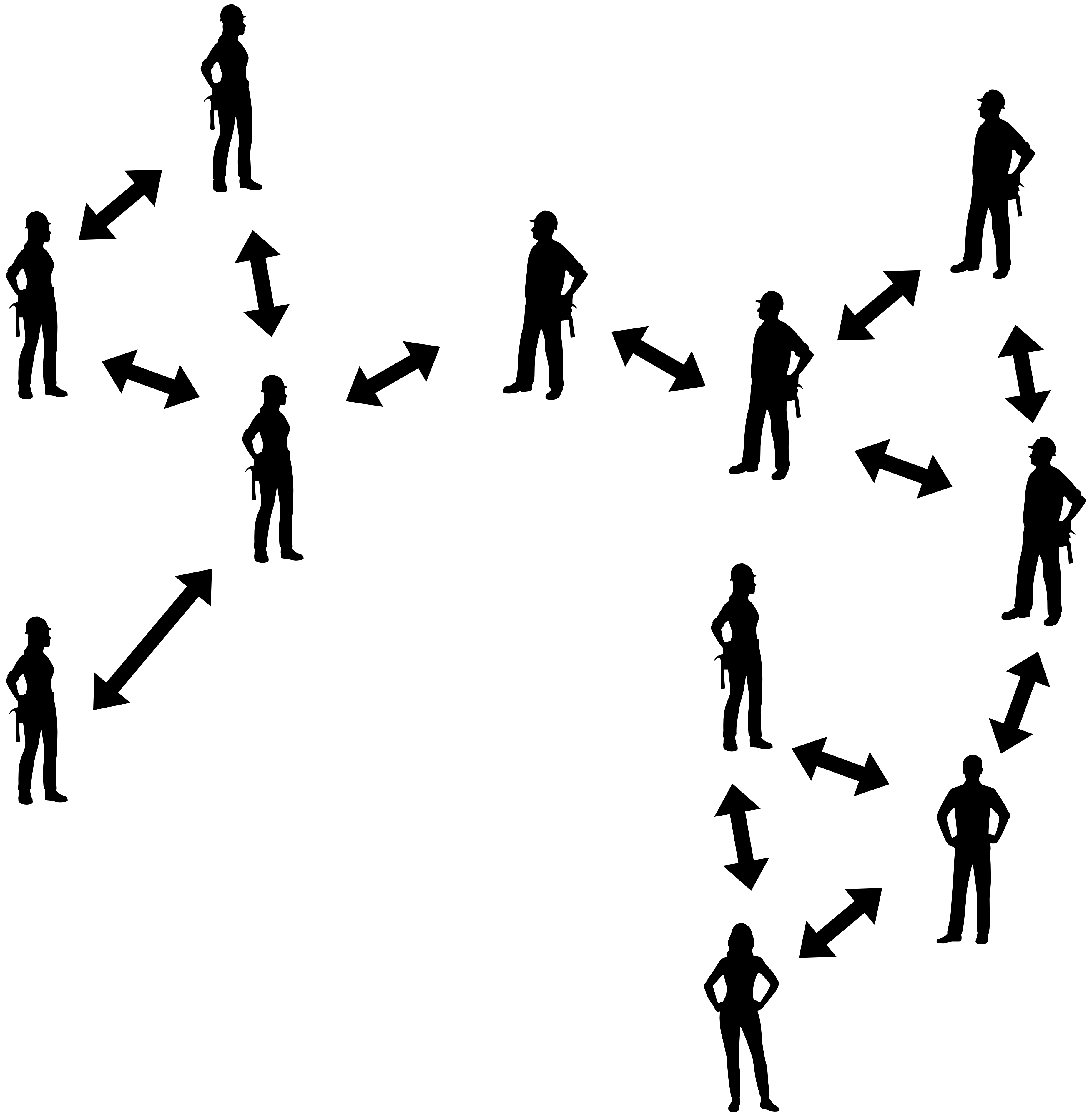
Preferential Attachment

[Albert and Barabasi 1999]

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Preferential Attachment

[Albert and Barabasi 1999]



Preferential Attachment

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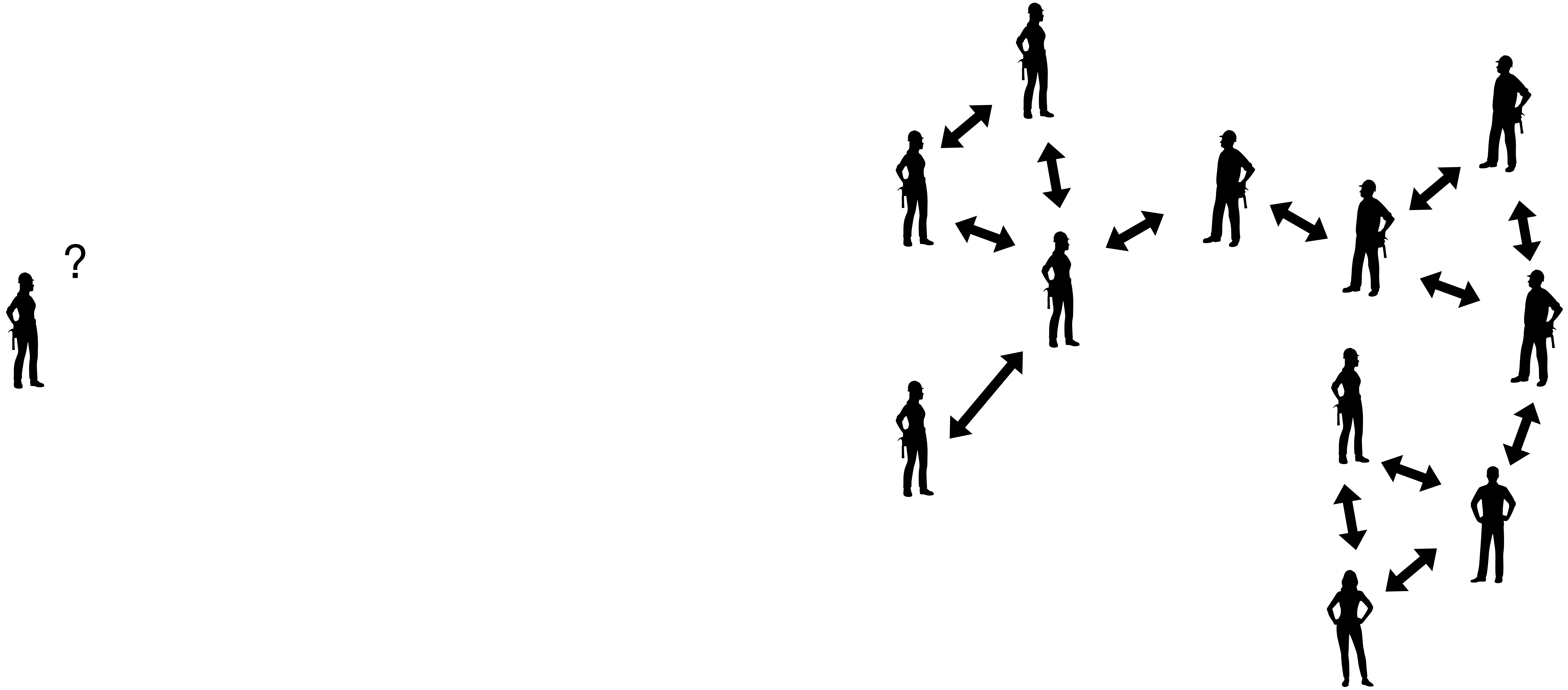

Preferential Attachment

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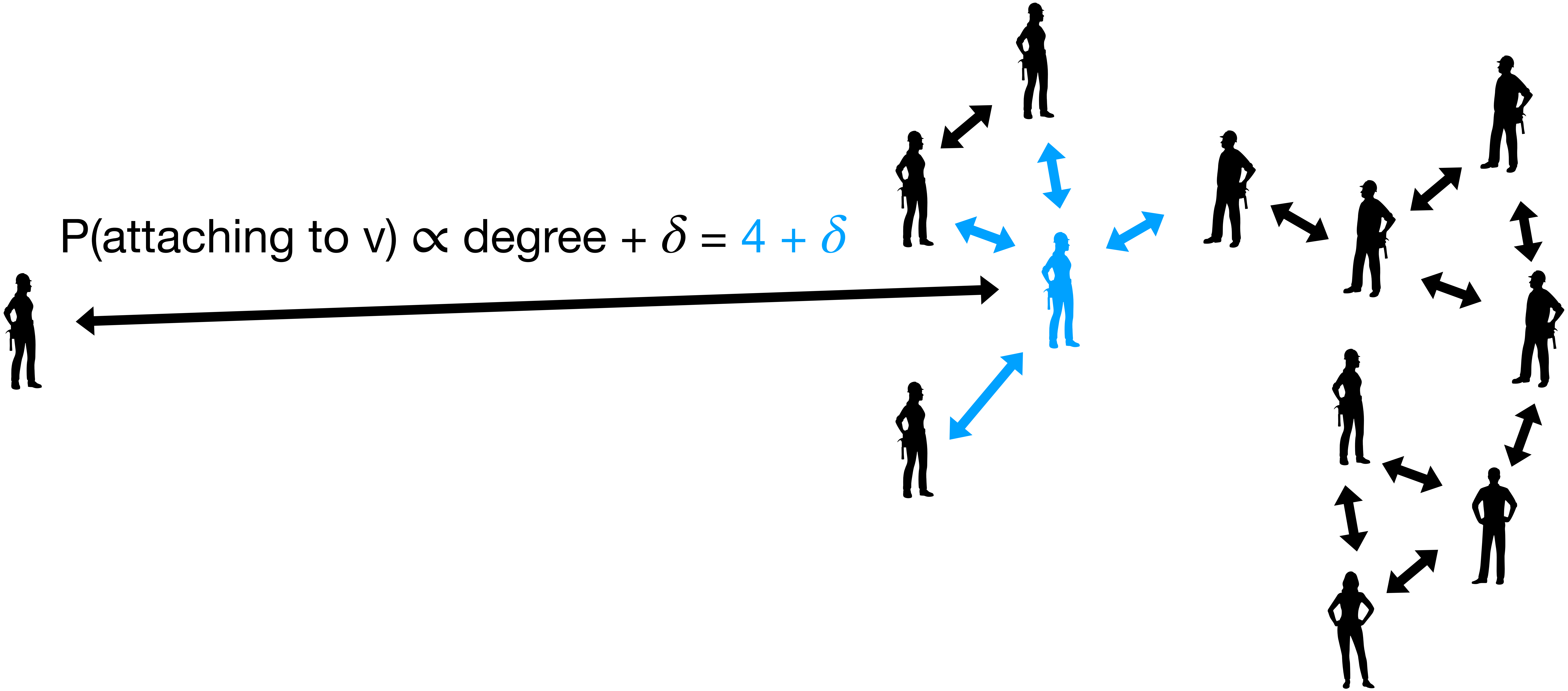
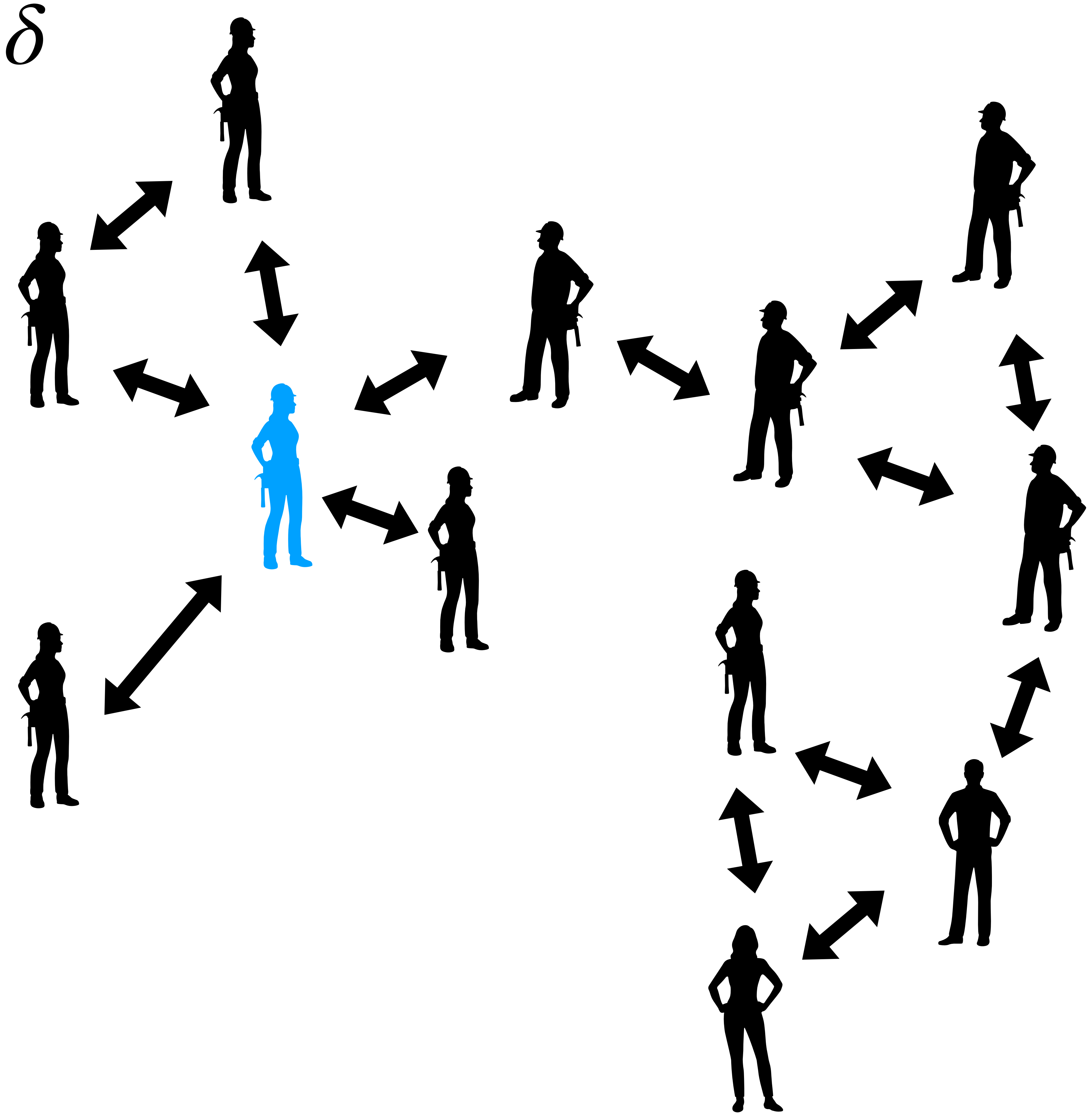


Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) $\propto$ degree + a tuning parameter δ



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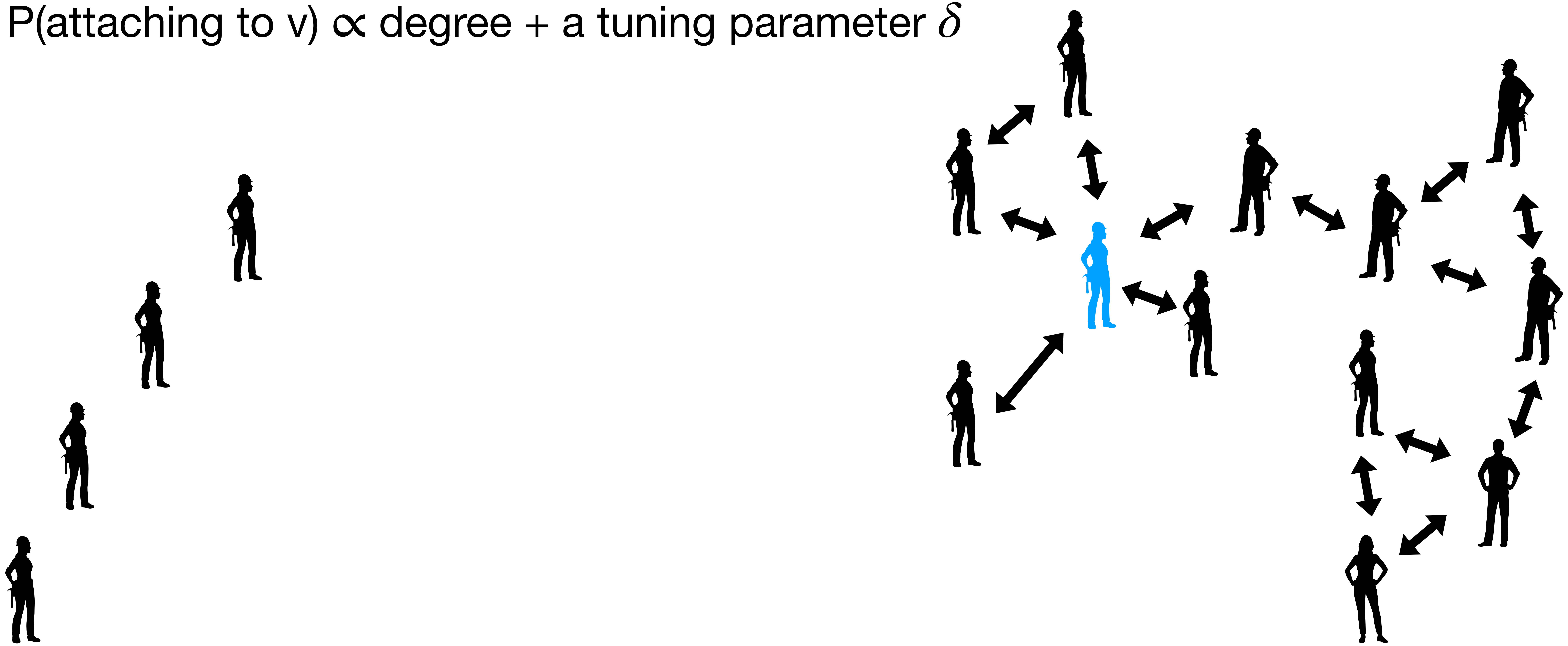

Preferential Attachment

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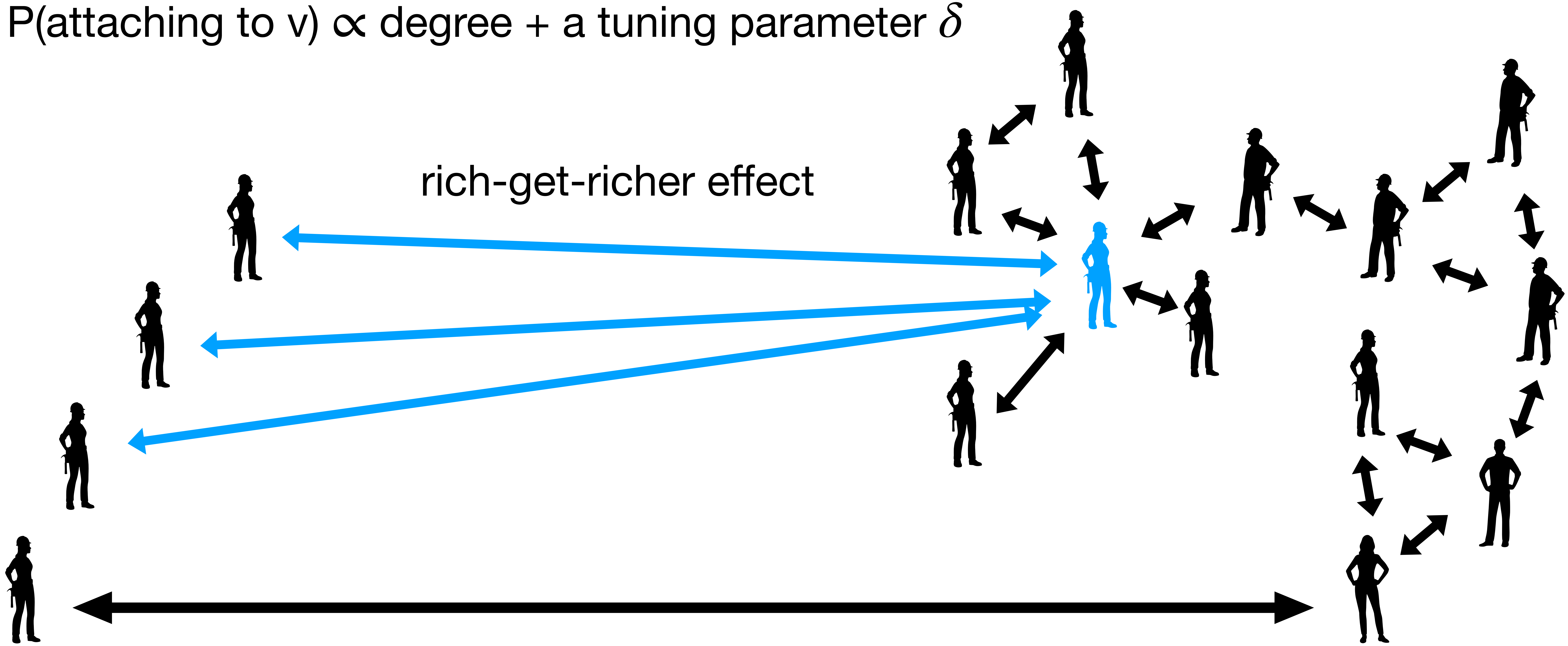


What do we know?

What do we know?

- Scale-freeness and Degree distribution
  [Barabasi and Albert 1999; Dorogovtsev, Mendes and Samukhin 2000; Krapivsky, Redner and Leyvraz 2000]

pmf of degree distribution

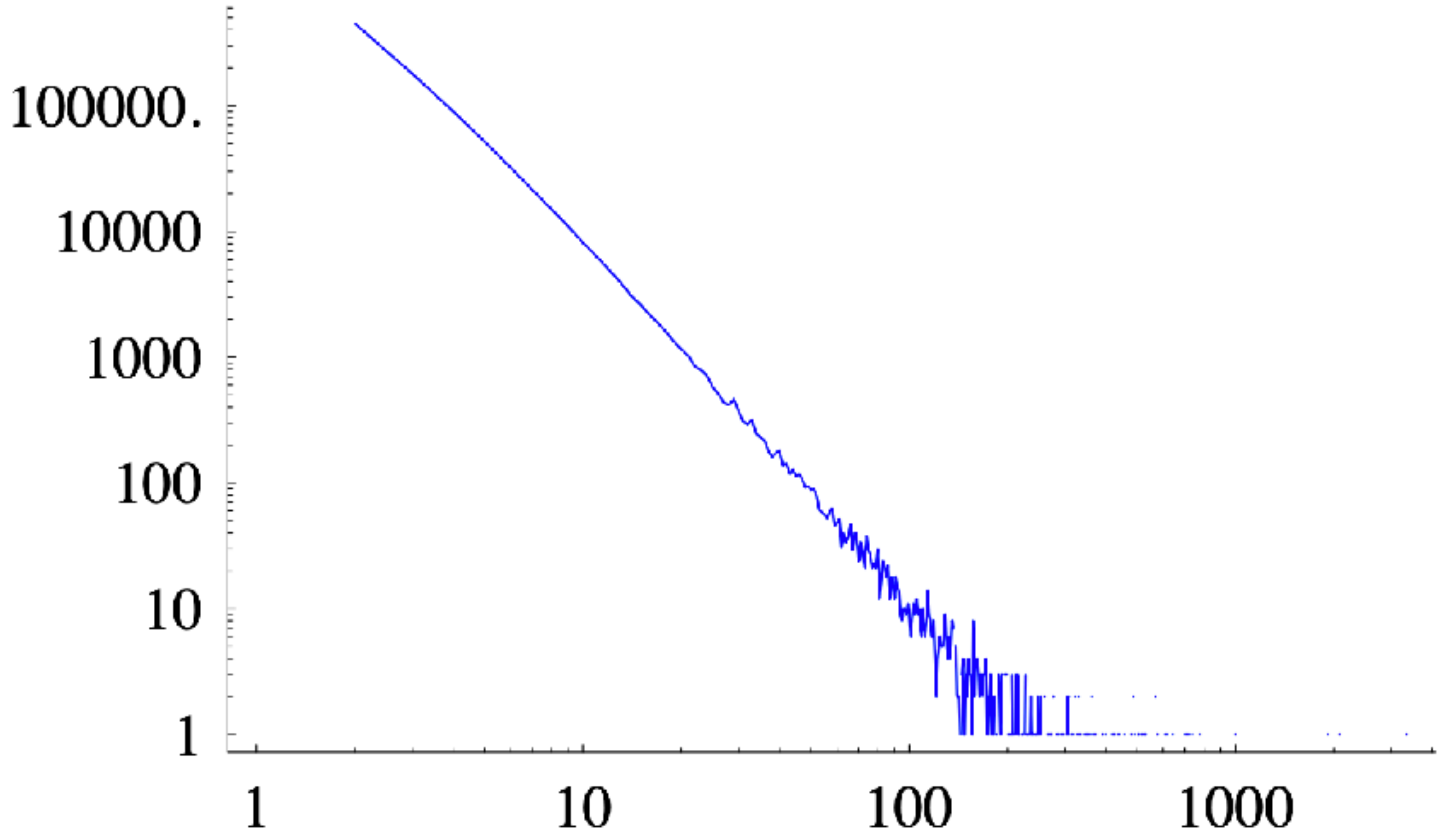


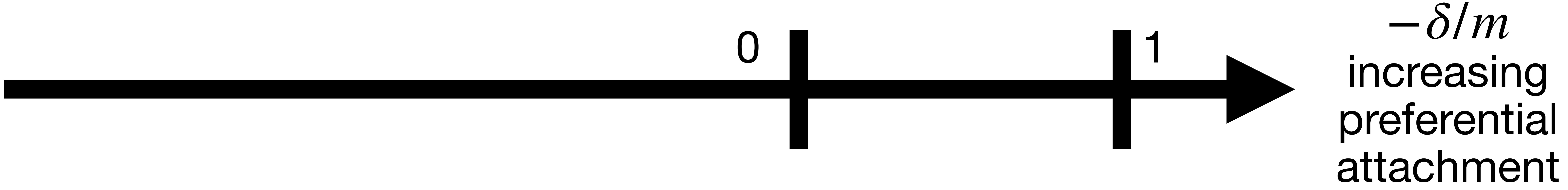
Fig 8.3 of R. Hofstad (2013).
Random Graphs and Complex Networks.
https://doi.org/10.1017/9781316779422

Phase transition

Recall

P(attaching to v) $\propto$ degree + δ

m = number of edges per new node



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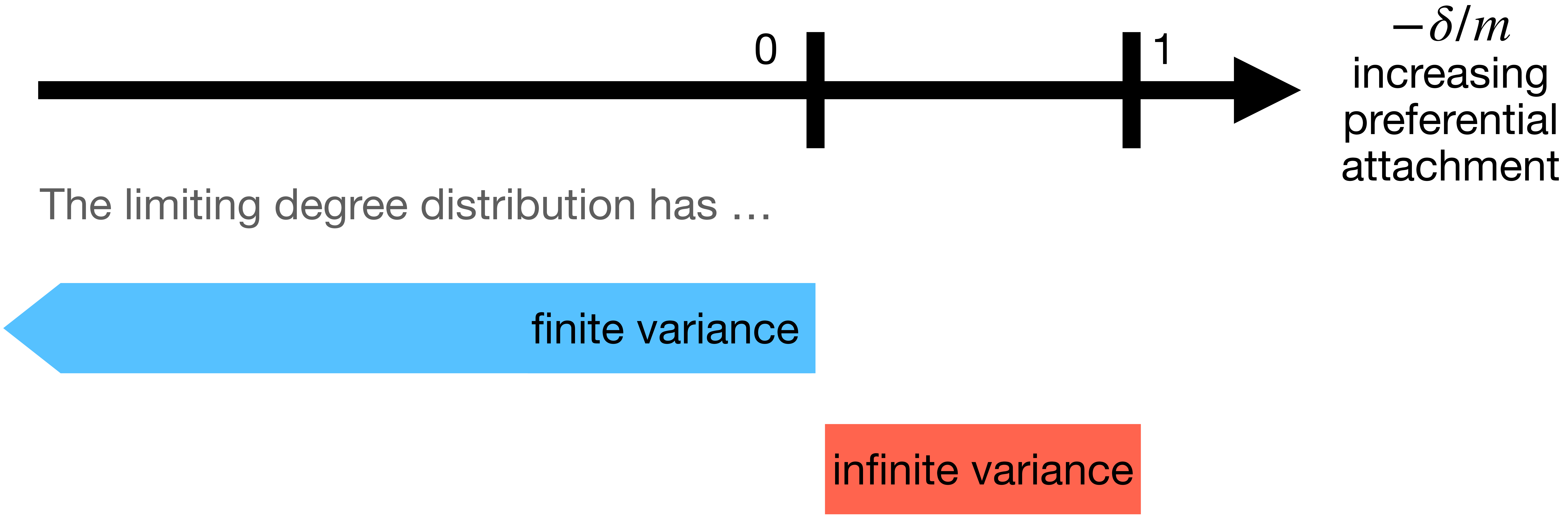
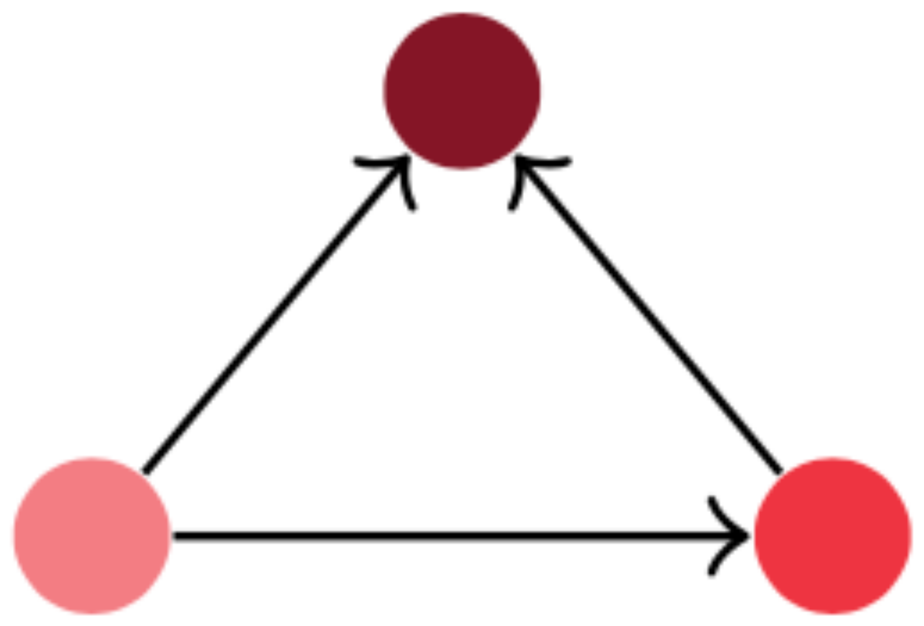


What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013, Garavaglia and Stegehuis 2019]



(a) $t^{(3-\tau)/(\tau-1)} \log(t)$

$\tau = 3 + \delta/m$

Fig 2 of A. Garavaglia and C. Stegehuis (2019). Subgraphs in Preferential Attachment Models. https://doi.org/10.1017/apr.2019.36

What do we know?

- subgraph counts [Garavaglia and Stegehuis 2019]

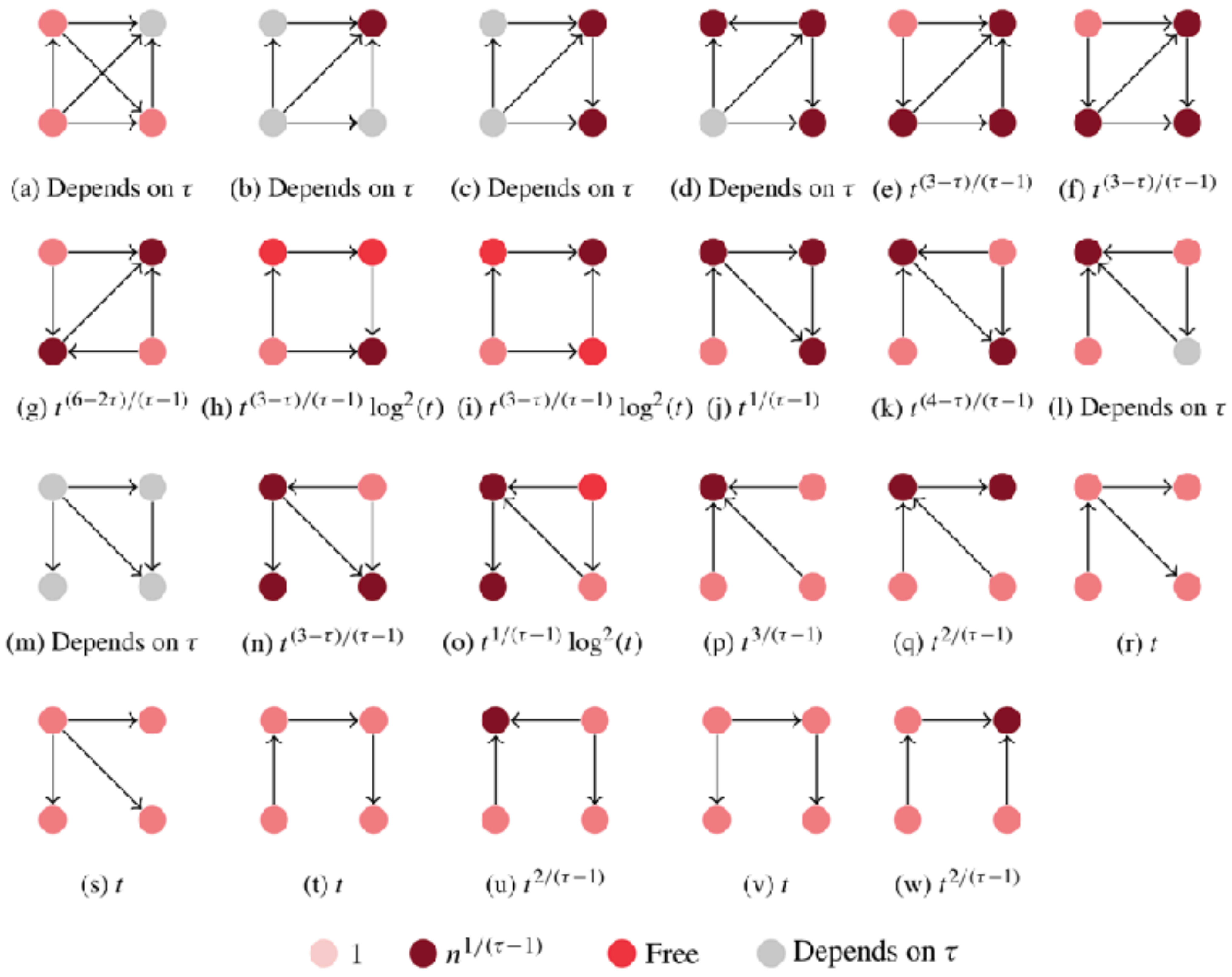
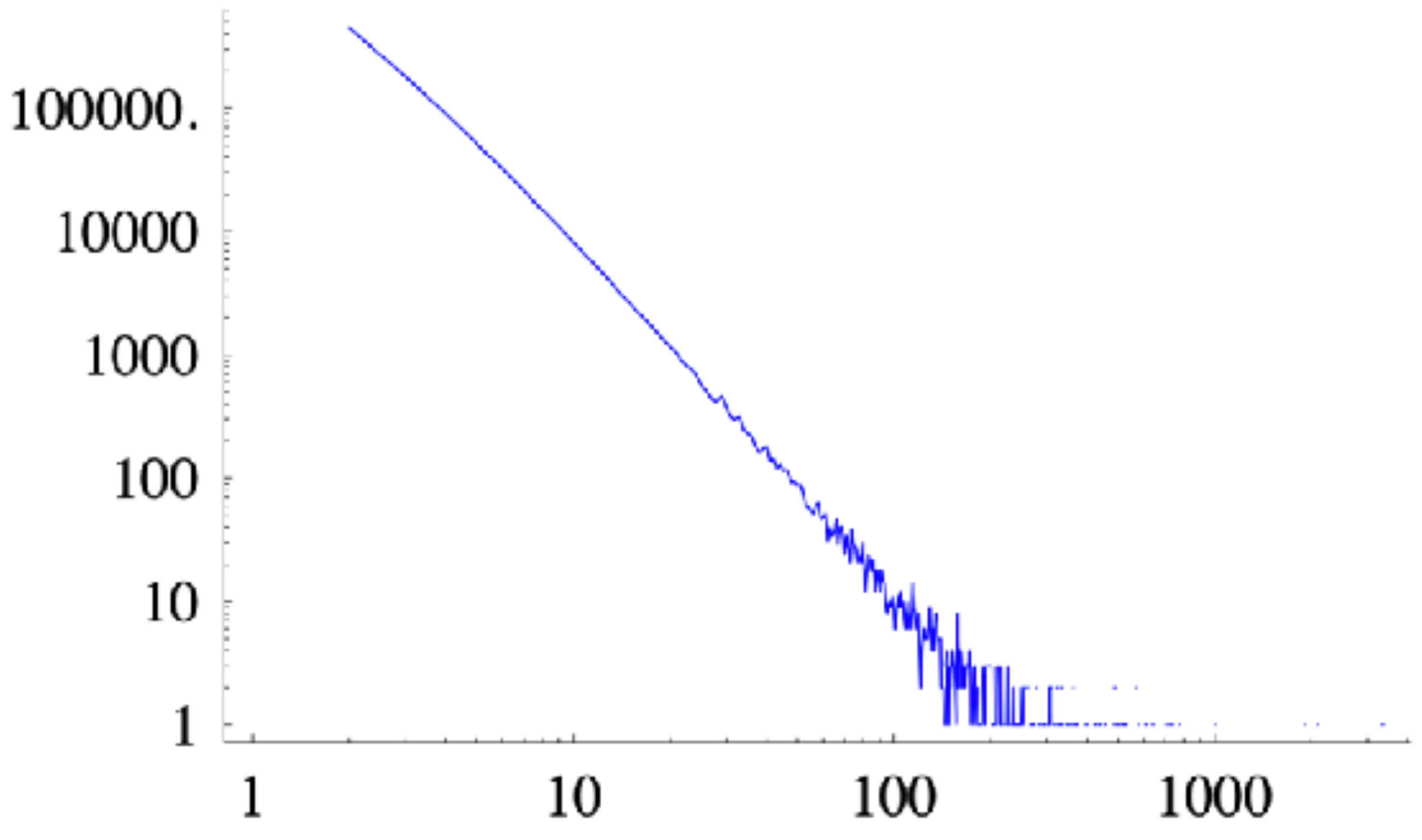
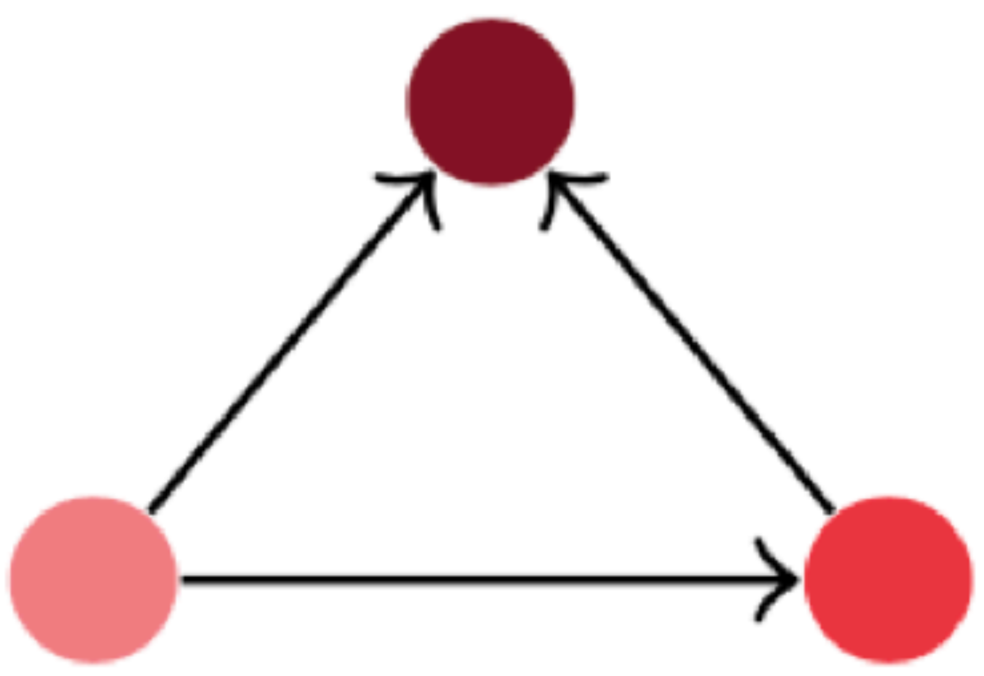


Fig 3 of A. Garavaglia and C. Stegehuis (2019). Subgraphs in Preferential Attachment Models. https://doi.org/10.1017/apr.2019.36

What do we know?

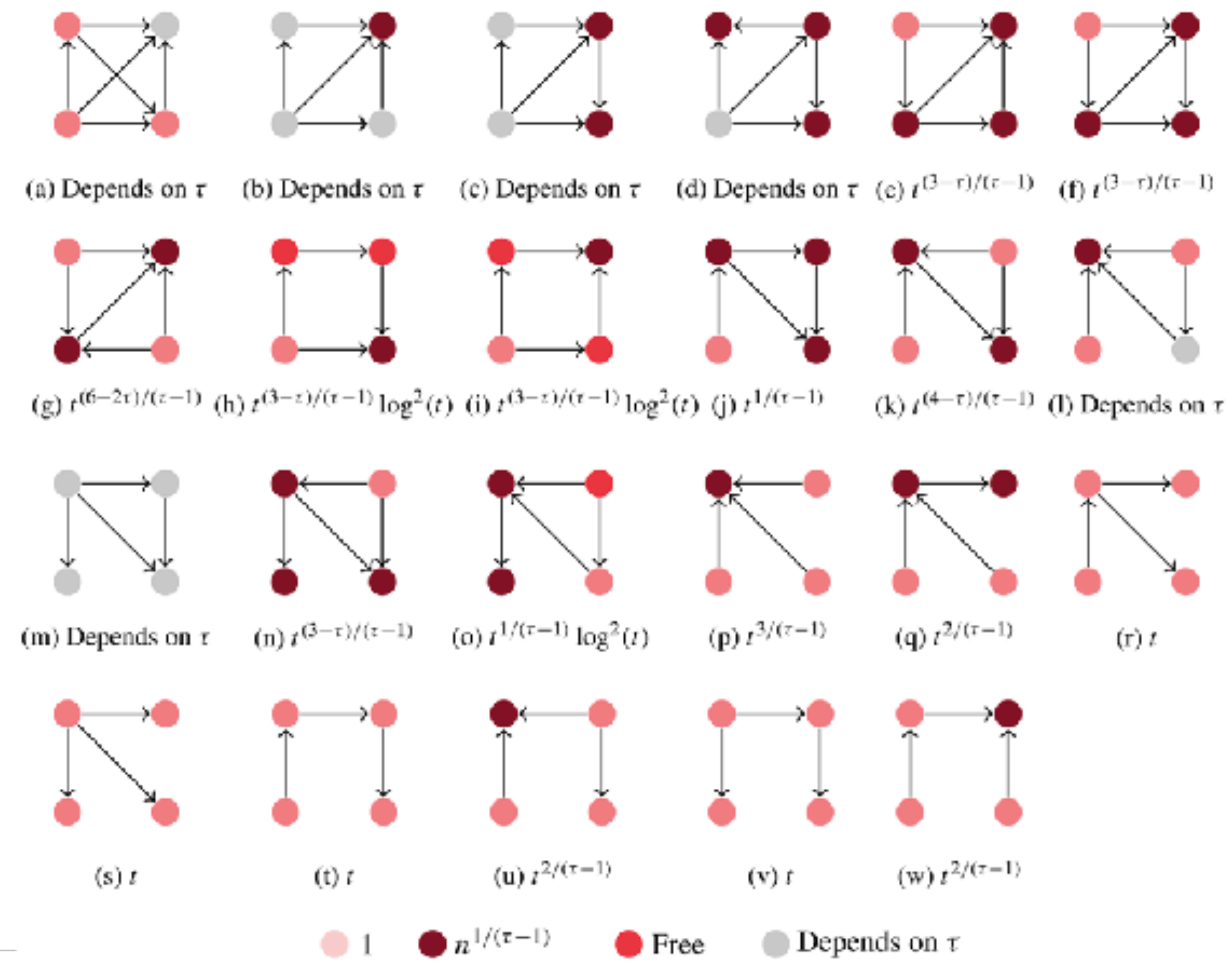


degree distribution



(a) $t^{(3-\tau)/(\tau-1)} \log(t)$

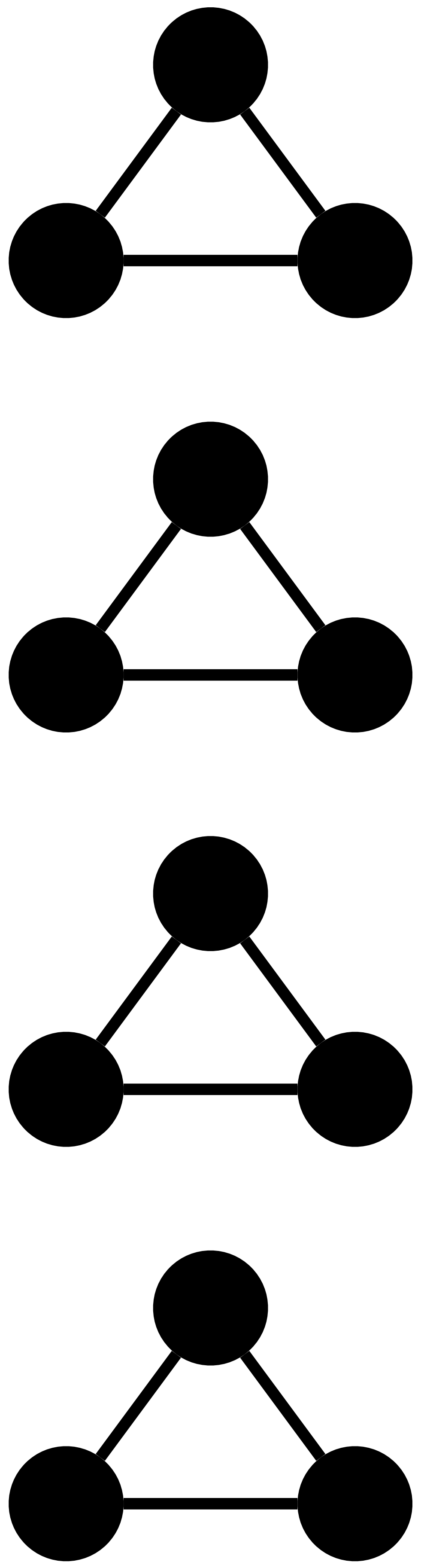
triangle counts

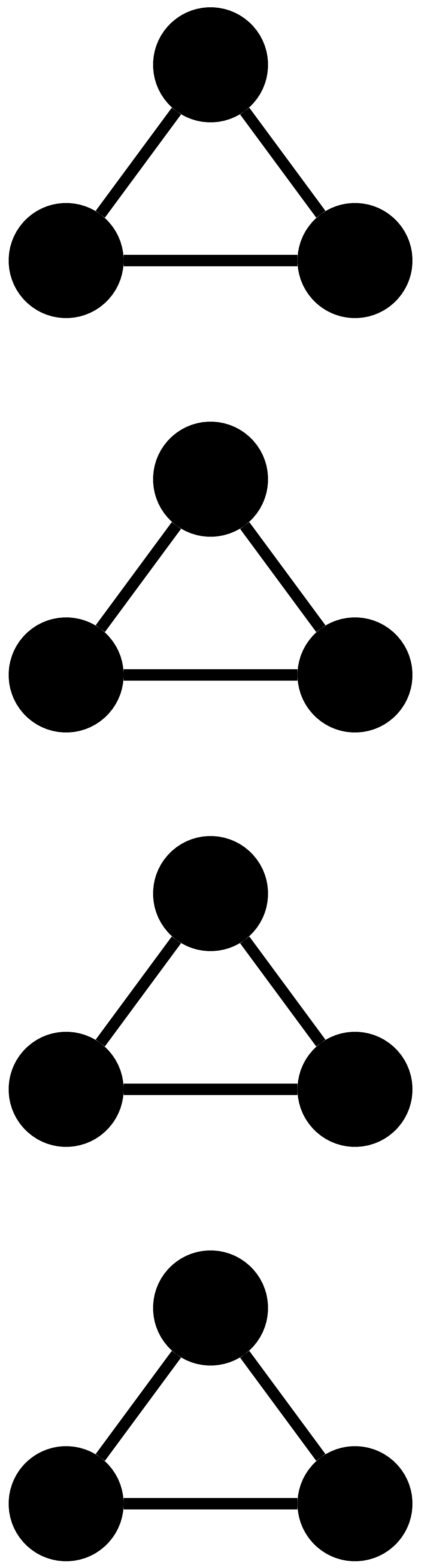


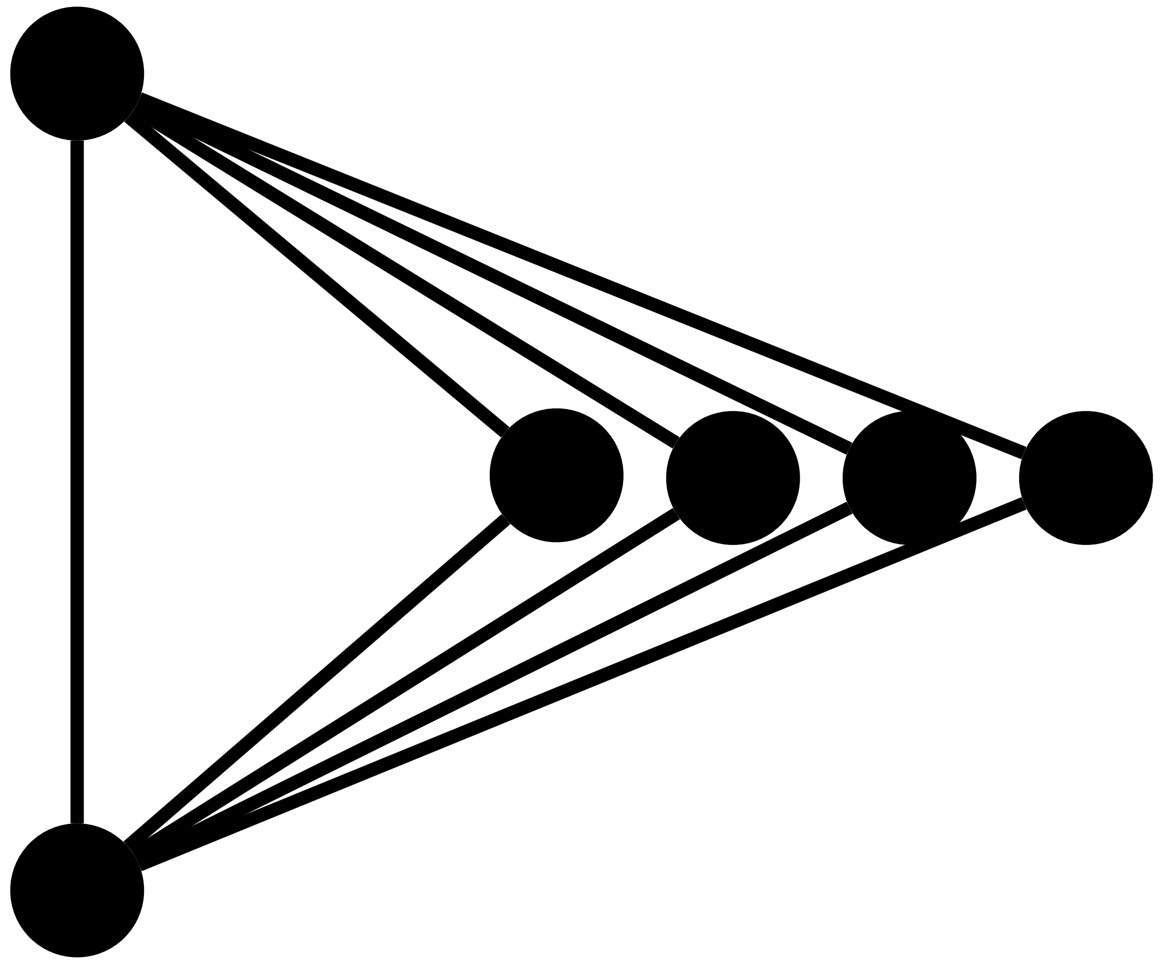
subgraph counts

What should we count?
And how?

