Topology of Scale-Free Graphs

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Topology of Scale-Free Graphs — Homology and Homotopy

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with AATRN network exclusive: thoughts on neuroscience

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So, scale-free graphs…

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

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- topological properties

(Stephen Coast
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- topological properties
- random fluctuation?

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- topological properties
- random fluctuation?

- —> random topology

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Agenda

random topology

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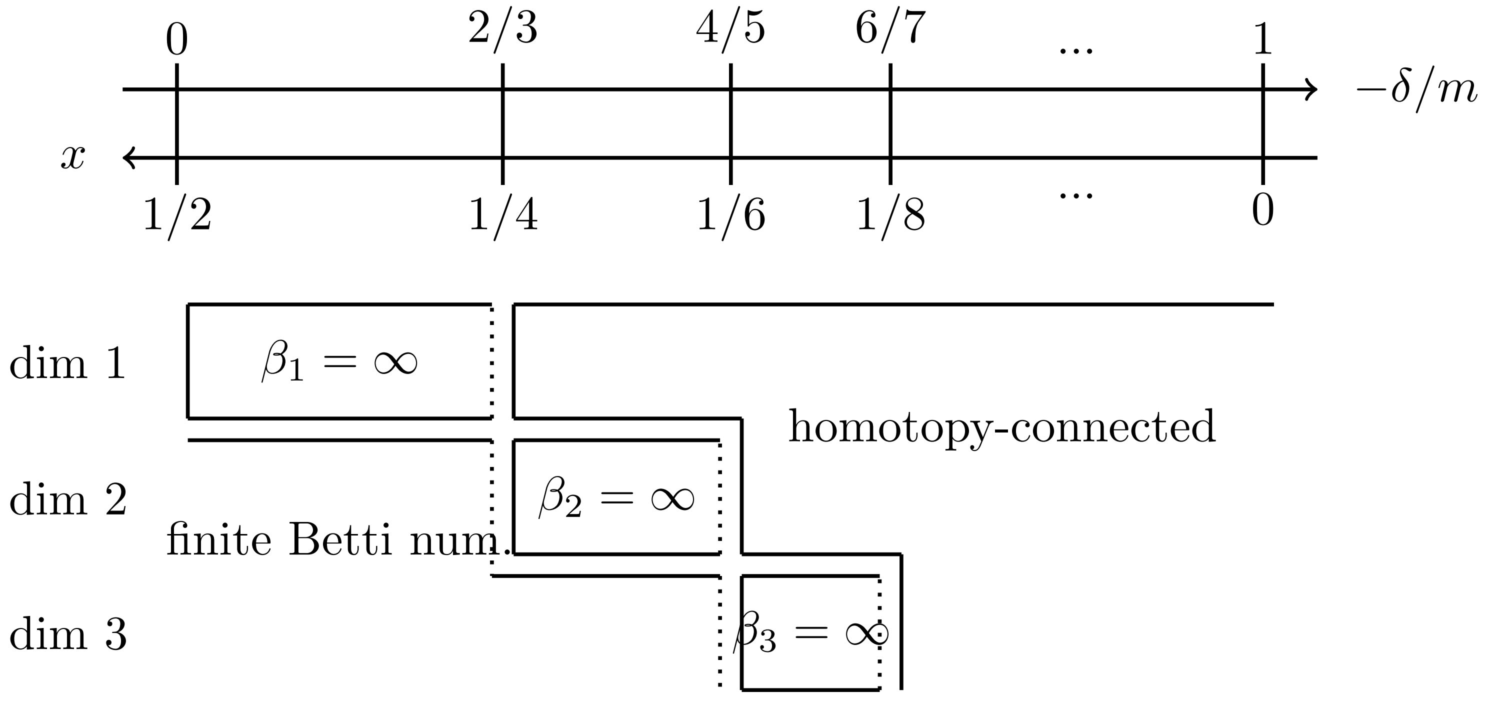
random topology