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)







II. Into Topology

Counting everything in every dimension all at once

……

Betti numbers count repeated connections “in all dimensions”.

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GOOD ✓

“correct” way to count things

Betti numbers count repeated connections “in all dimensions”.

GOOD ✔

“correct” way to count things

homological algebra

Betti numbers count repeated connections “in all dimensions”.

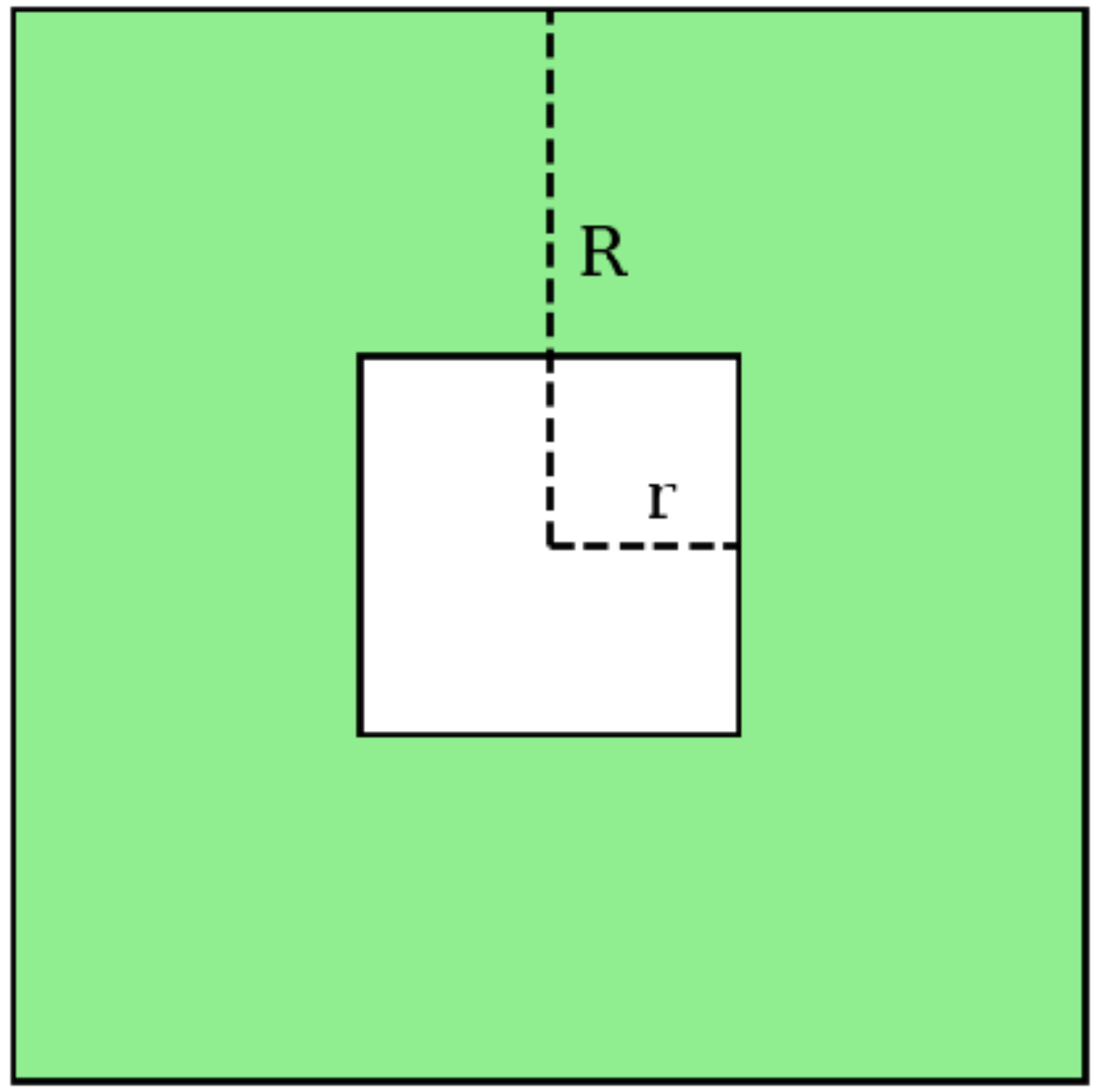
GOOD	**BAD**
“correct” way to count things	hard to write down
homological algebra	hard to do

Betti numbers β_k

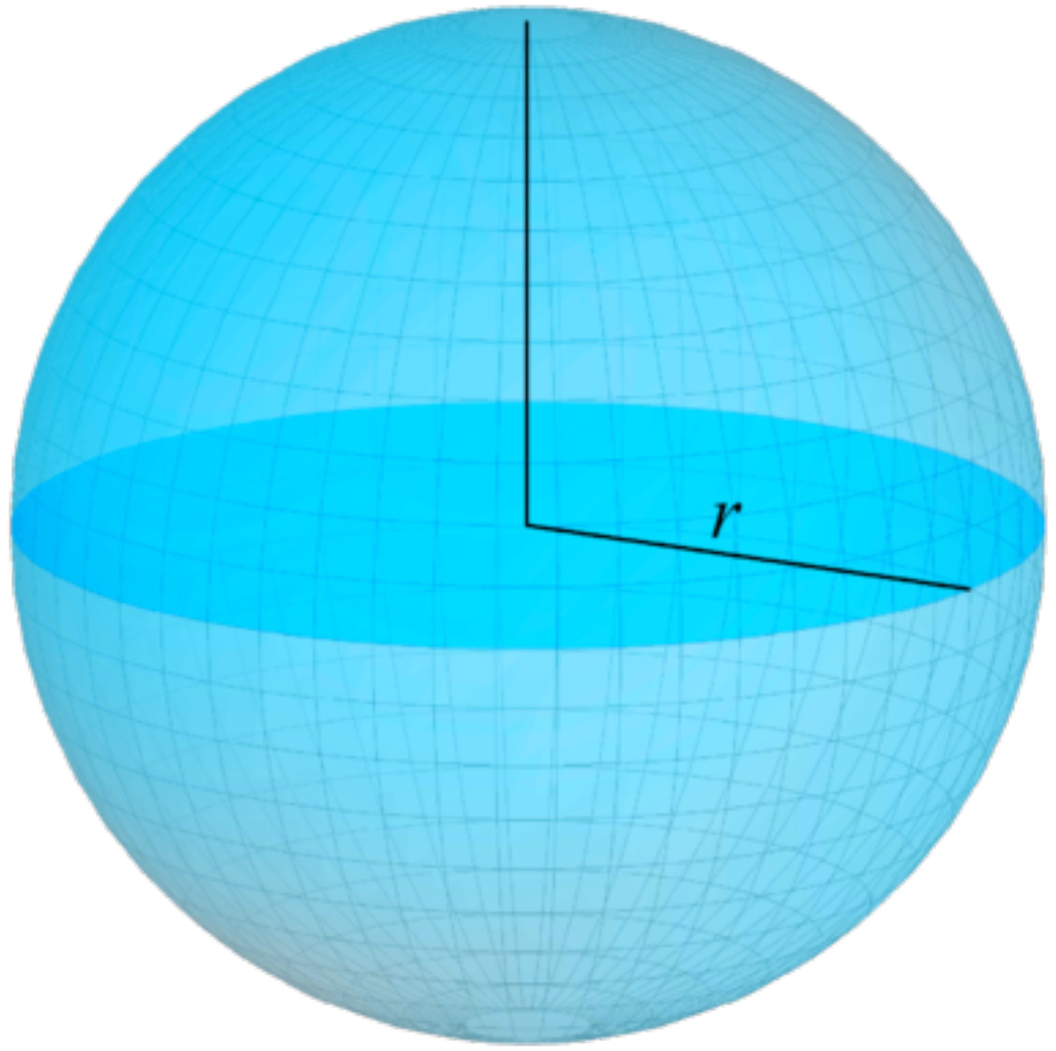
- Repeated connections?
- Holes?

Betti numbers β_k

Count of Holes



$\beta_1 = 1$: 1 loop

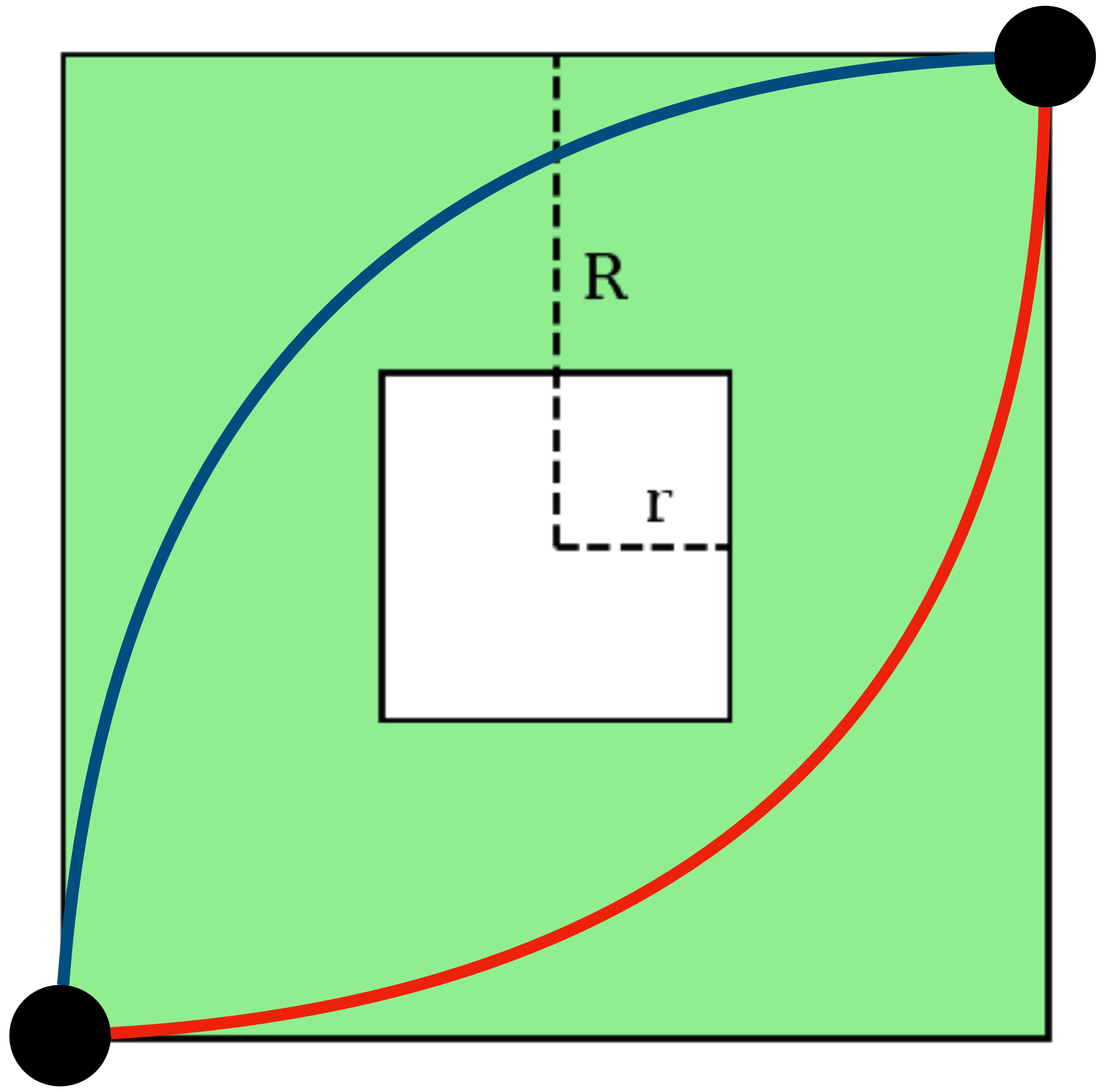


$\beta_1 = 0$: 0 loop

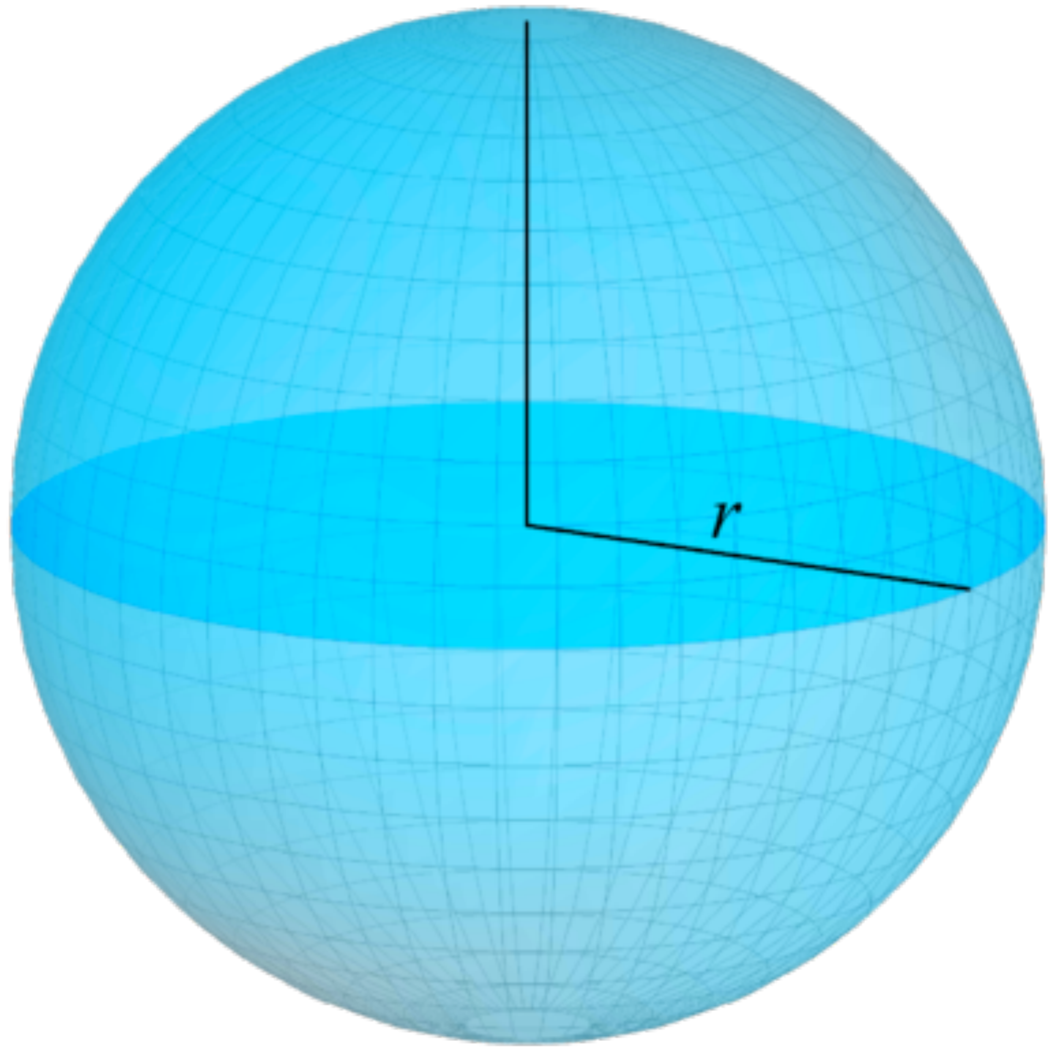
$\beta_2 = 1$: 1 cavity

Betti numbers

Count of Repeated Connections



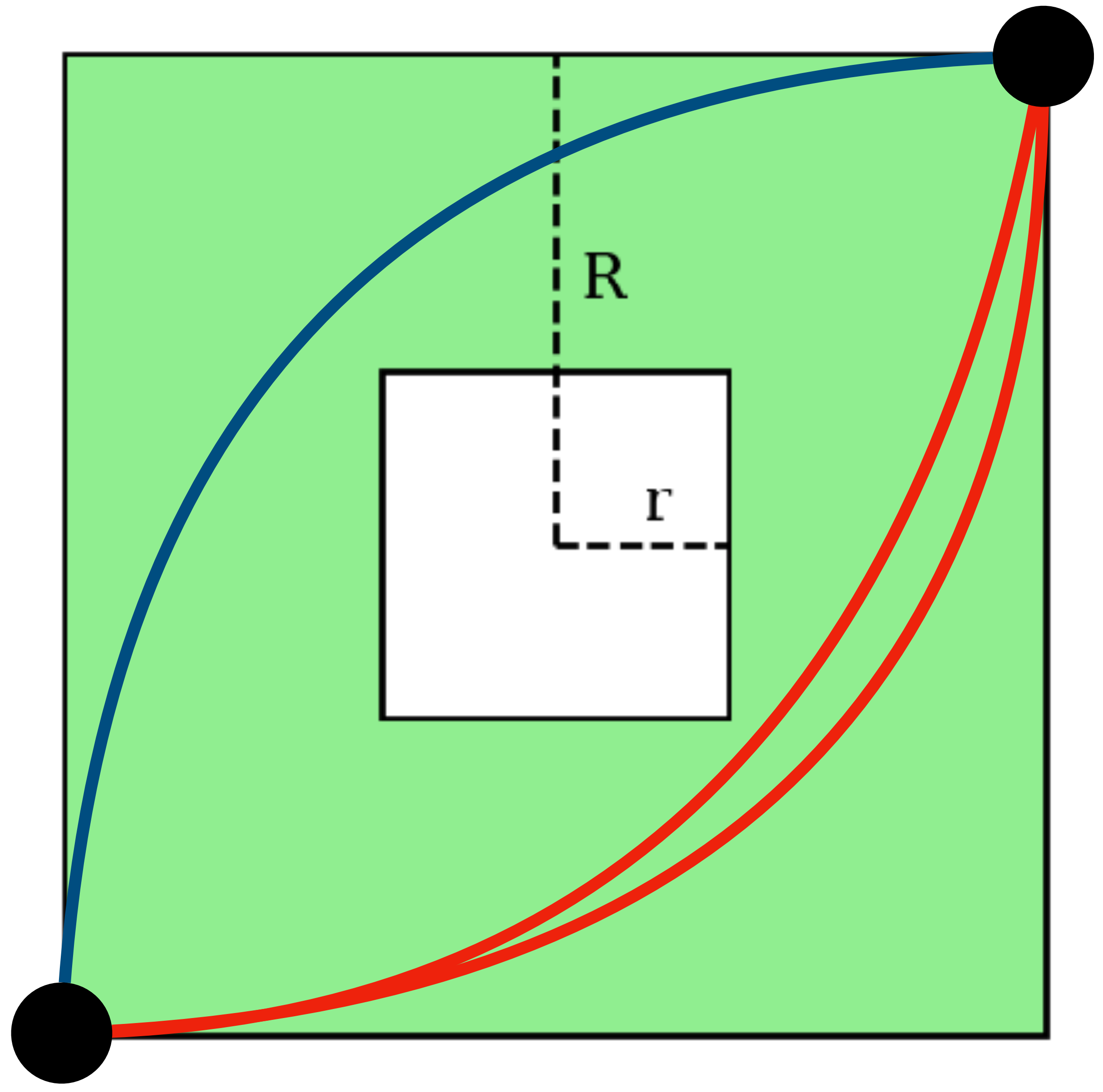
1 alternative path



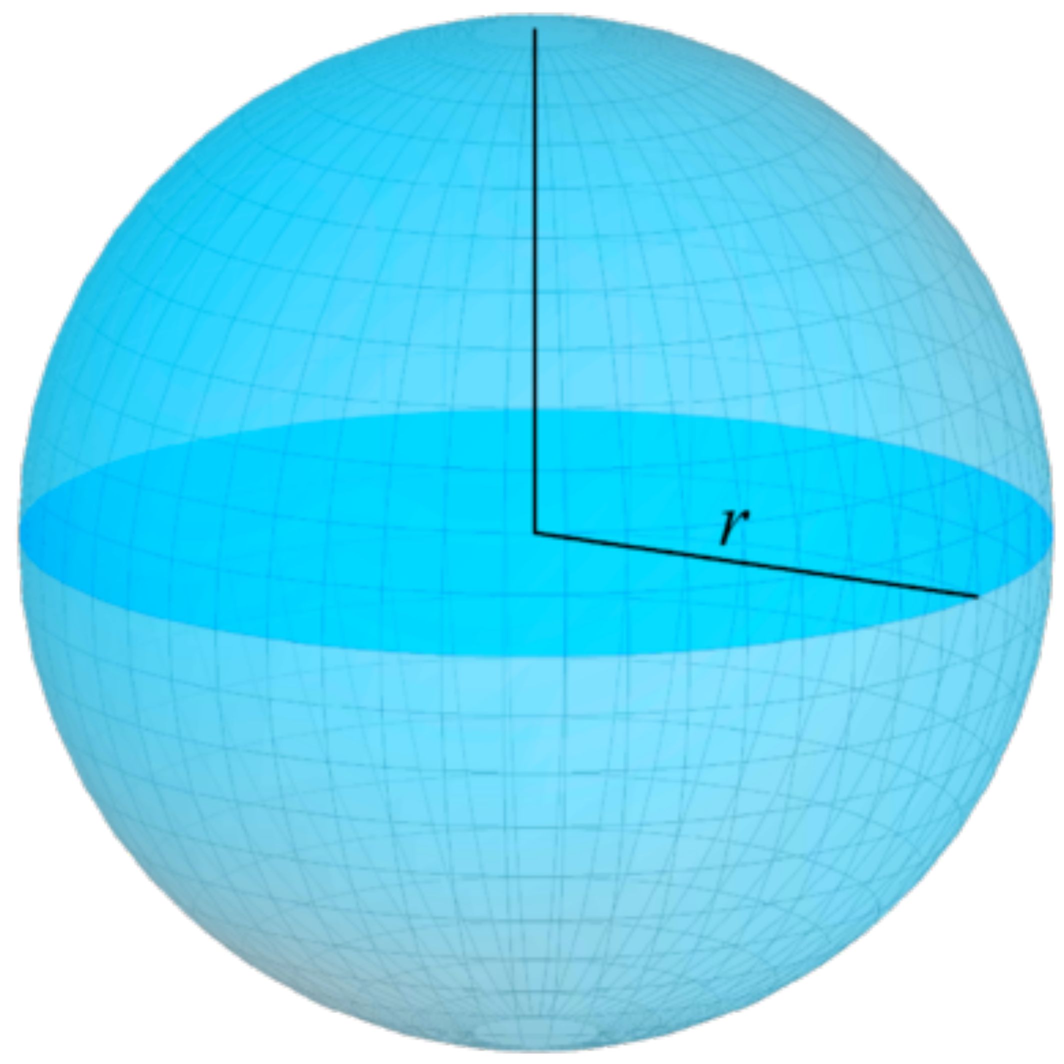
0 loop
1 cavity

Betti numbers

Count of (Independent) Repeated Connections



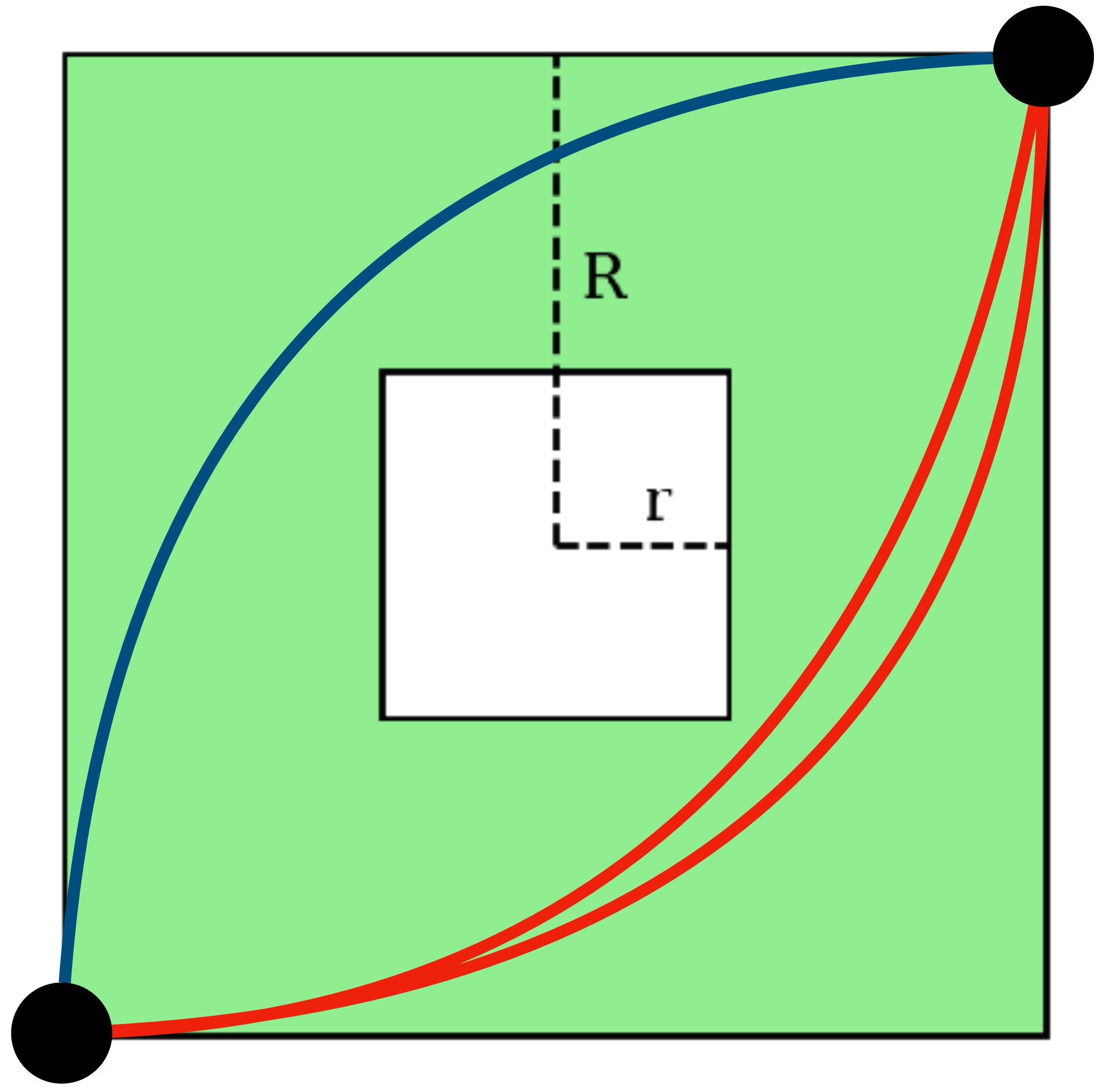
1 alternative path



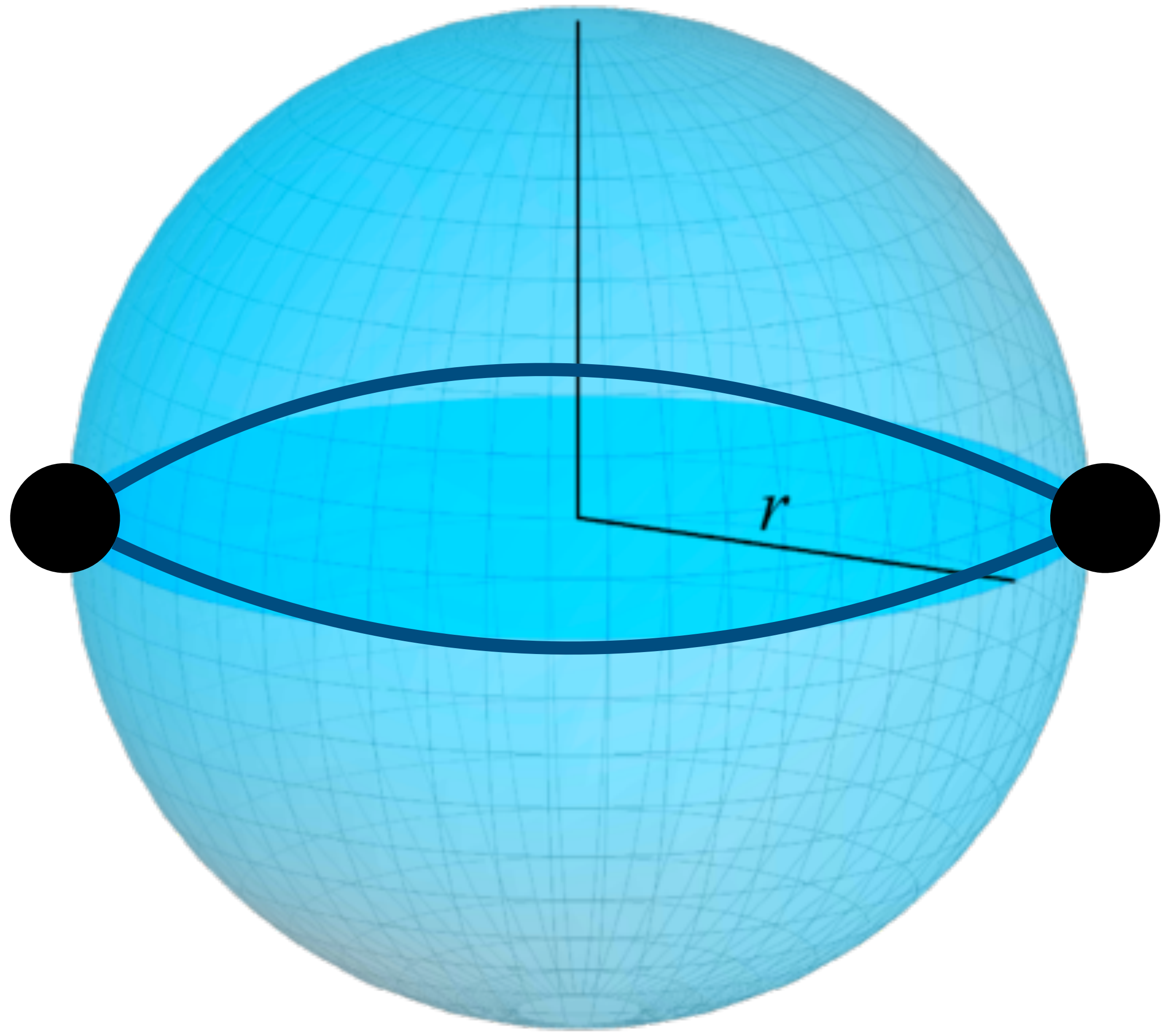
0 loop
1 cavity

Betti numbers

Count of (Independent) Repeated Connections



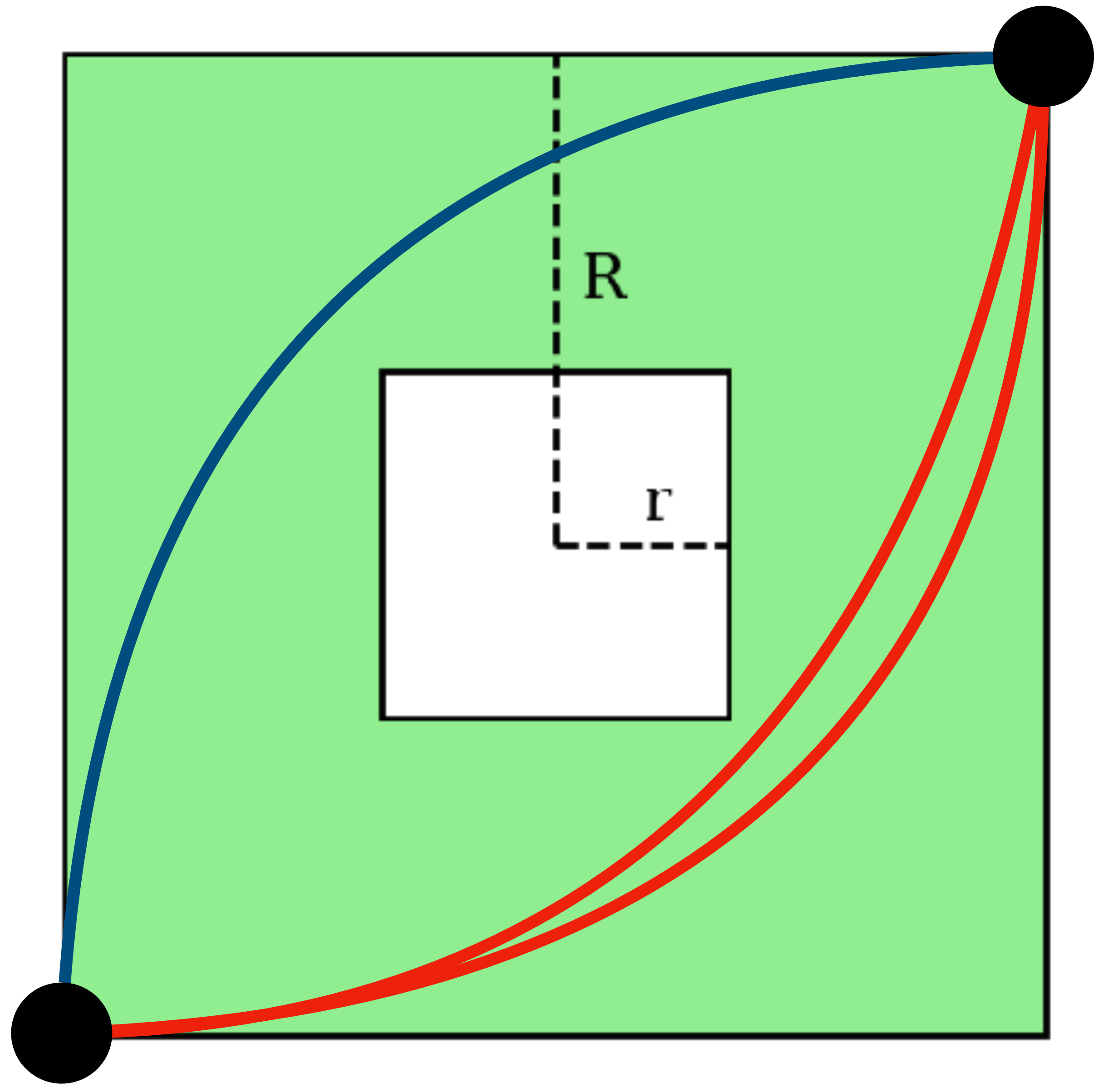
1 alternative path



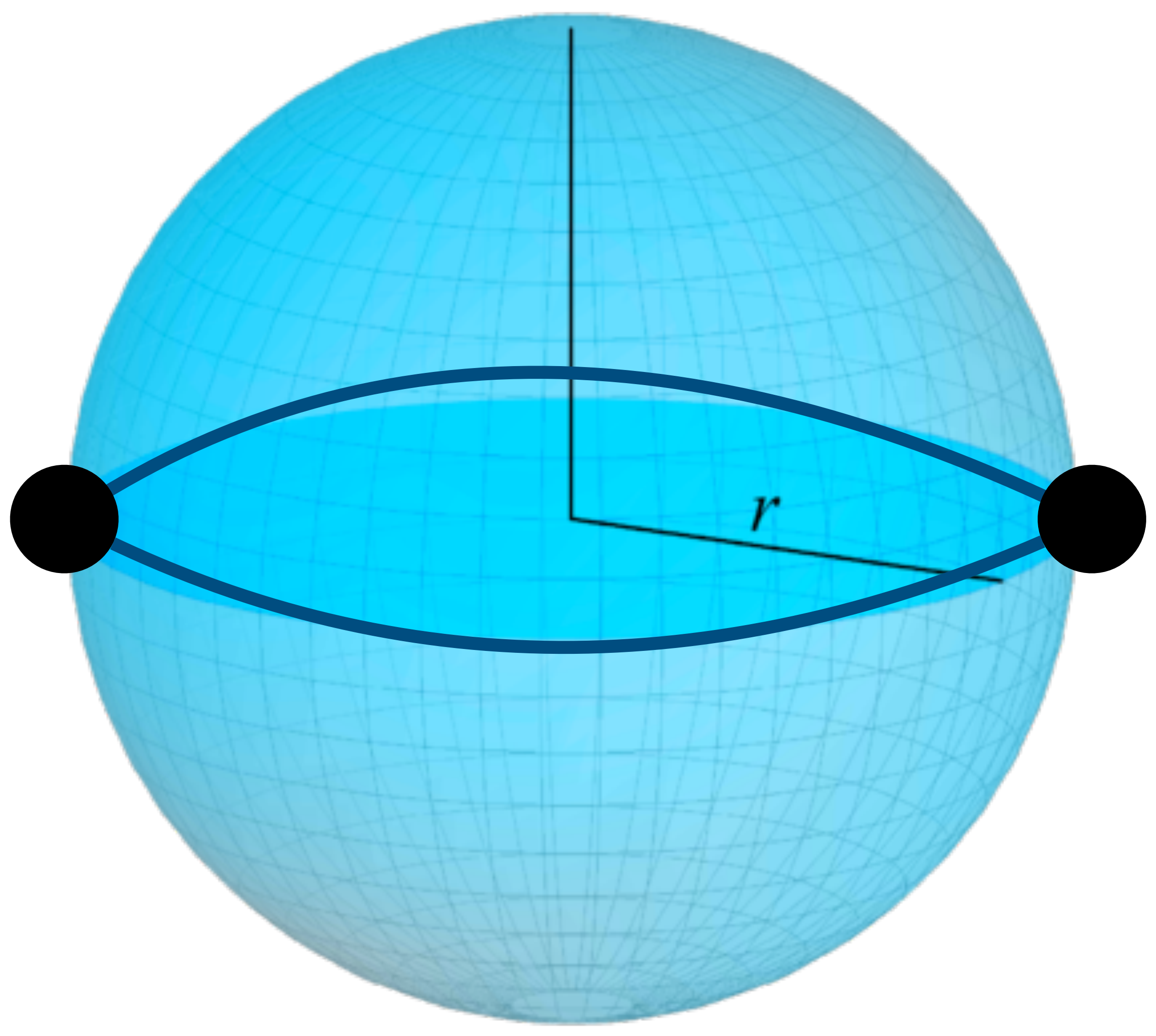
0 alternative path (slide through upper hemisphere)
1 cavity

Betti numbers

Count of (Independent) Repeated Connections

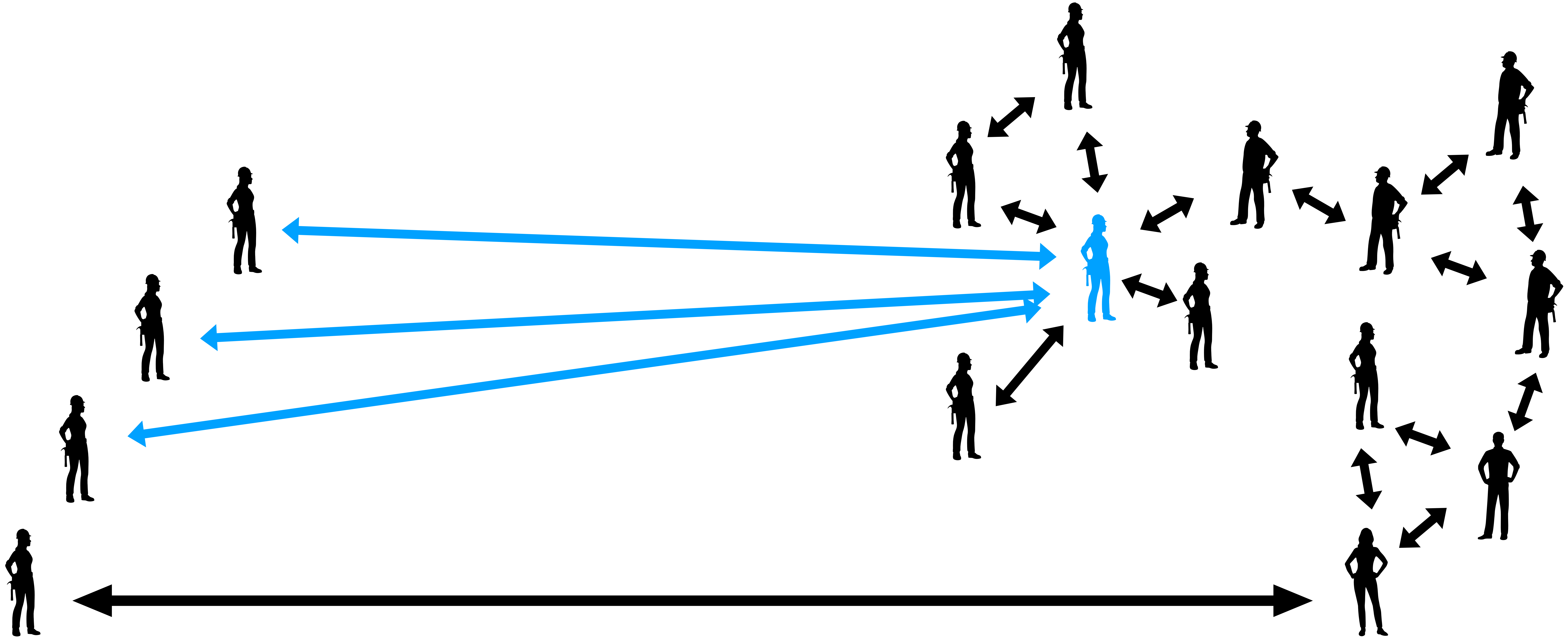


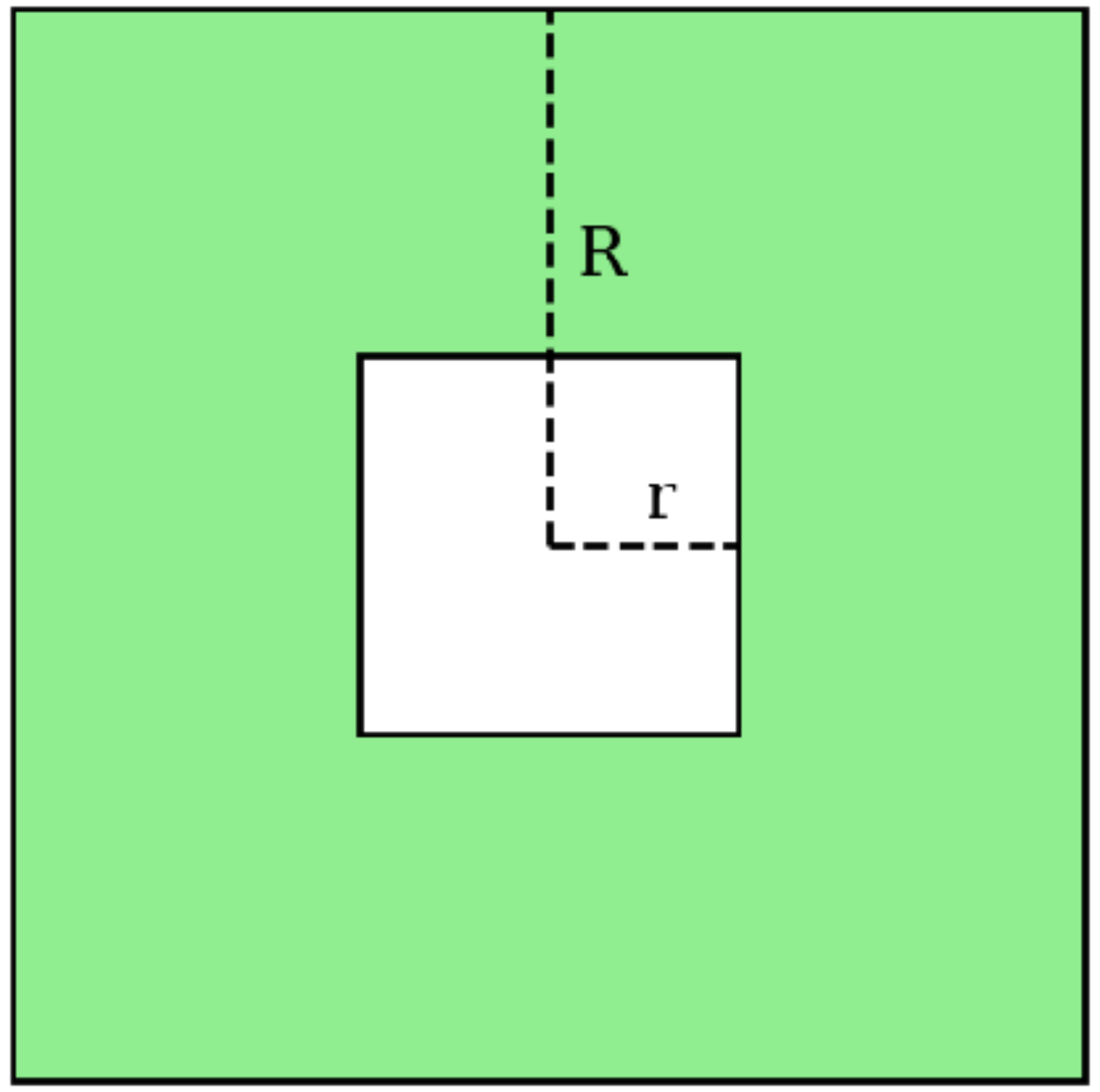
1 alternative path

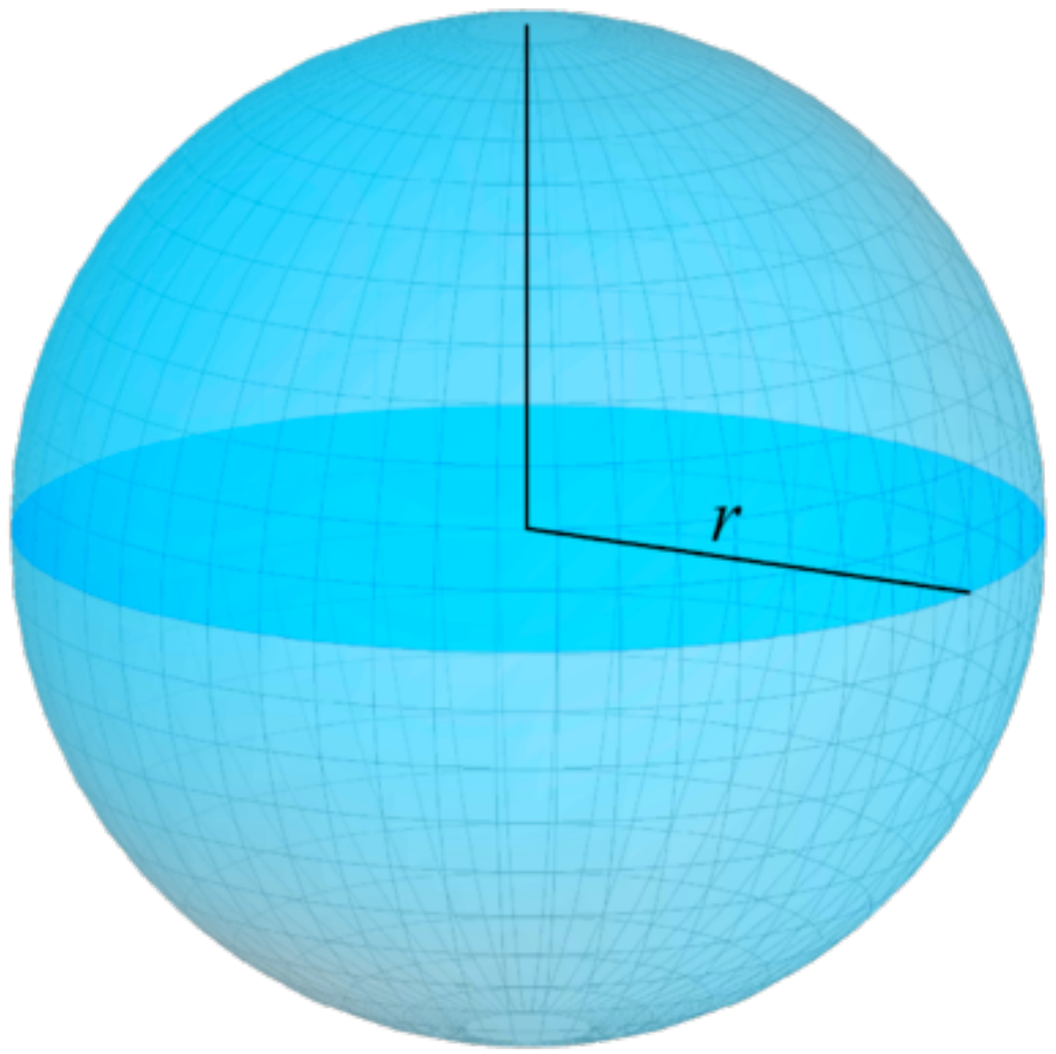


0 alternative path (slide through upper hemisphere)
1 alternative way to slide a path

Betti numbers count repeated connections “in all dimensions”.

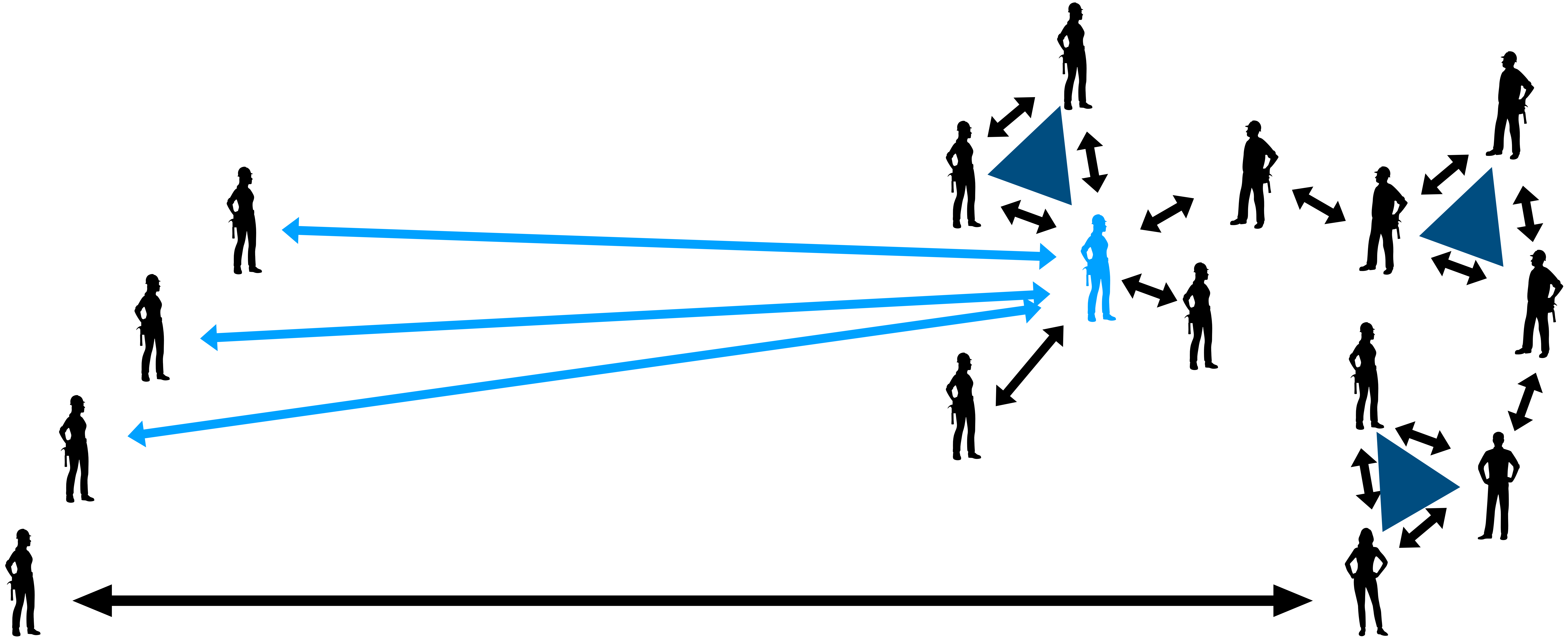



R
r


r

Clique Complex

aka Flag Complex

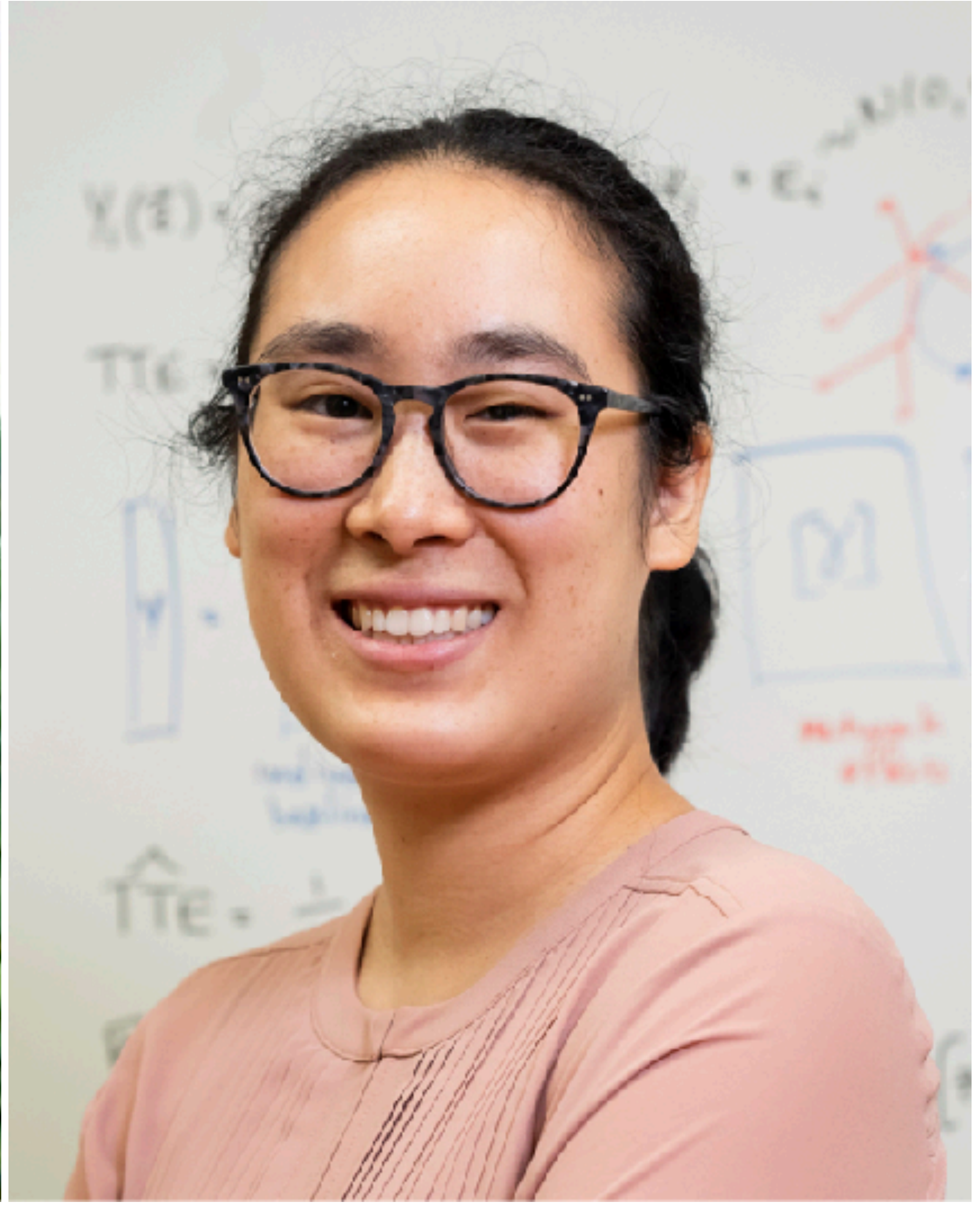


III Topology of Preferential Attachment

x

My Lovely Collaborators

Avhan Misra



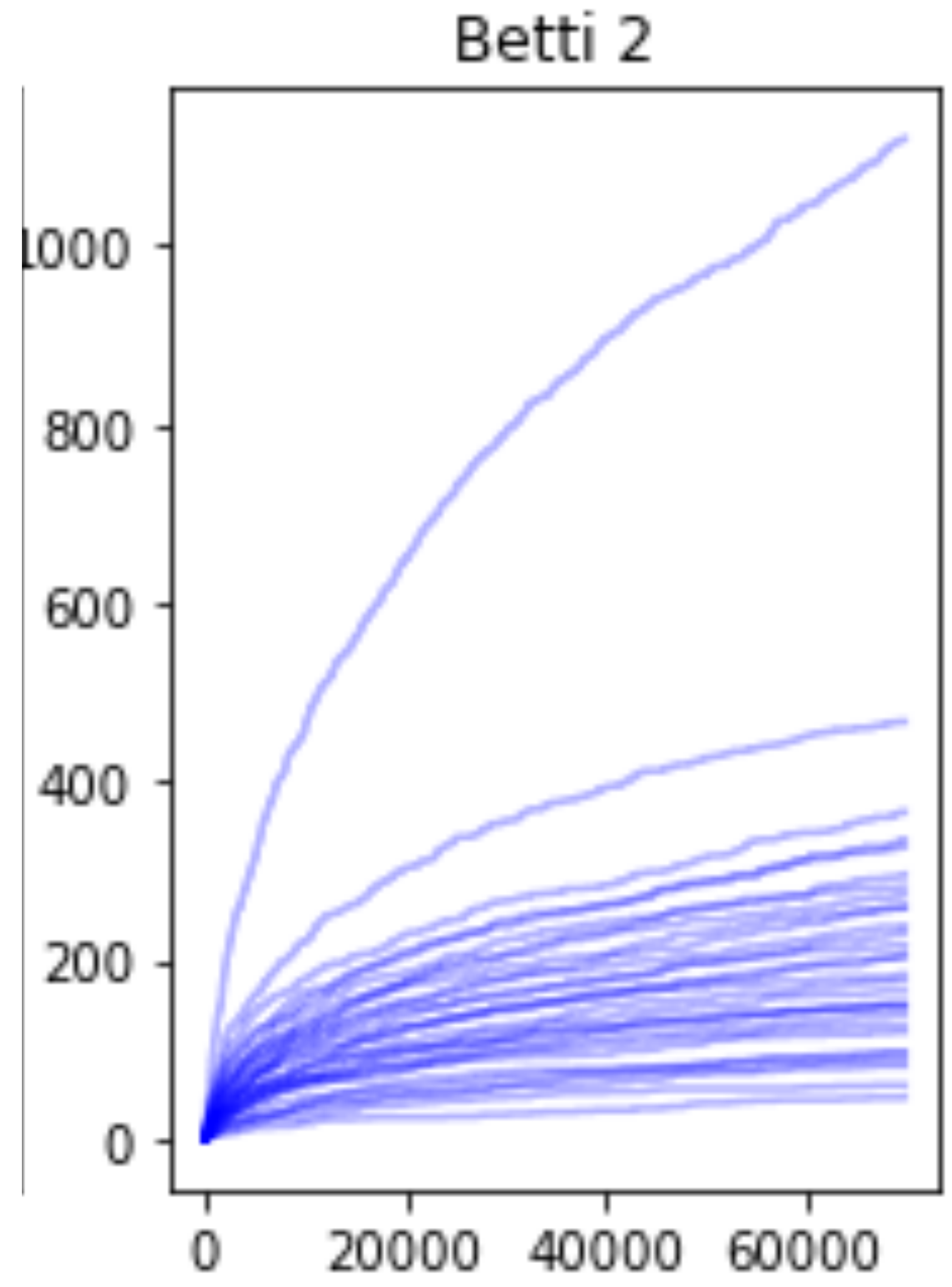
Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

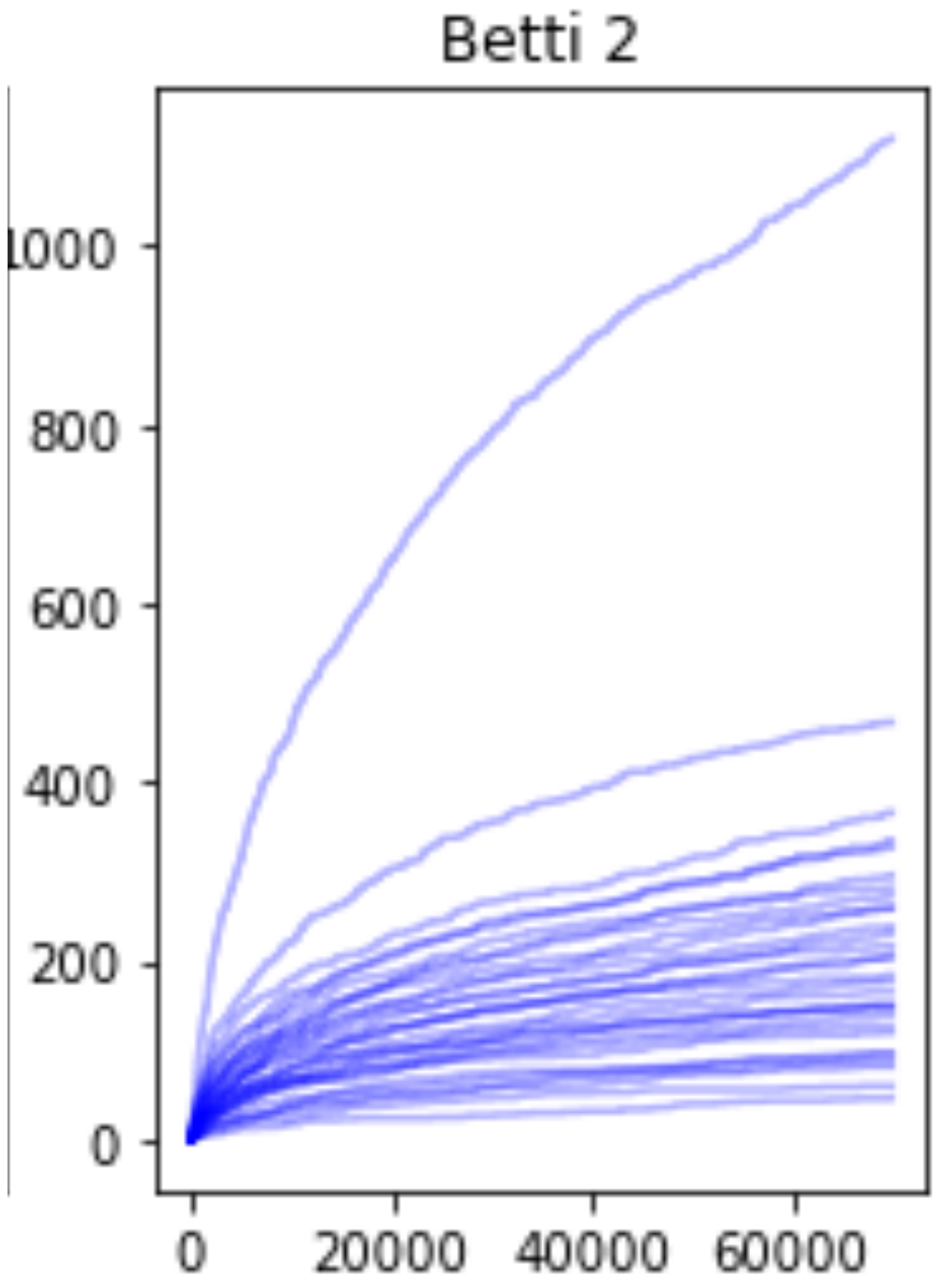
Betti Number β_q

Betti Number β_q



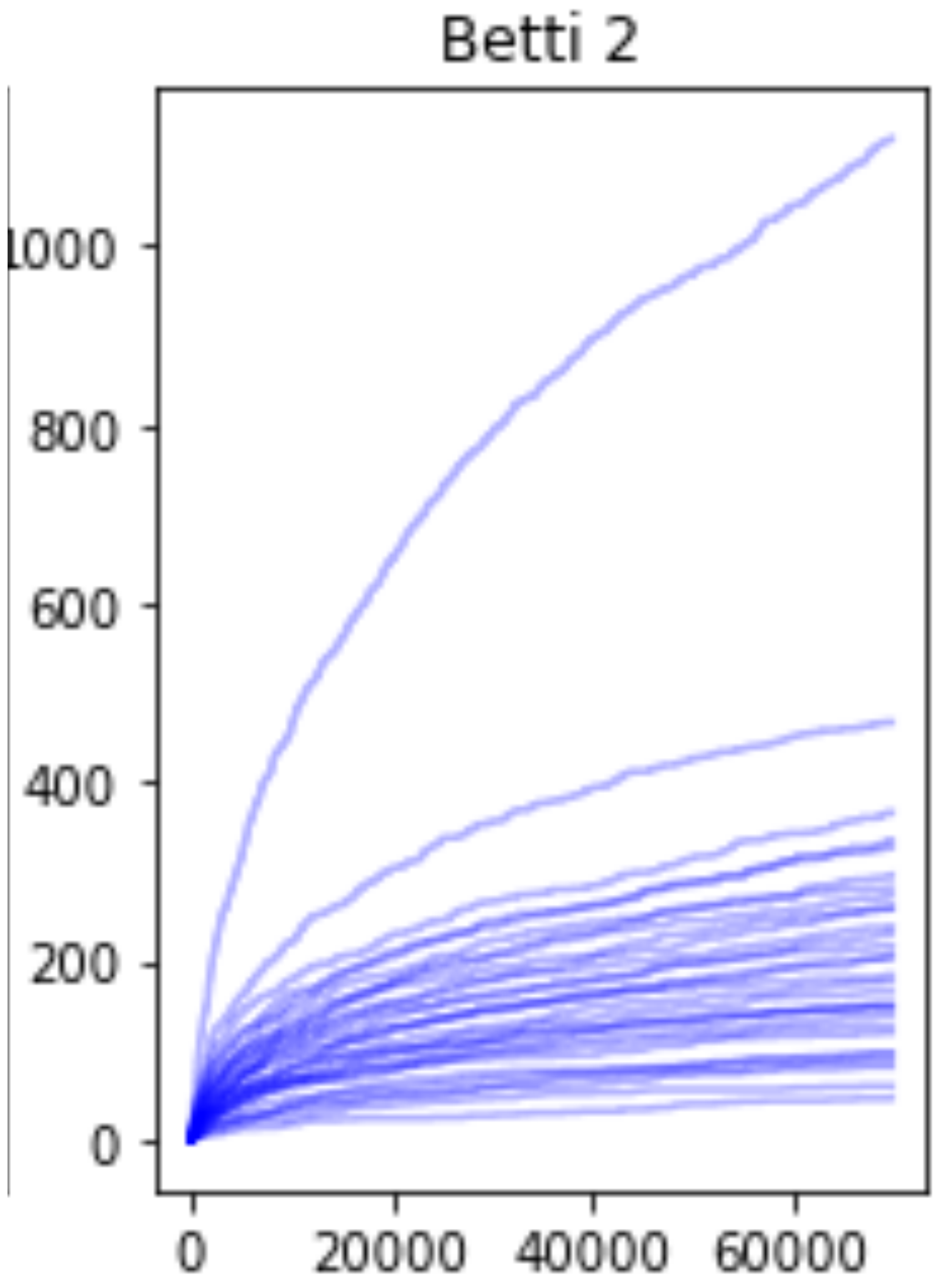
Betti Number β_q

- increasing trend



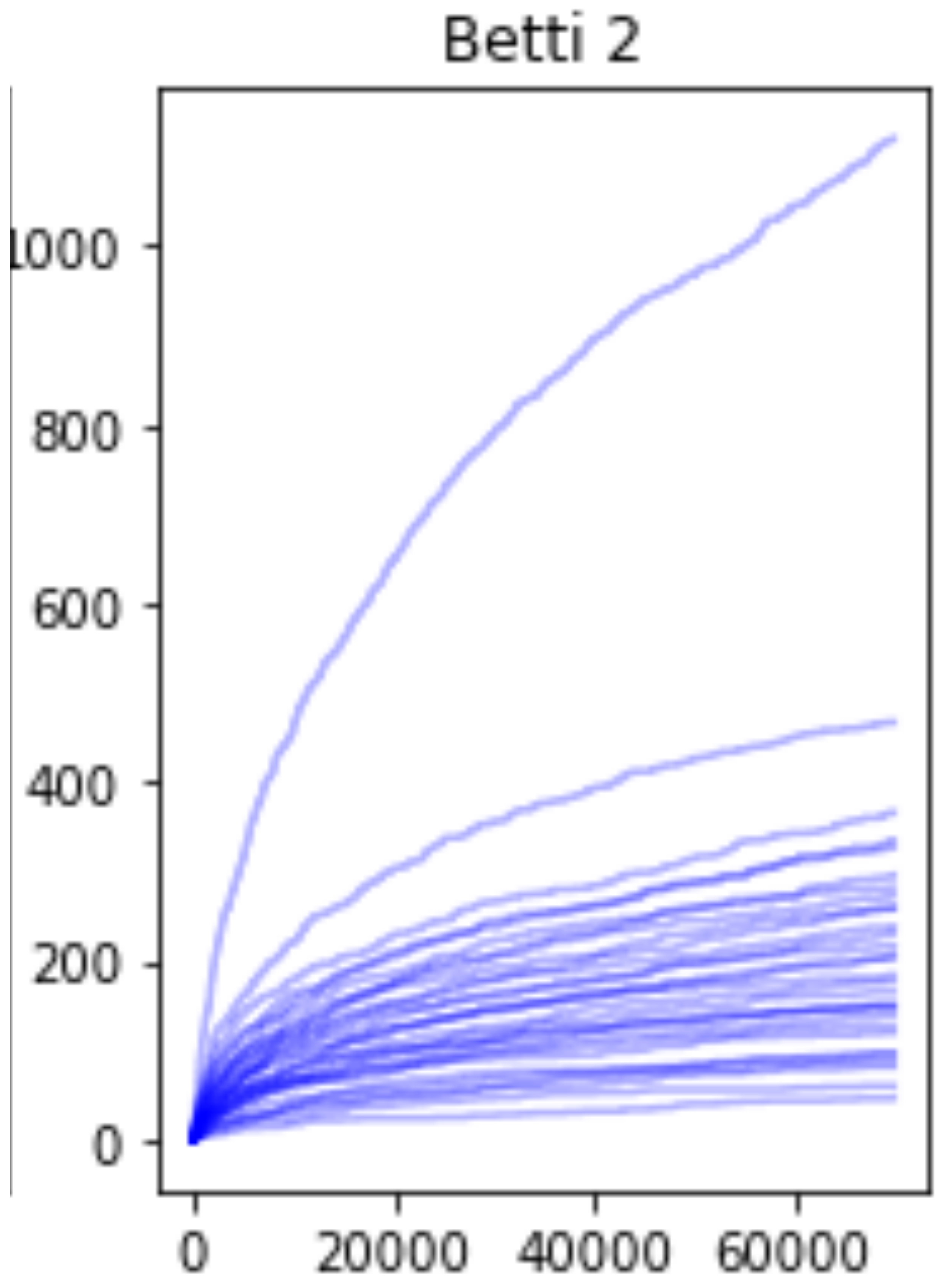
Betti Number β_q

- increasing trend
- concave growth



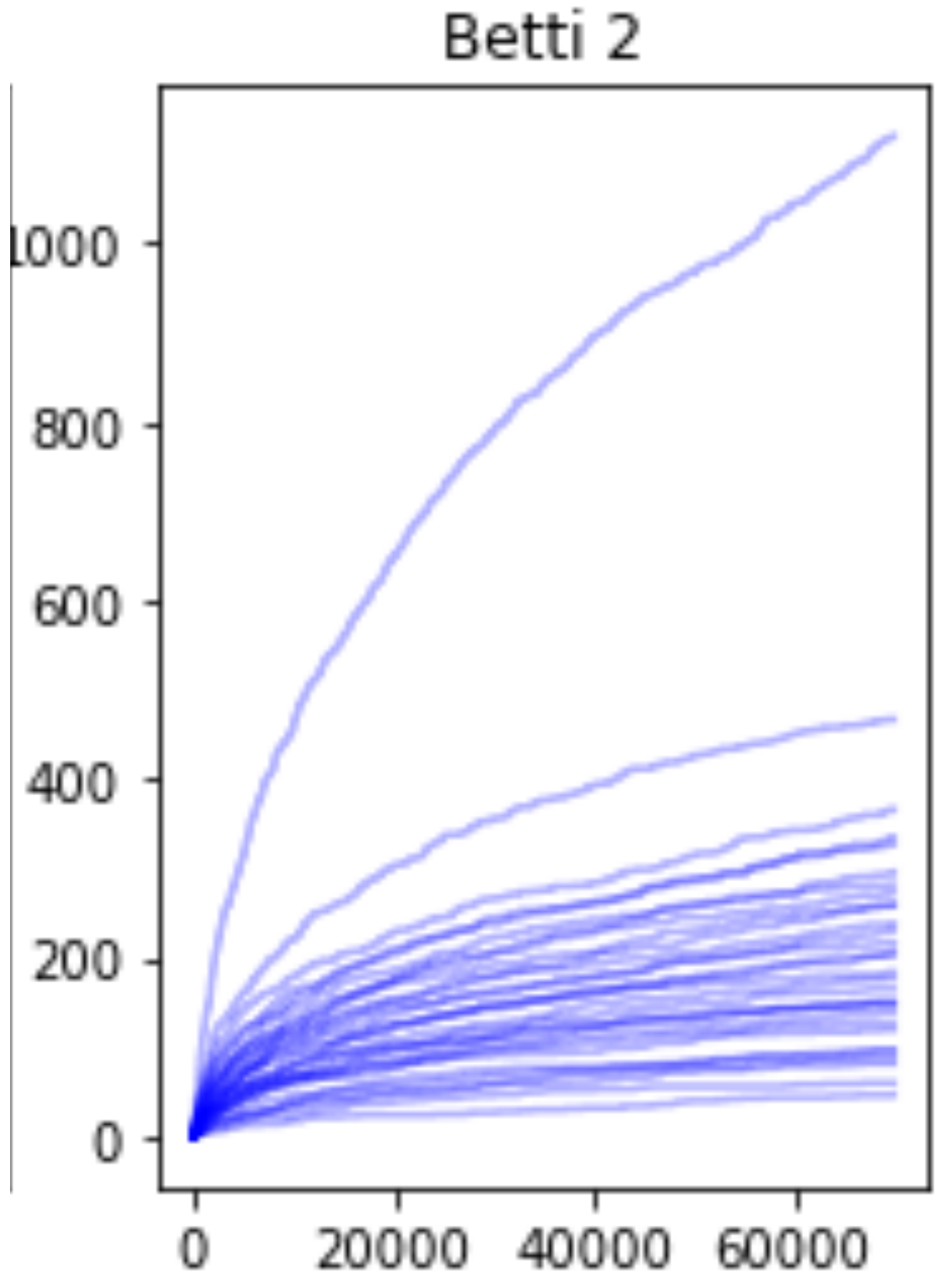
Betti Number β_q

- increasing trend
- concave growth
- outlier



Betti Number β_q

- With probability at least $1 - \varepsilon$,
- $c_\varepsilon(\text{num of nodes}^{\boxed{1-4x}}) \leq \beta_2 \leq C_\varepsilon(\text{num of nodes}^{\boxed{1-4x}})$
  - $x = 1 - \dfrac{1}{2 + \delta/m} \in (0,1)$ decreases with the preferential attachment strength
    - $P[T \text{ attaches to } i] \propto T^{-x}$



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  - $x = 1 - \dfrac{1}{2 + \delta/m} \in (0,1)$ decreases with the preferential attachment strength
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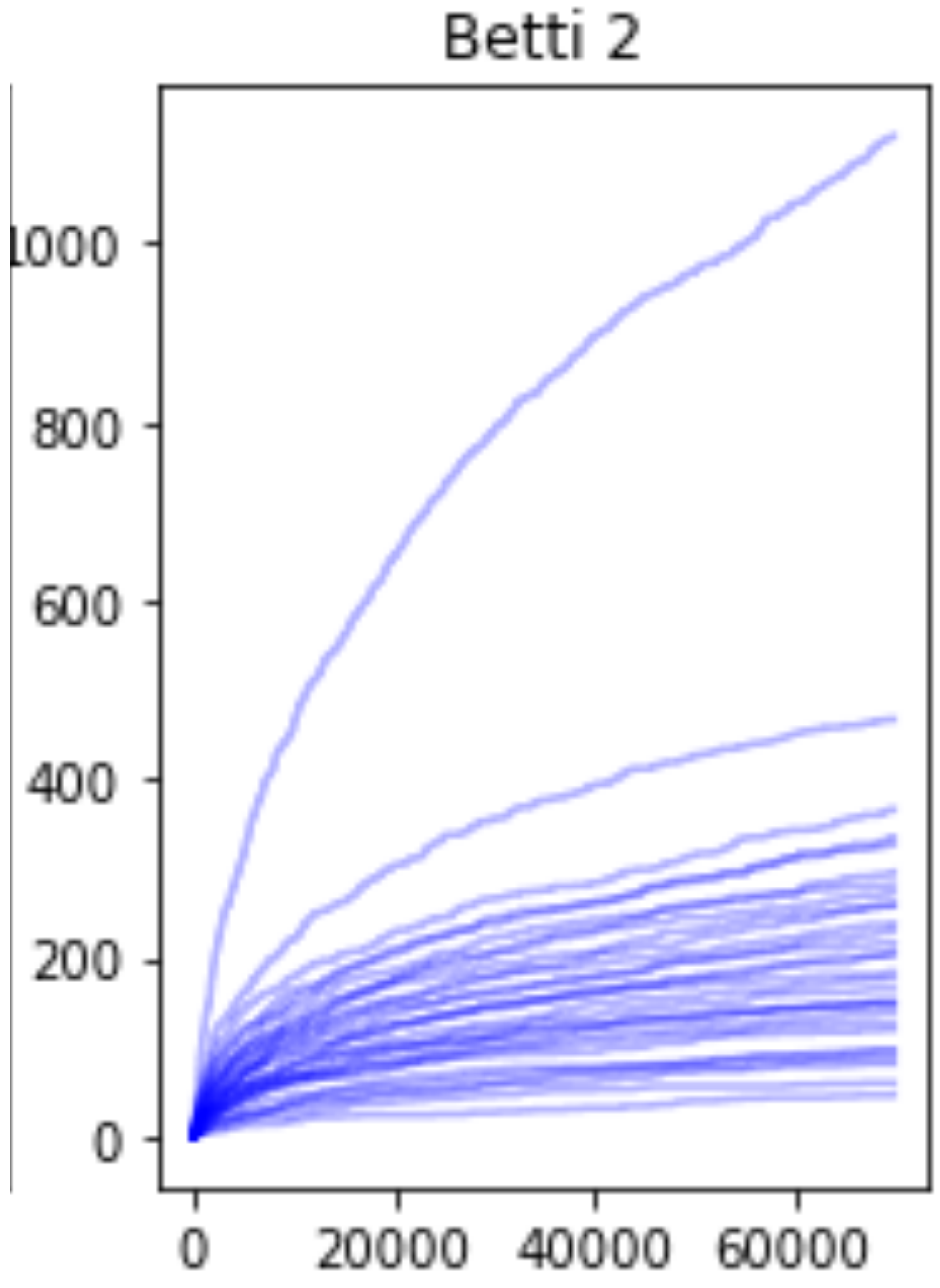
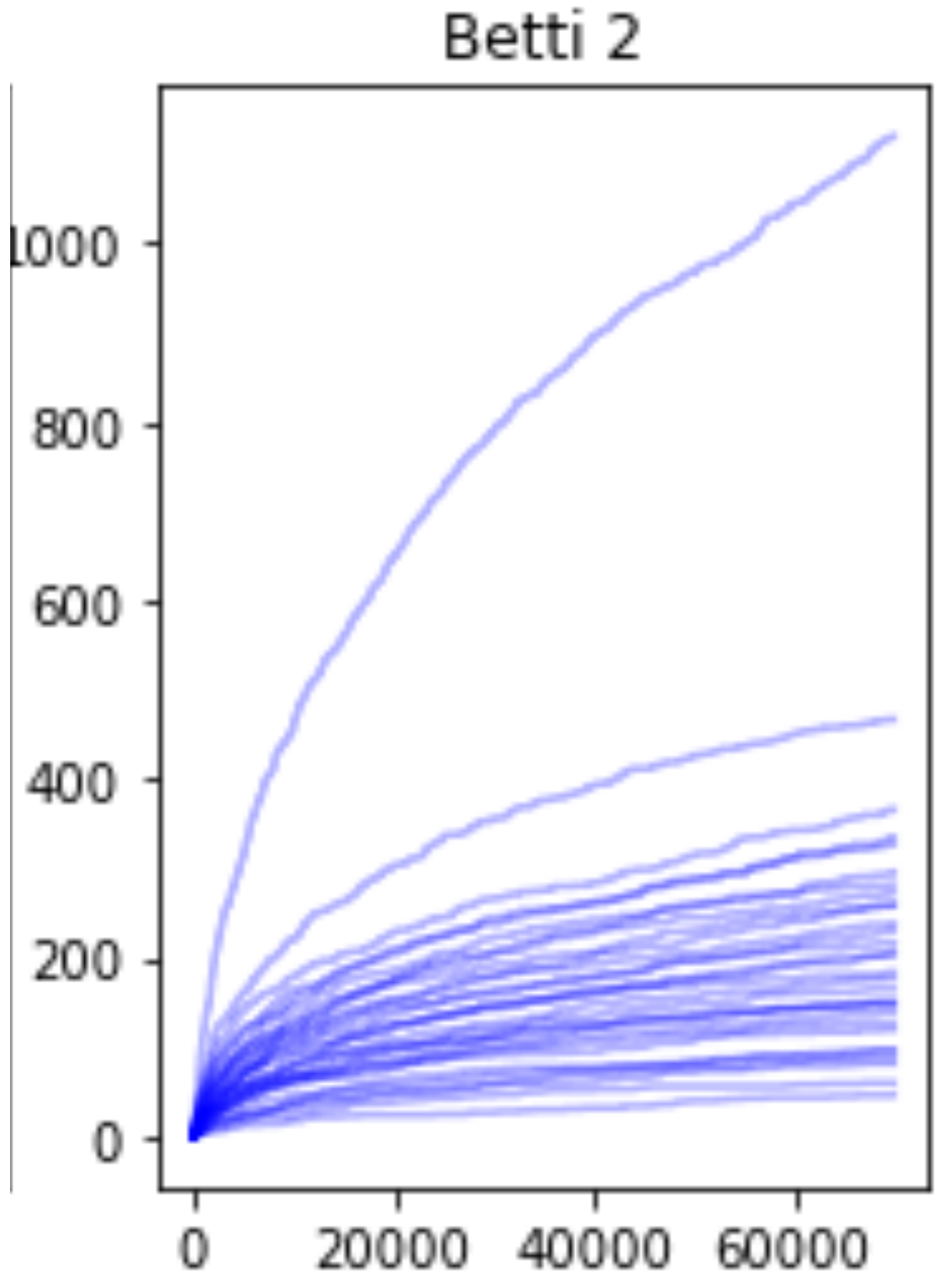
  - If $\boxed{1 - 4x} < 0$, then $\beta_2 \leq C_\varepsilon$.



Betti Number β_q

- With probability at least $1-\varepsilon$,
- $c_\varepsilon(\text{num of nodes}^{1-\boxed{4}x}) \leq \beta_2 \leq C_\varepsilon(\text{num of nodes}^{1-\boxed{4}x})$
  - $x = 1 - \dfrac{1}{2+\delta/m} \in (0,1)$ decreases with the preferential attachment strength
    - $P[T \text{ attaches to } i] \propto T^{-x}$
  - If $1-4x < 0$, then $\beta_2 \leq C_\varepsilon$.
- $c_\varepsilon(\text{num of nodes}^{1-\boxed{2q}x}) \leq \beta_q \leq C_\varepsilon(\text{num of nodes}^{1-\boxed{2q}x})$ for $q \geq 2$.