preferential attachment

Agenda

random topology

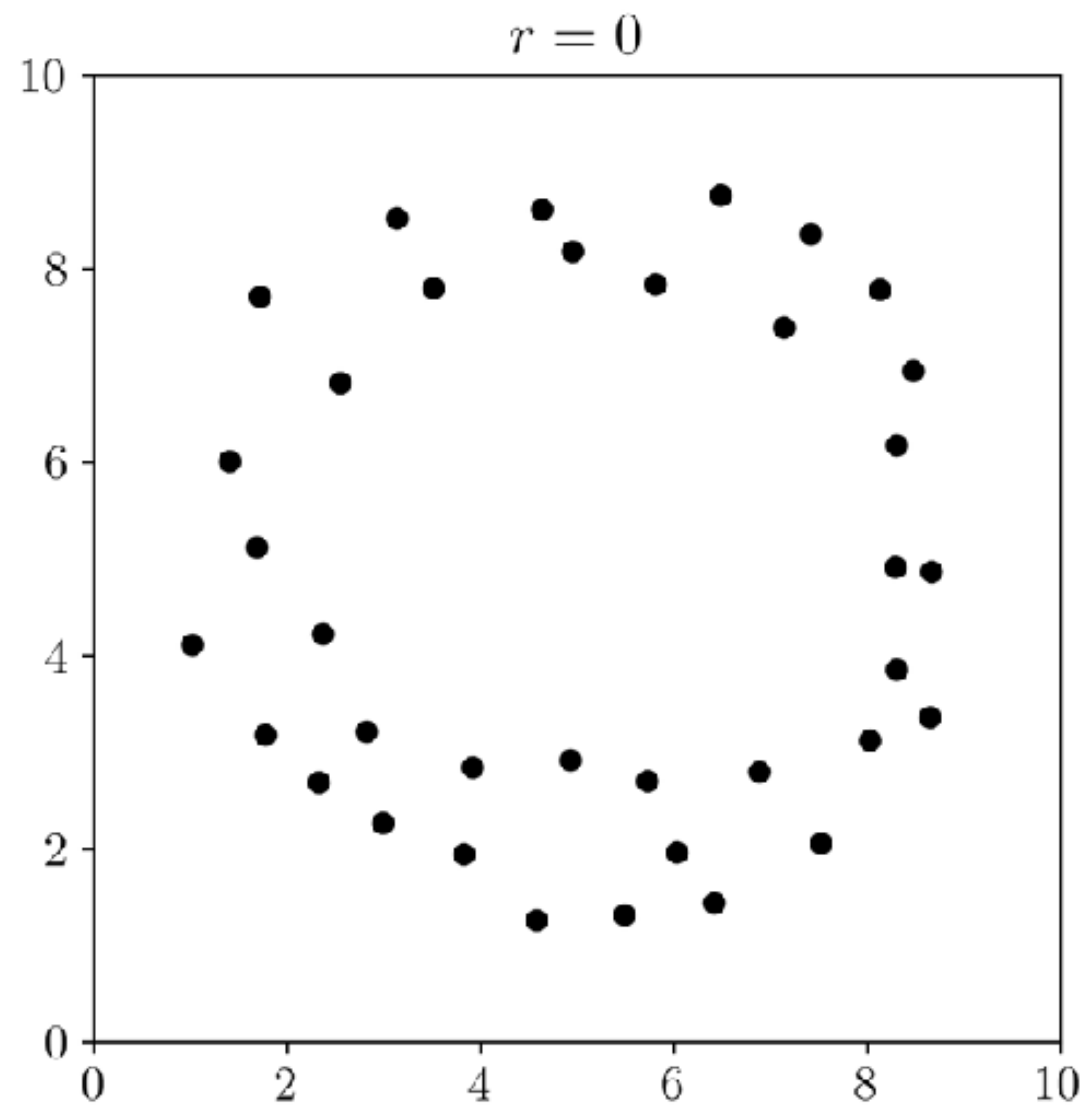
preferential attachment

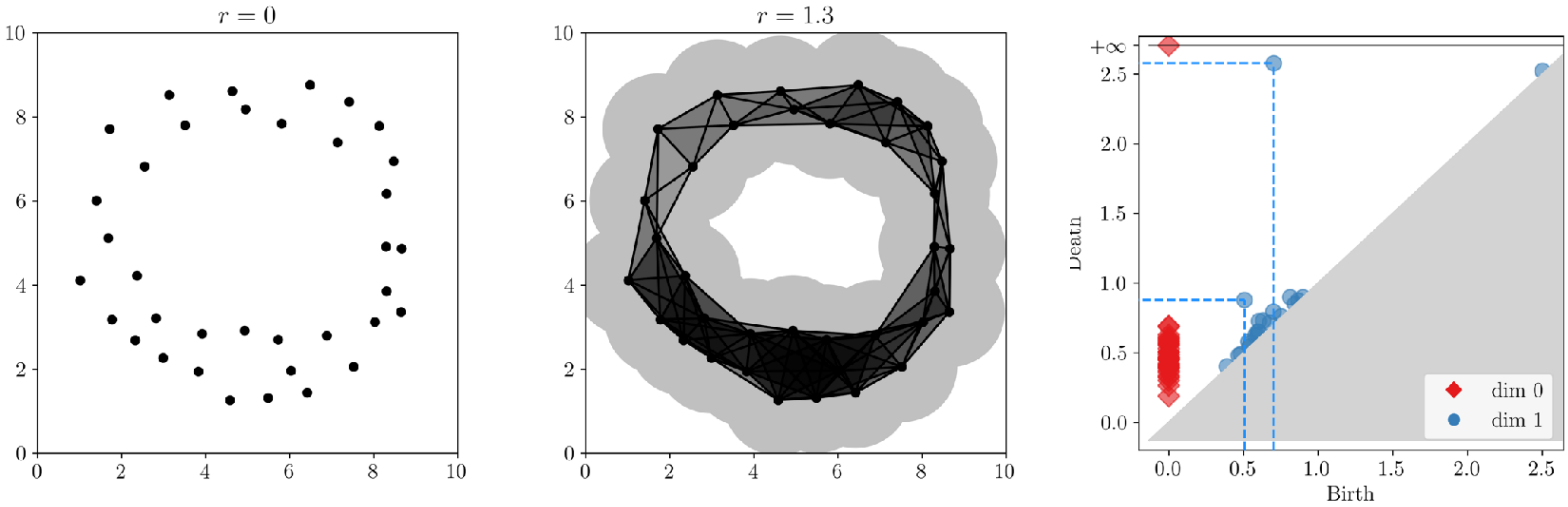


our result

I. A Probabilist's Apology