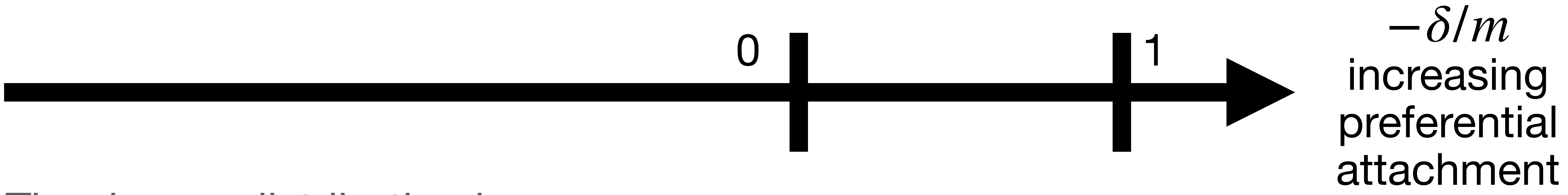


Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



The degree distribution has …

Phase transition

Recall
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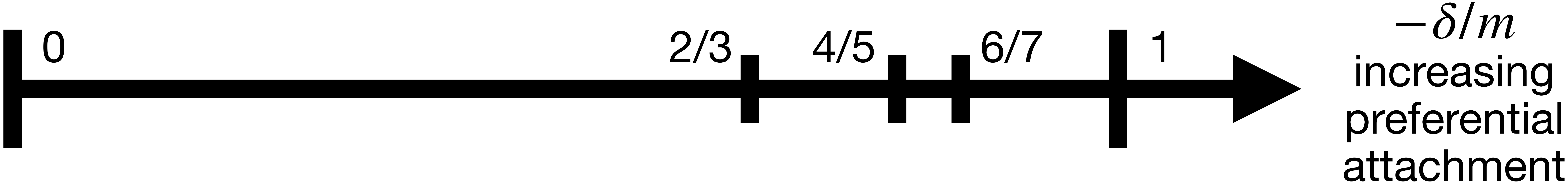

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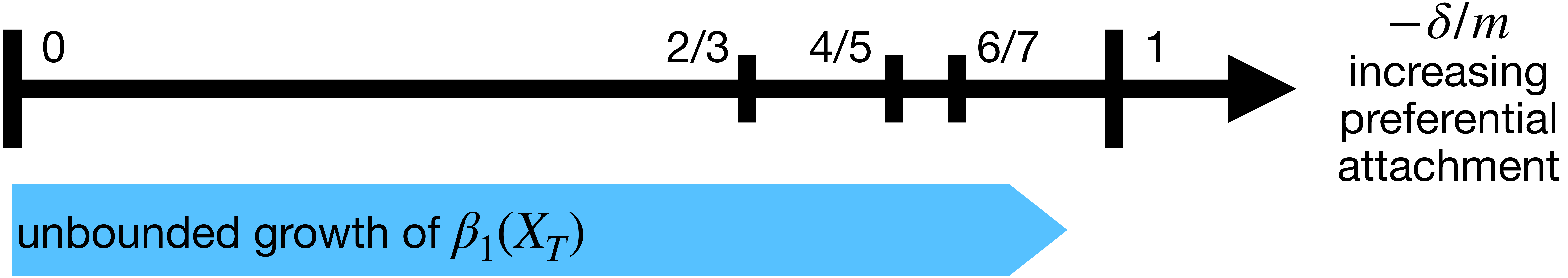

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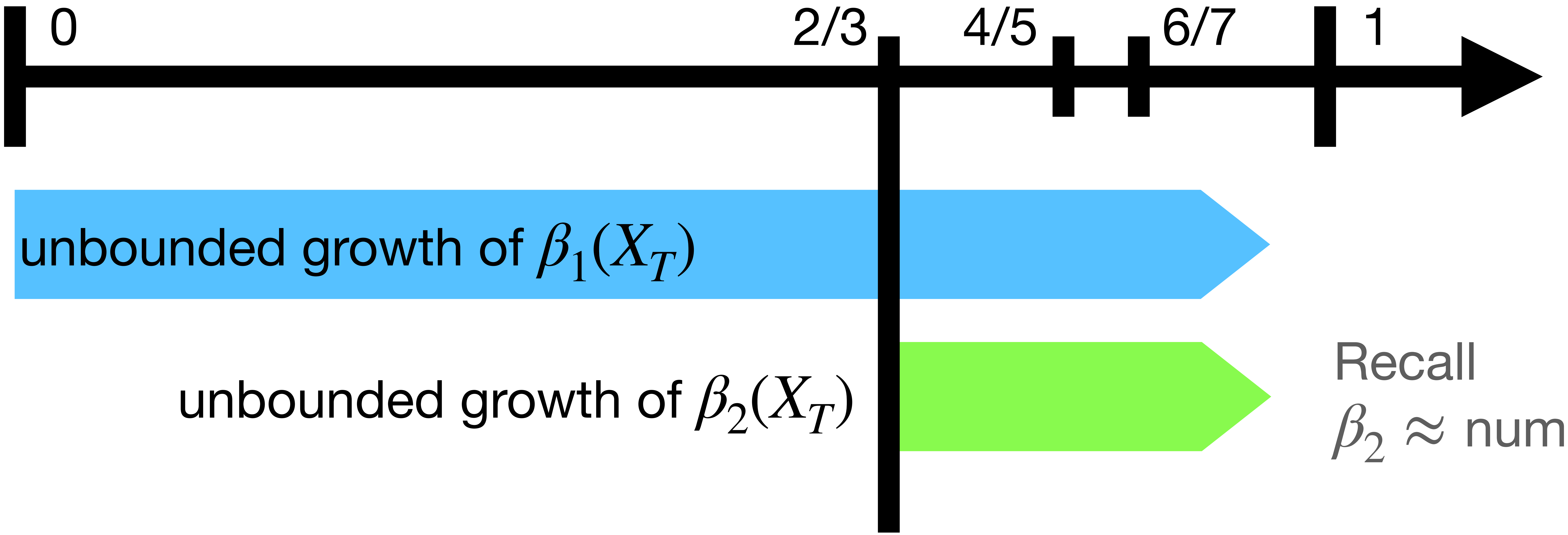

$-\delta/m$
increasing
preferential
attachment

Recall
$\beta_2 \approx$ num of nodes^{1-4x}

Phase transition

Recall

P(attaching to v) ∝ degree + δ

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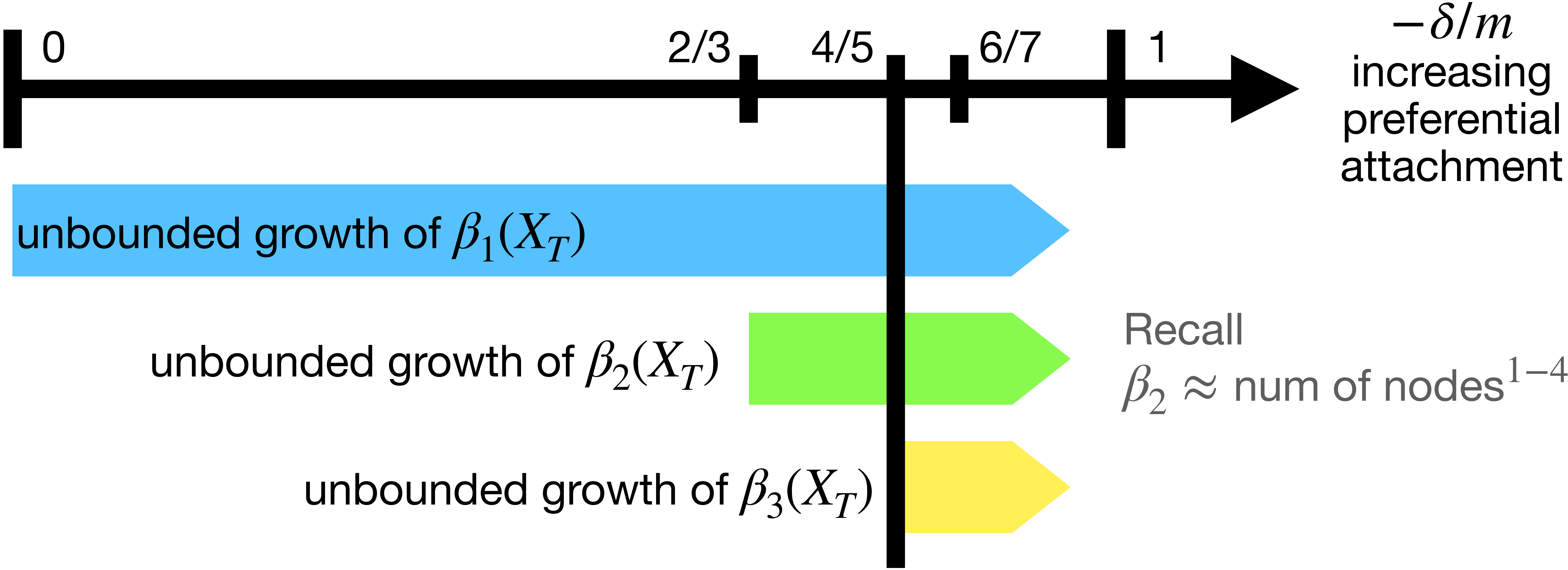

Recall

$\beta_2 \approx$ num of nodes^{1-4x}

Phase transition

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P(attaching to v) $\propto$ degree + δ
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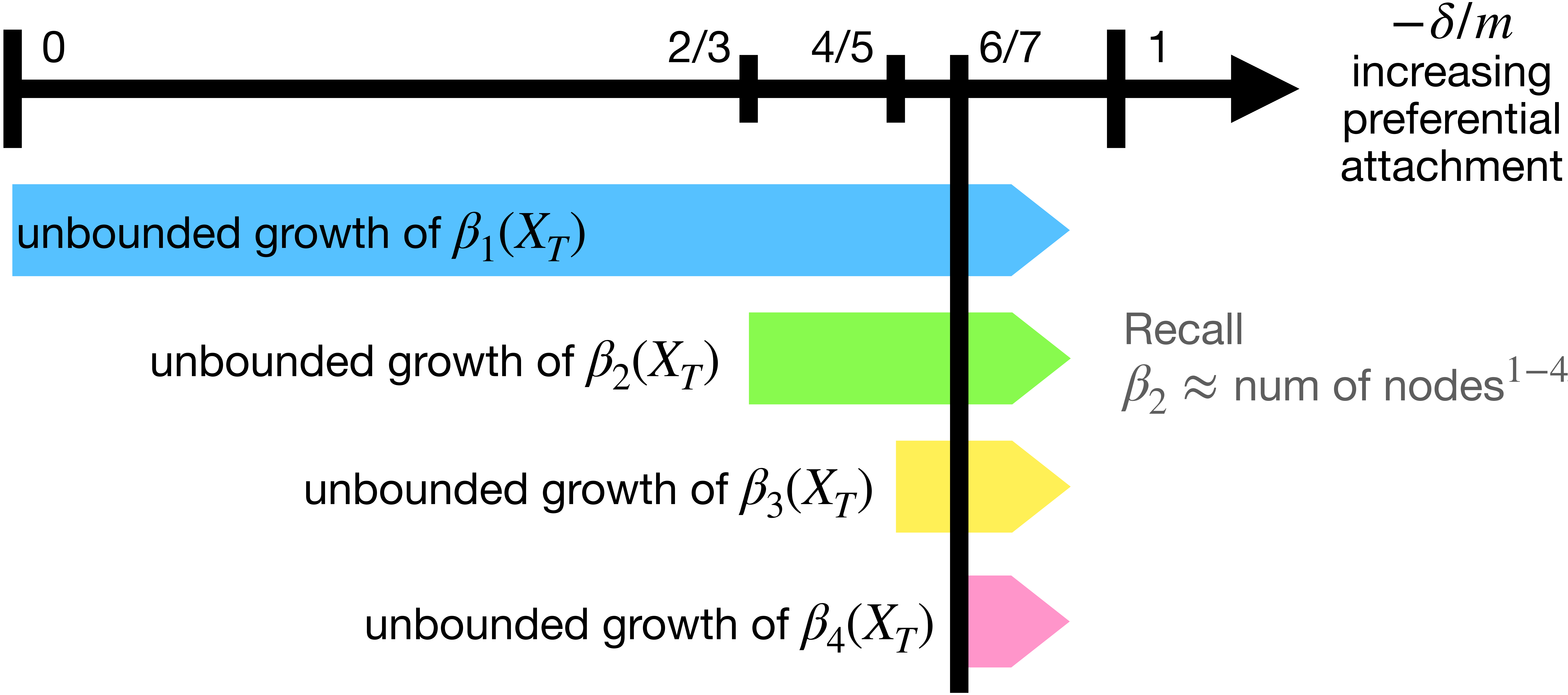


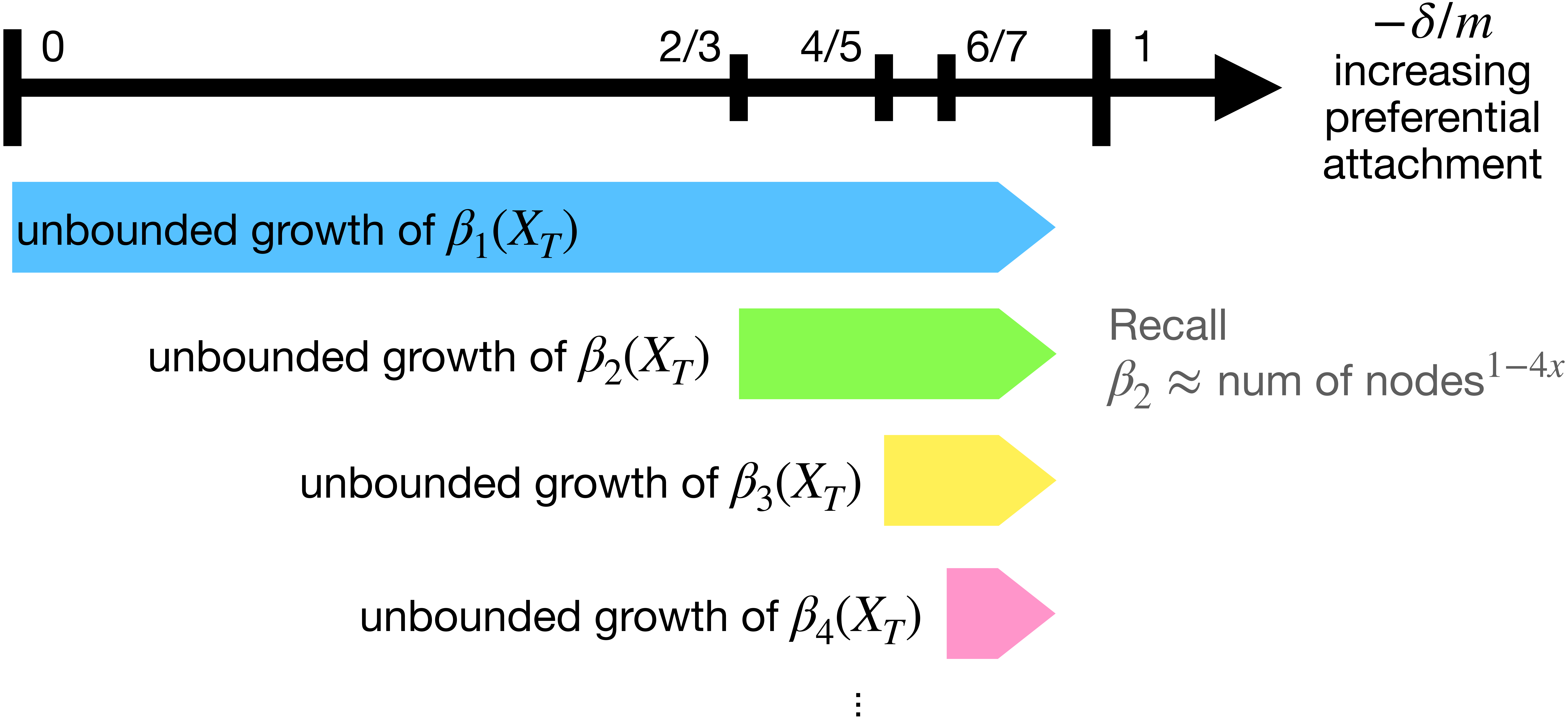
Recall
$\beta_2 \approx$ num of nodes^{1-4x}

Phase transition

Recall

P(attaching to v) $\propto$ degree + δ

m = number of edges per new node



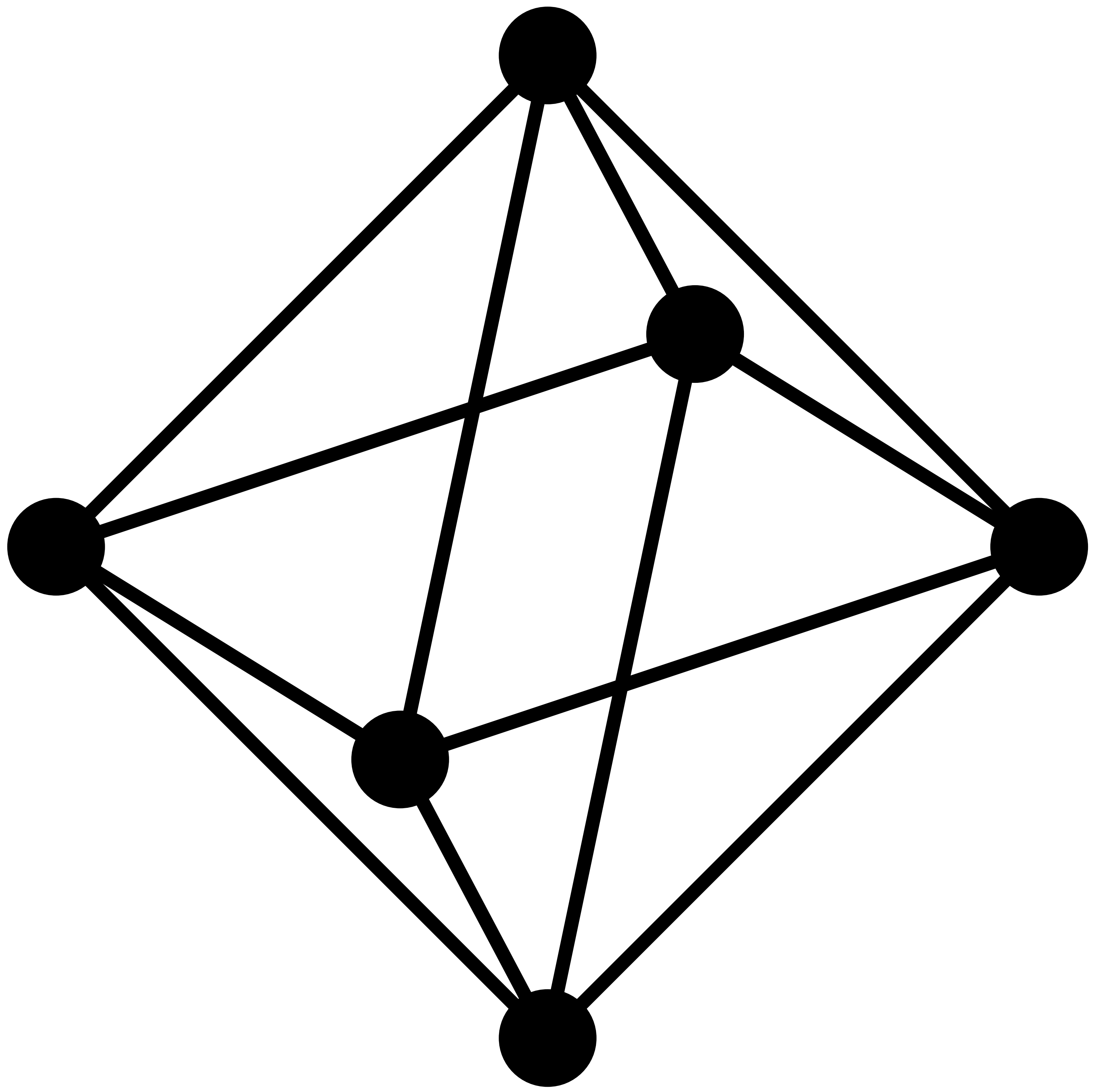
Recall

$\beta_2 \approx$ num of nodes^{1-4x}

Theorem: $\beta_2 \approx$ num of nodes^{1-4x}

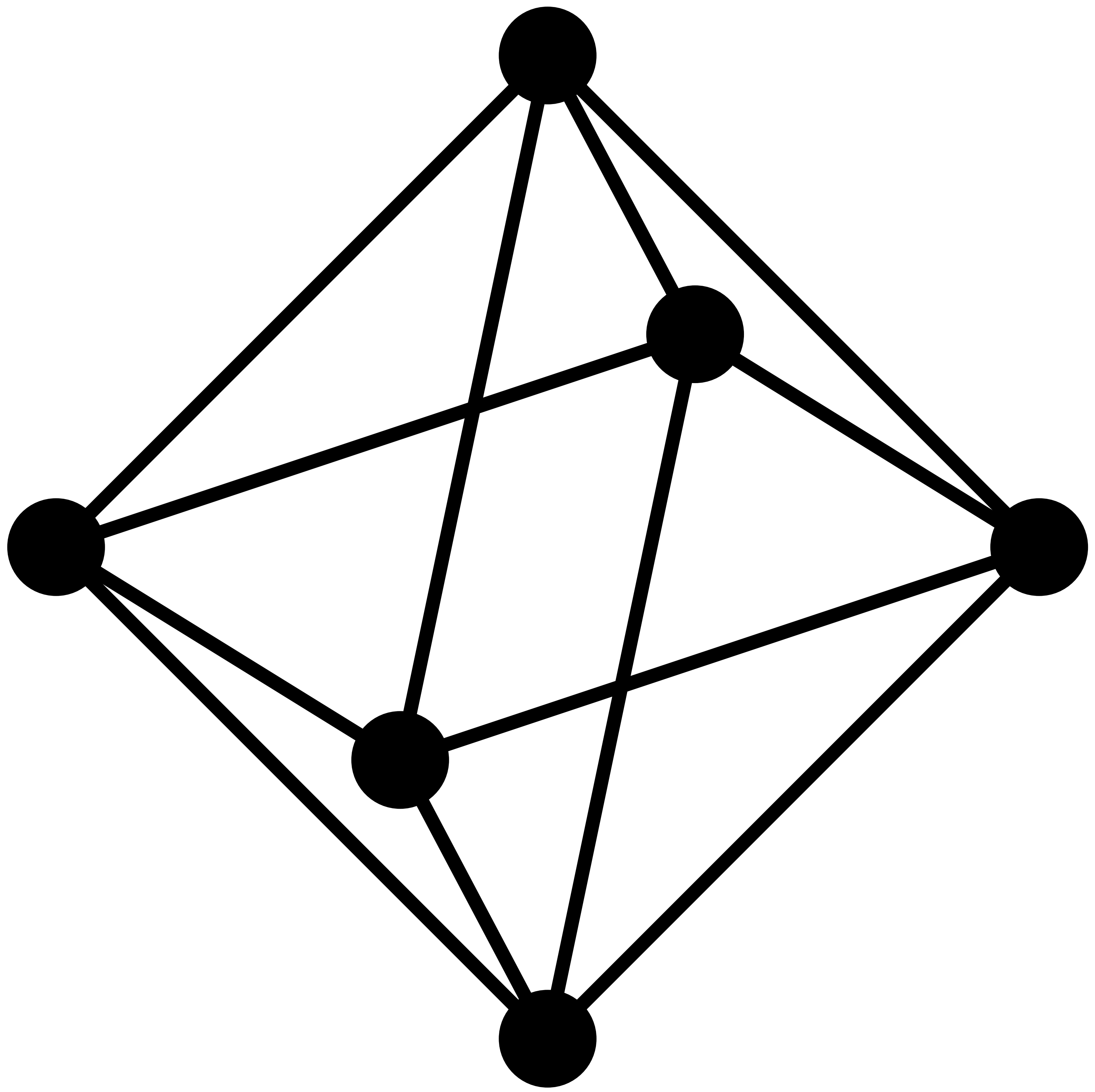
Proof?

Proof of $\beta_2 \approx$ num of nodes^{1-4x}

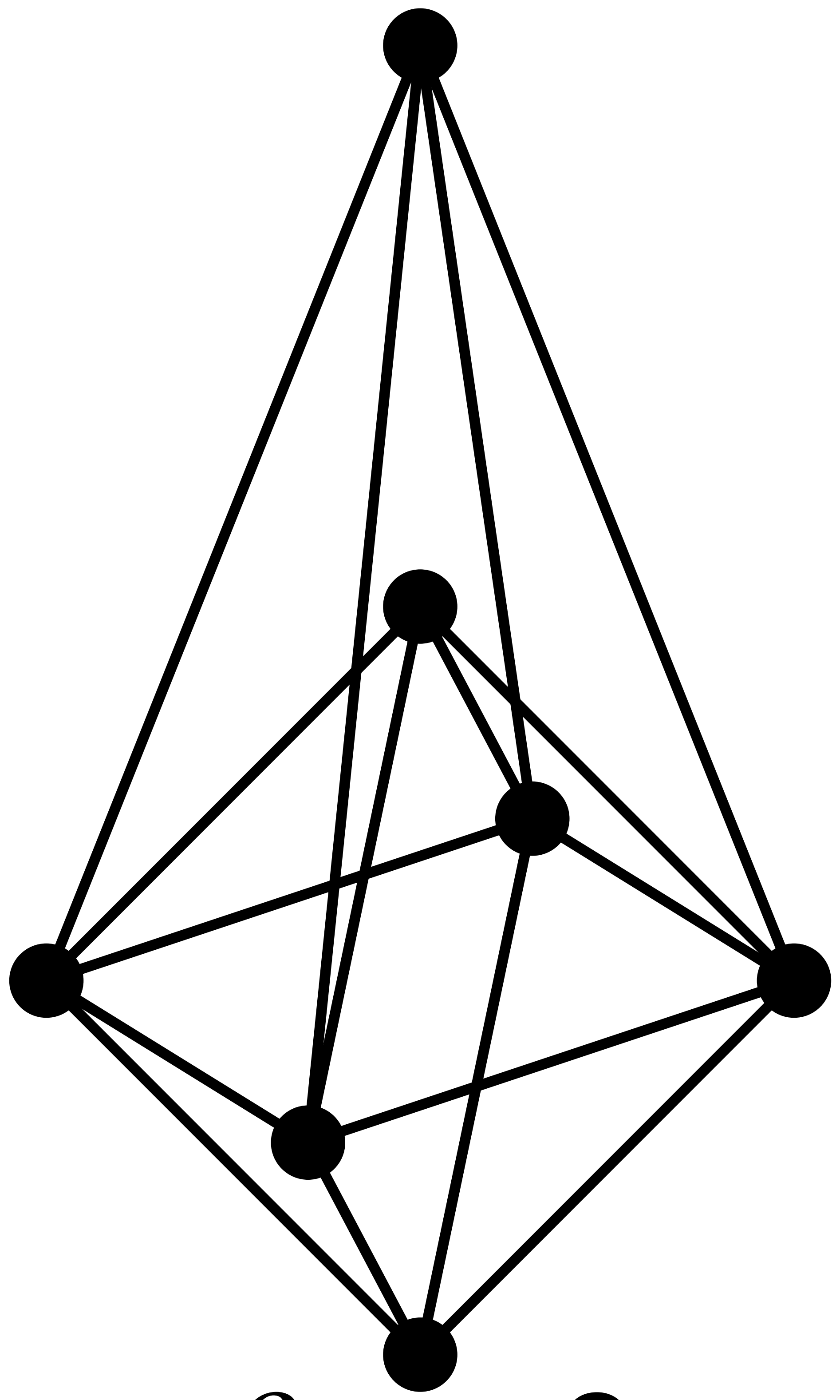


$\beta_2 = 1$

Proof of $\beta_2 \approx$ num of nodes^{1-4x}

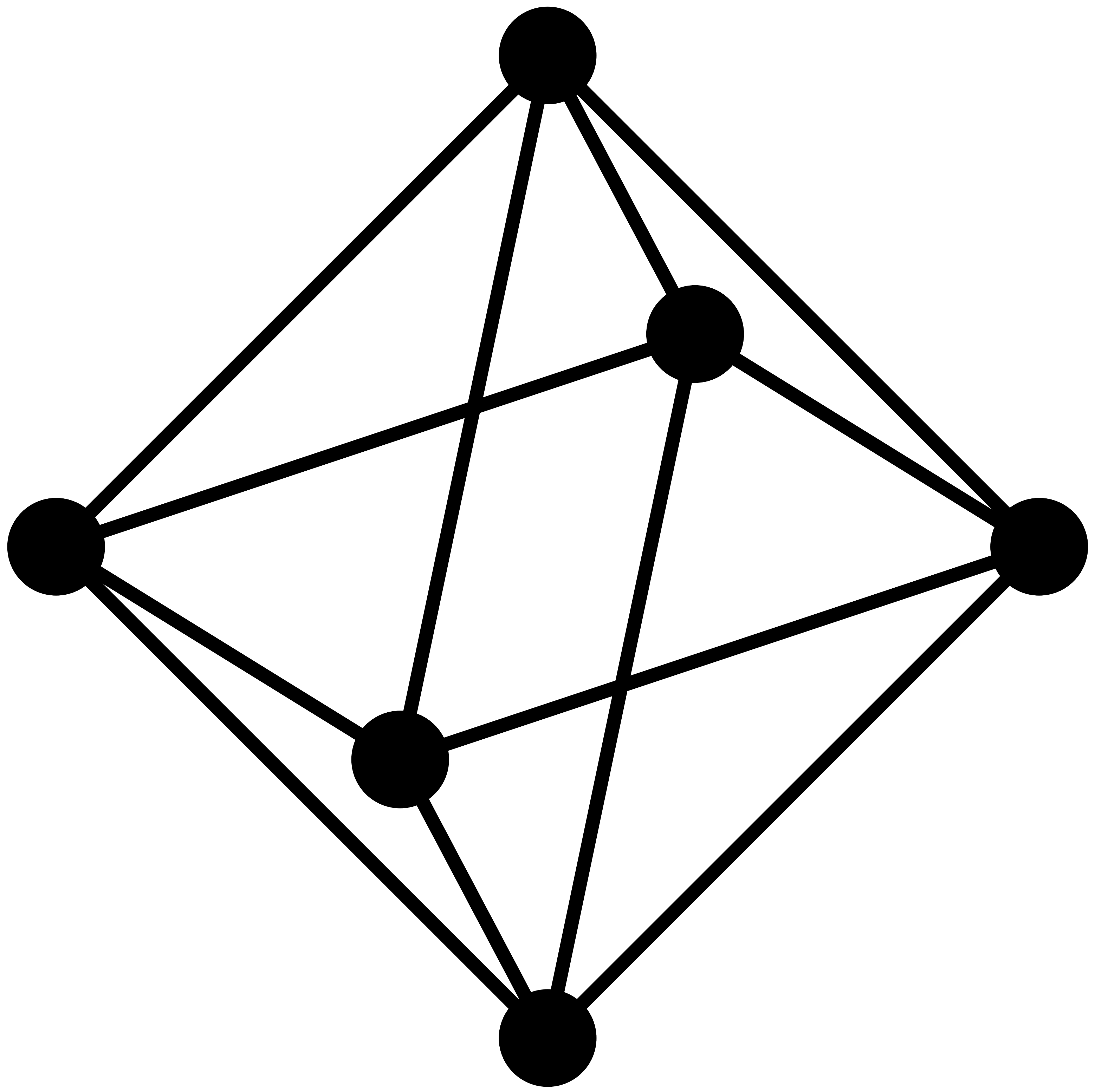


$\beta_2 = 1$

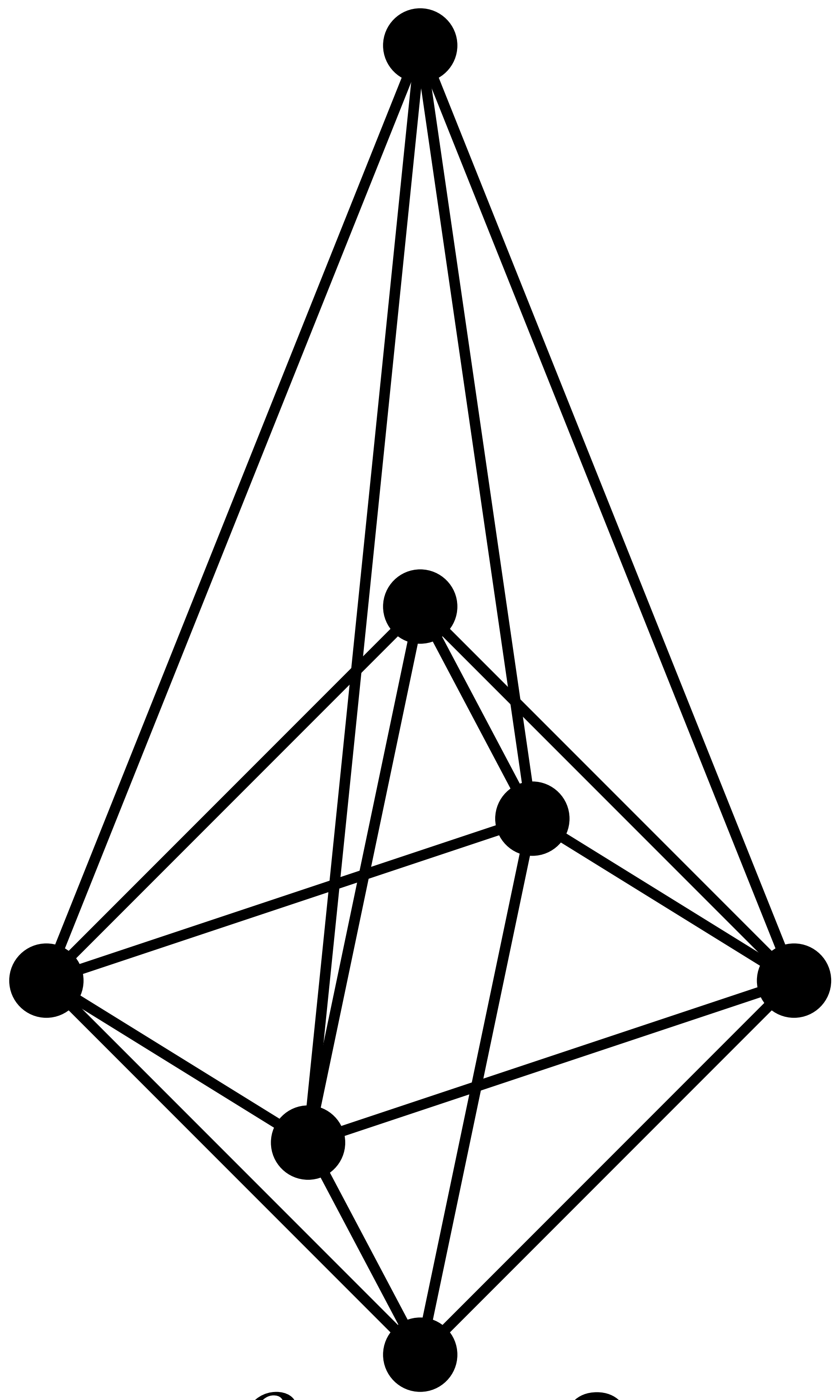


$\beta_2 = 2$

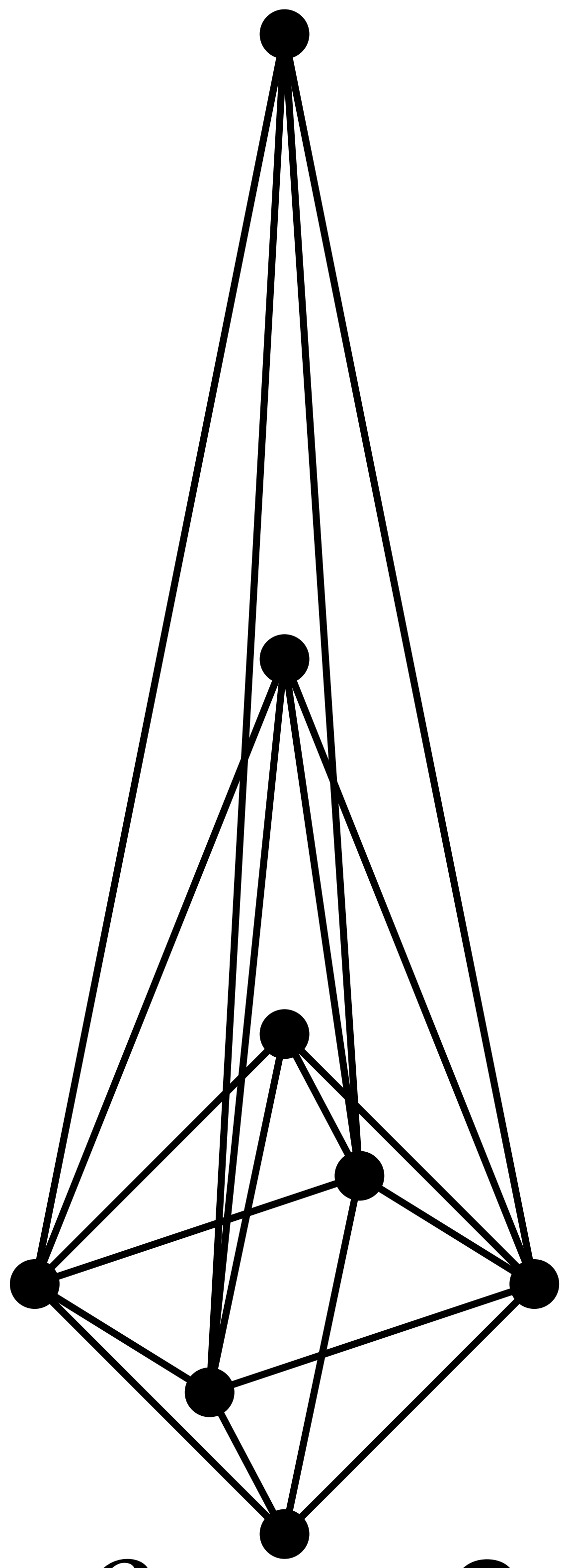
Proof of $\beta_2 \approx$ num of nodes^{1-4x}



$\beta_2 = 1$



$\beta_2 = 2$



$\beta_2 = 3$

Subtleties

- Need homological algebra to relate Betti numbers with counts (a mapping cone argument)
-

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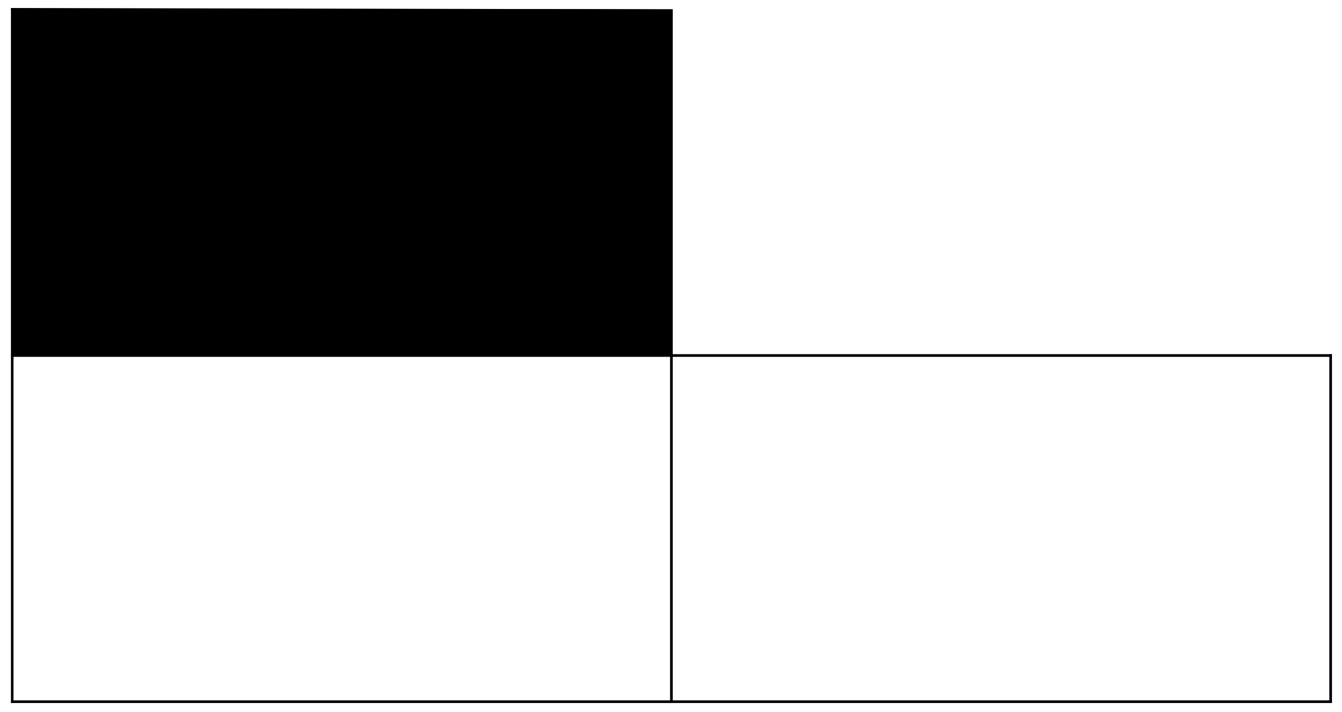
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs

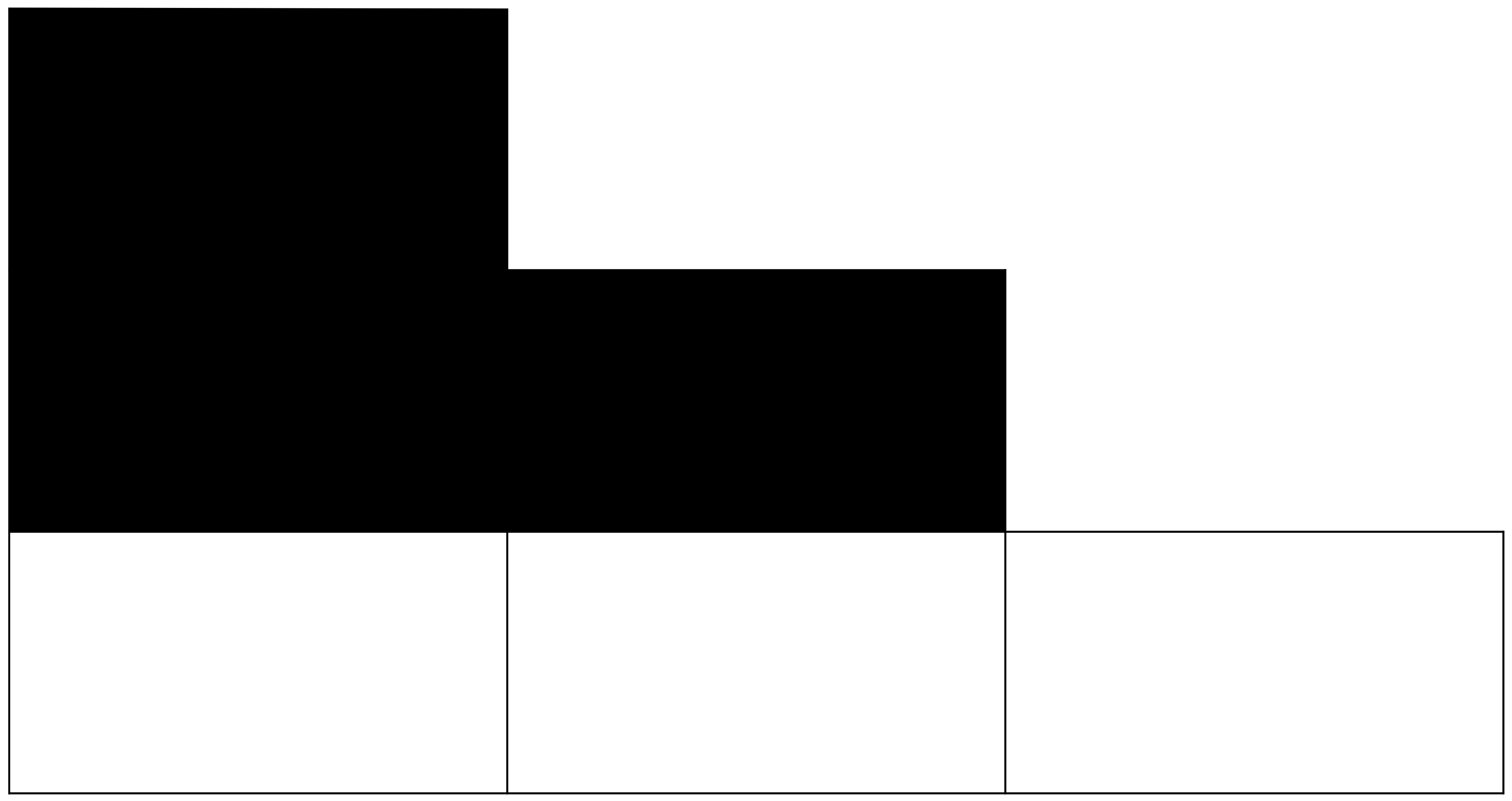
Homotopy-Connectivity?

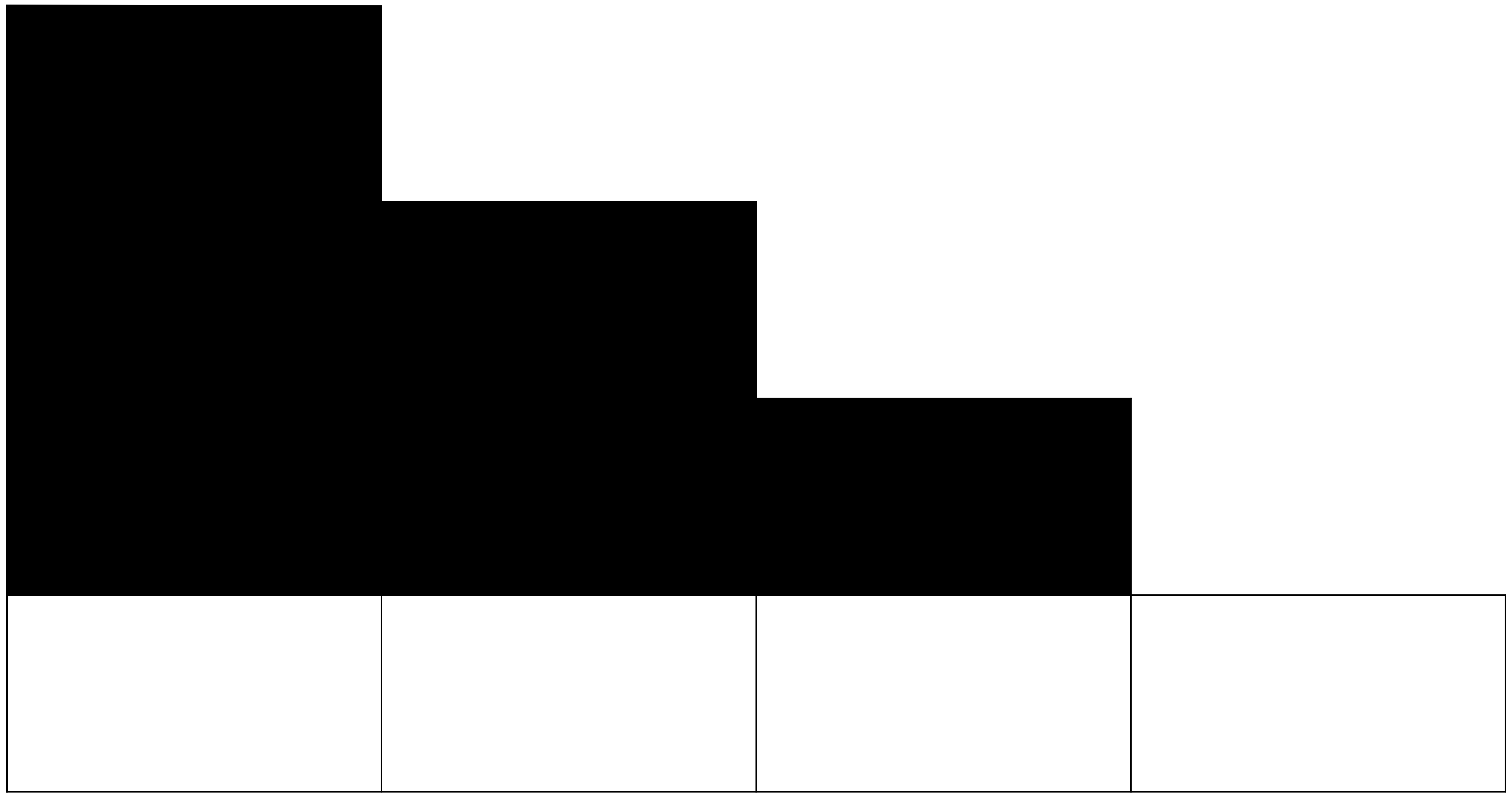
Homotopy-Connectivity?

$$\beta_2 \approx \text{num of nodes}^{1-4x}$$

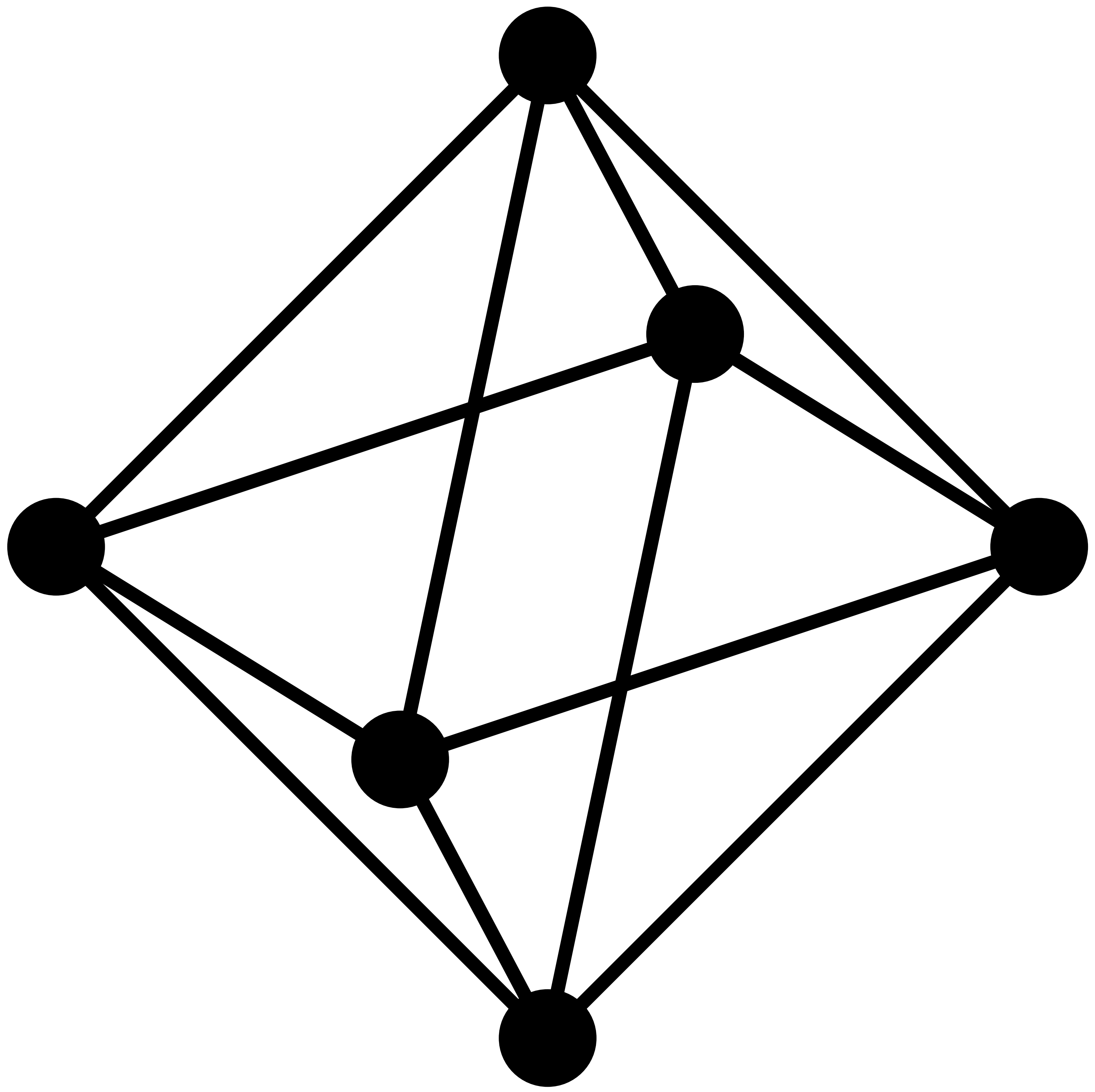
Pass to infinity



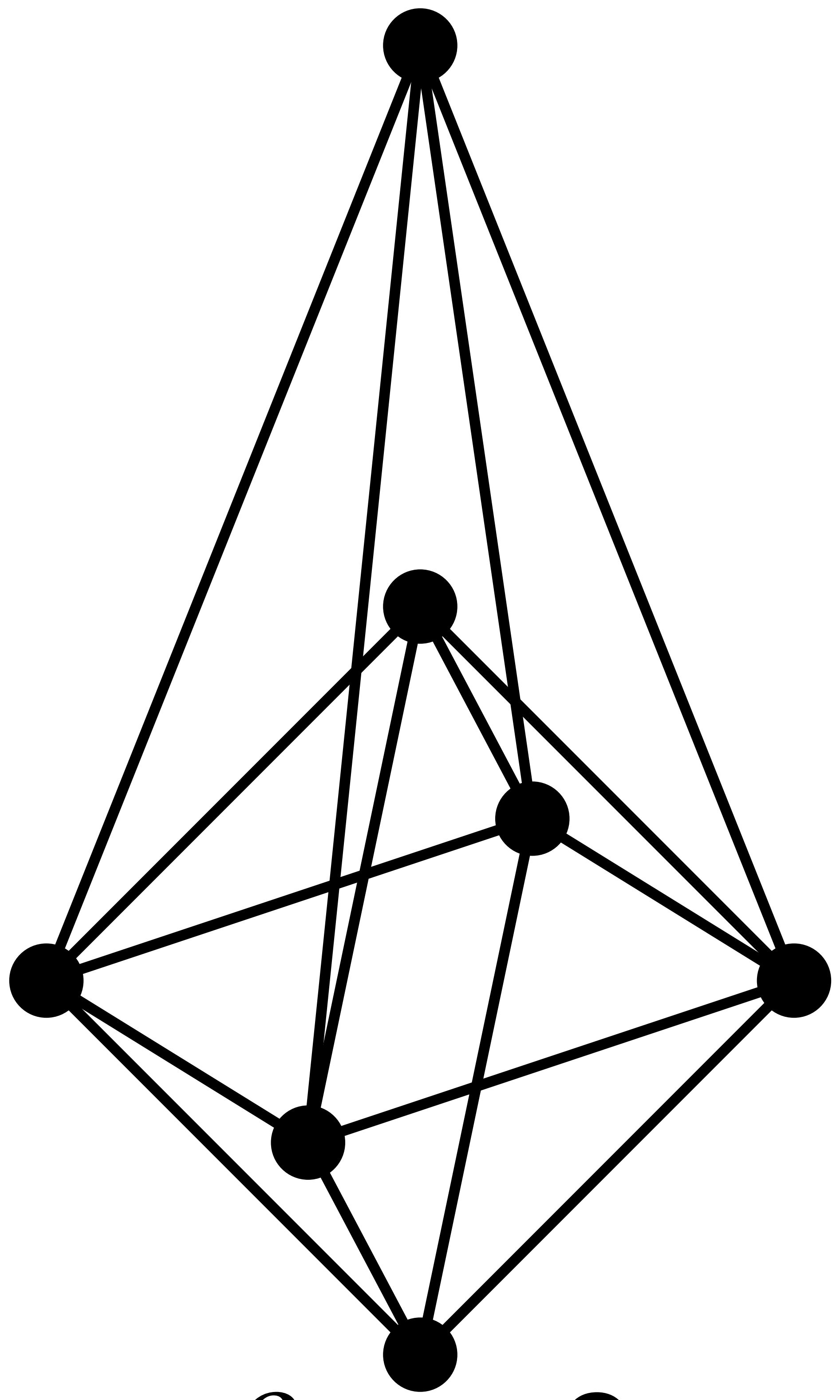




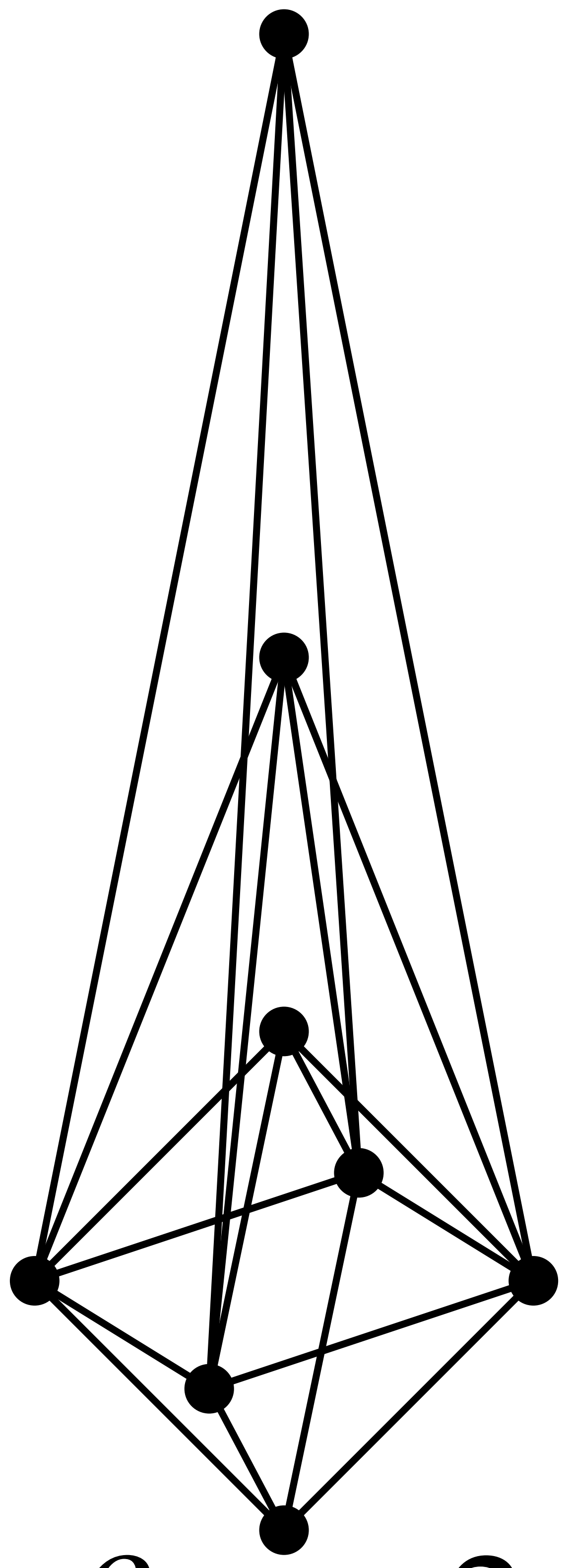
Will all of these be filled in at infinity?