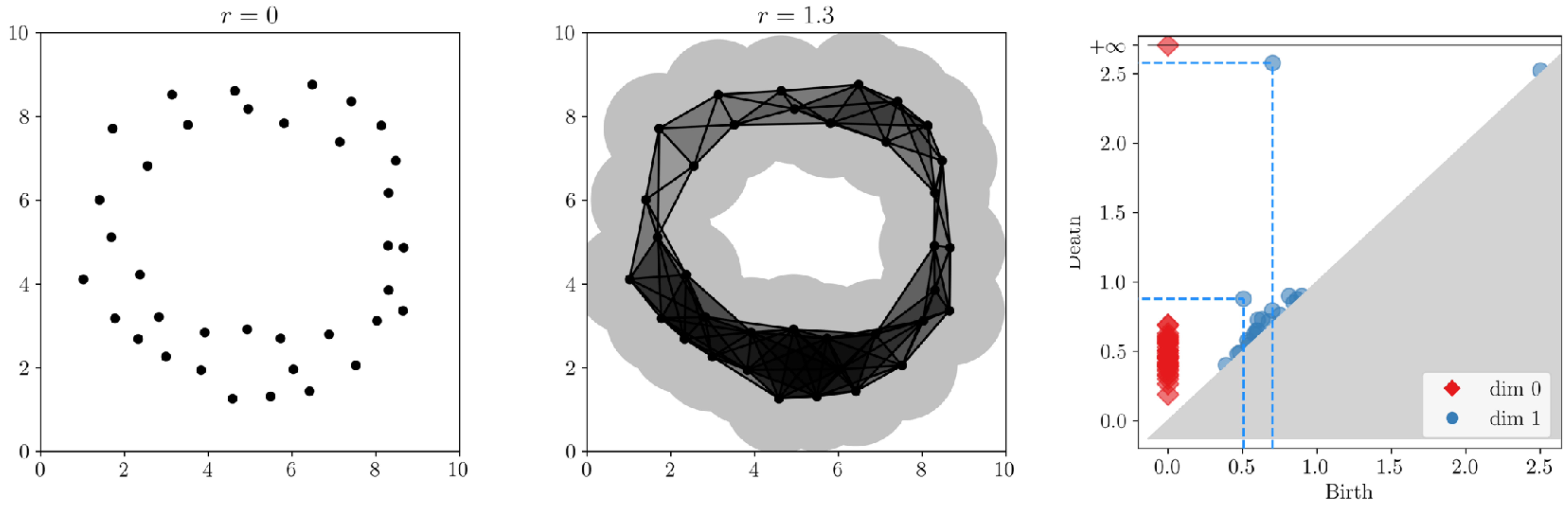
Why Random Topology and What we Know



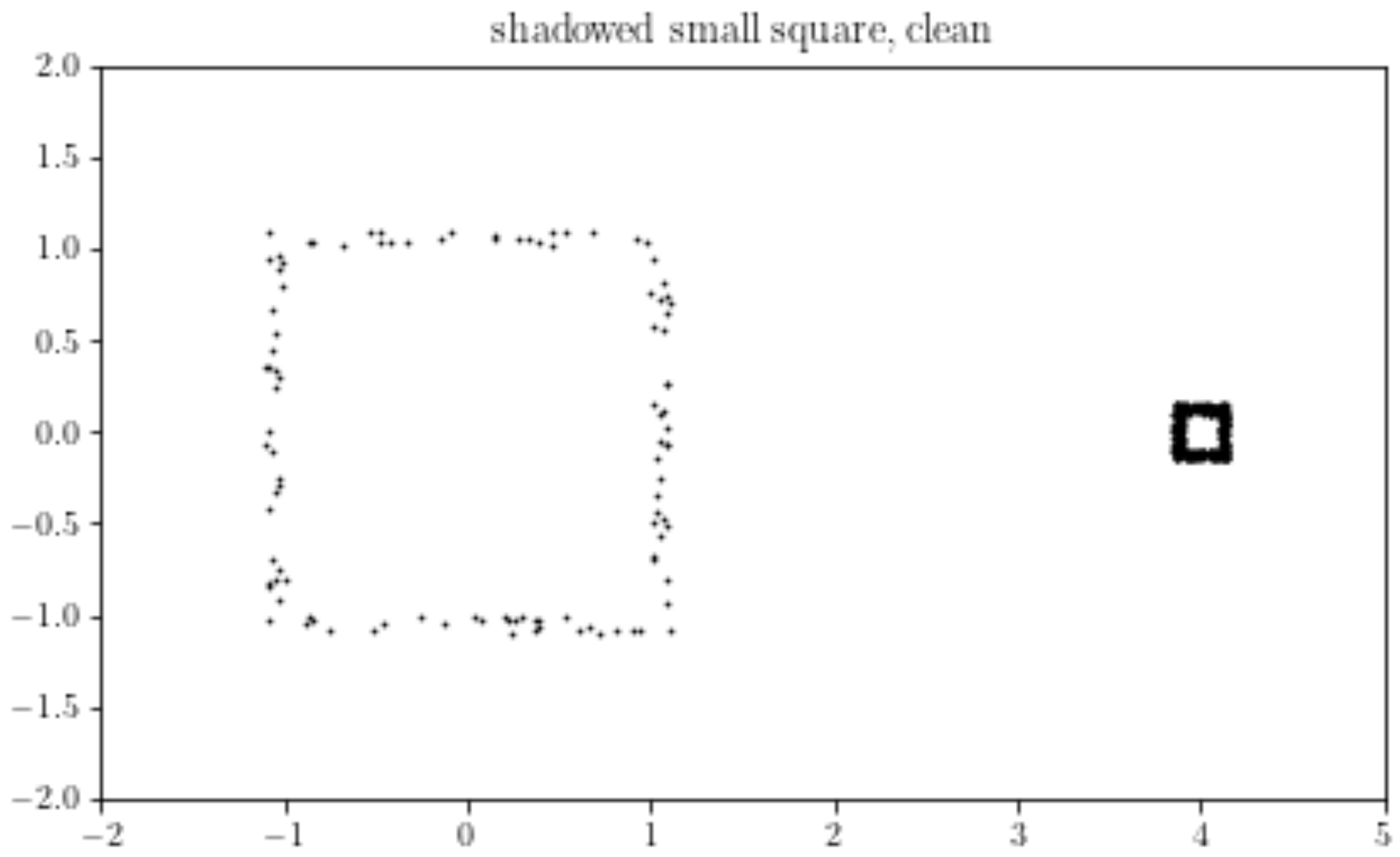


plots generated by Andrey Yao

Size is Signal

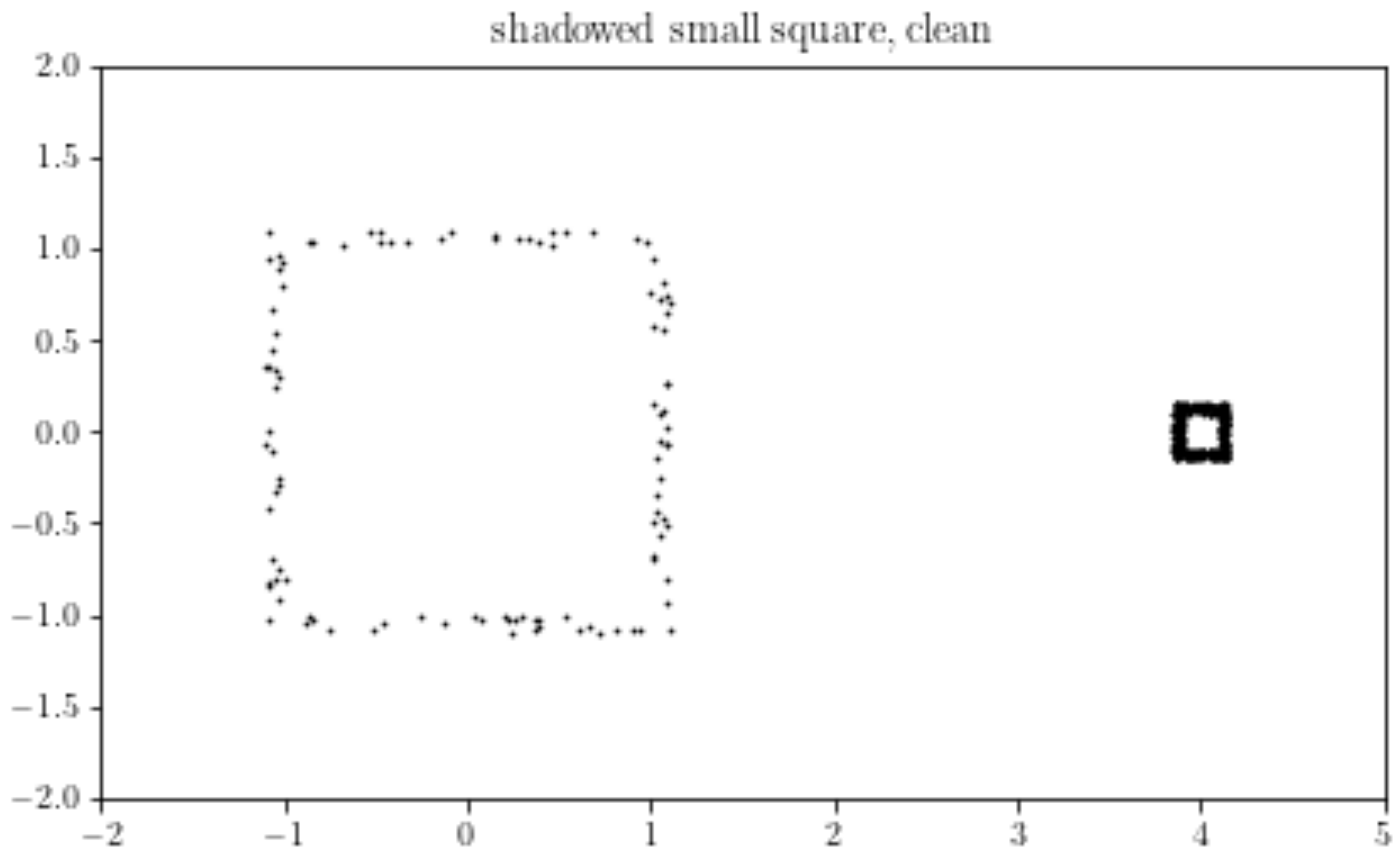


Or is it?

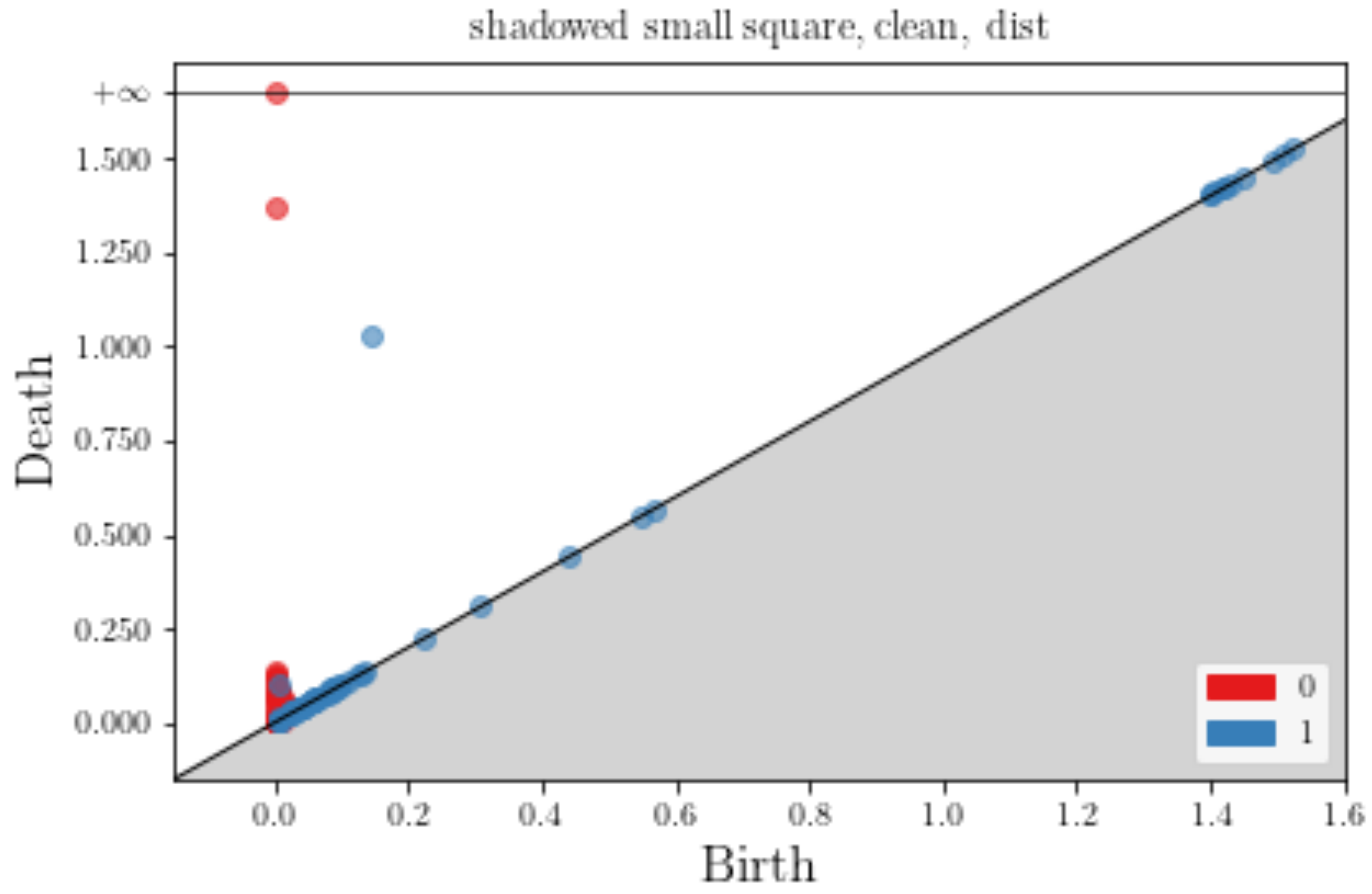


shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5

Or is it?



shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5



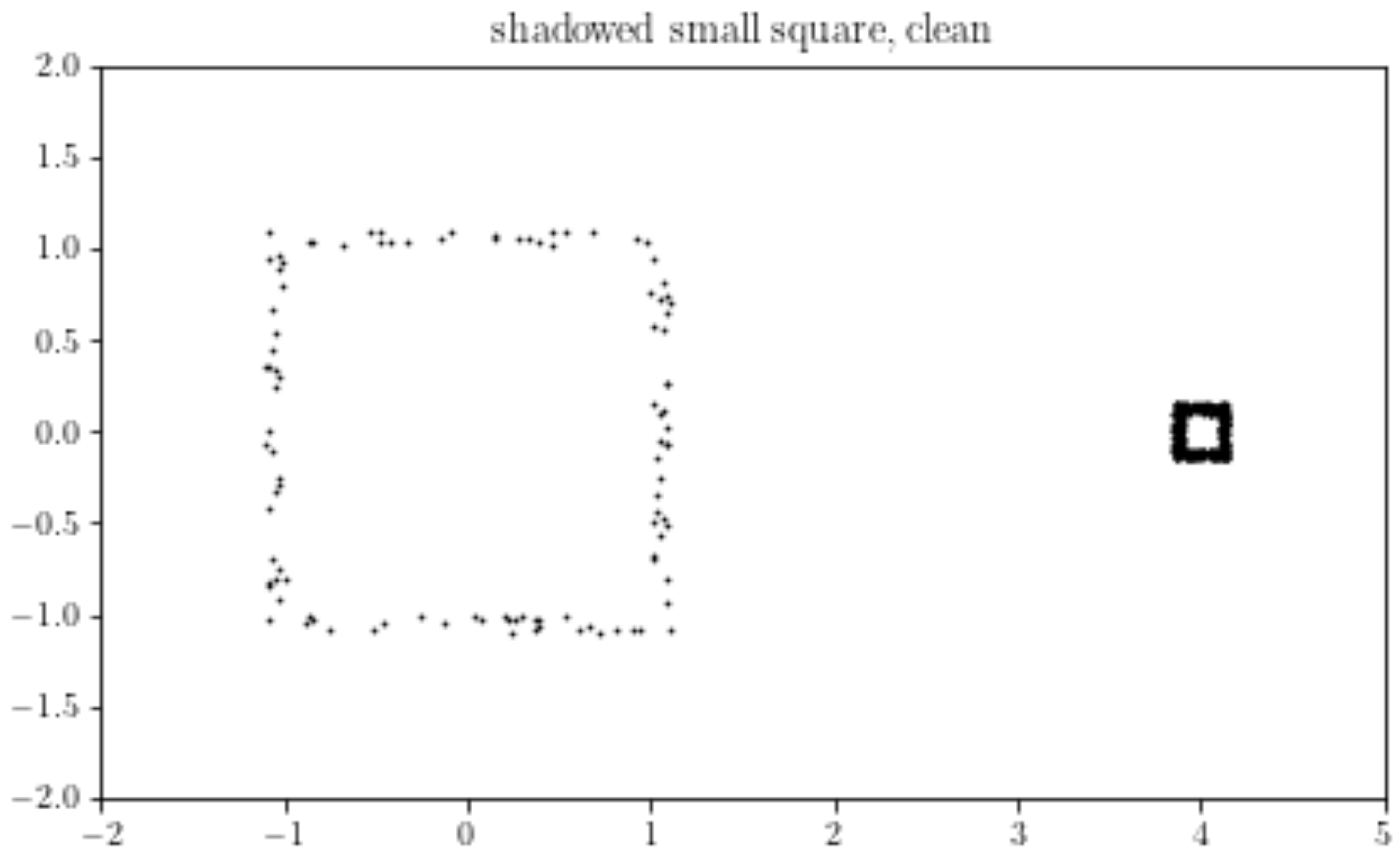
shadowed small square, clean, dist
+∞
1.500
1.250
1.000
0.750
0.500
0.250
0.000
Death
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Birth
0
1