$\beta_2 = 1$



$\beta_2 = 2$



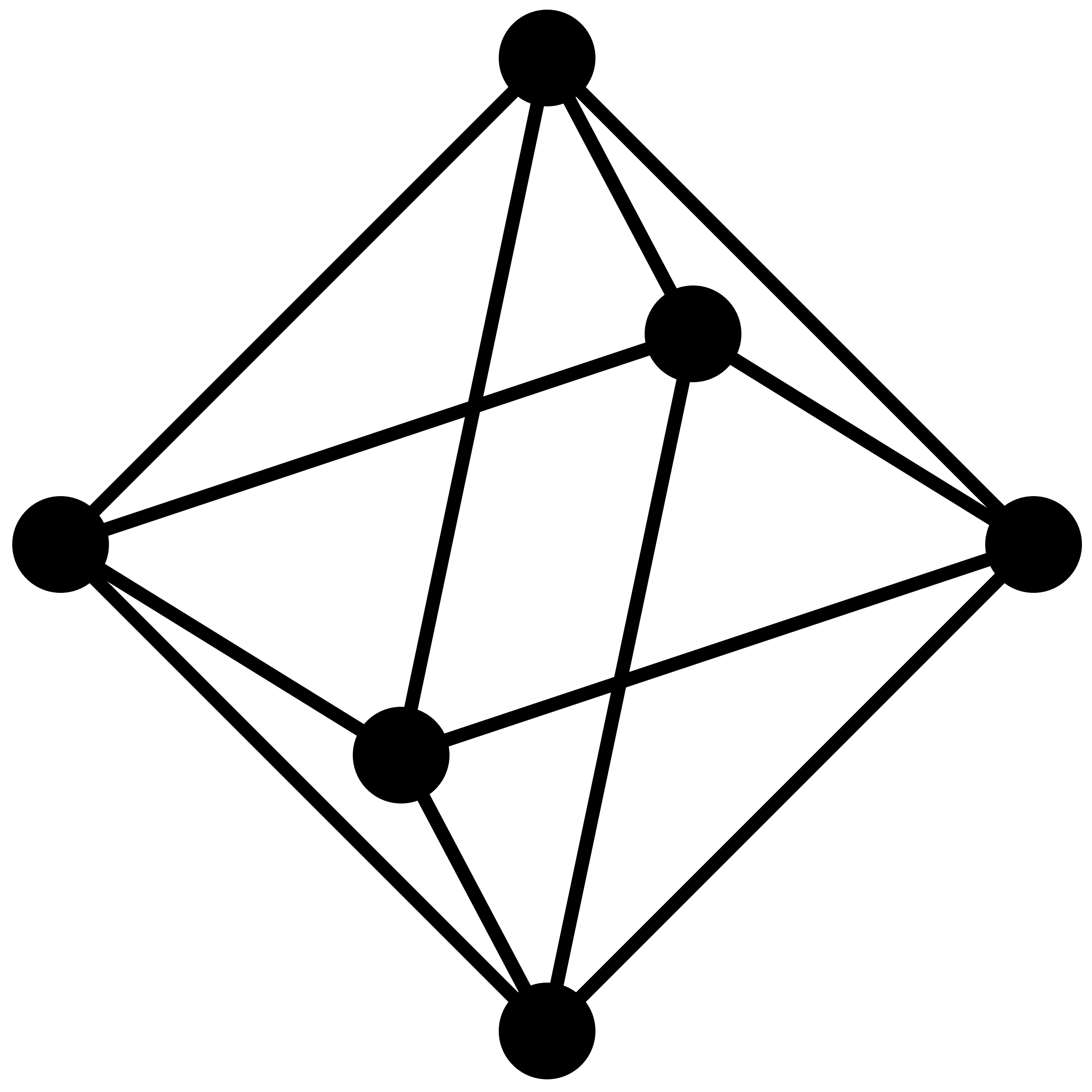
$\beta_2 = 3$

[Barmak 2023]

- A clique complex is q-homotopy-connected
- if every collection of $2(q + 1)$ nodes has a common neighbor.

[Barmak 2023]

- A clique complex is q-homotopy-connected
- if every collection of 2(q + 1) nodes has a common neighbor.



Homotopy-Connected

- Almost surely, the infinite preferential attachment complex
- is q-homotopy-connected if $x \leq \dfrac{1}{2(q+1)}$

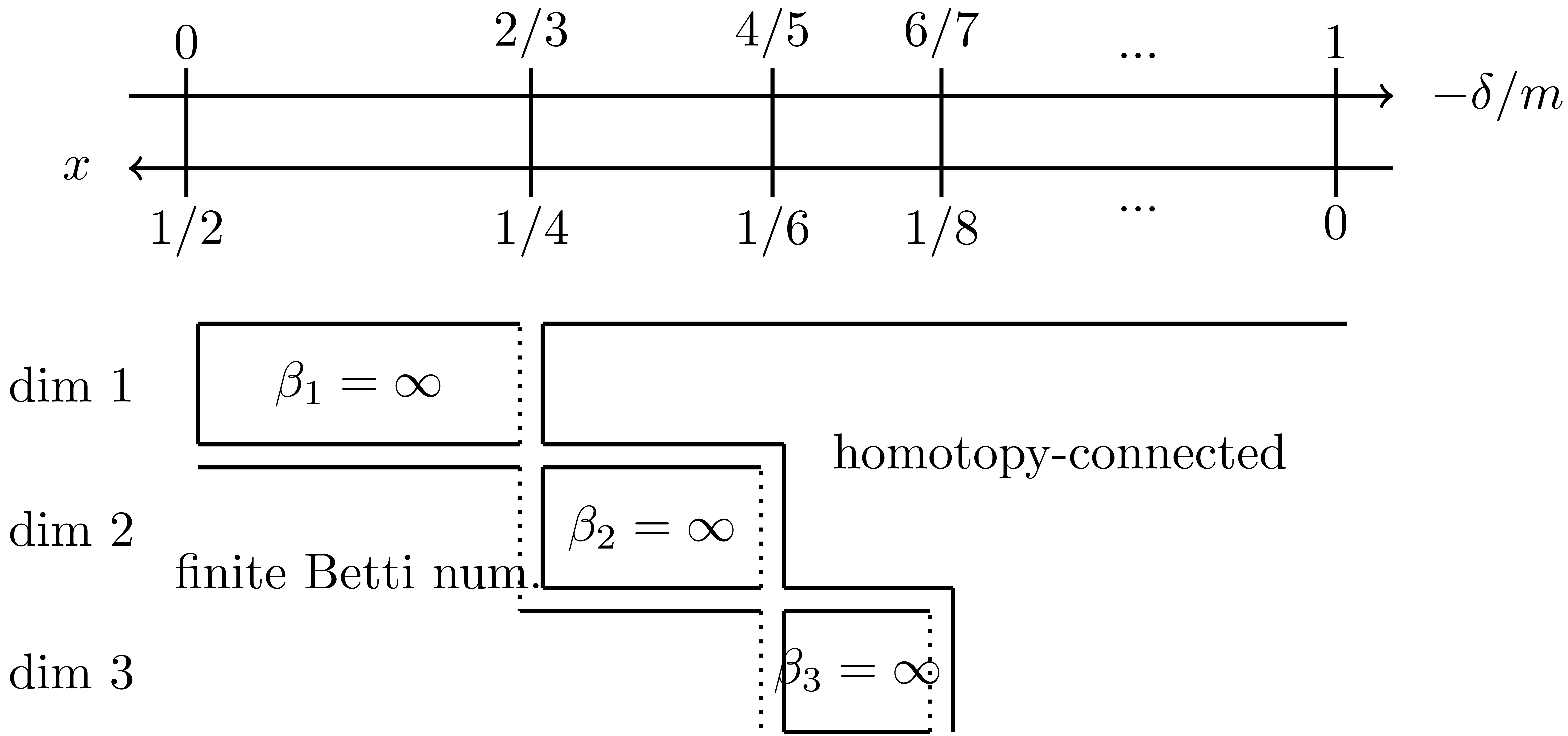
Recall:
$x \in (0,1/2)$ decreases with
the preferential attachment strength
$P[T$ attaches to $i] \propto T^{-x}$

Homotopy-Connected

- Almost surely, the infinite preferential attachment complex
- is q-homotopy-connected if $x \leq \dfrac{1}{2(q+1)}$
- has infinite Betti number at dimension q if $\dfrac{1}{2(q+1)} < x \leq \dfrac{1}{2q}$

Recall:
$x \in (0,1/2)$ decreases with
the preferential attachment strength
$P[T$ attaches to $i] \propto T^{-x}$

Phase Transition

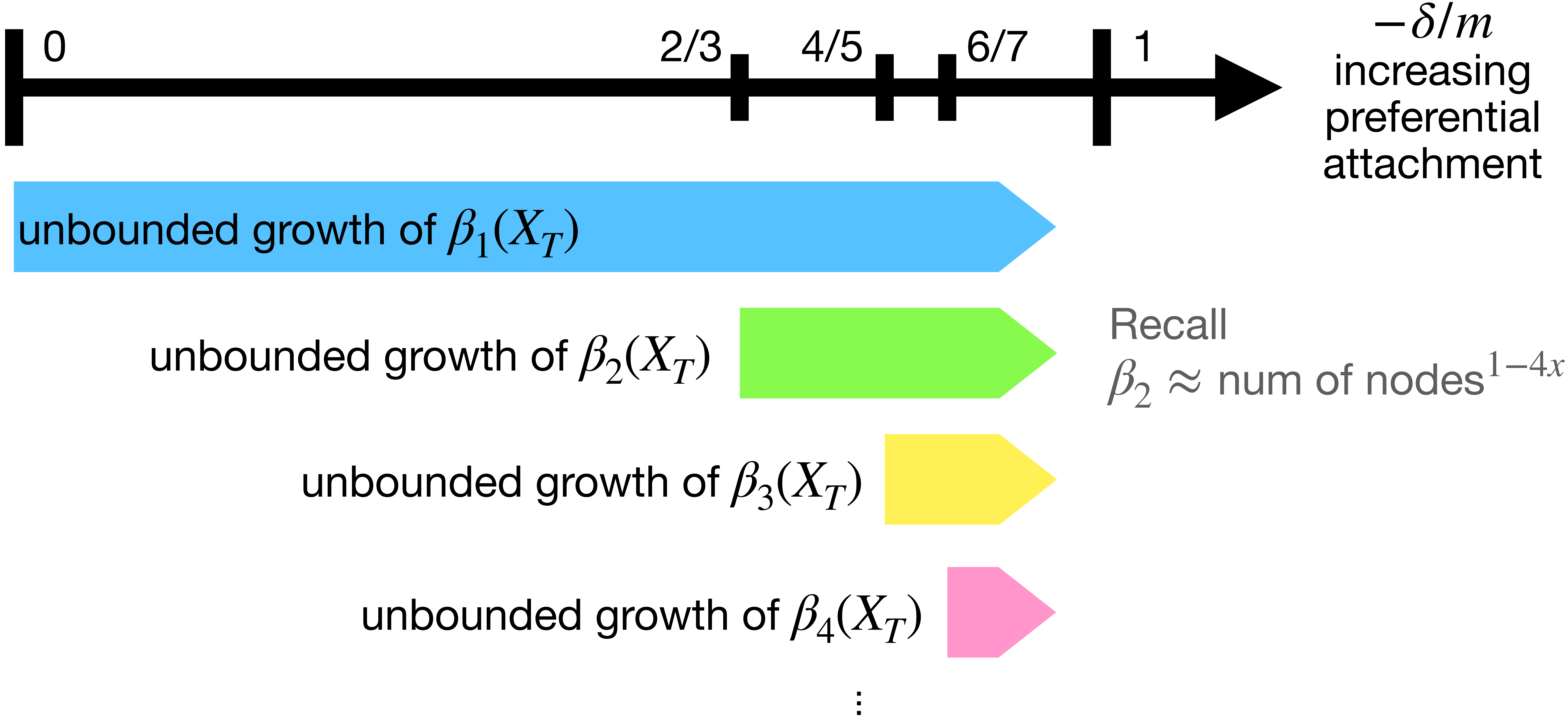


Phase transition

Recall

P(attaching to v) $\propto$ degree + δ

m = number of edges per new node



Recall

$\beta_2 \approx$ num of nodes^{1-4x}

IV. What lies ahead

order of magnitude of Betti numbers

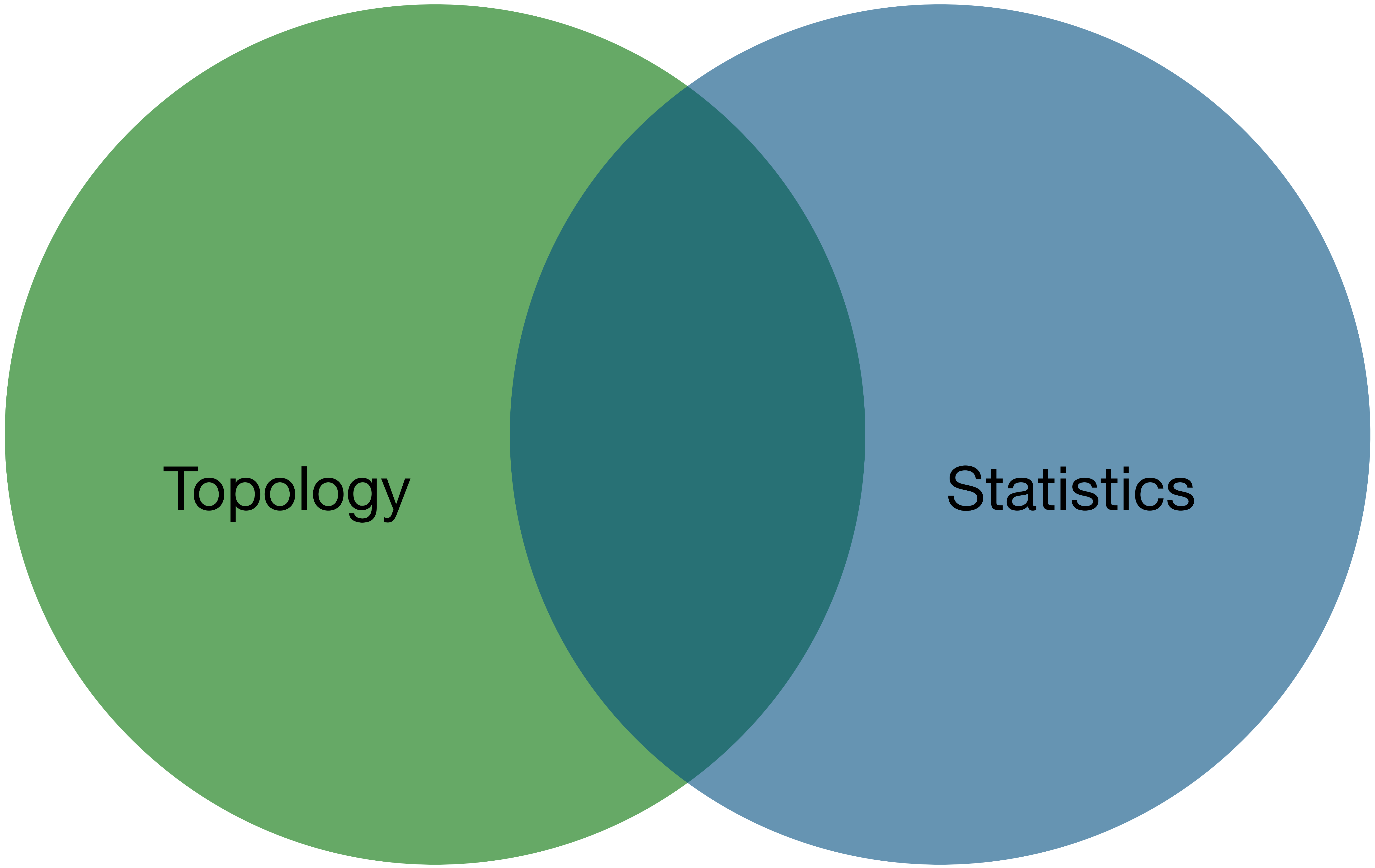
homotopy connectedness

order of magnitude of Betti numbers

homotopy connectedness

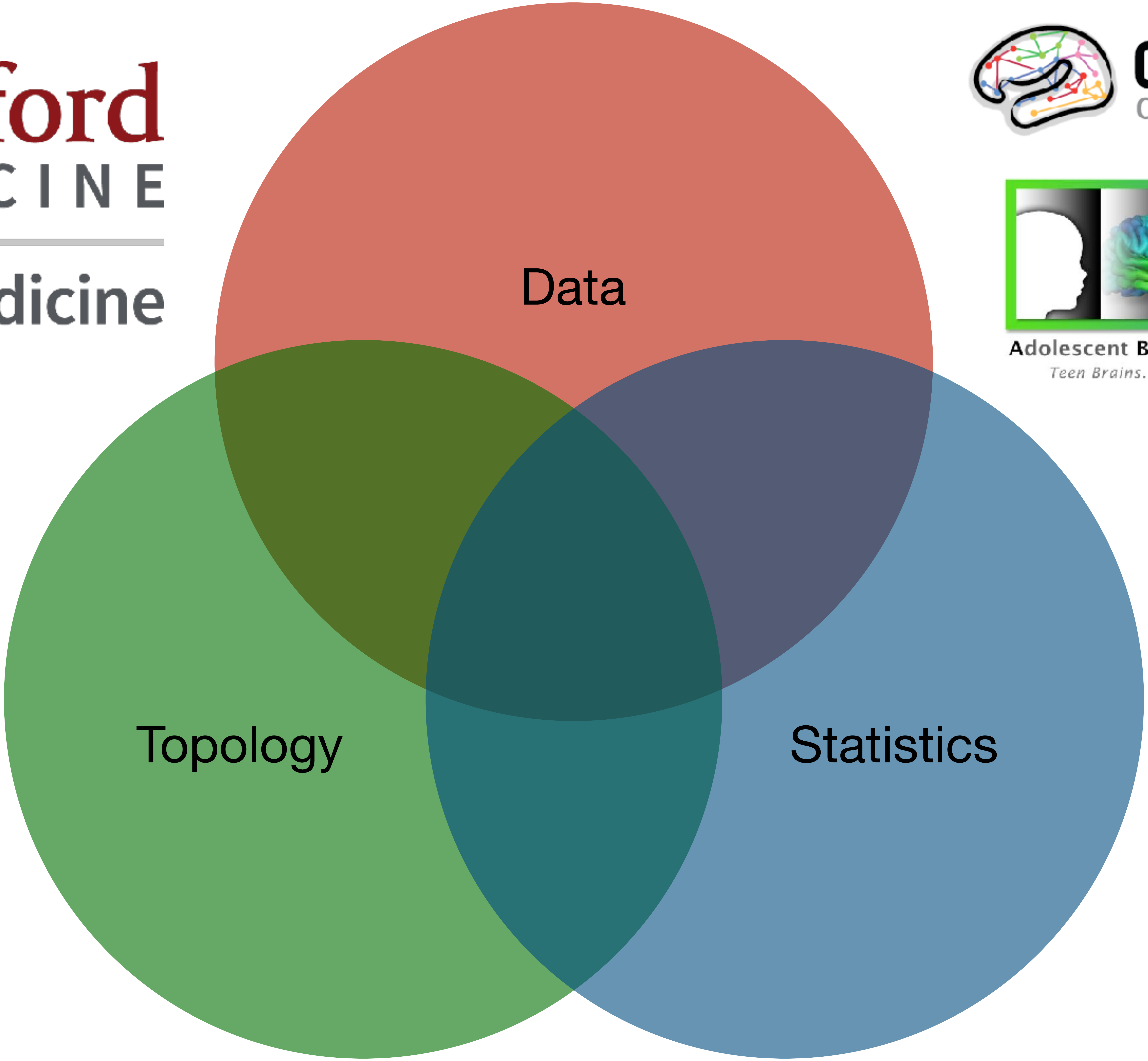
more precise estimates?

other inhomogeneous complexes?


Topology
Statistics

Stanford MEDICINE

School of Medicine



CONNECTOME COORDINATION FACILITY

Adolescent Brain Cognitive Development

Teen Brains. Today's Science. Brighter Future.

What did we learn today?

- Random topology is cool.

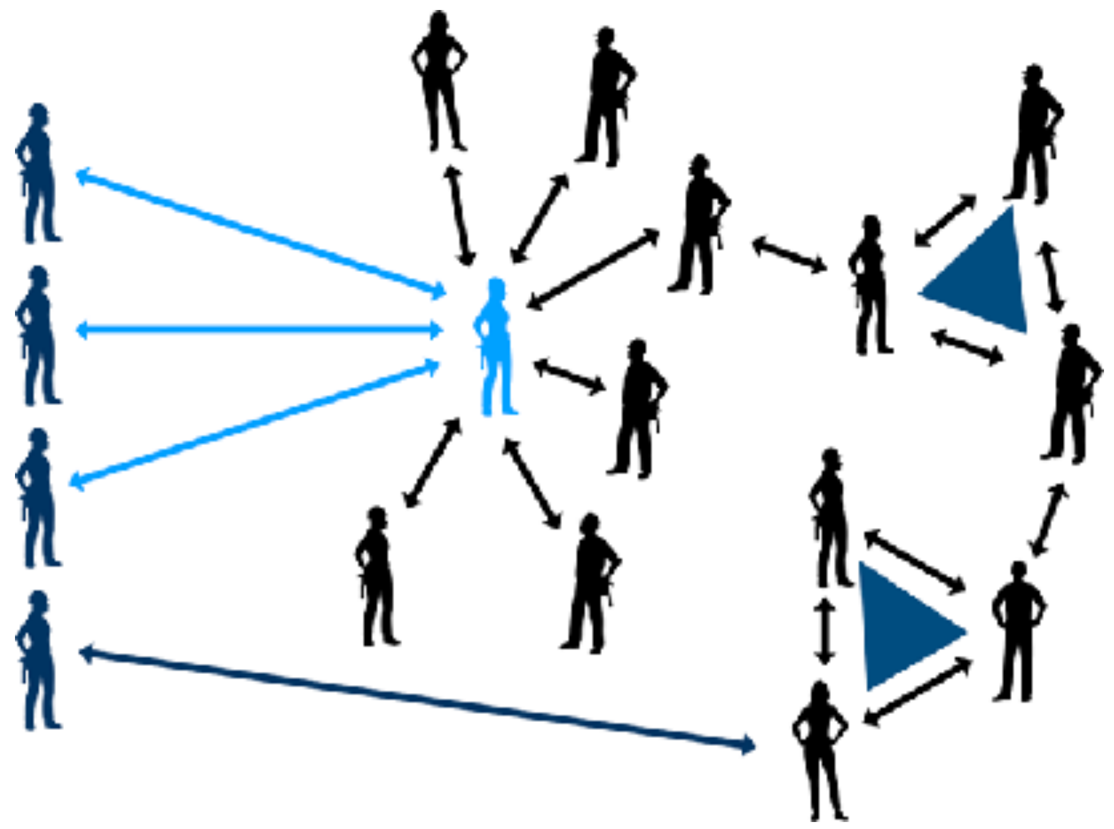
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.

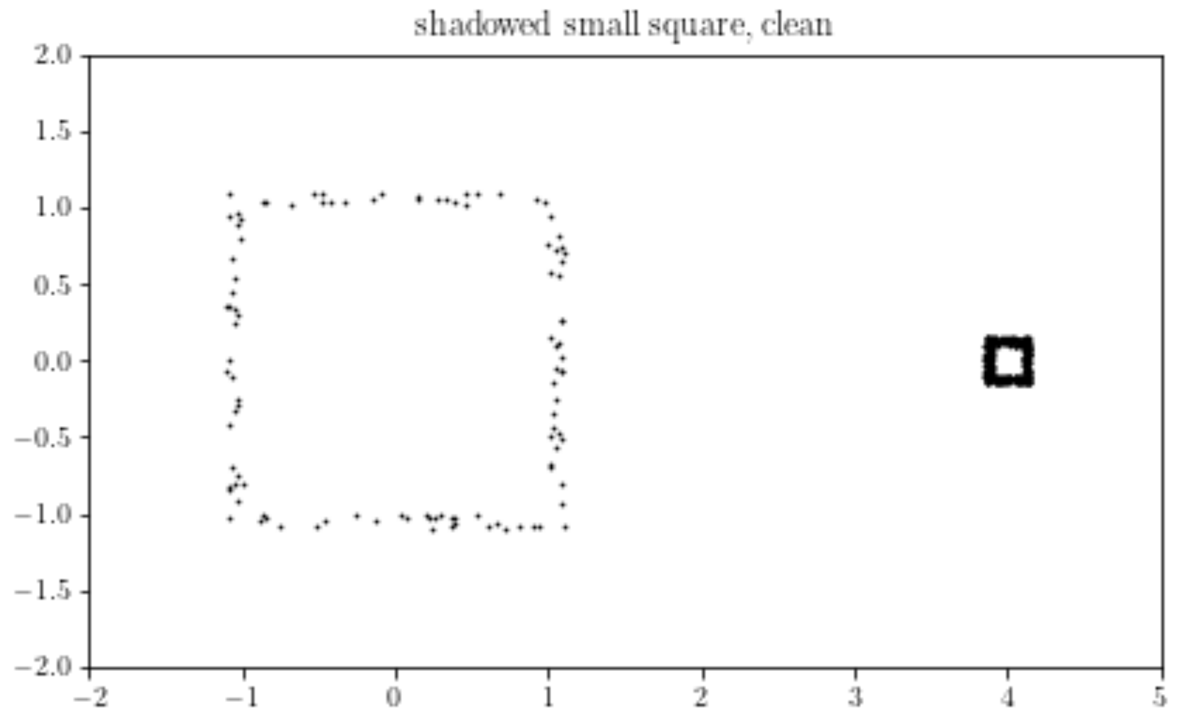
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

Chunyin Siu **Stanford University, School of Medicine**
siuc@stanford.edu



paper (homology, homotopy)

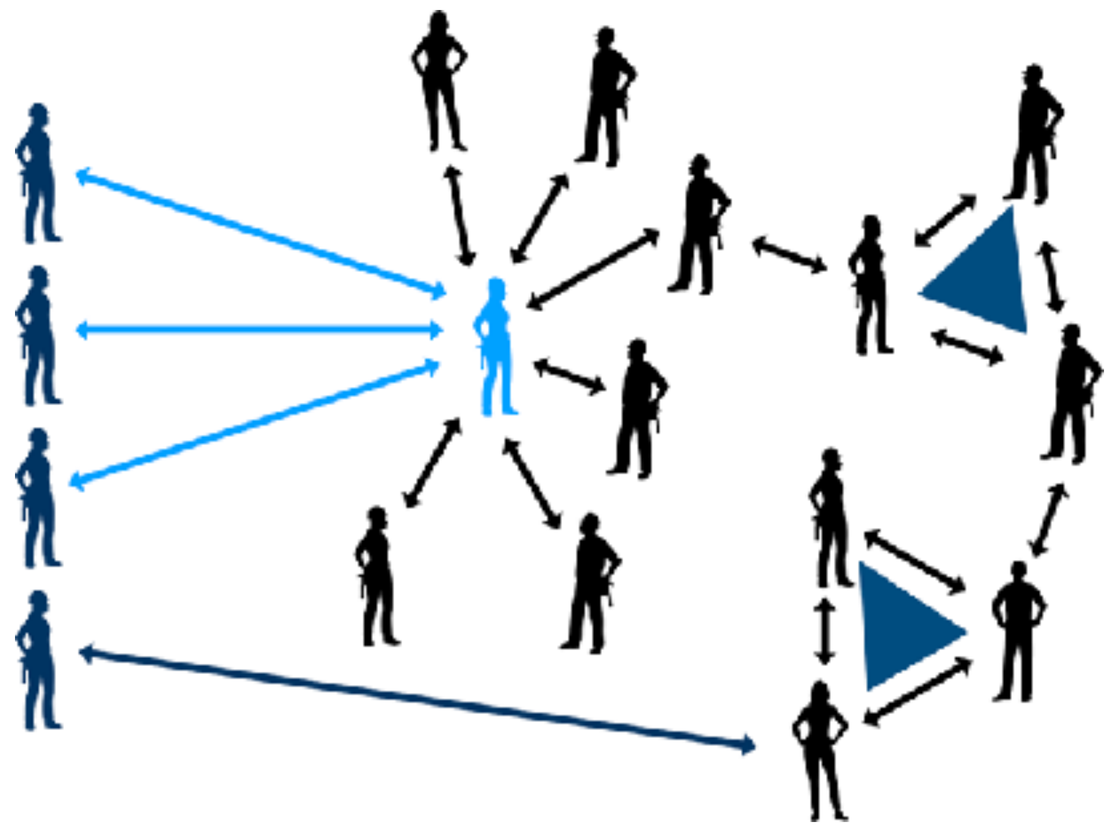


my video about small holes

Thank you!

Chunyin Siu   Stanford University, School of Medicine

siuc@stanford.edu



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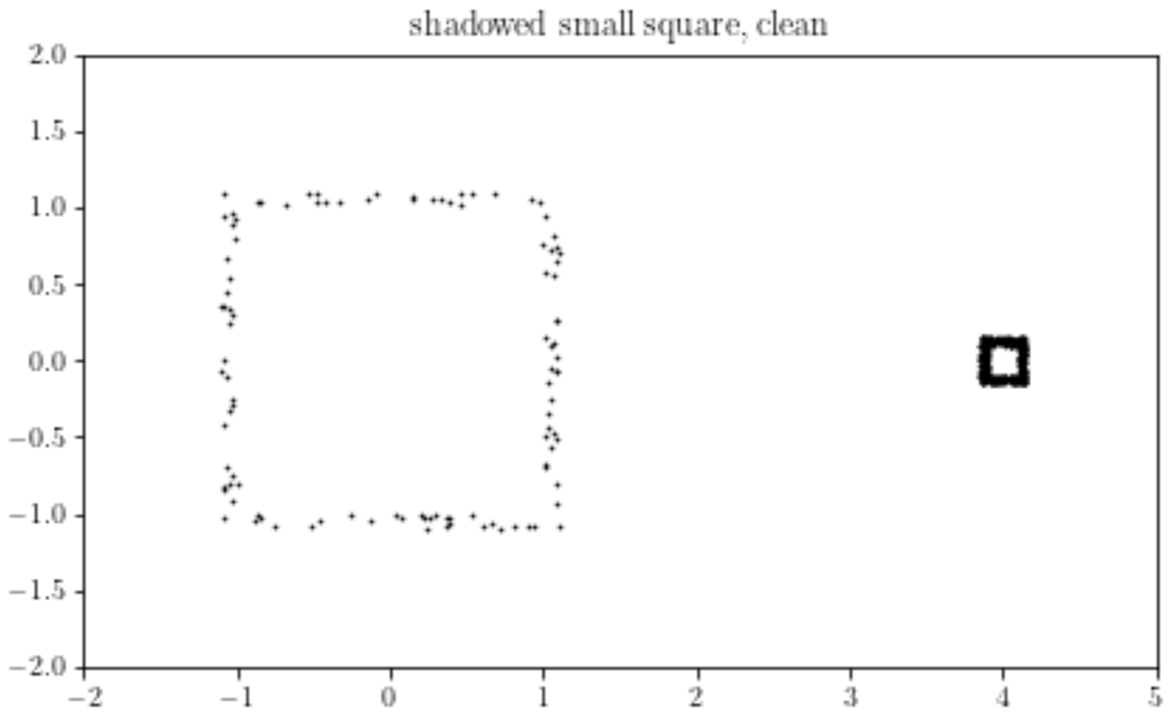
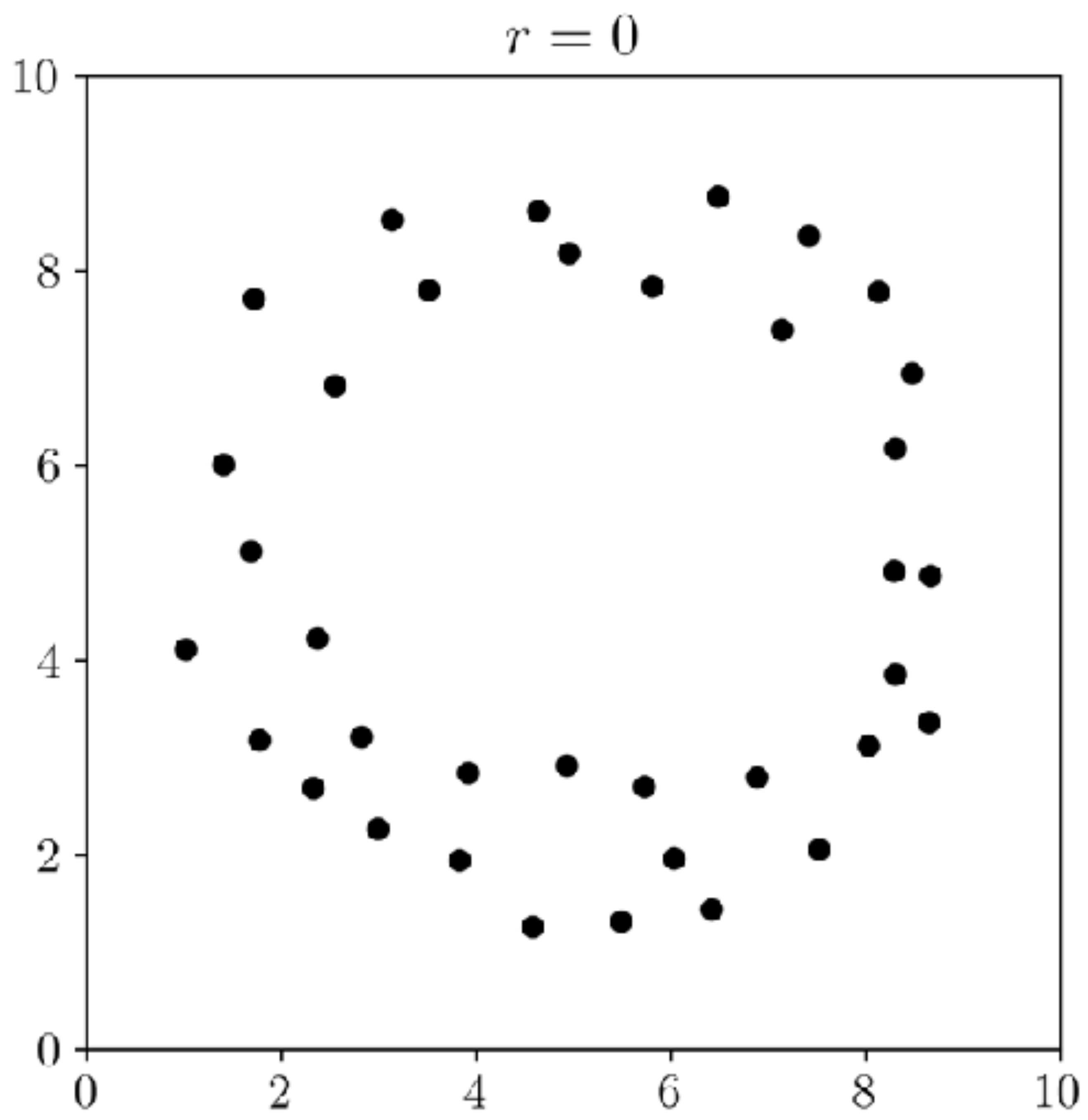
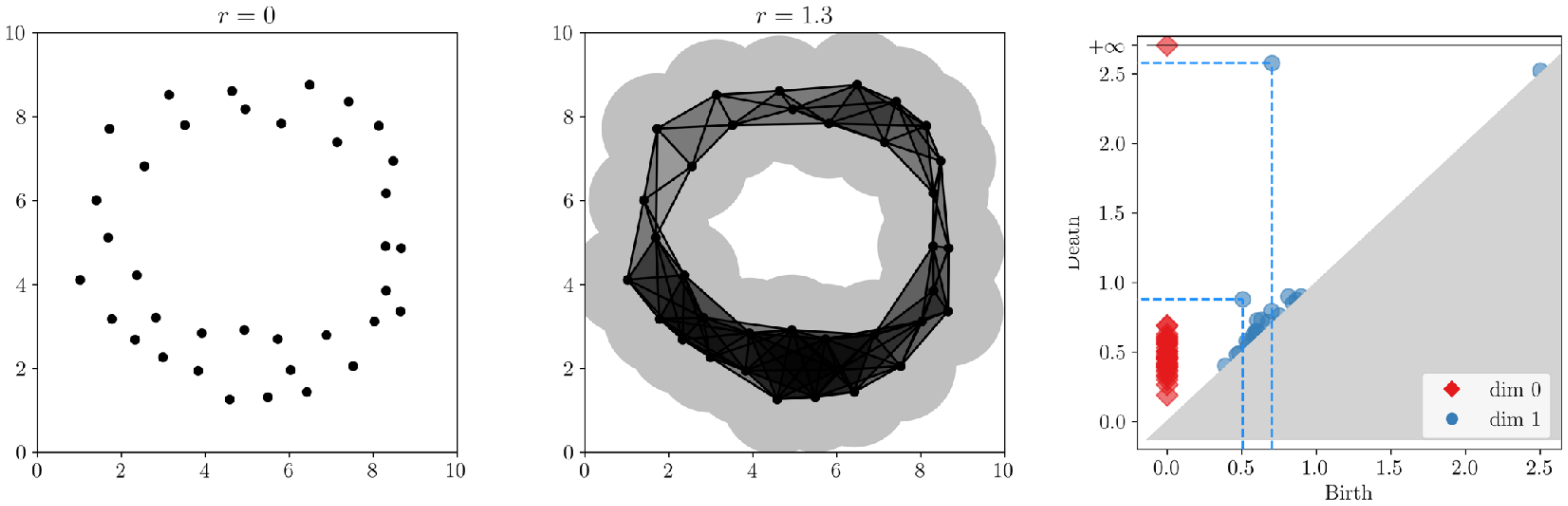


my video about small holes

I. A Probabilist’s Apology

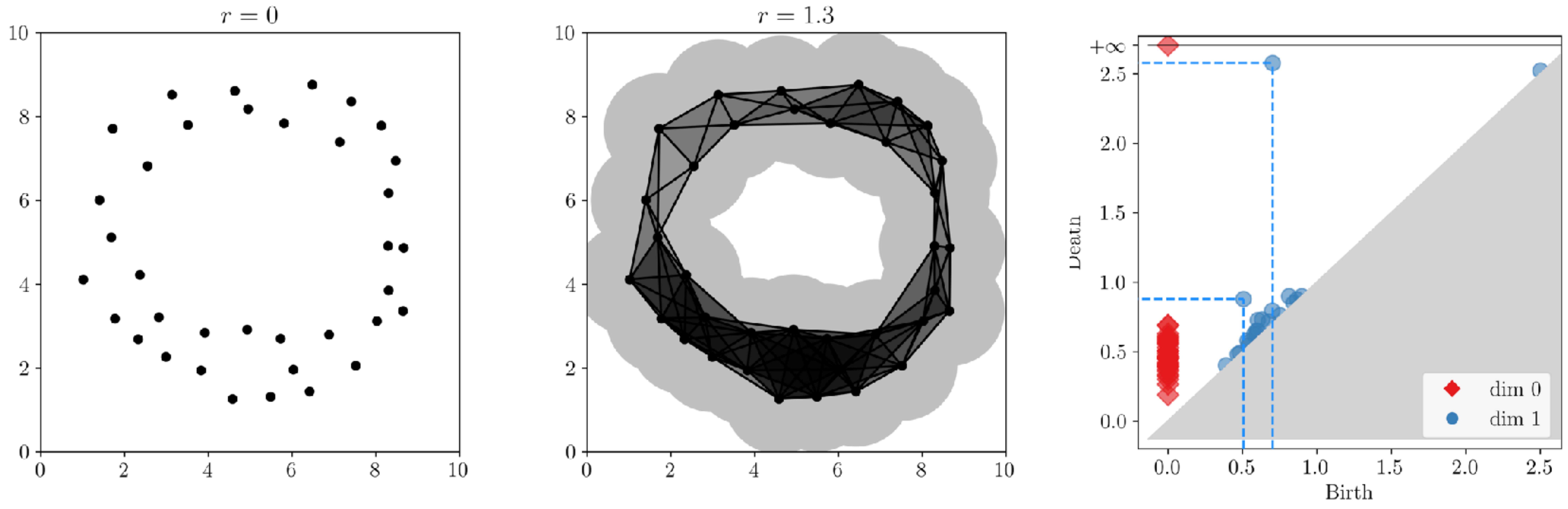
Why Random Topology


$r = 0$
0
2
4
6
8
10

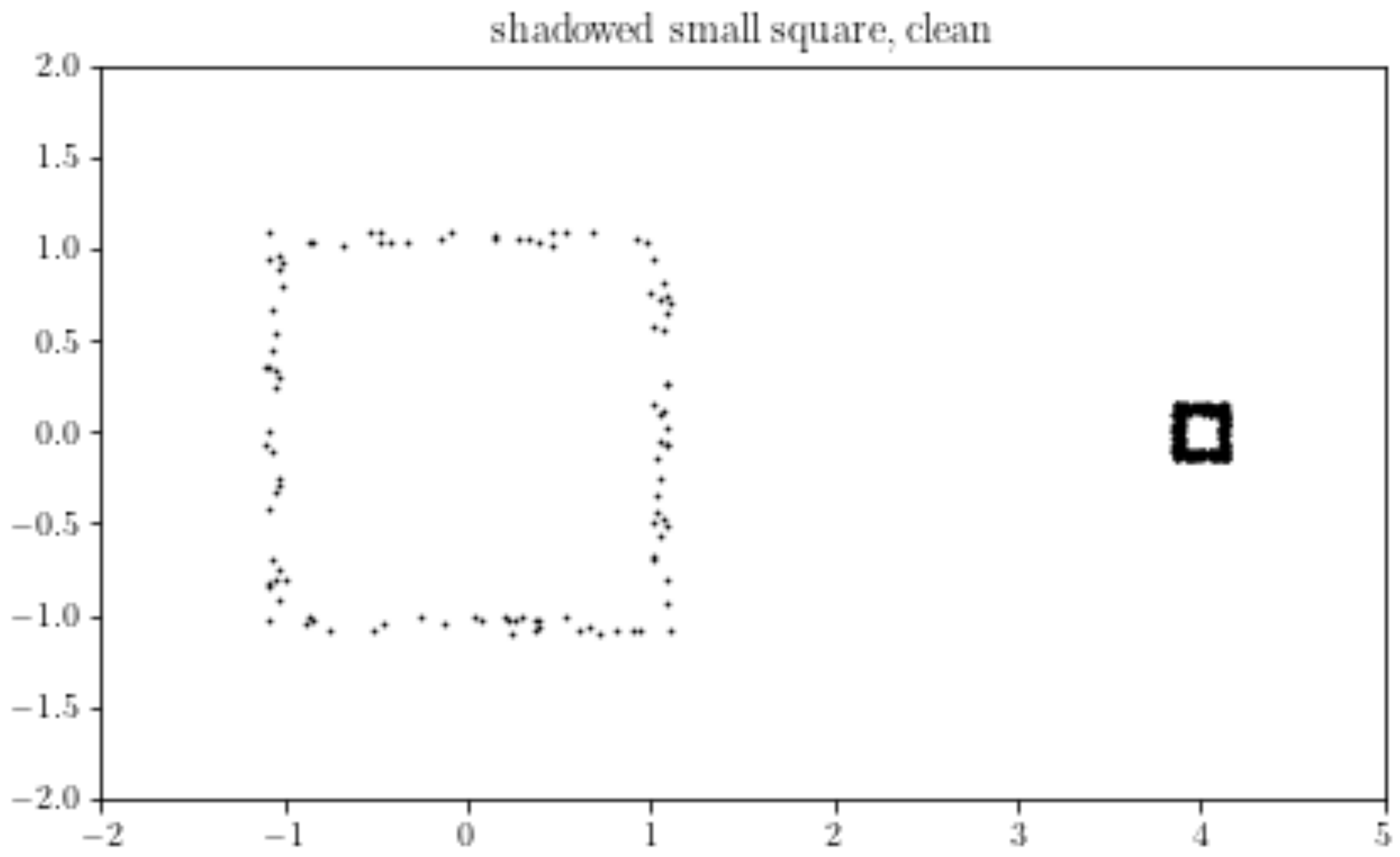


plots generated by Andrey Yao

Size is Signal

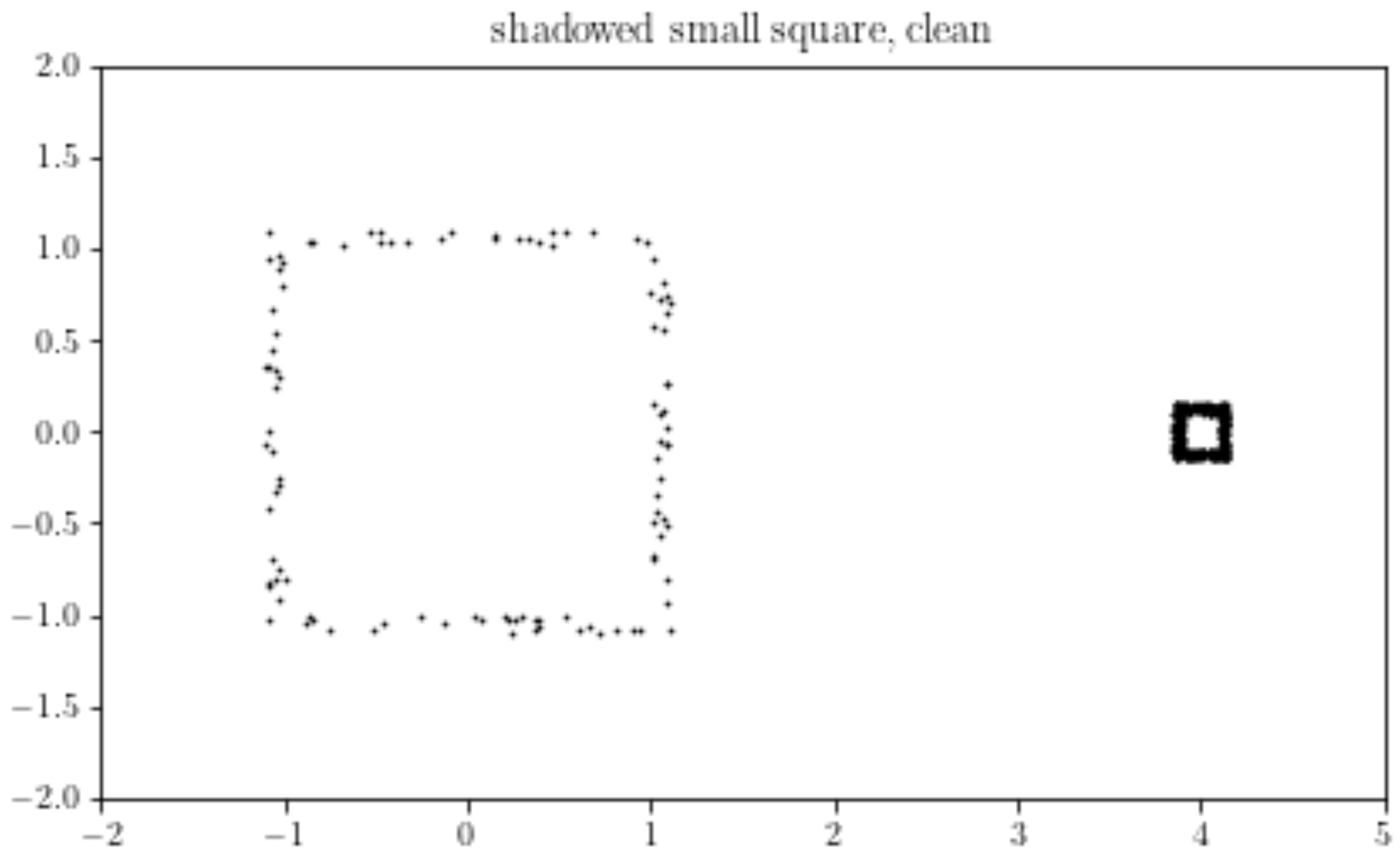


Or is it?

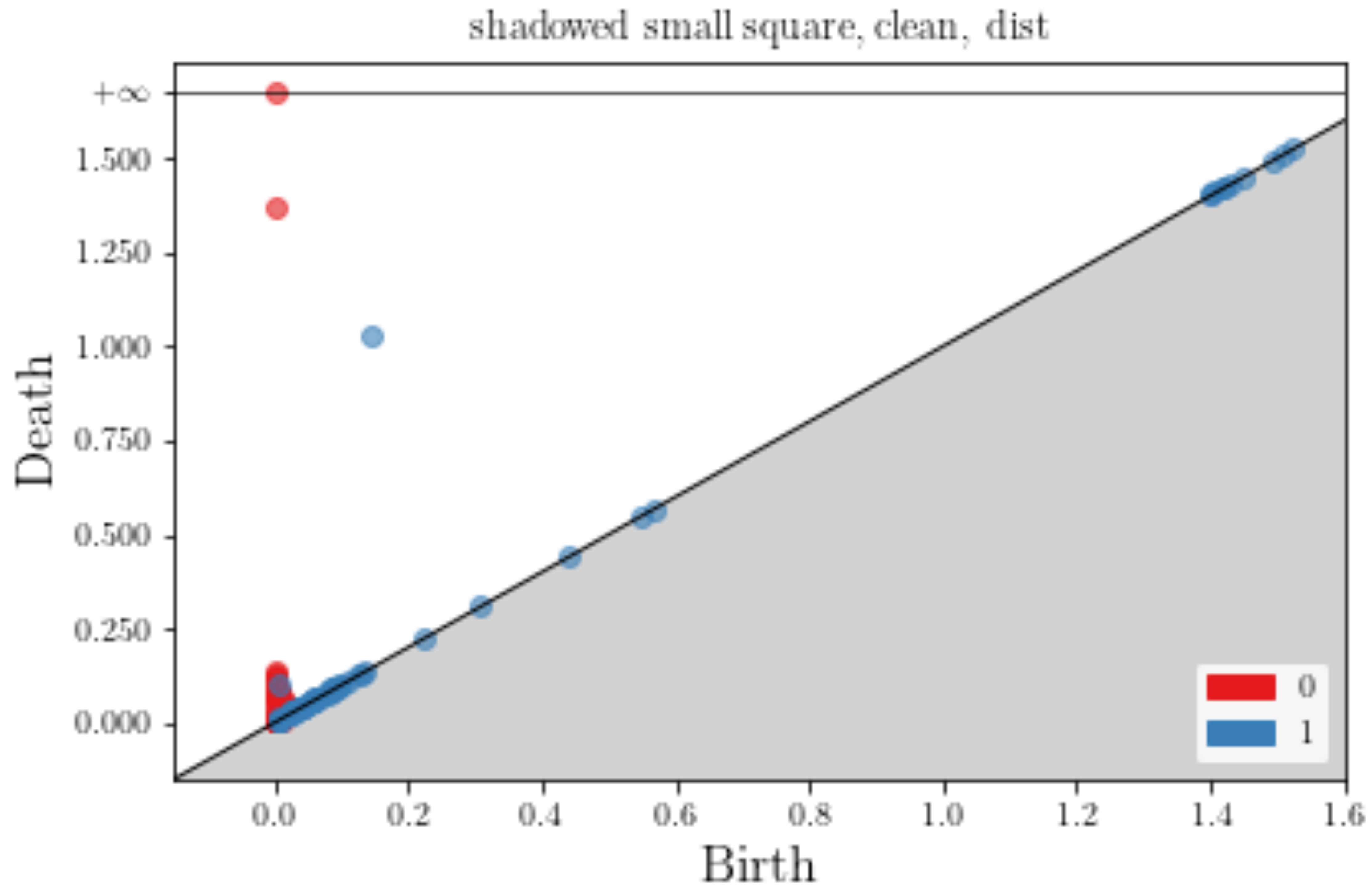


shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5

Or is it?



shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5



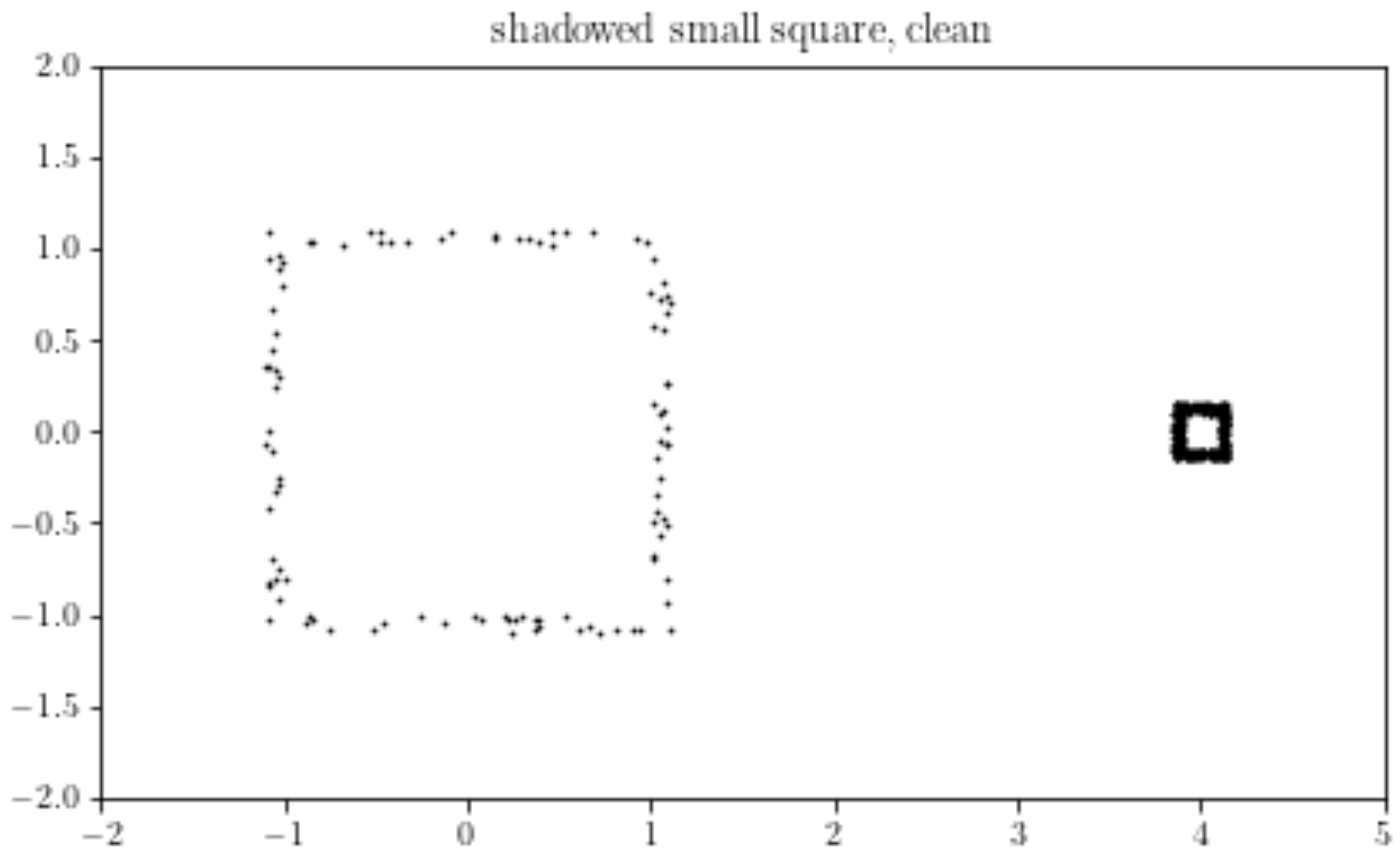
shadowed small square, clean, dist
+∞
1.500
1.250
1.000
0.750
0.500
0.250
0.000
Death
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Birth
0
1

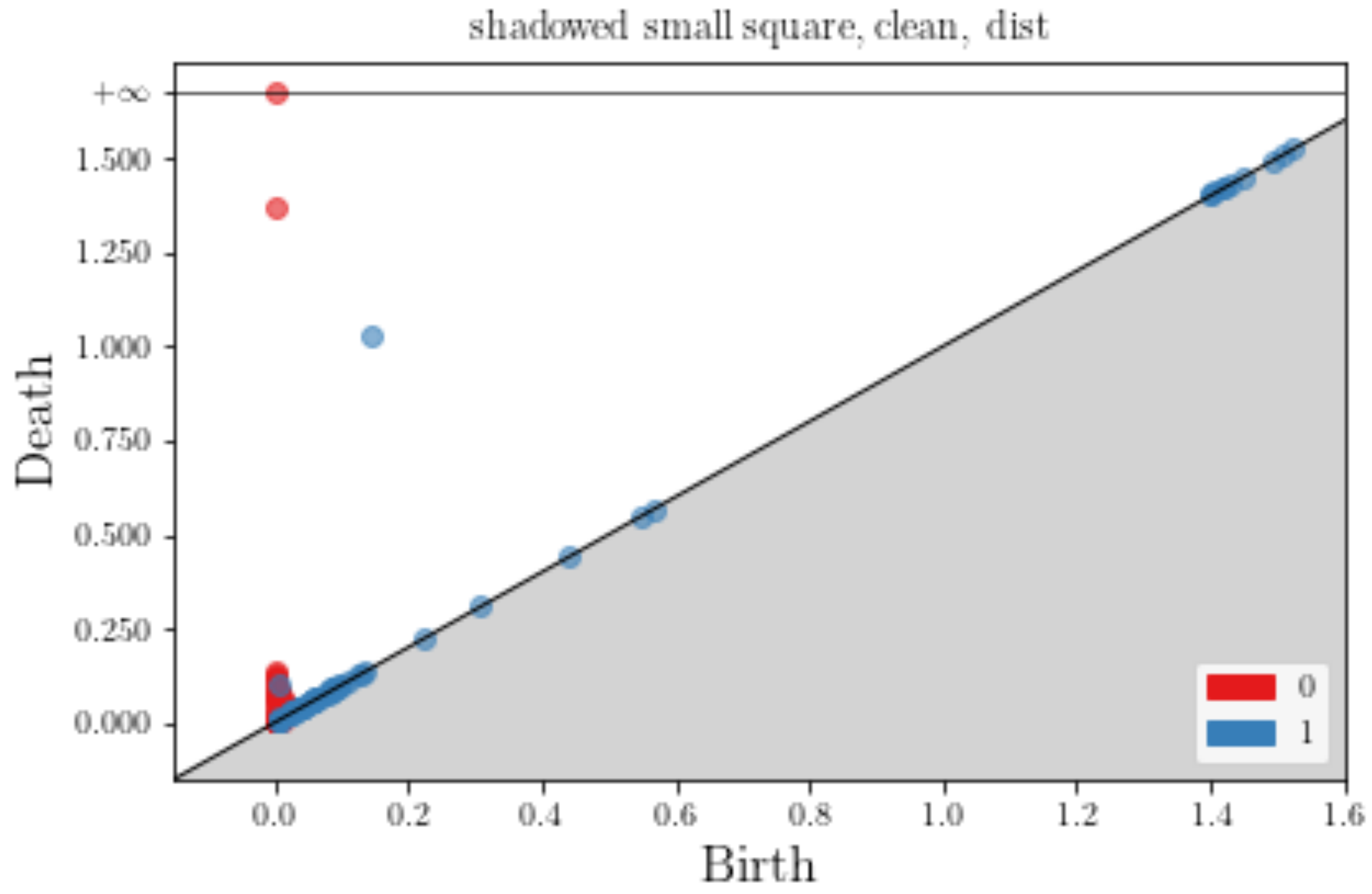
Size is Signal?

Surprise

~~Size~~ is Signal.

Random points don’t do that.





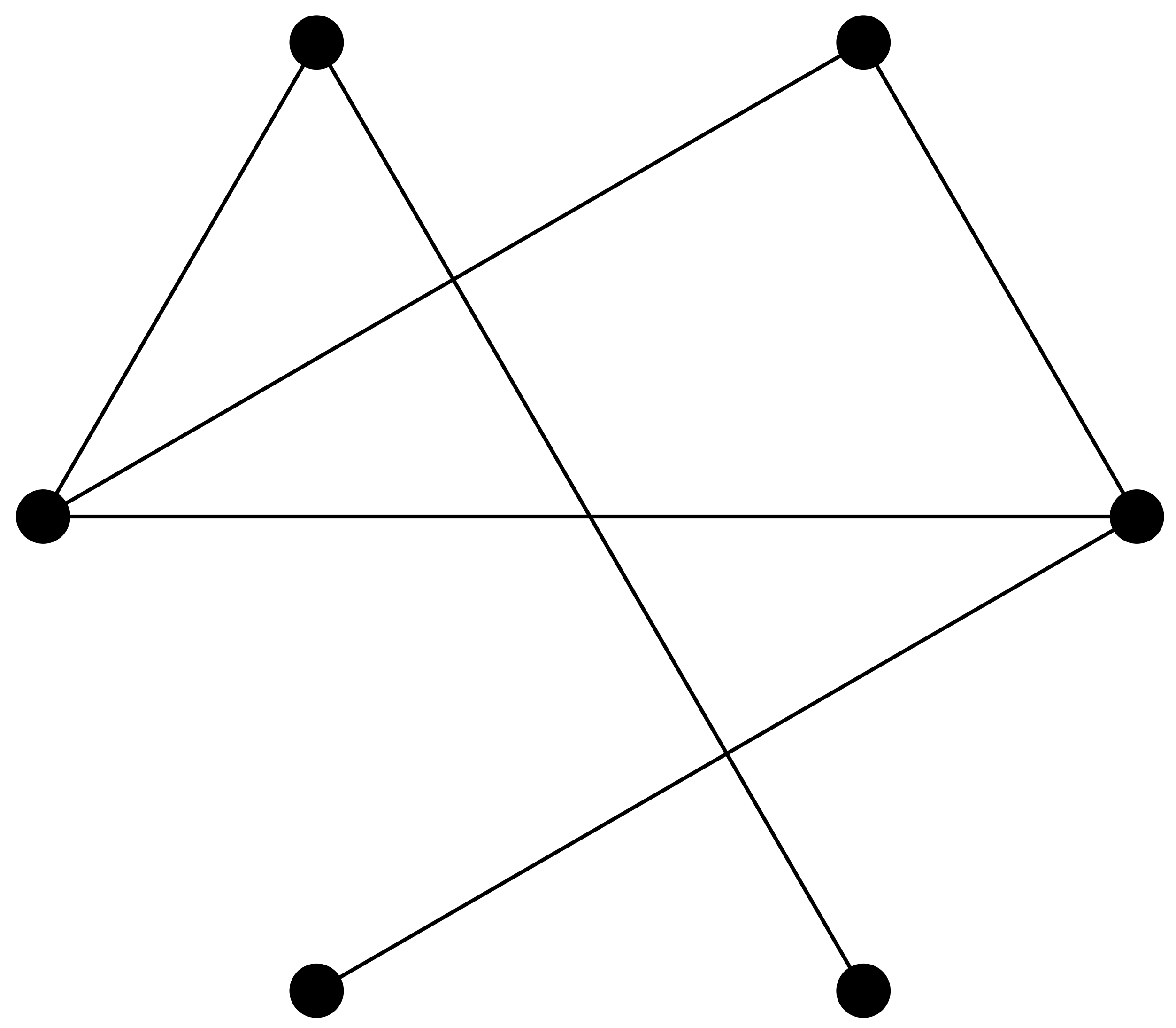
Signal is what is not random.