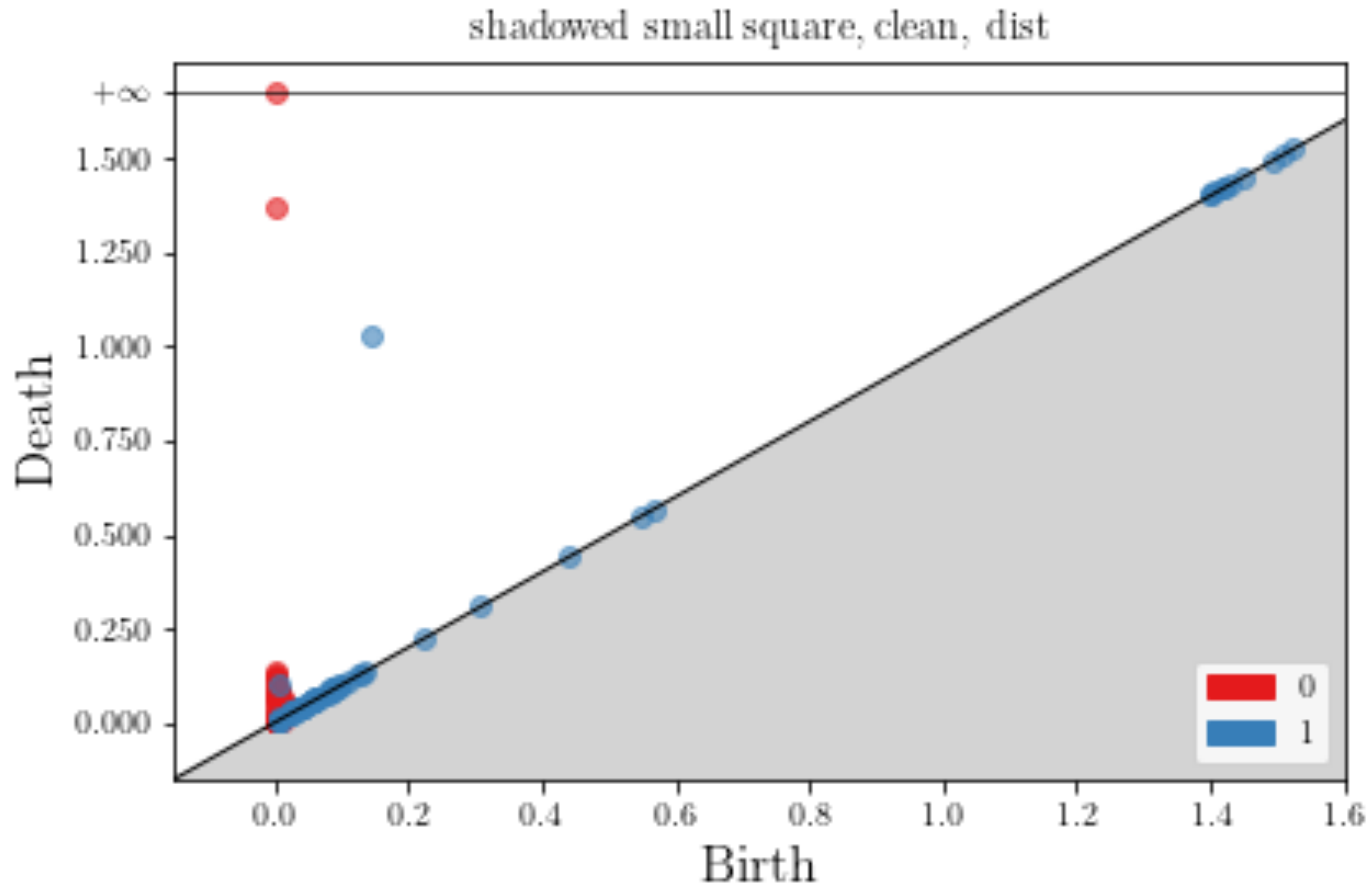
Size is Signal?

Surprise

~~Size~~ is Signal.

Random points don't do that.

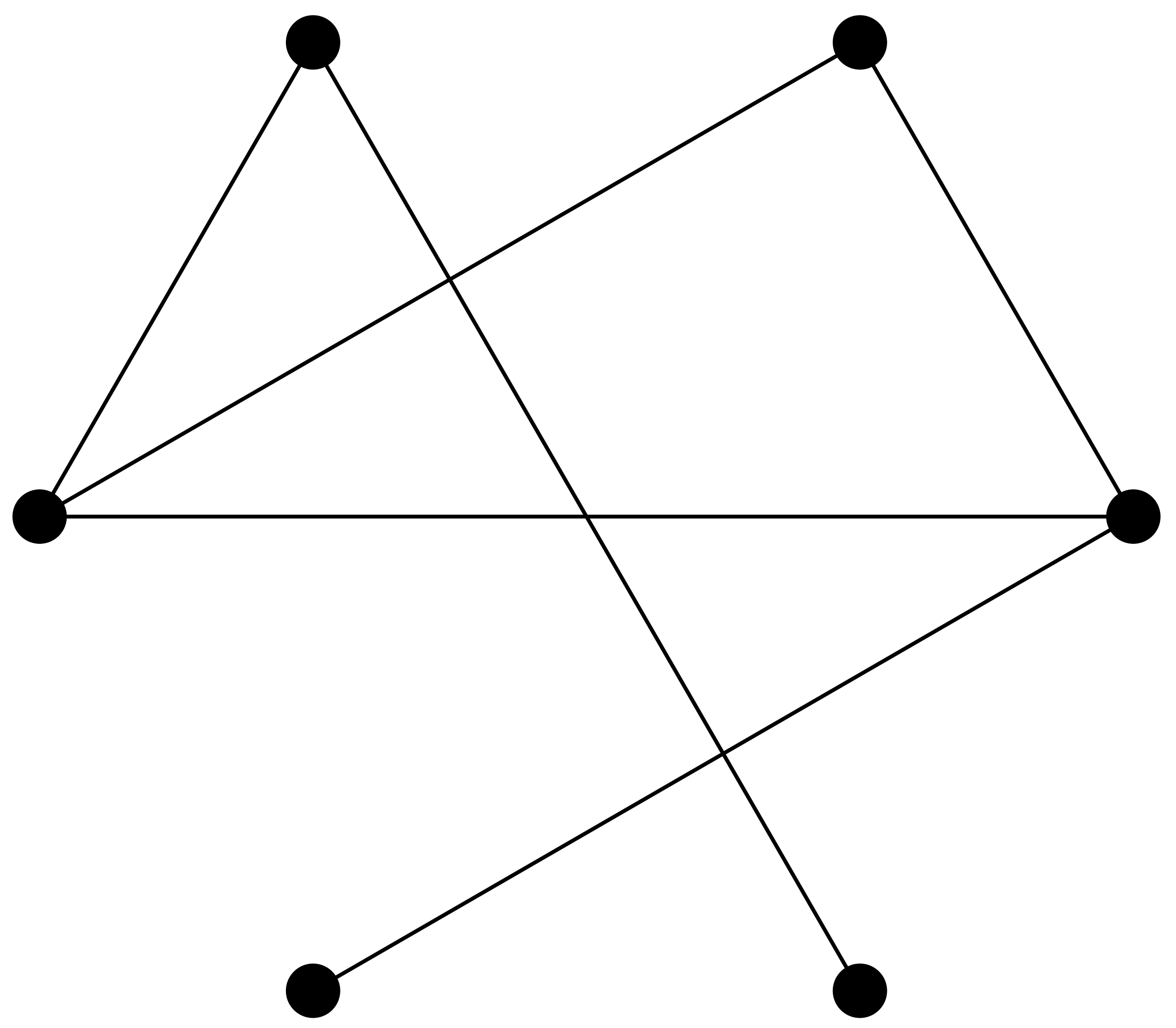




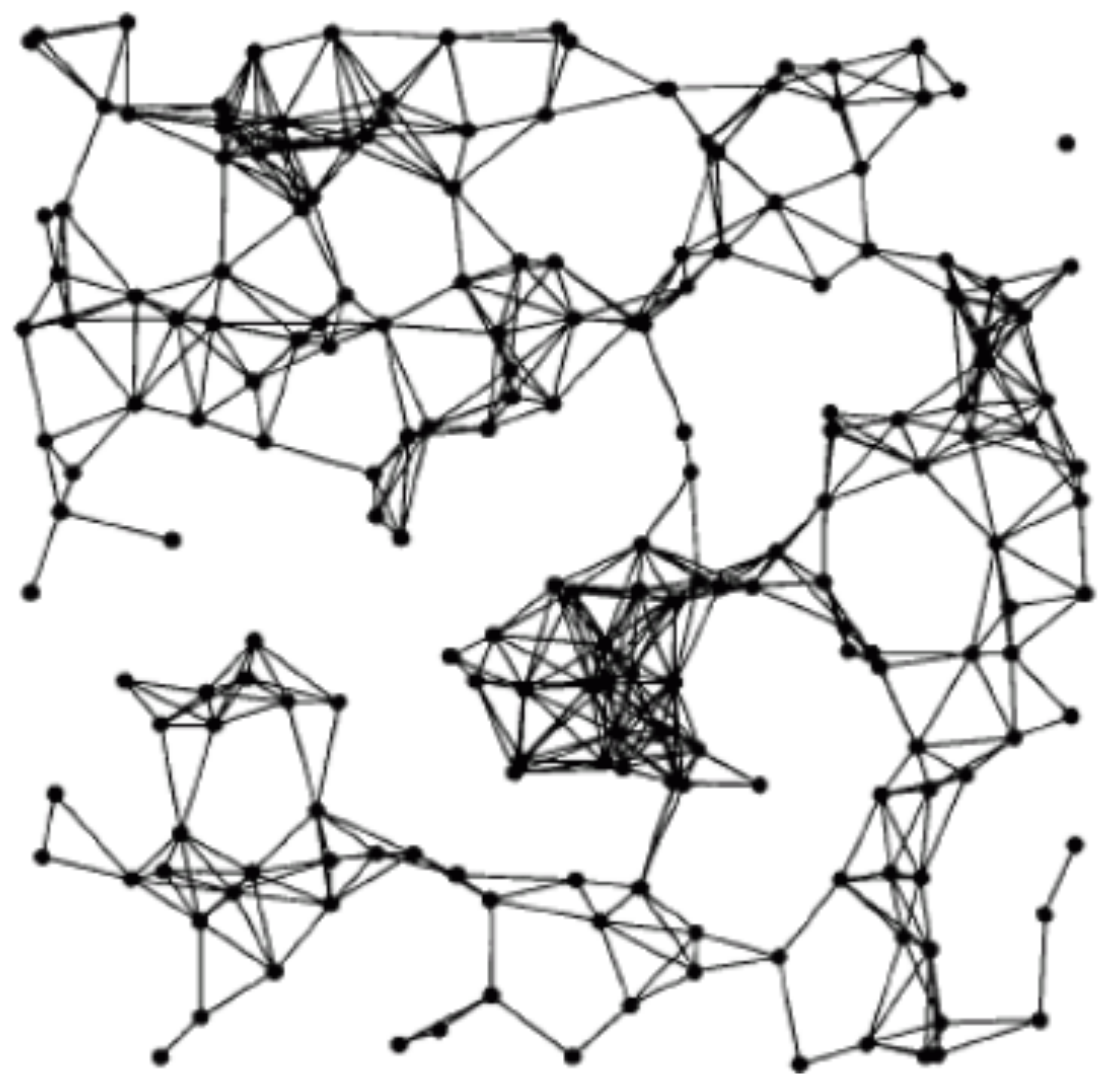
Signal is what is not random.