Signal is what is not **random.**
So what is **random?**

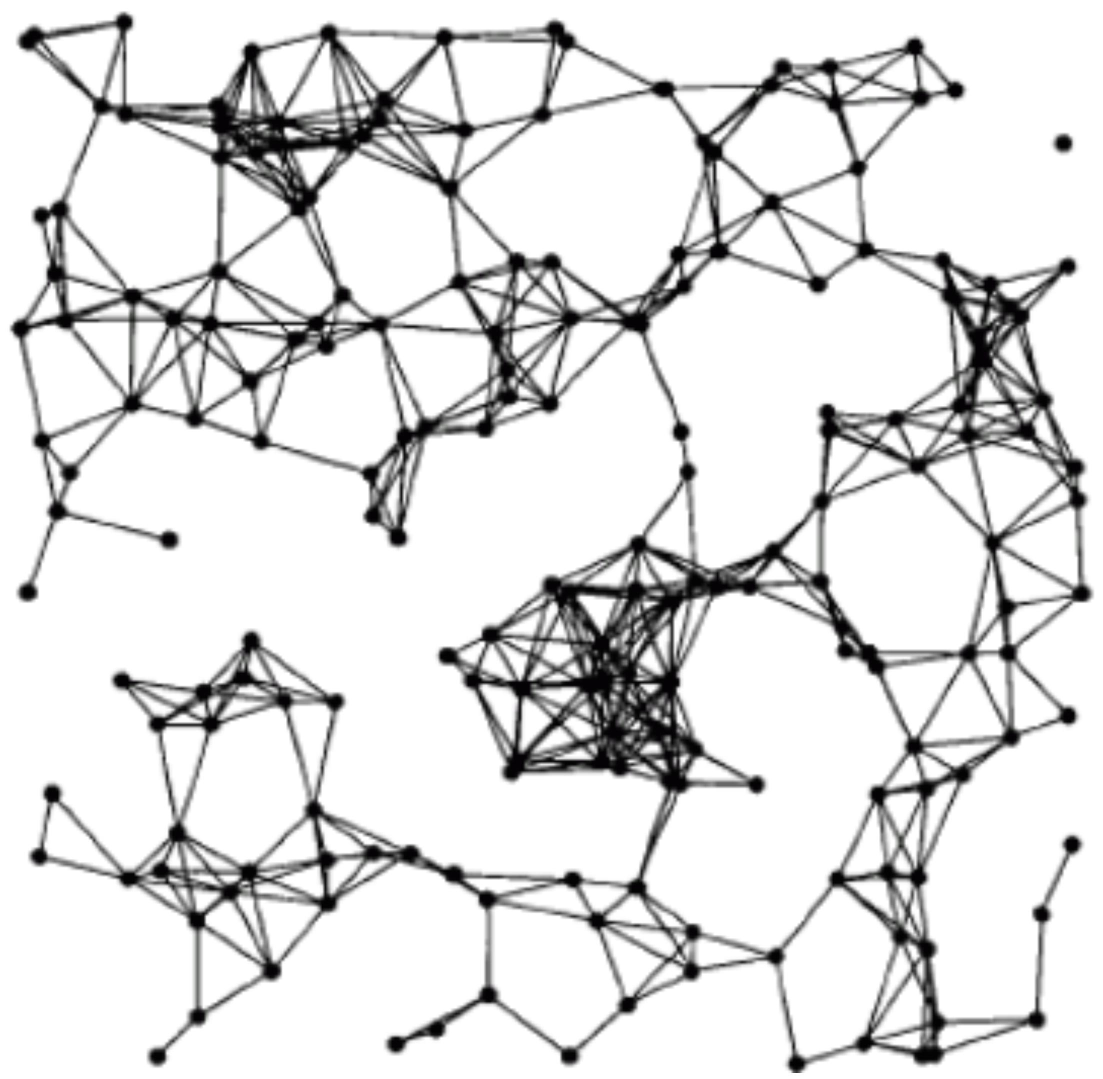
II. Random Walk in the Literature

What Random Topologists Already Know

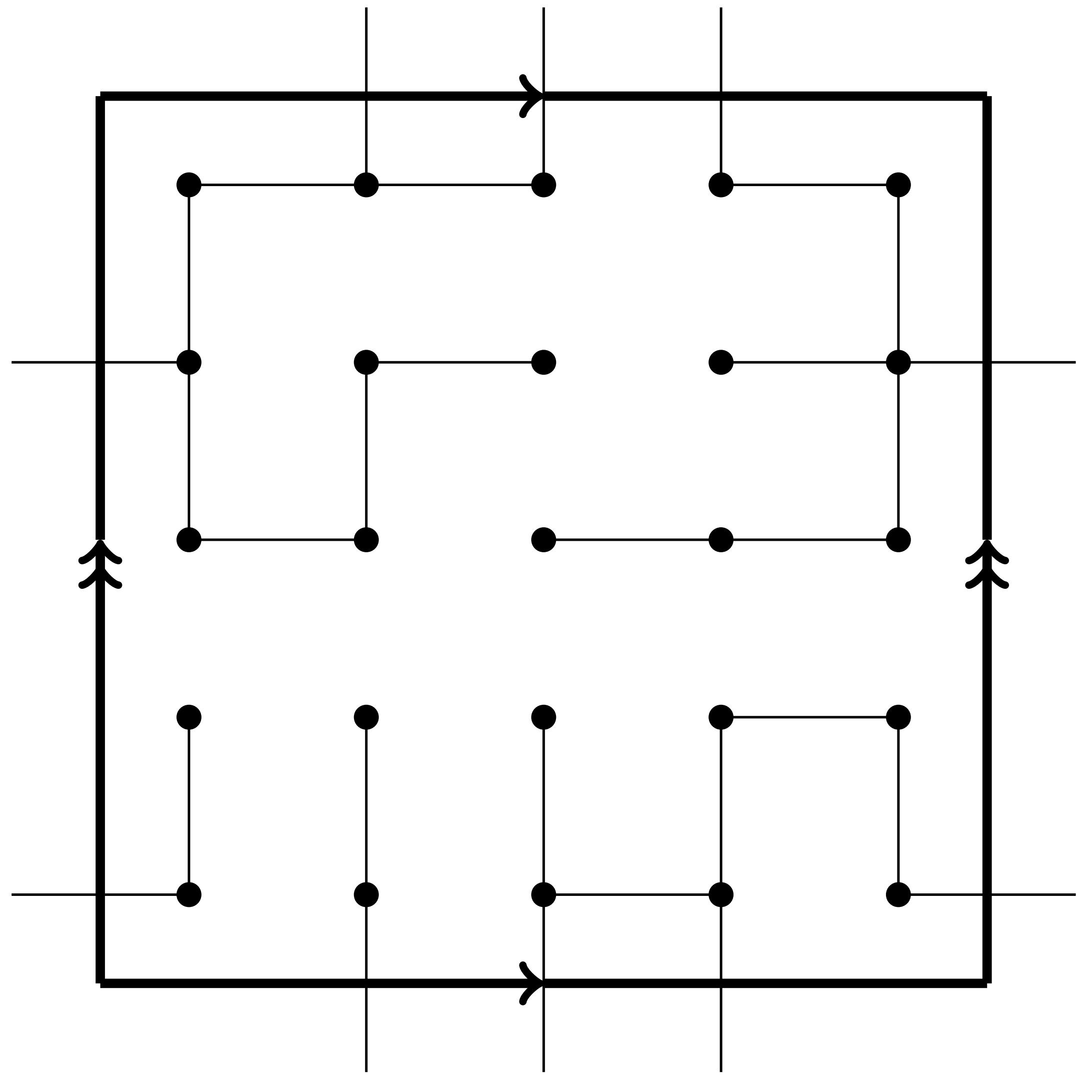
Tapas at Random Topology



Erdo-Renyi Complexes

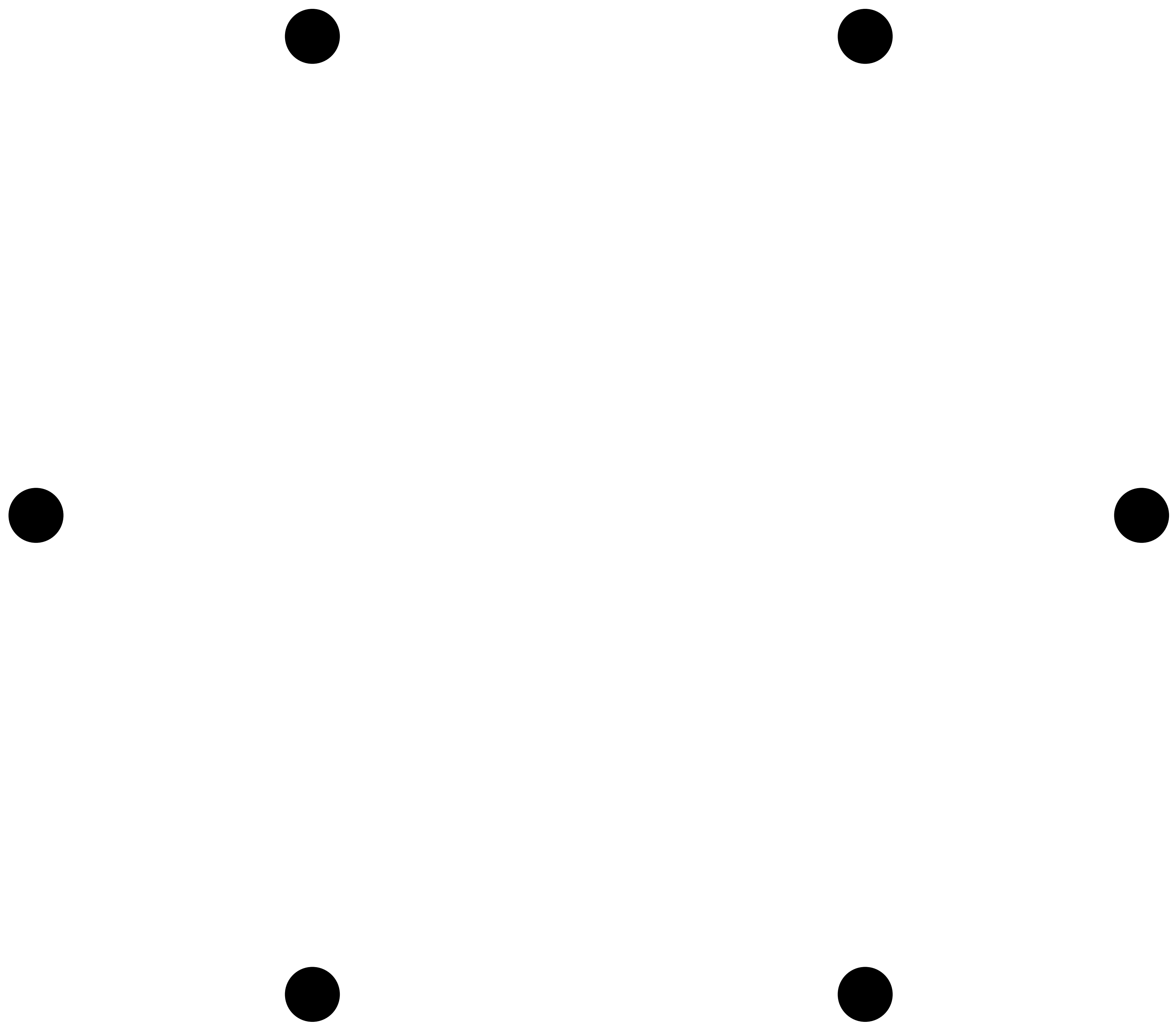


Geometric Complexes

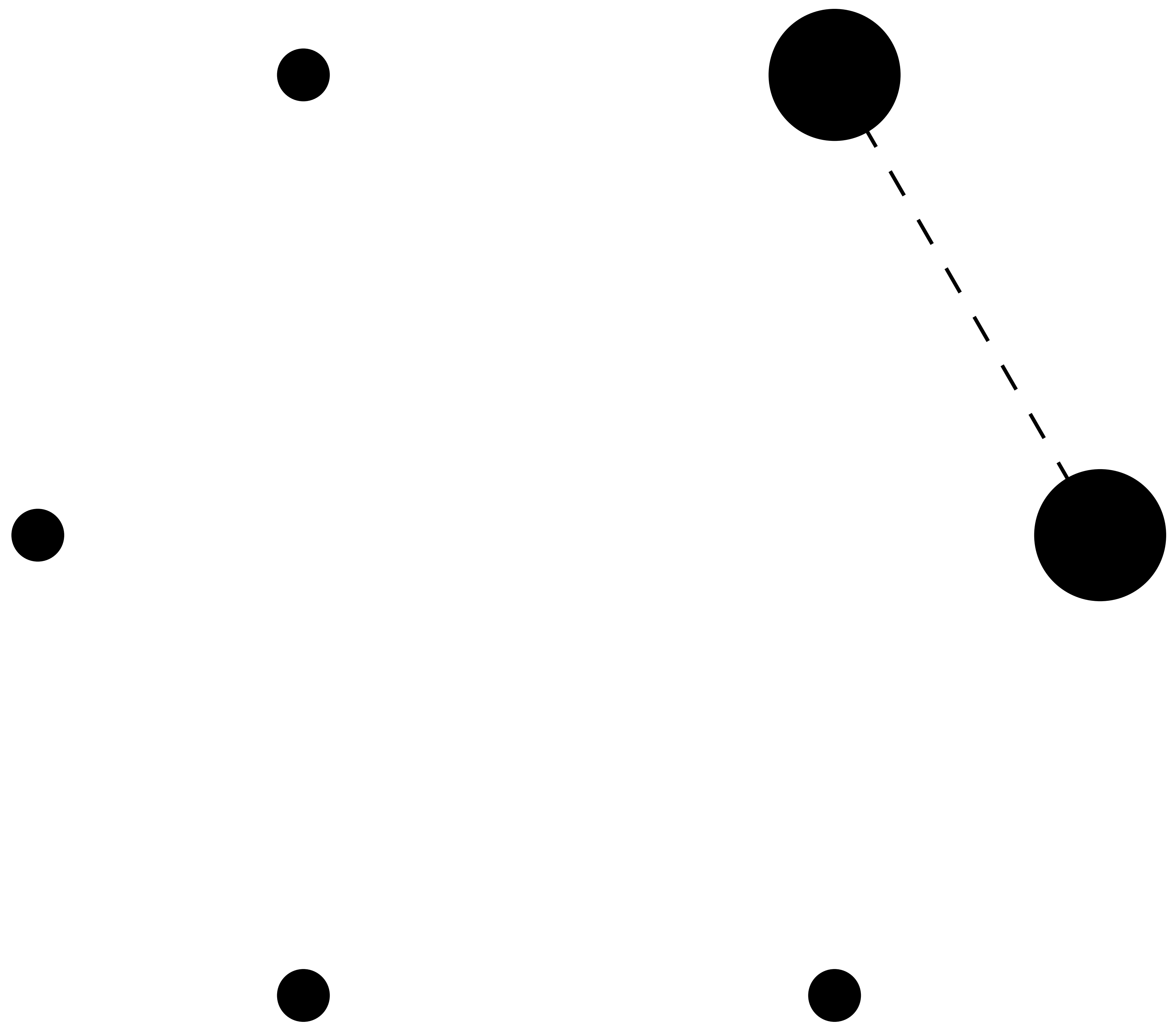


Topological Percolation

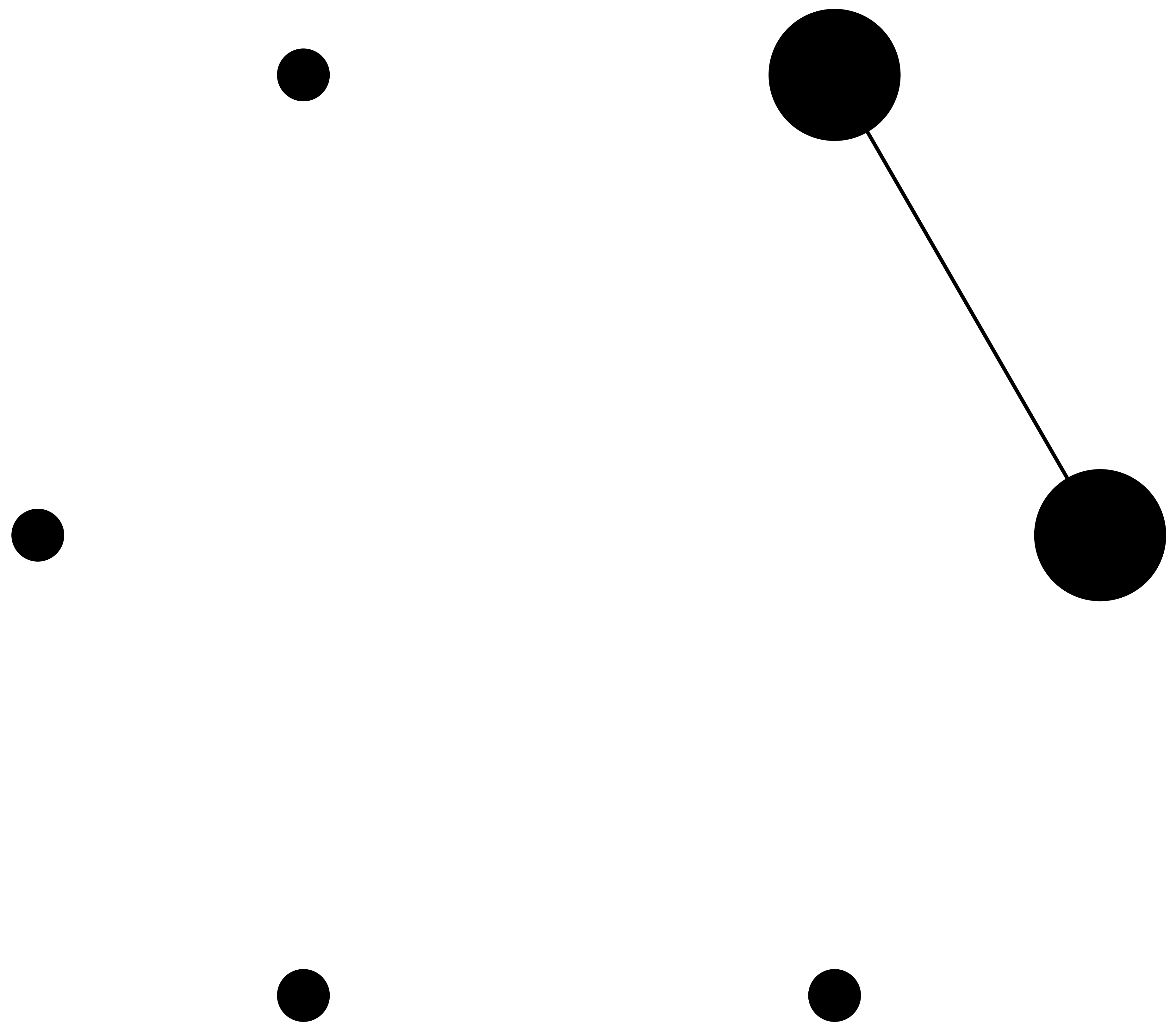
Erdos-Renyi graphs



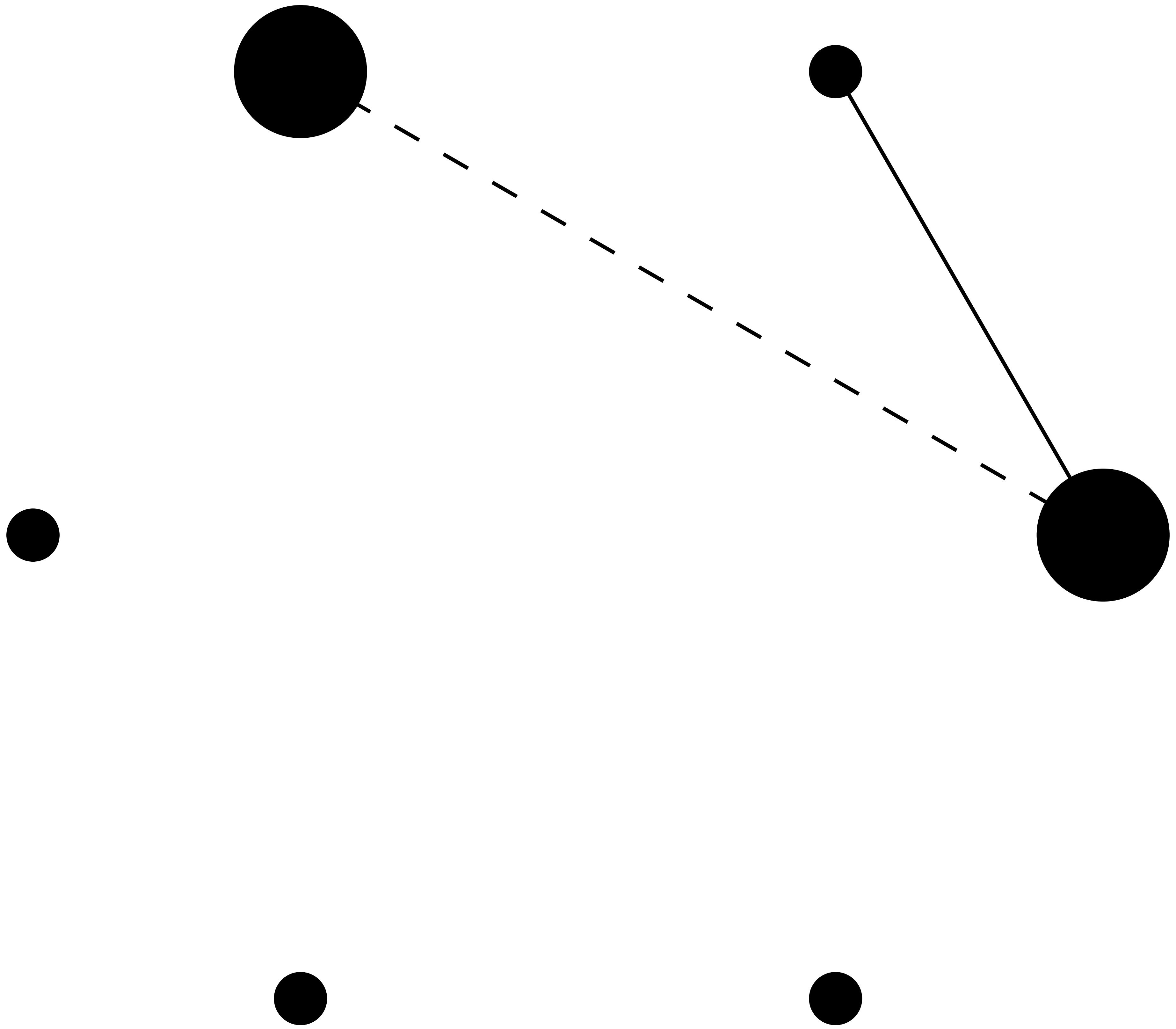
Erdos-Renyi graphs



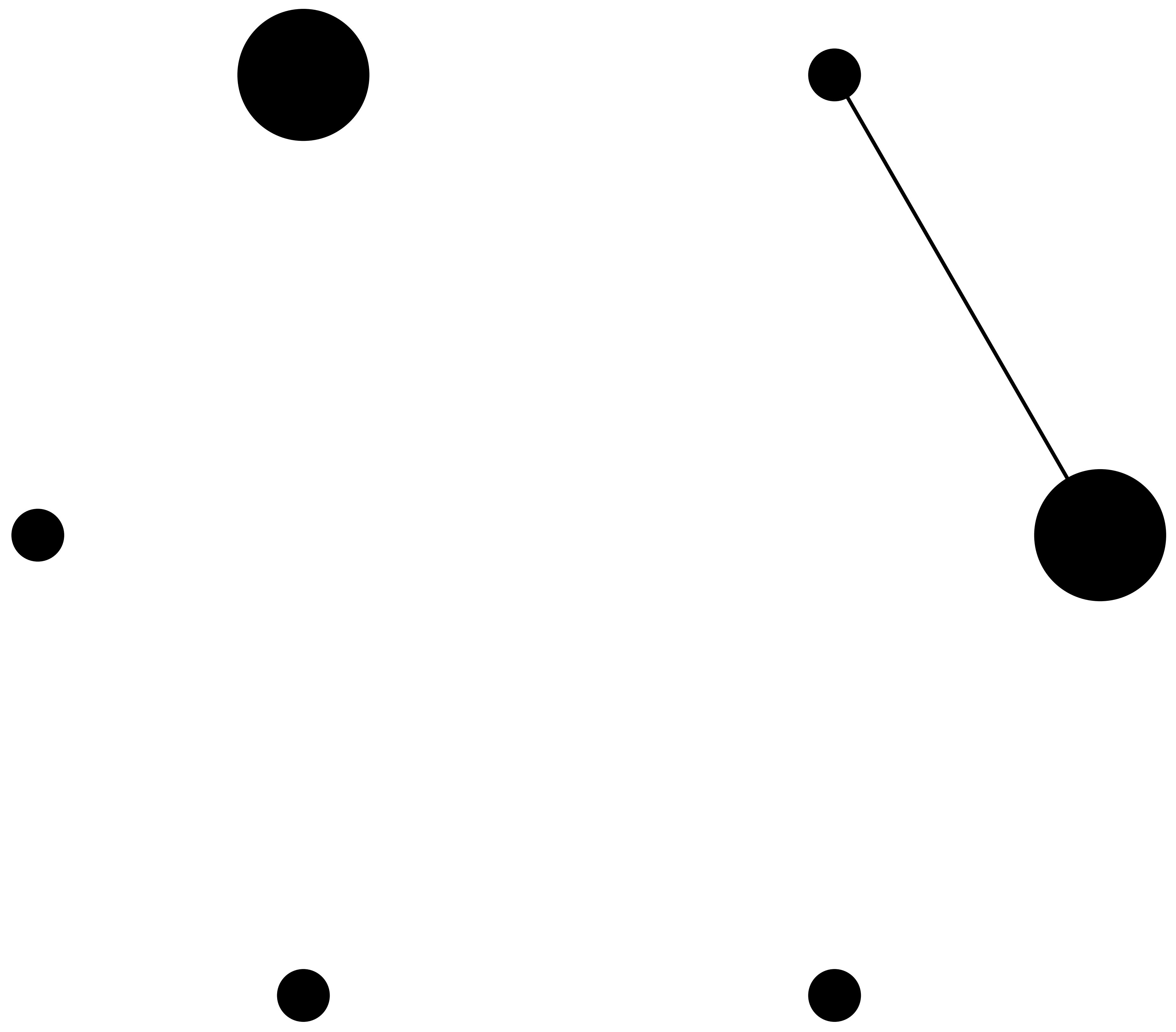
Erdos-Renyi graphs



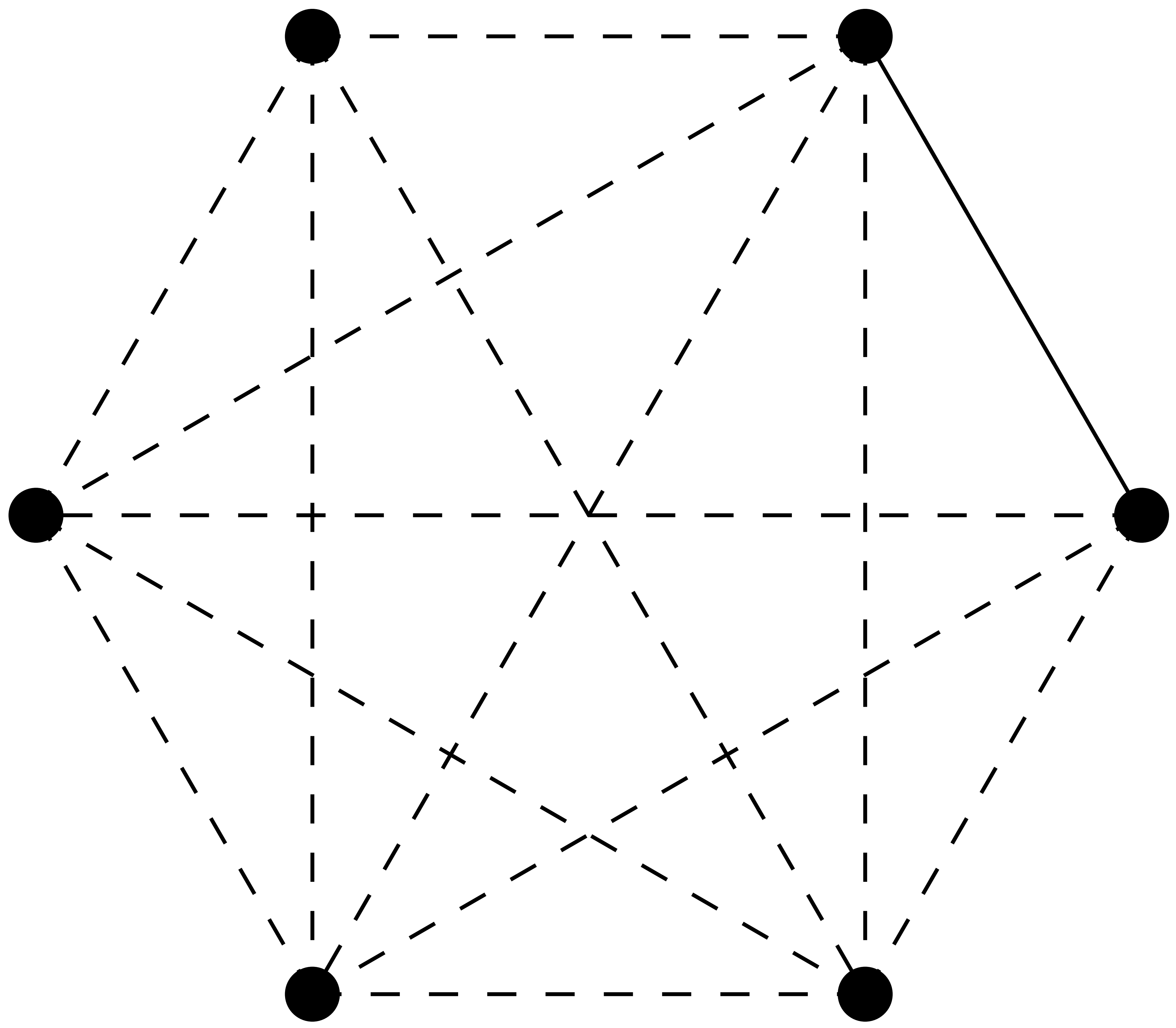
Erdos-Renyi graphs



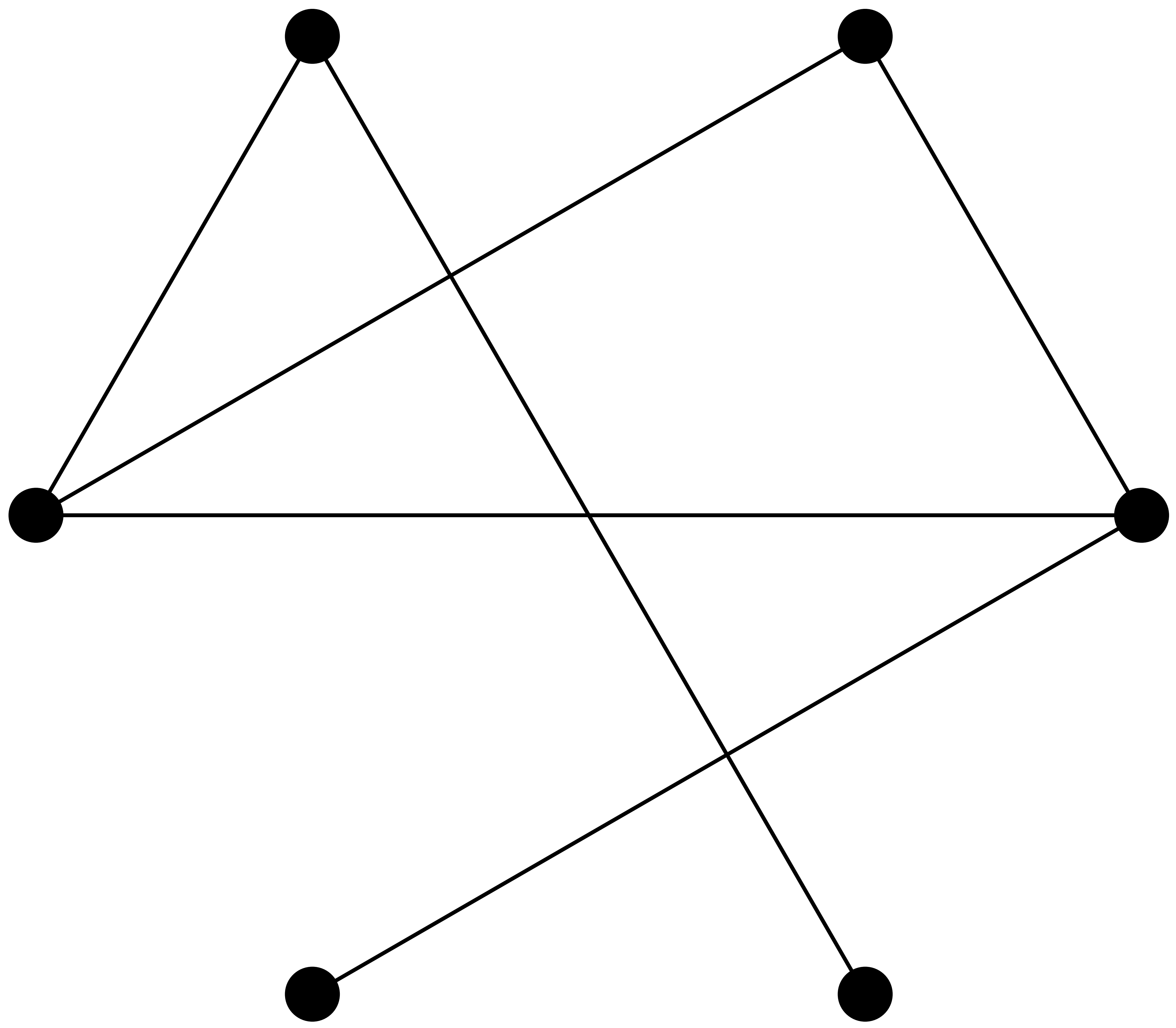
Erdos-Renyi graphs



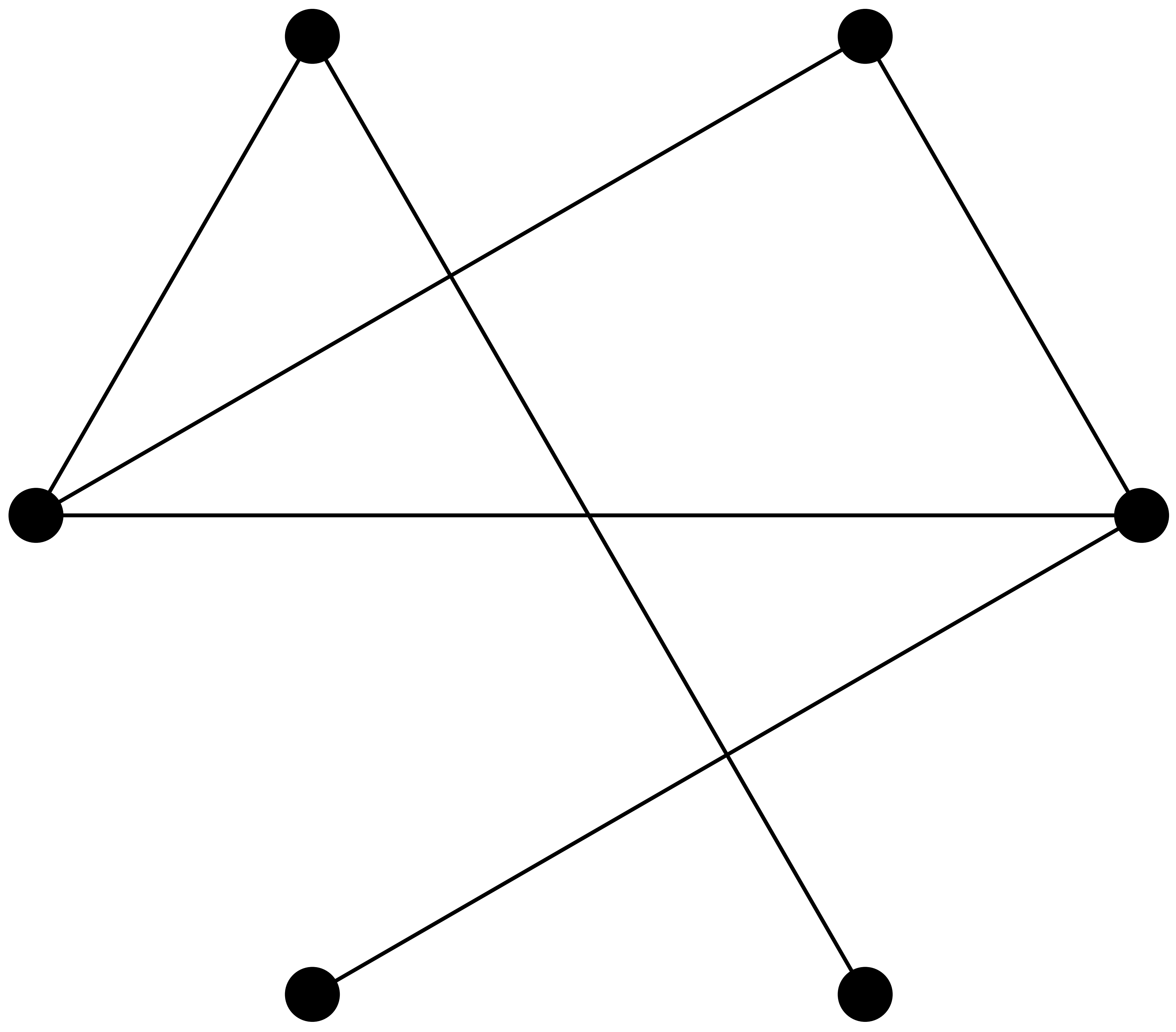
Erdos-Renyi graphs



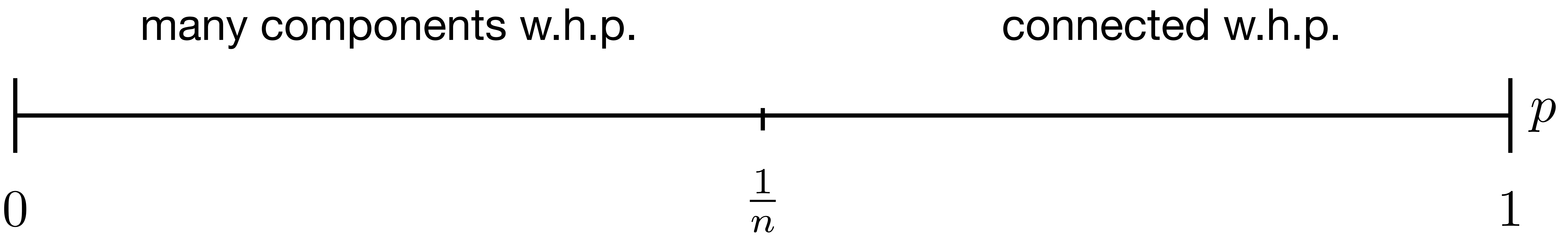
Erdos-Renyi graphs



Erdos-Renyi graphs

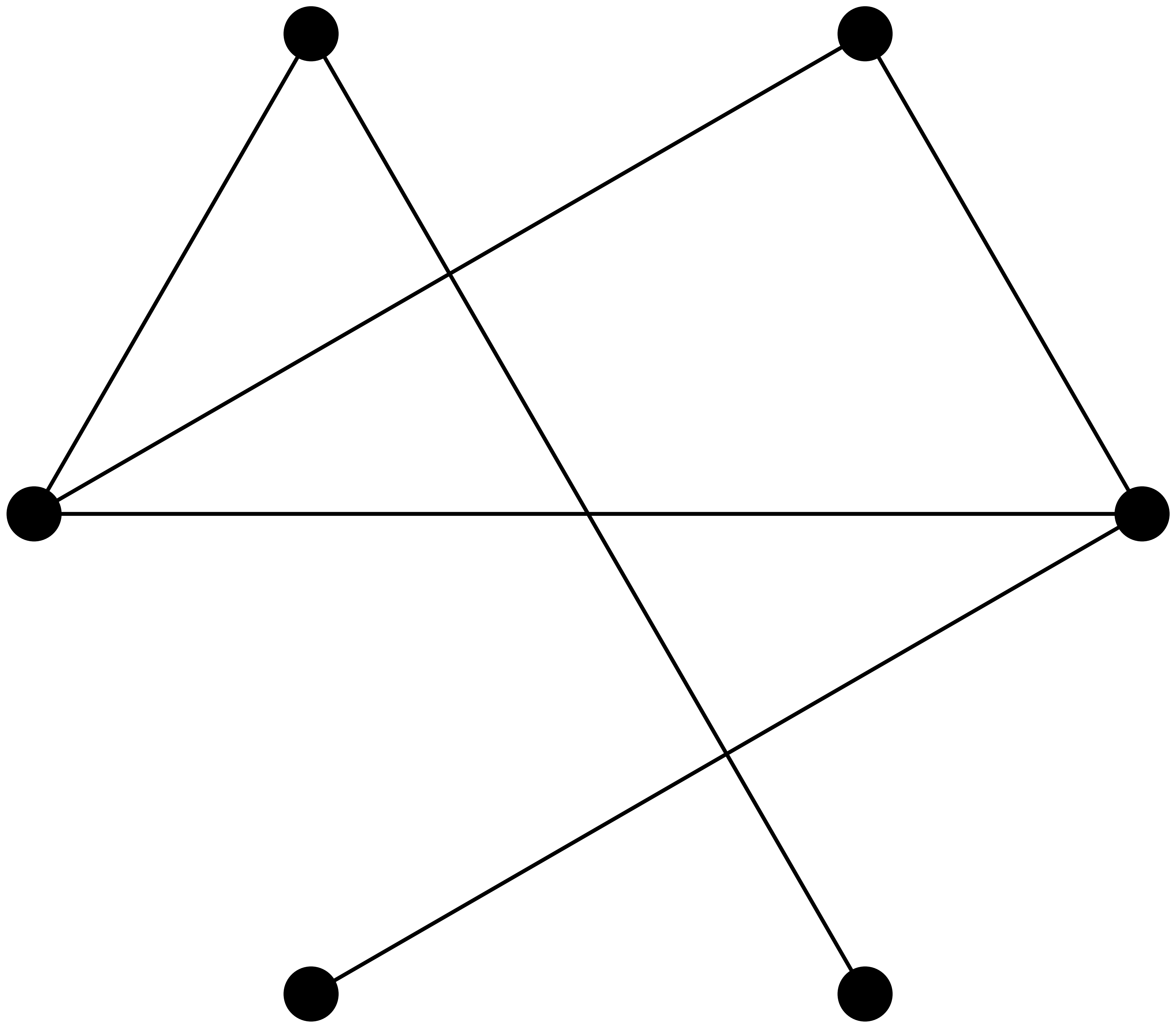


Phase Transition
[Erdos-Renyi 1960]

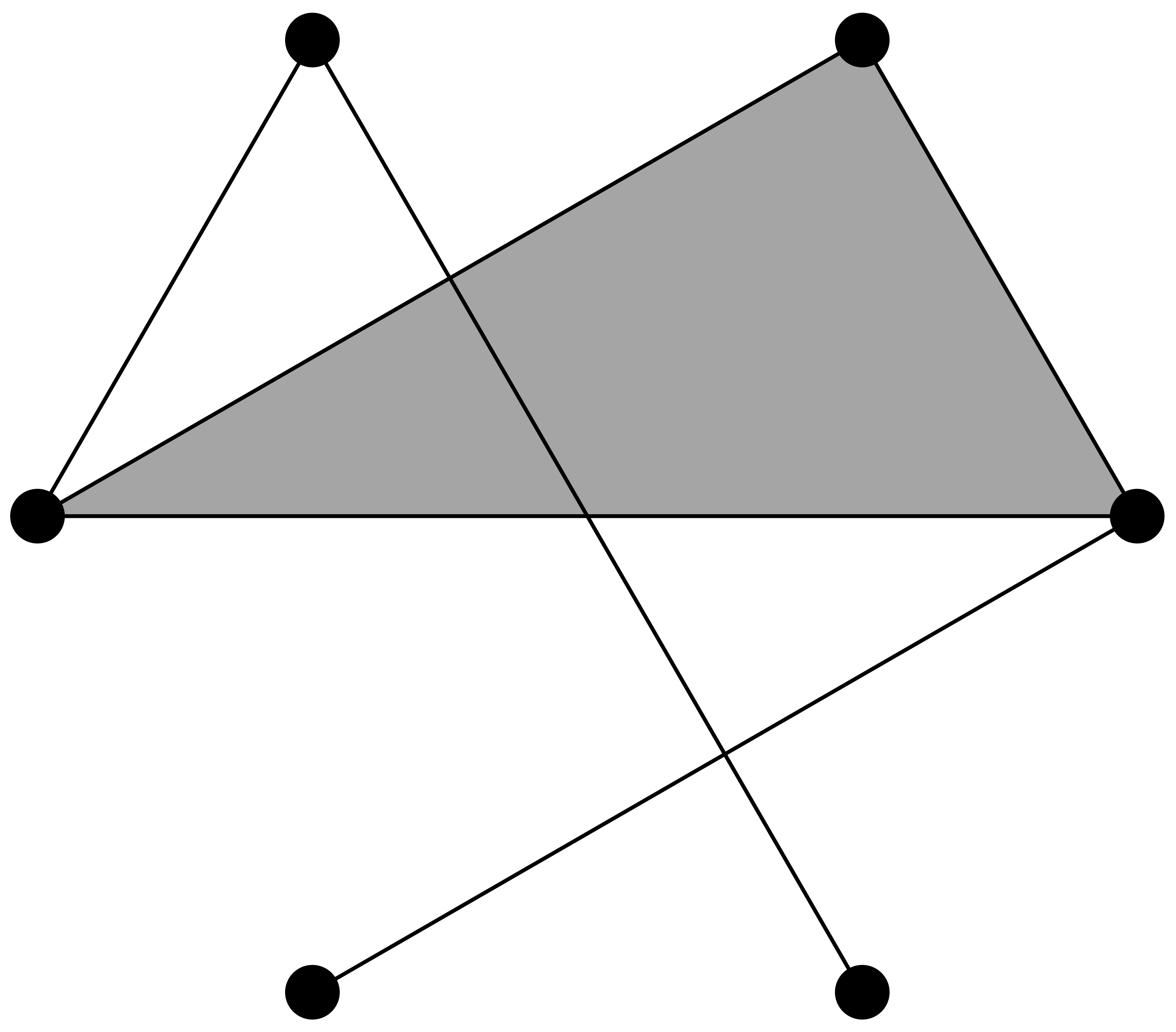


all log terms and constants forgone

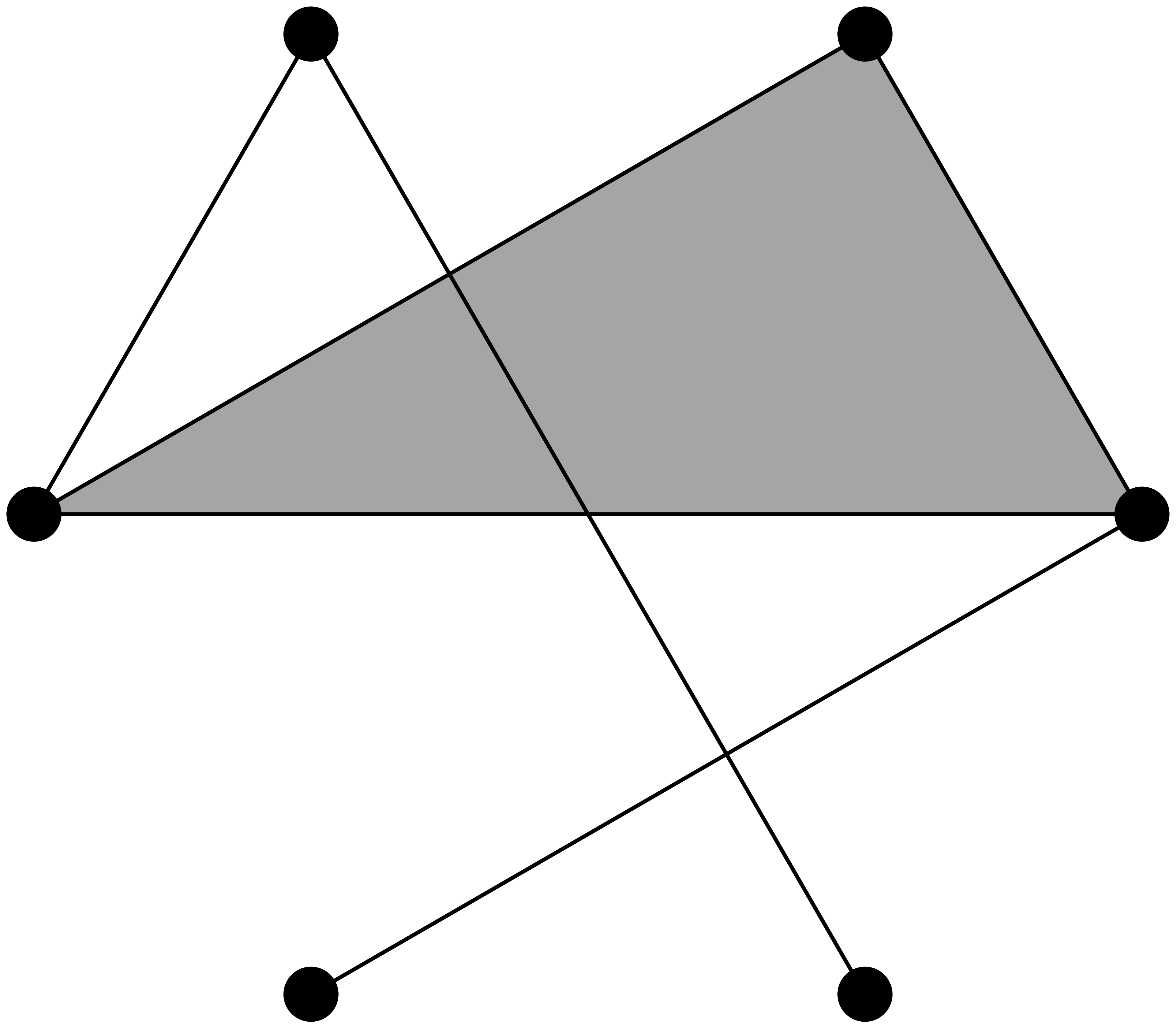
Erdos-Renyi Clique Complex

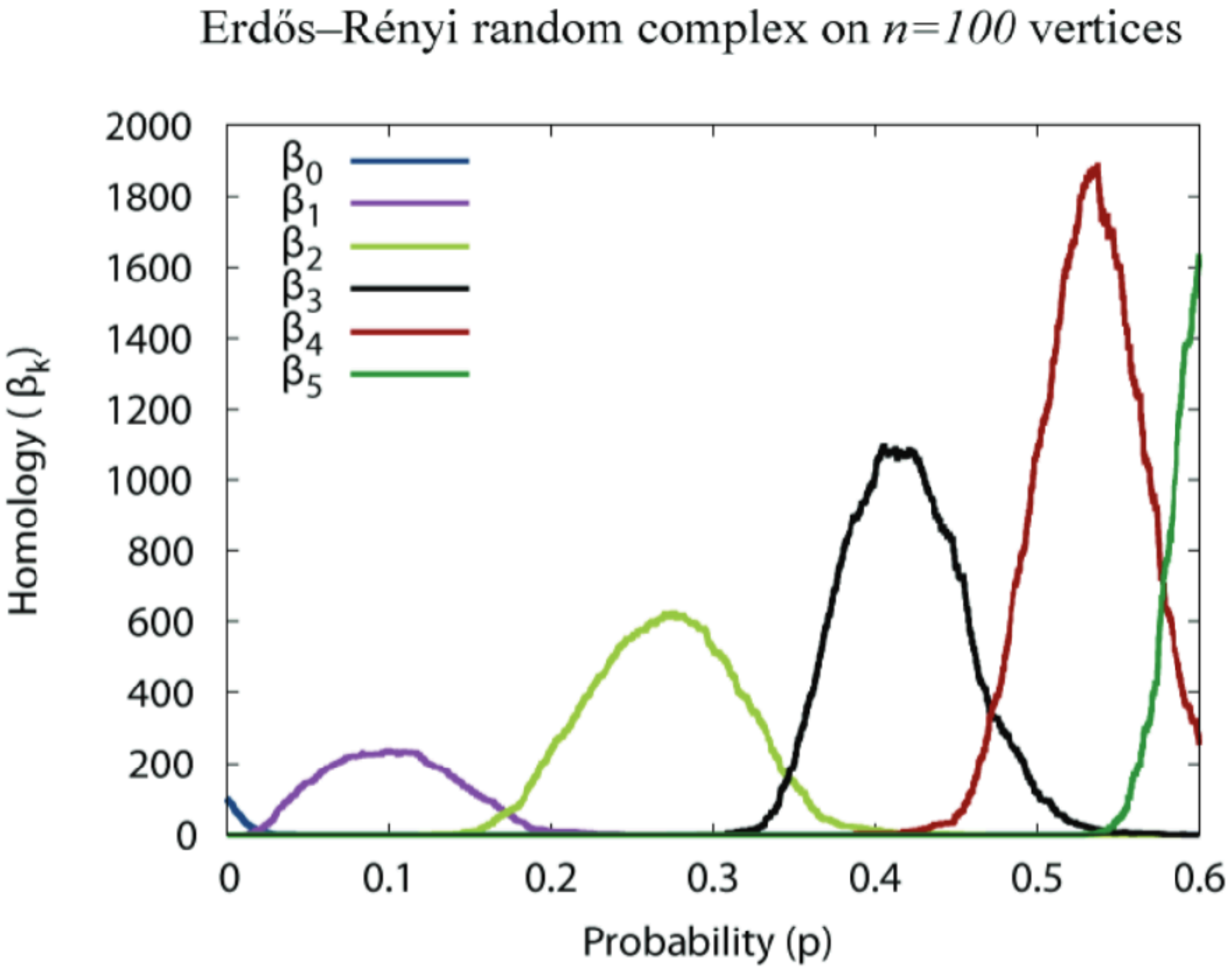


Erdos-Renyi Clique Complex



Betti Numbers

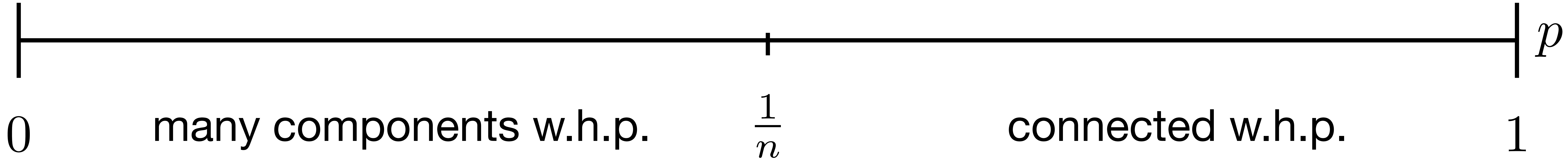




computation and plotting done by Zomorodian

Phase Transition

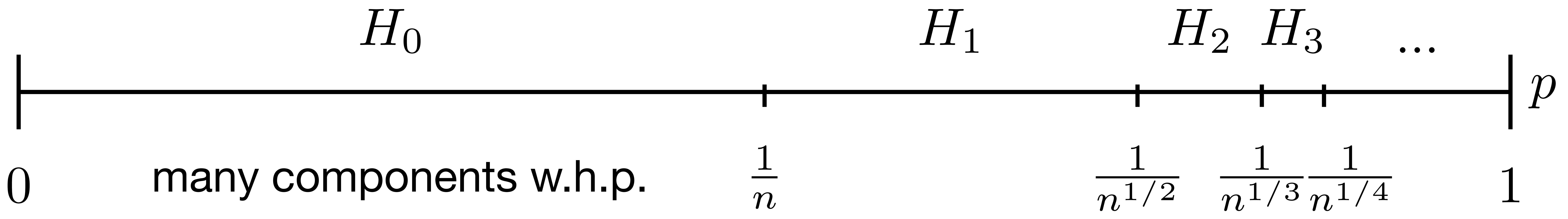
[Erdos-Renyi 1960]



all log terms and constants forgone

Phase Transition

[Kahle 2009, 2014]

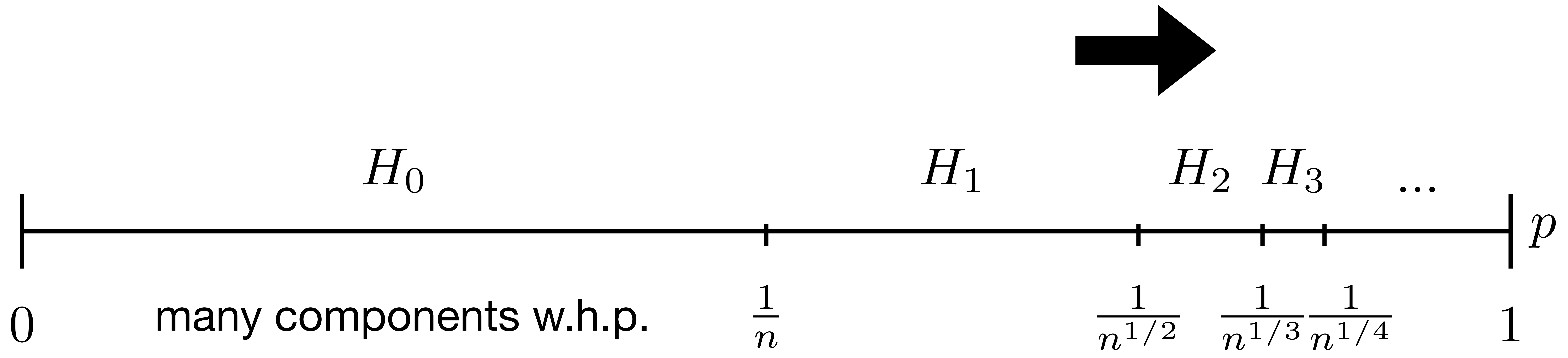


all log terms and constants forgone

Phase Transition

[Kahle 2009, 2014]

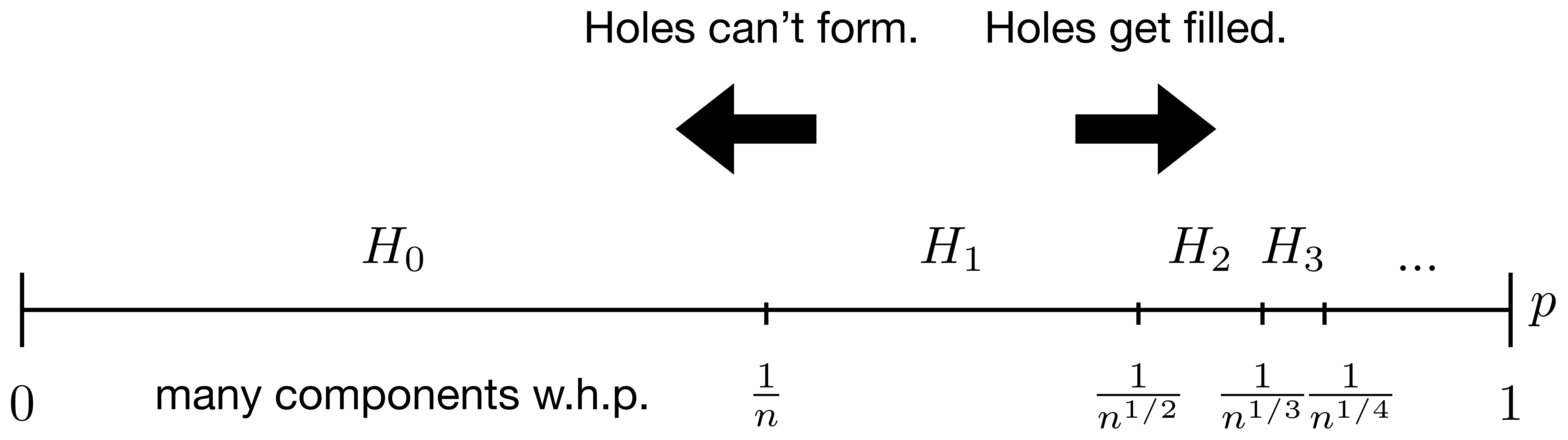
Holes get filled.



all log terms and constants forgone

Phase Transition

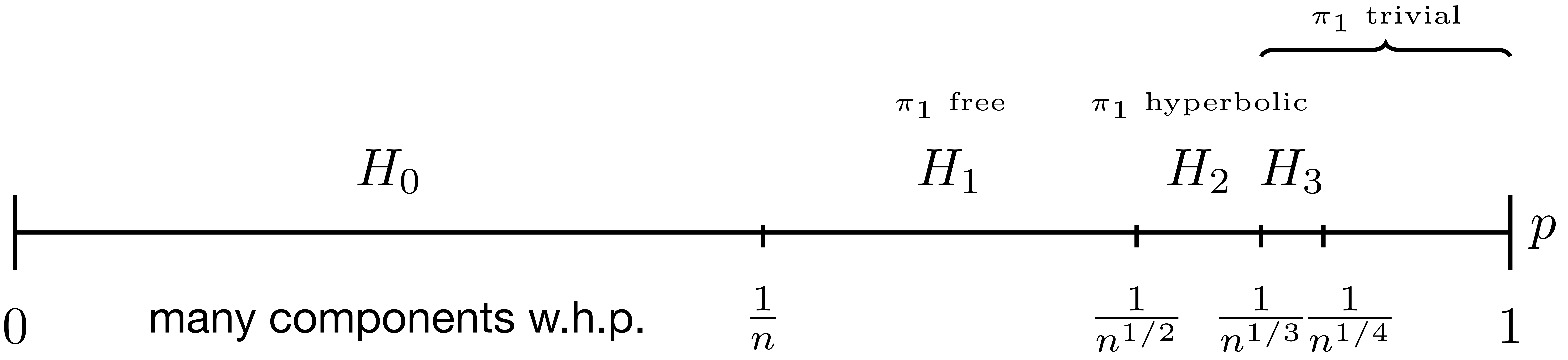
[Kahle 2009, 2014]



all log terms and constants forgone

Fundamental Group

[Kahle 2009, Babson 2012, Costa-Farber-Horak 2015]



all log terms and constants forgone

Geometric Complexes

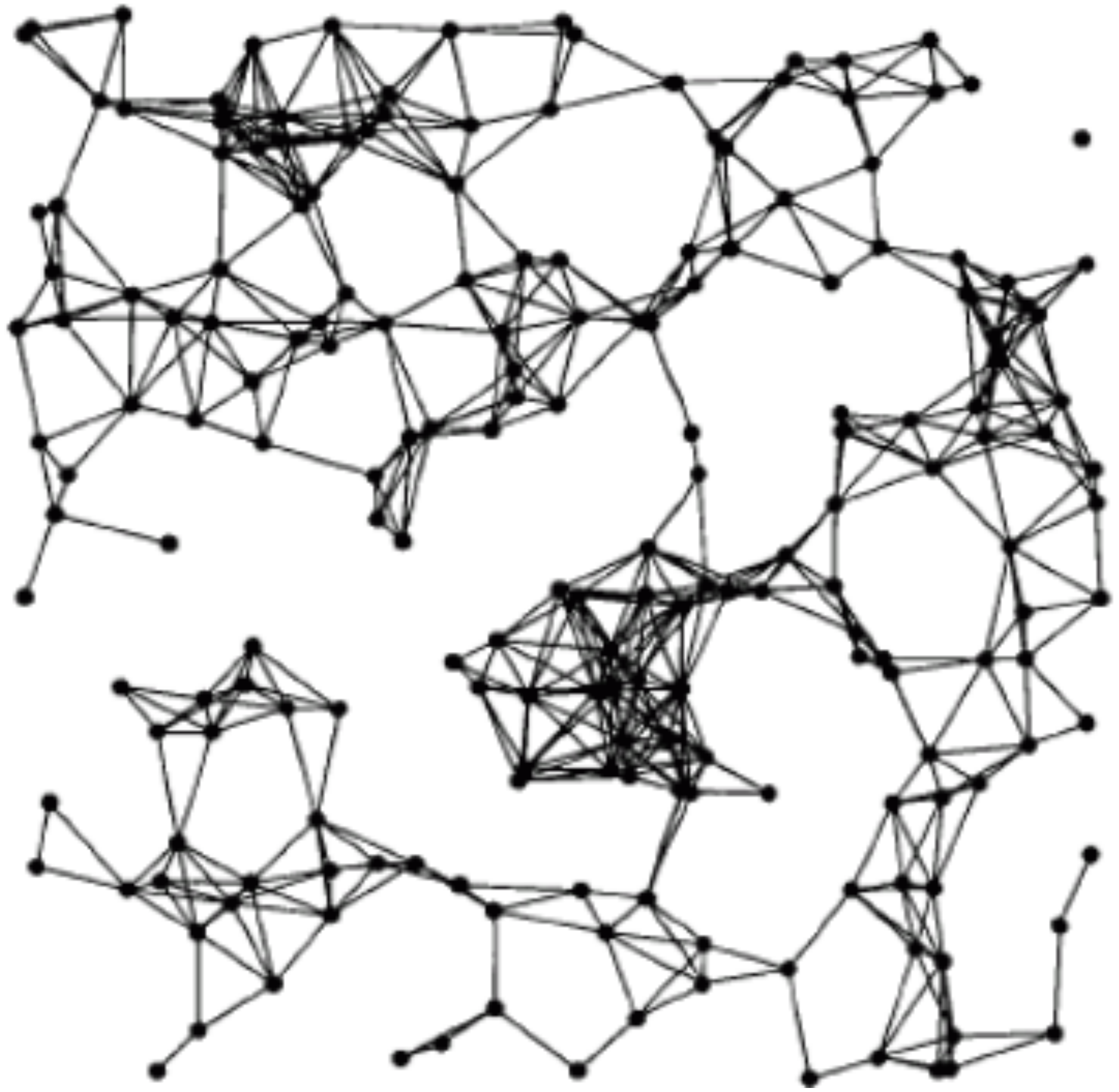


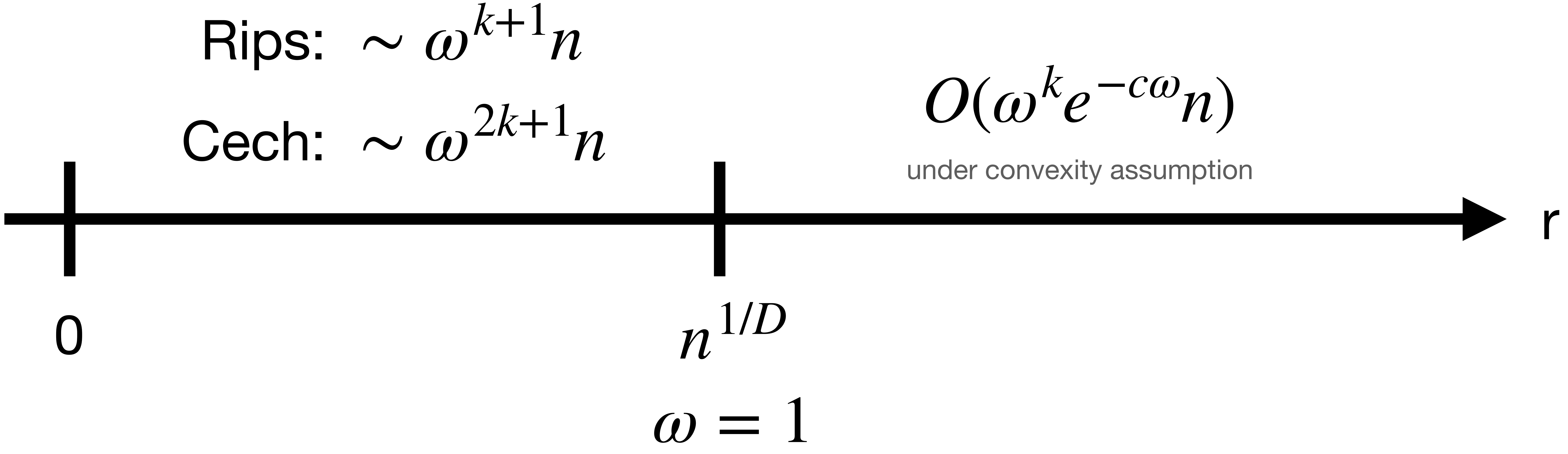
image credit: Penrose

Expected Betti numbers at dimension k

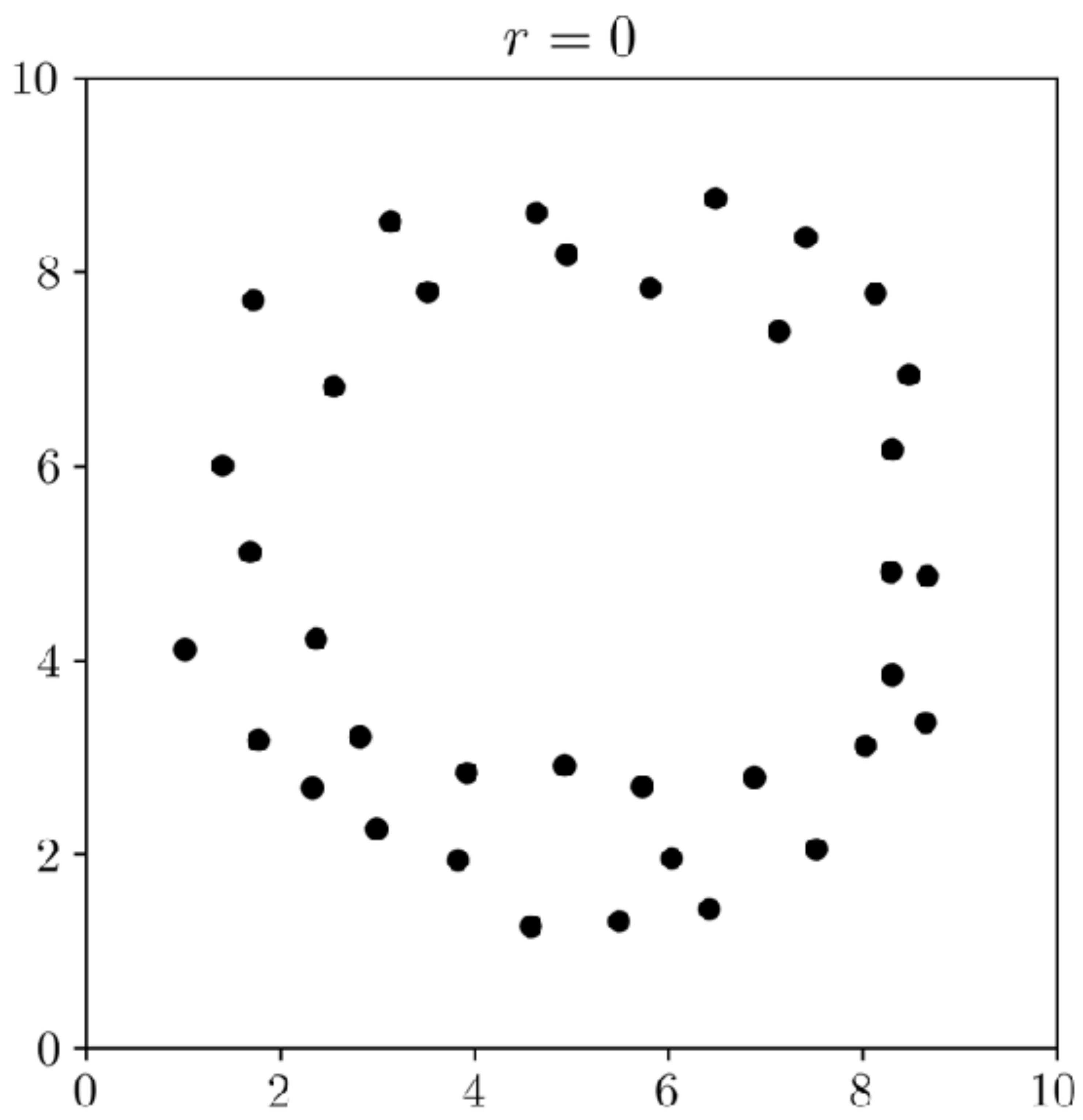
- Let $\omega = nr^D$, where D is the ambient dimension

Expected Betti numbers at dimension k
[Kahle 2011]