Signal is what is not **random.**
So what is **random?**

Examples in Random Topology

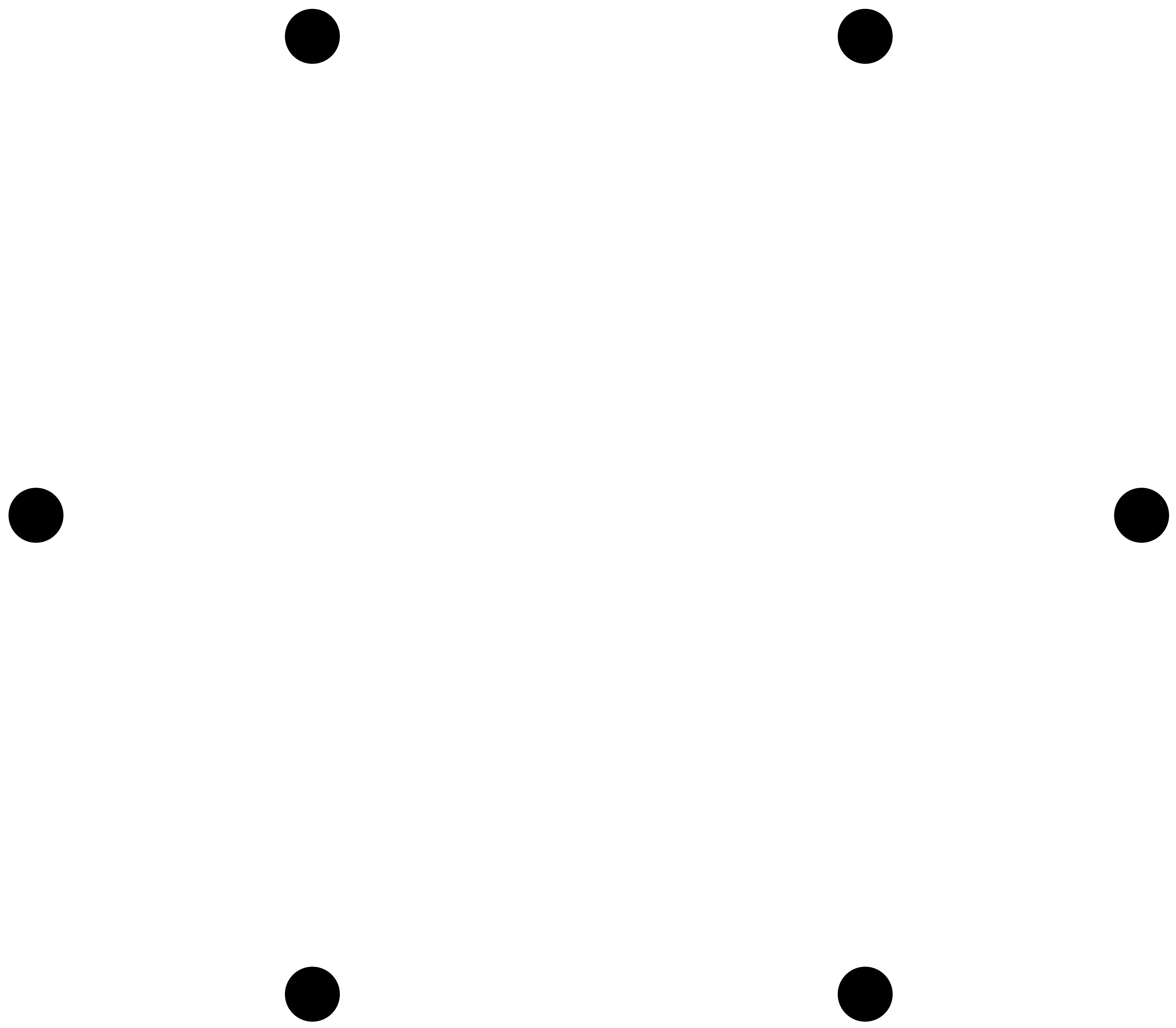


Erdo-Renyi Complexes

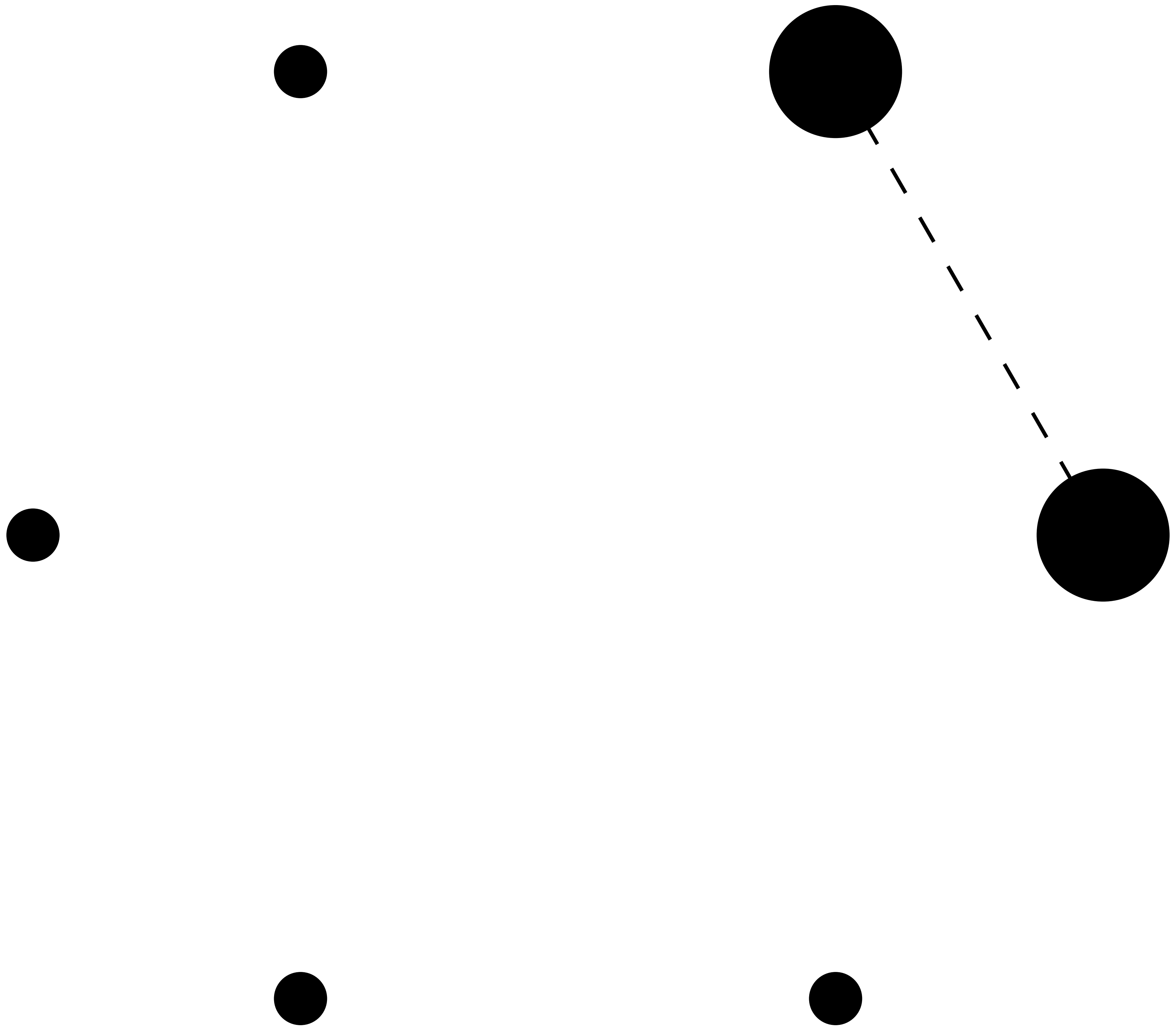


Geometric Complexes