- Let $\omega = nr^D$, where D is the ambient dimension



Maximally Persistent Cycles



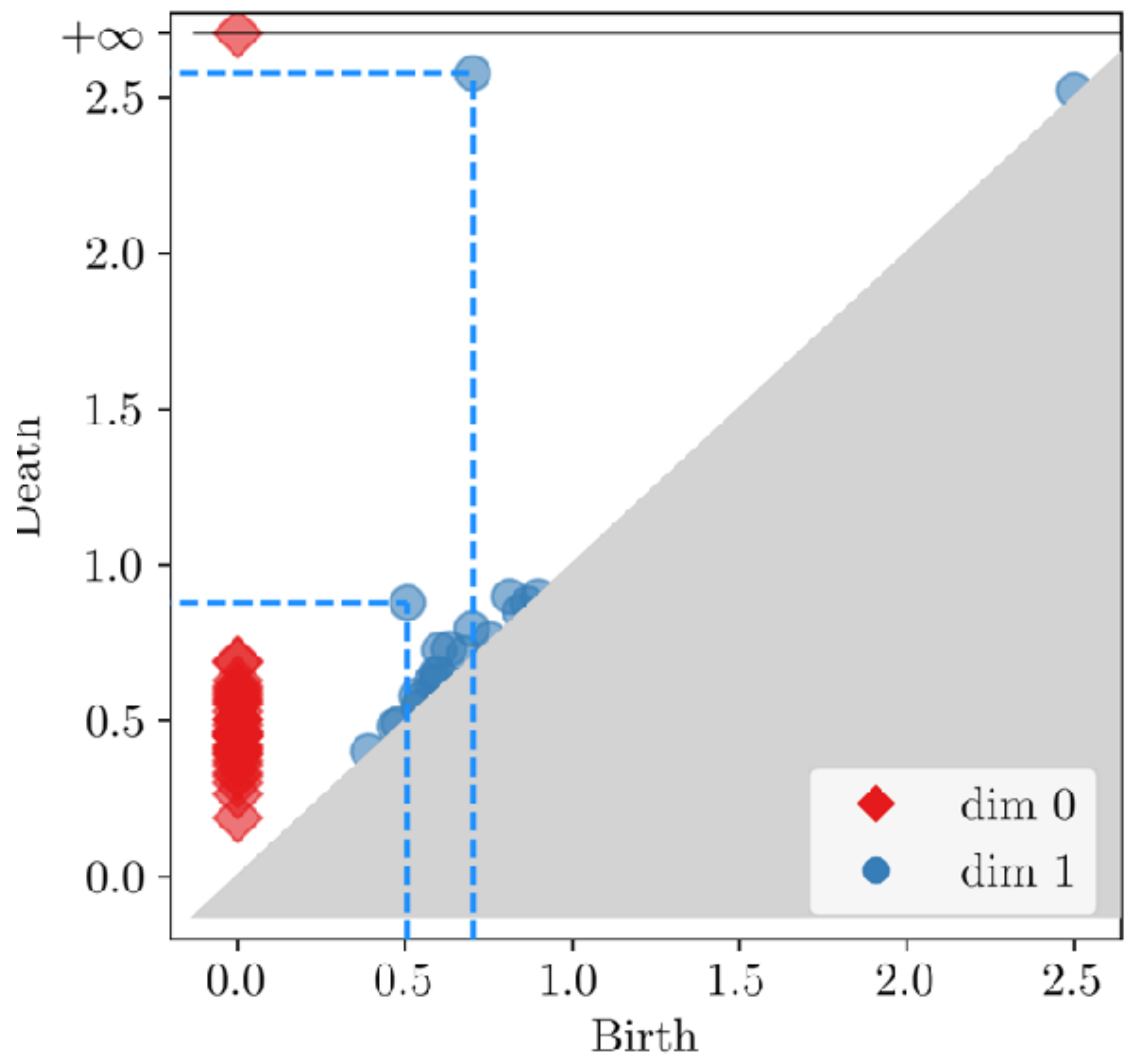
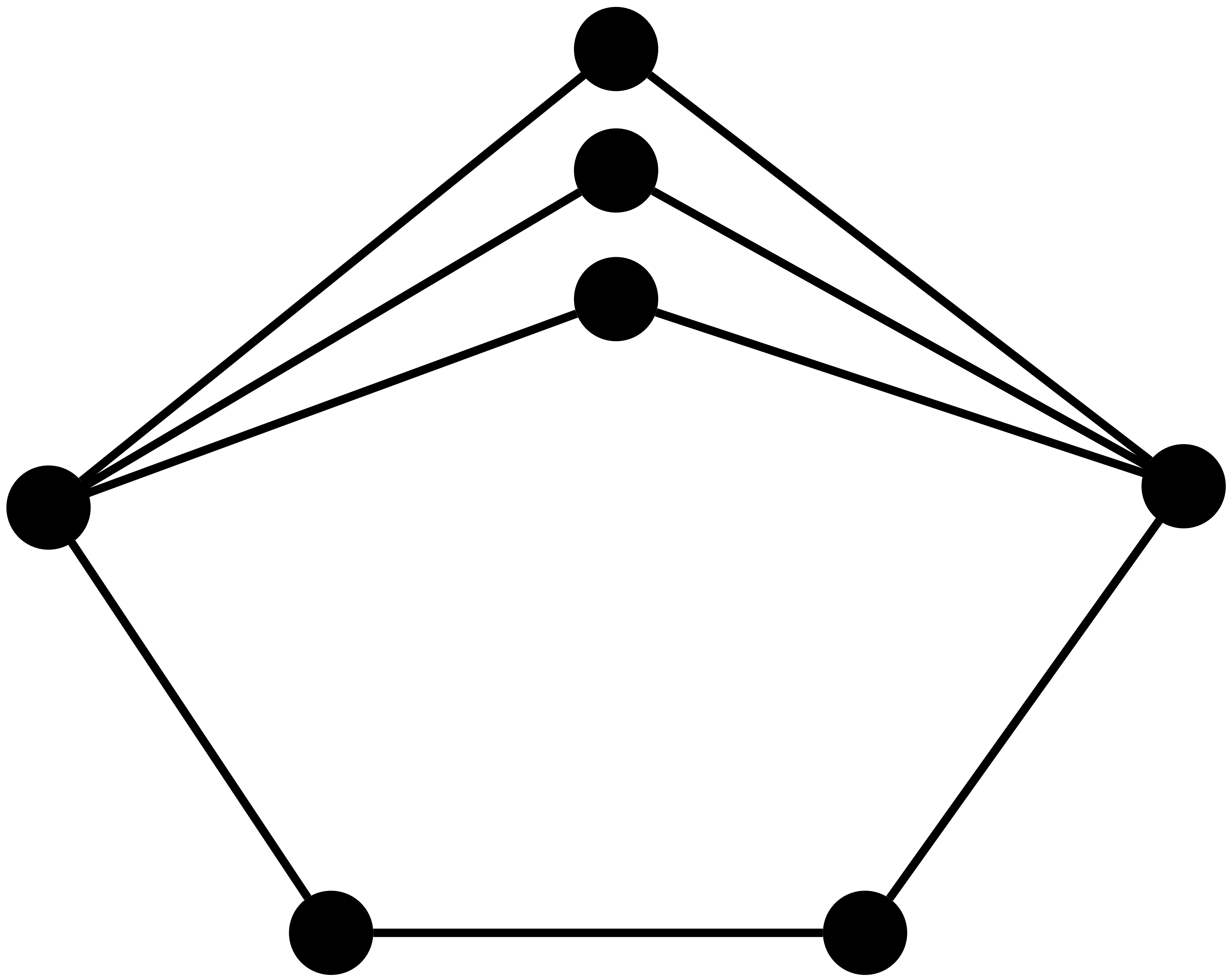
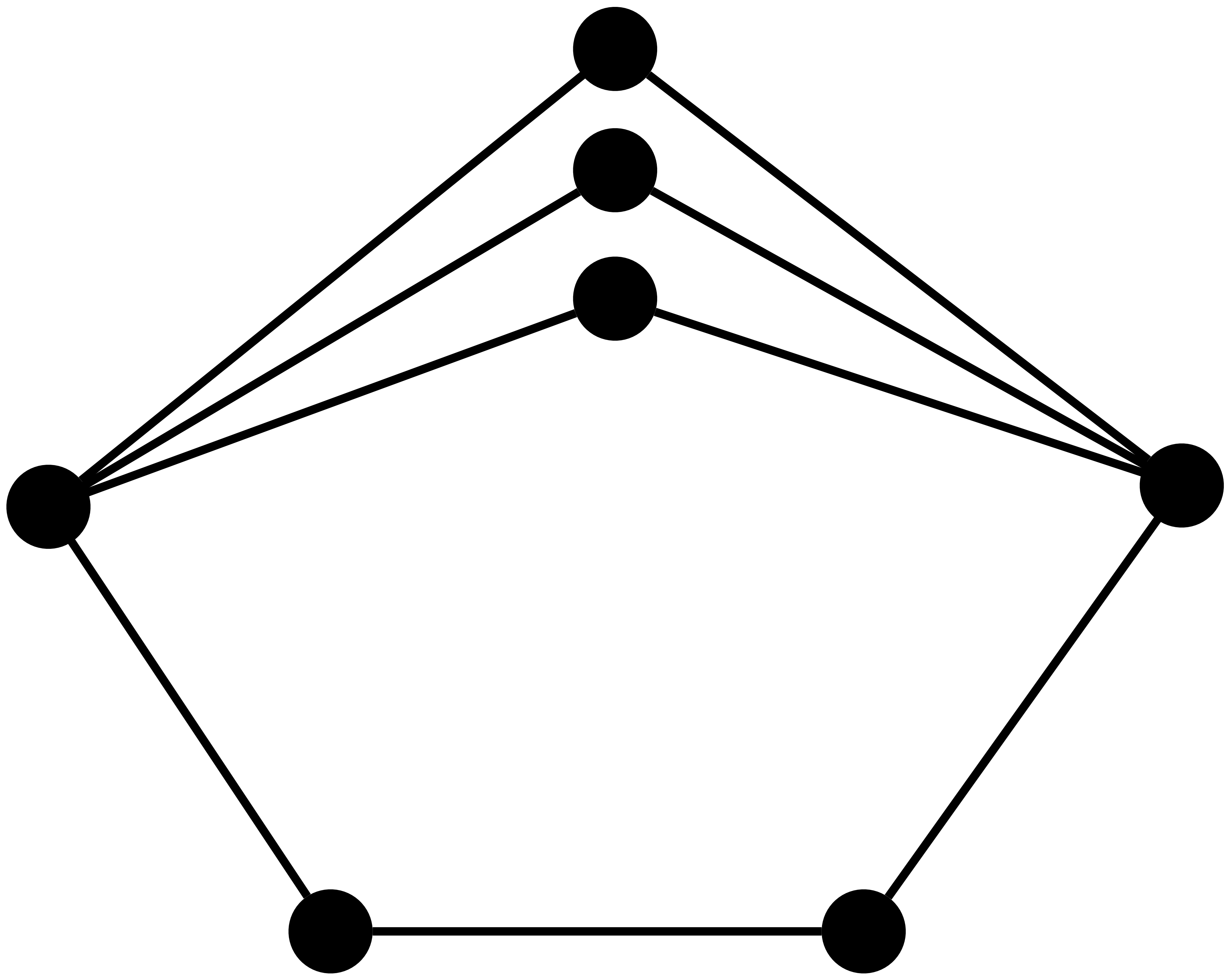


image credit: Andrey Yao

Paths from left to right?

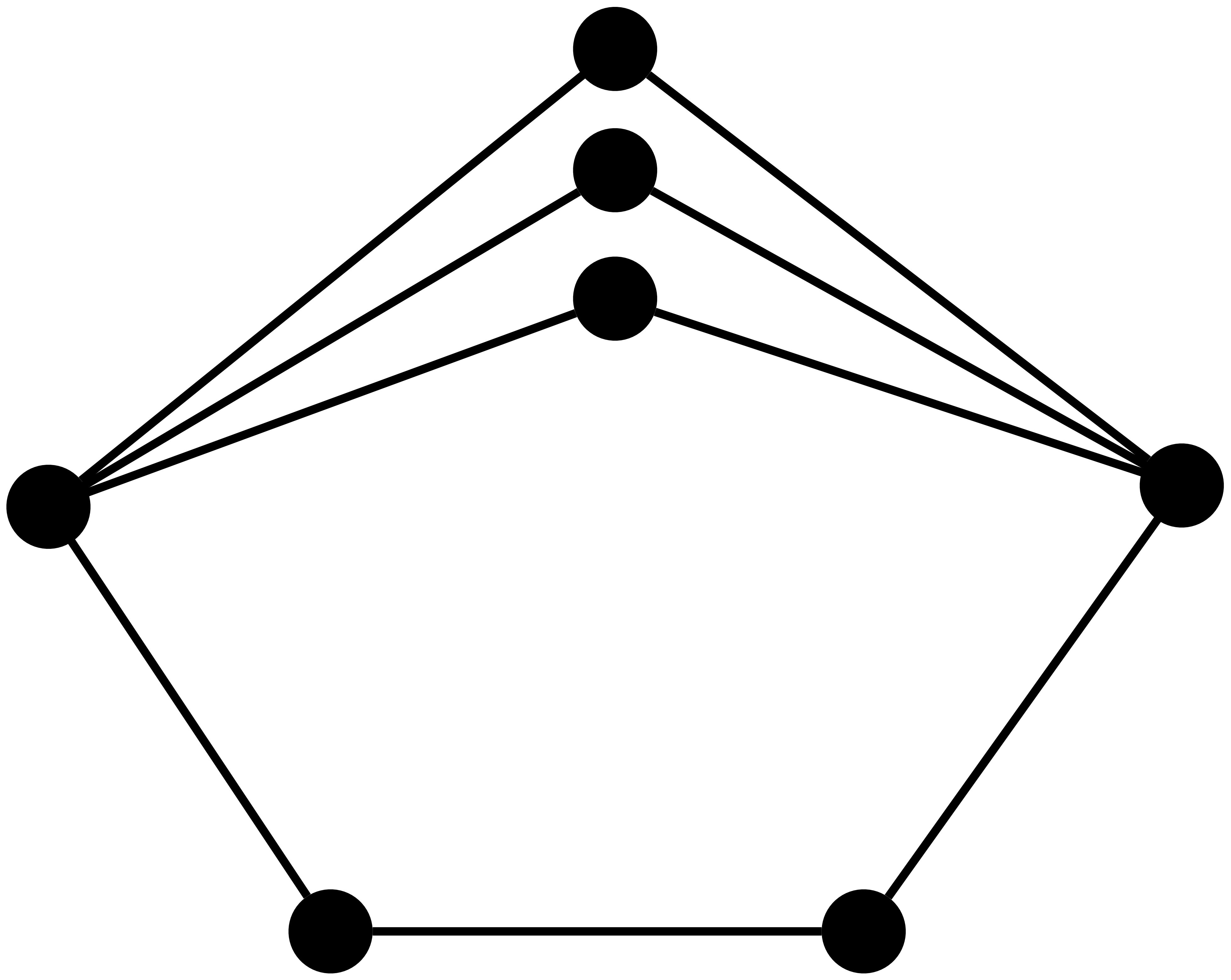


Paths from left to right?



- backtracking?

Paths from left to right?



- backtracking?
- concatenating with loops?

Maximally Persistent Cycles

n points in expectation

k-cycle

Maximally Persistent Cycles

[Bobrowski-Kahle-Skraba 2017]

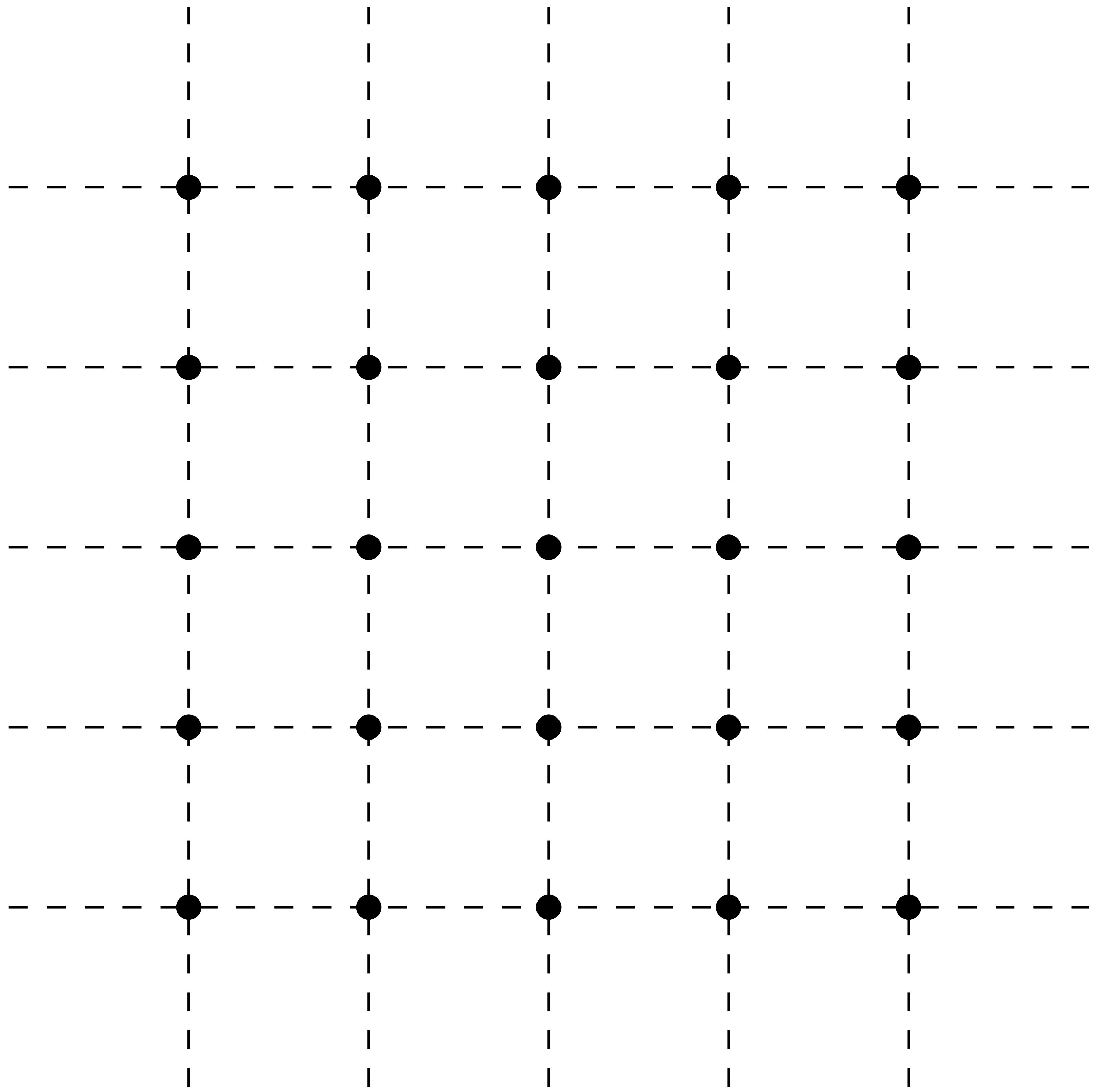
n points in expectation

k-cycle

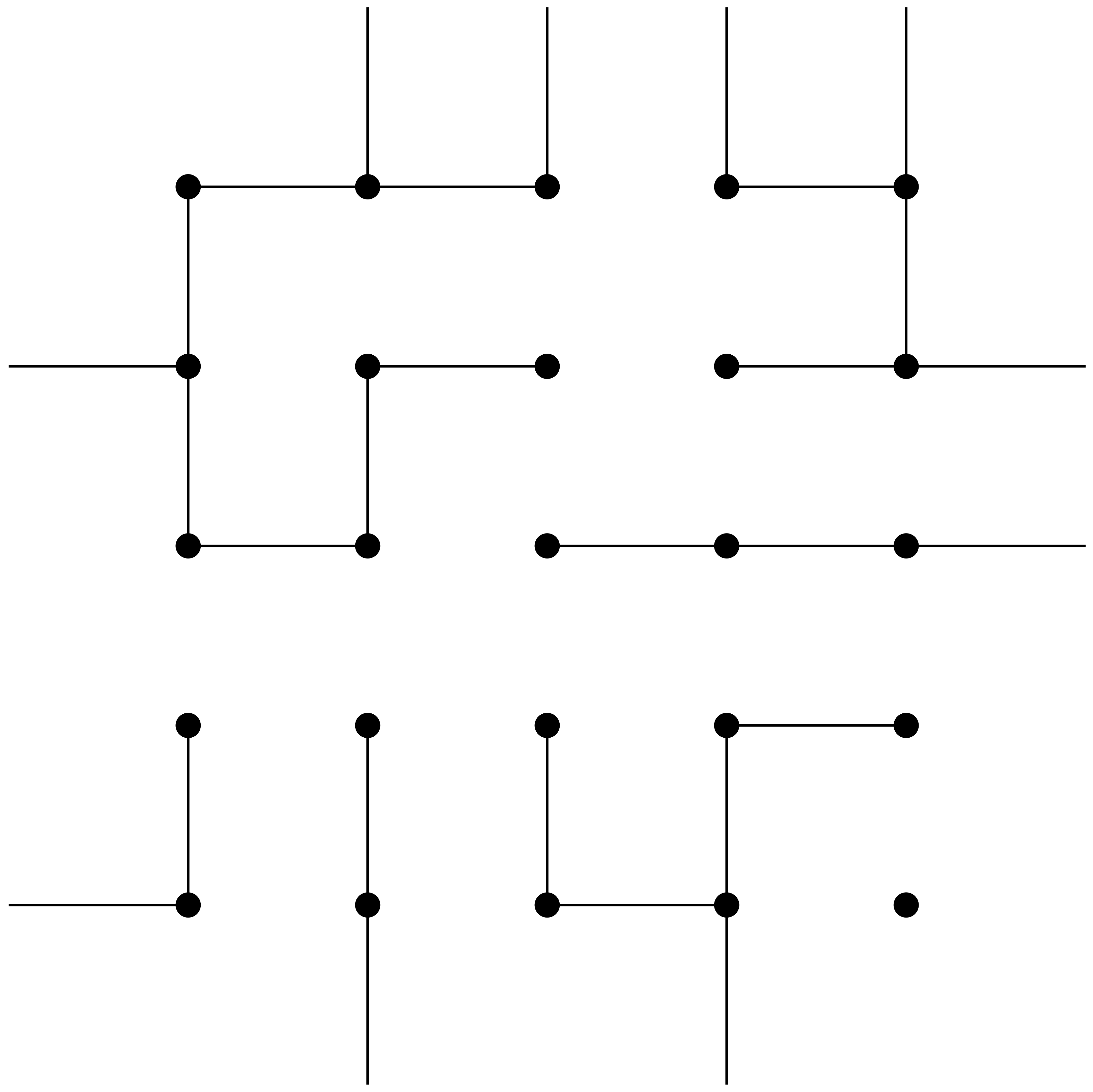
$$c\left(\frac{\log n}{\log\log n}\right)^{1/k} \leq \text{max persistence} \leq C\left(\frac{\log n}{\log\log n}\right)^{1/k}$$

a.a.s.

Bernoulli Bond Percolation

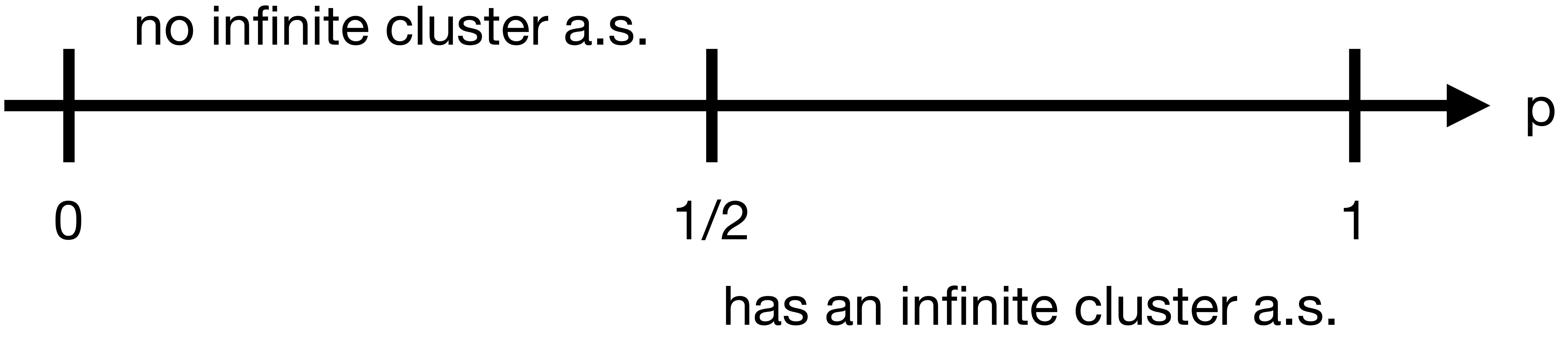


Bernoulli Bond Percolation



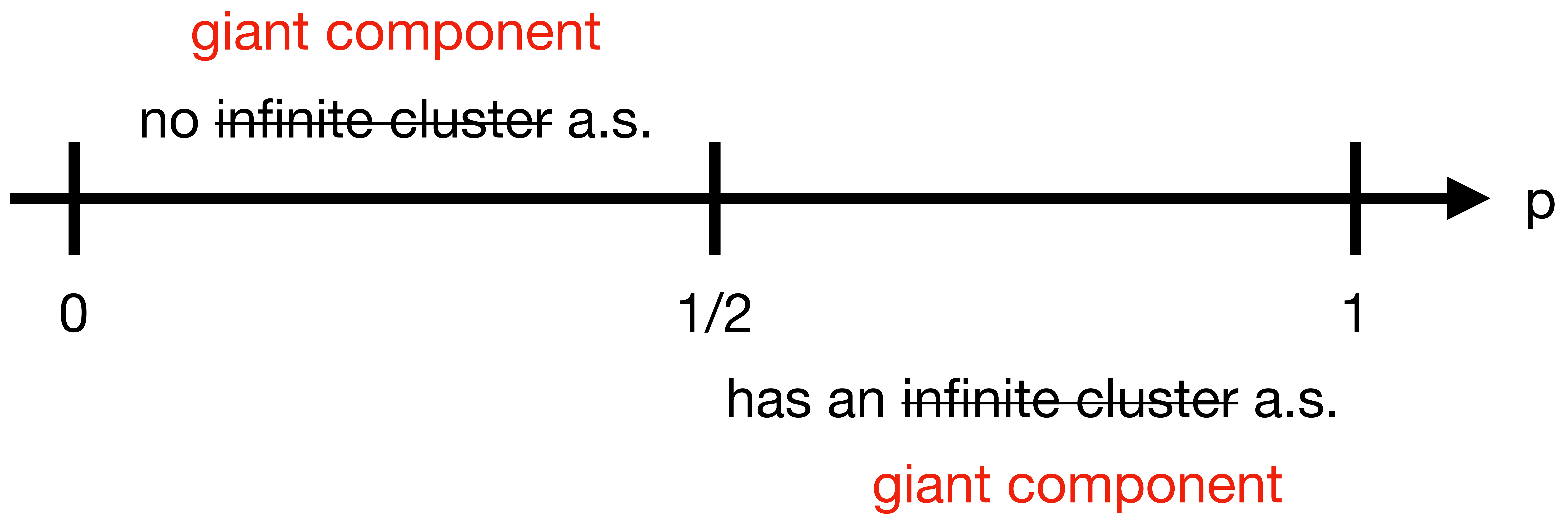
Phase Transition

[Harris 1960, Kesten 1980]



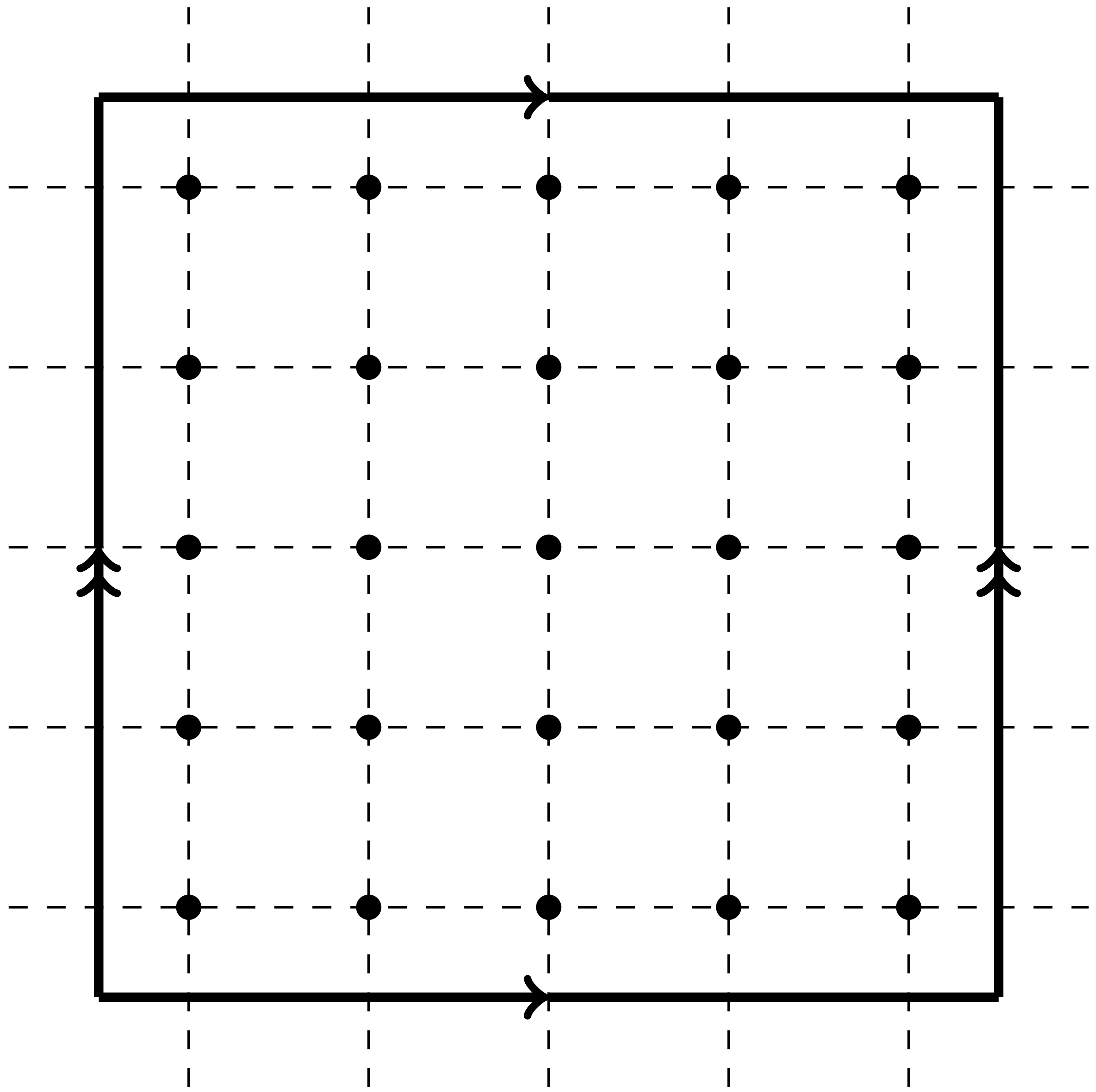
Phase Transition

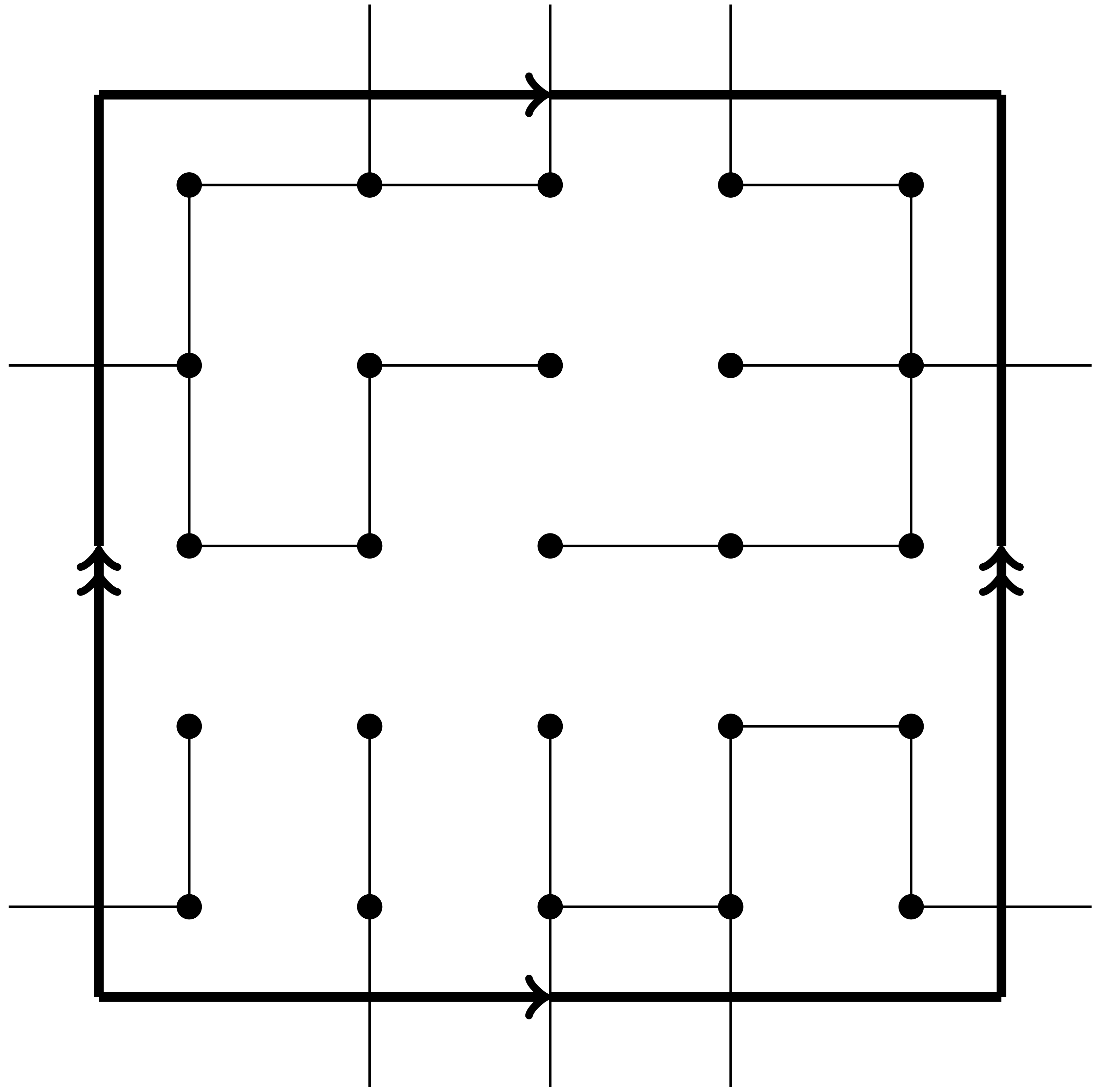
[Harris 1960, Kesten 1980]



Giant Cycles?

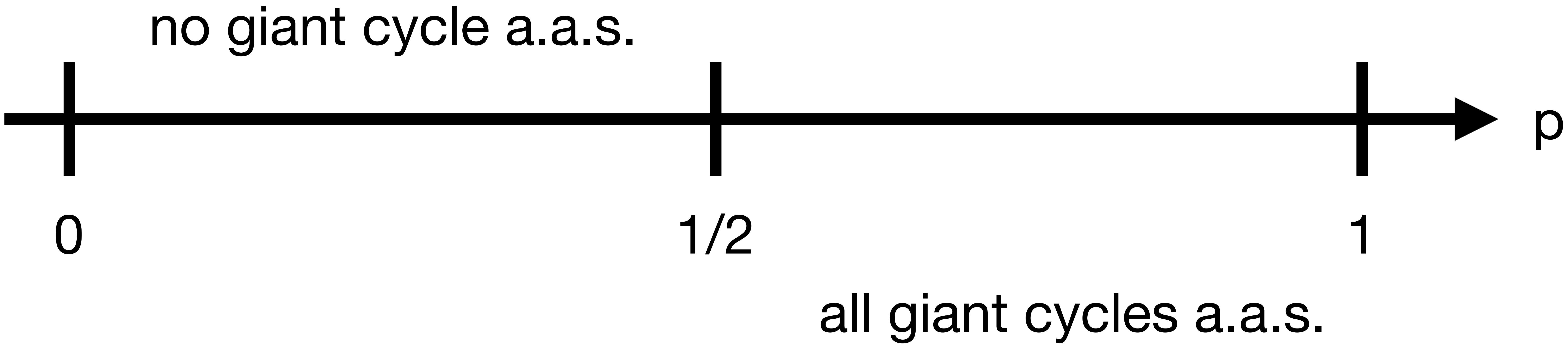
Bernoulli Bond Percolation



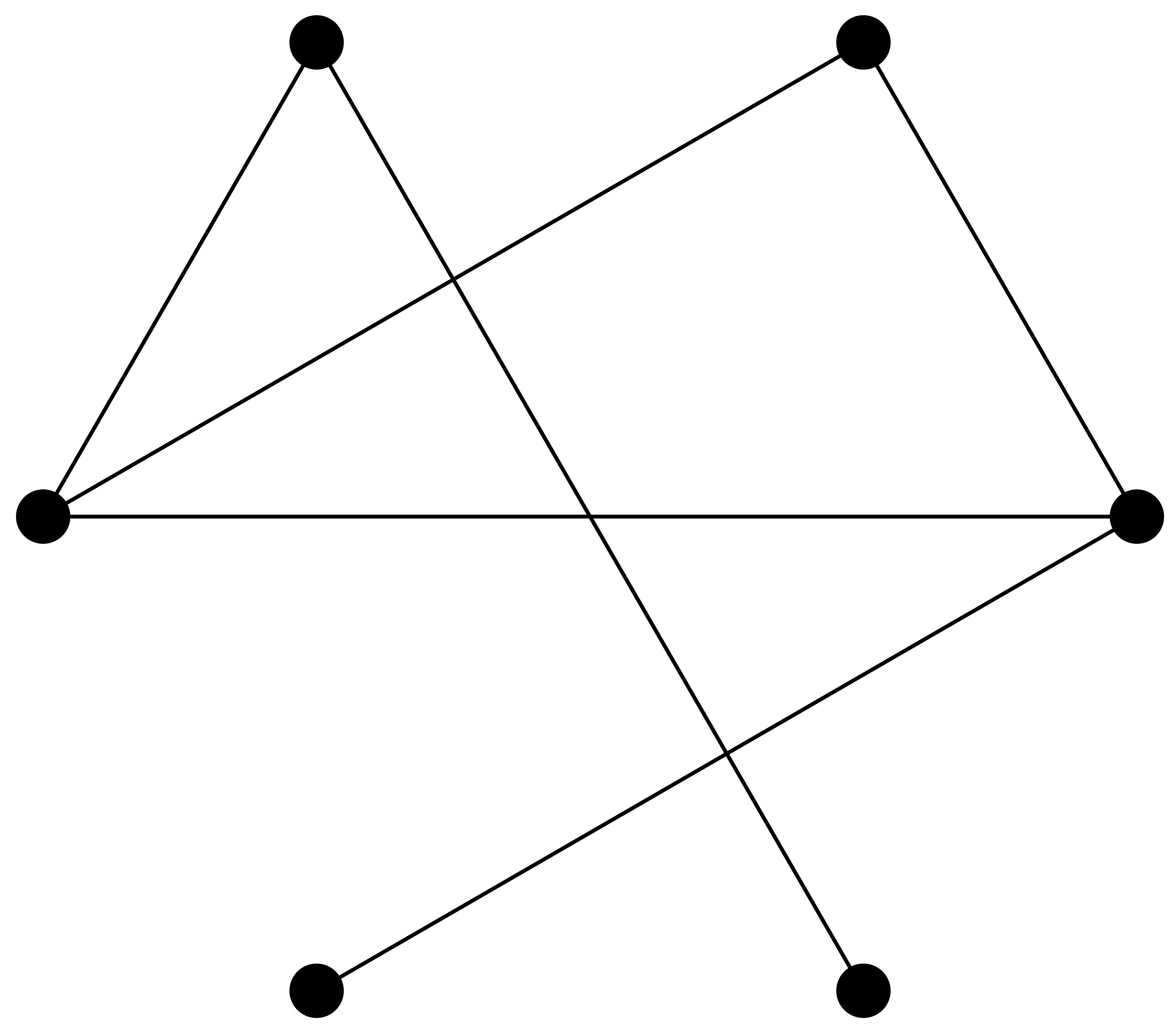


Phase Transition

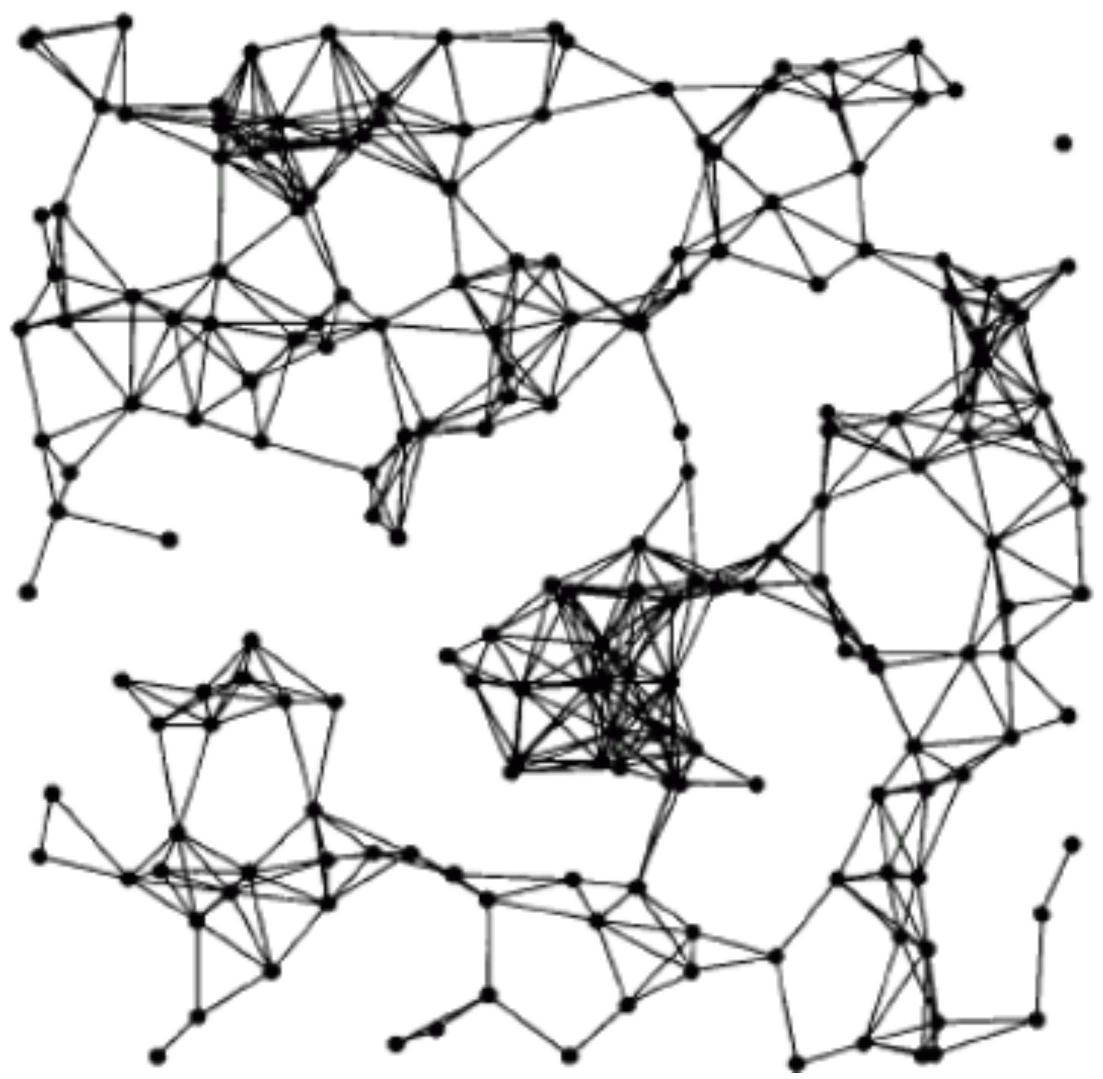
[Duncan-Kahle-Schweinhart, 2021]



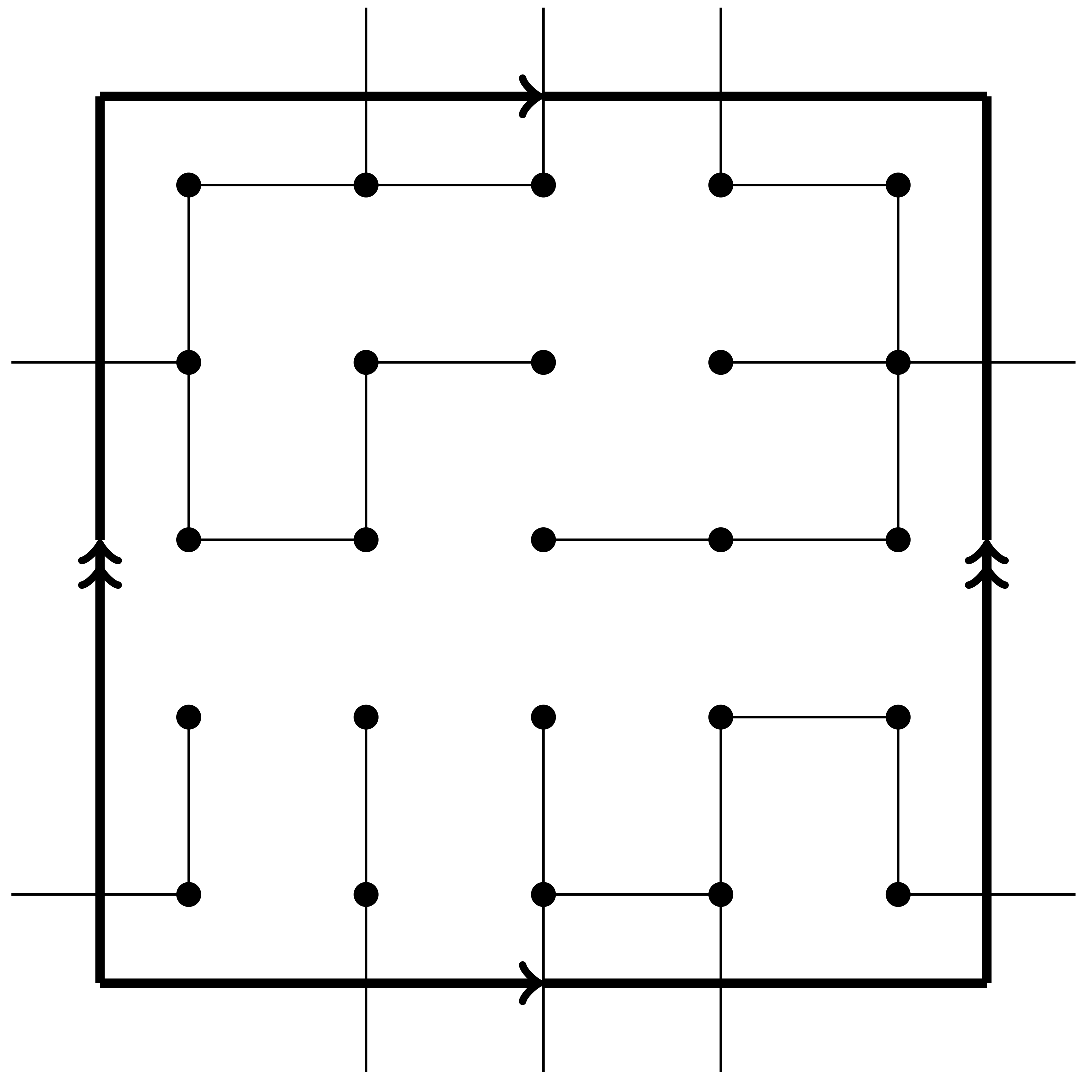
Tapas at Random Topology



Erdo-Renyi Complexes



Geometric Complexes



Topological Percolation