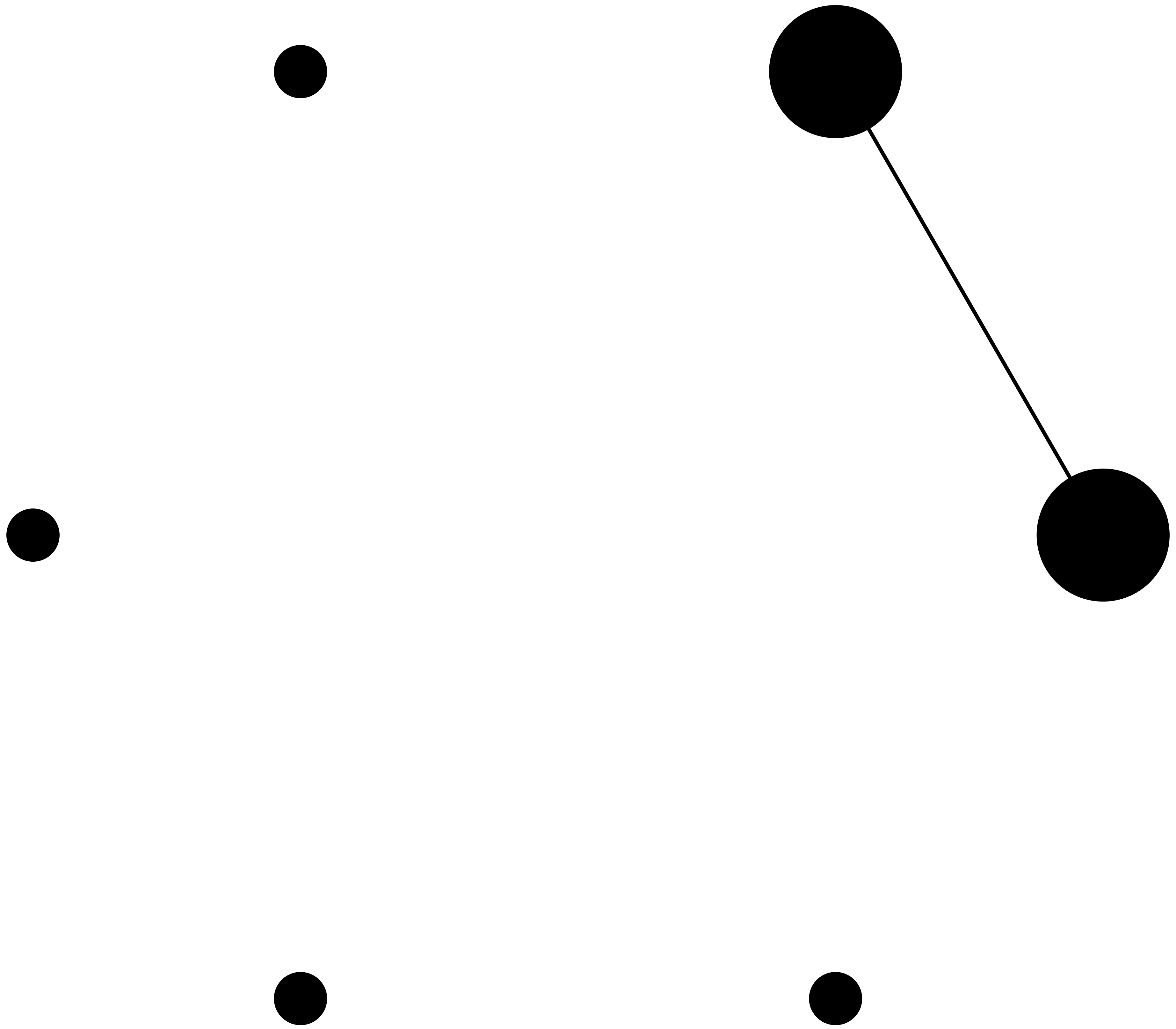
Erdos-Renyi graphs



Erdos-Renyi graphs



Erdos-Renyi graphs



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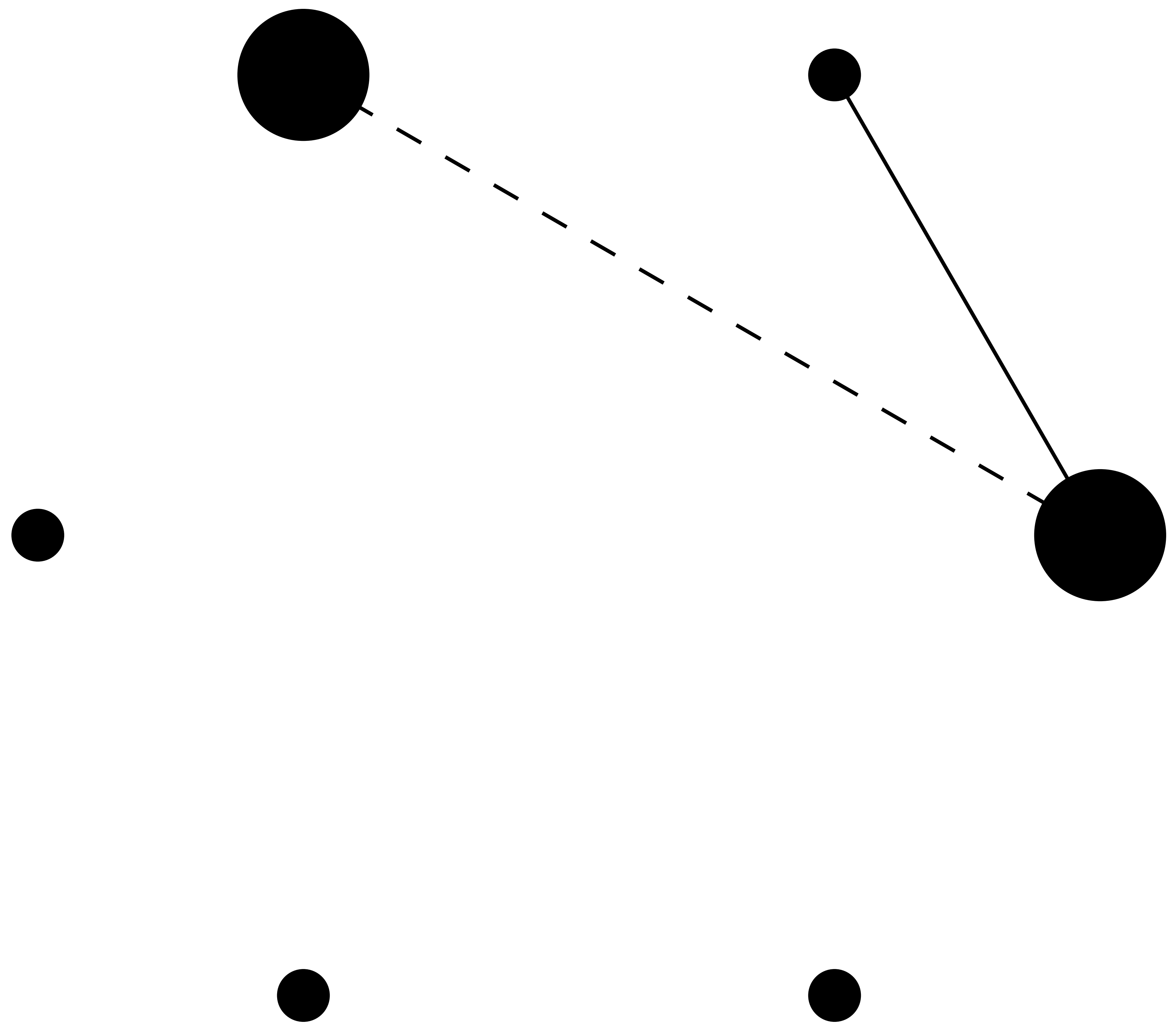

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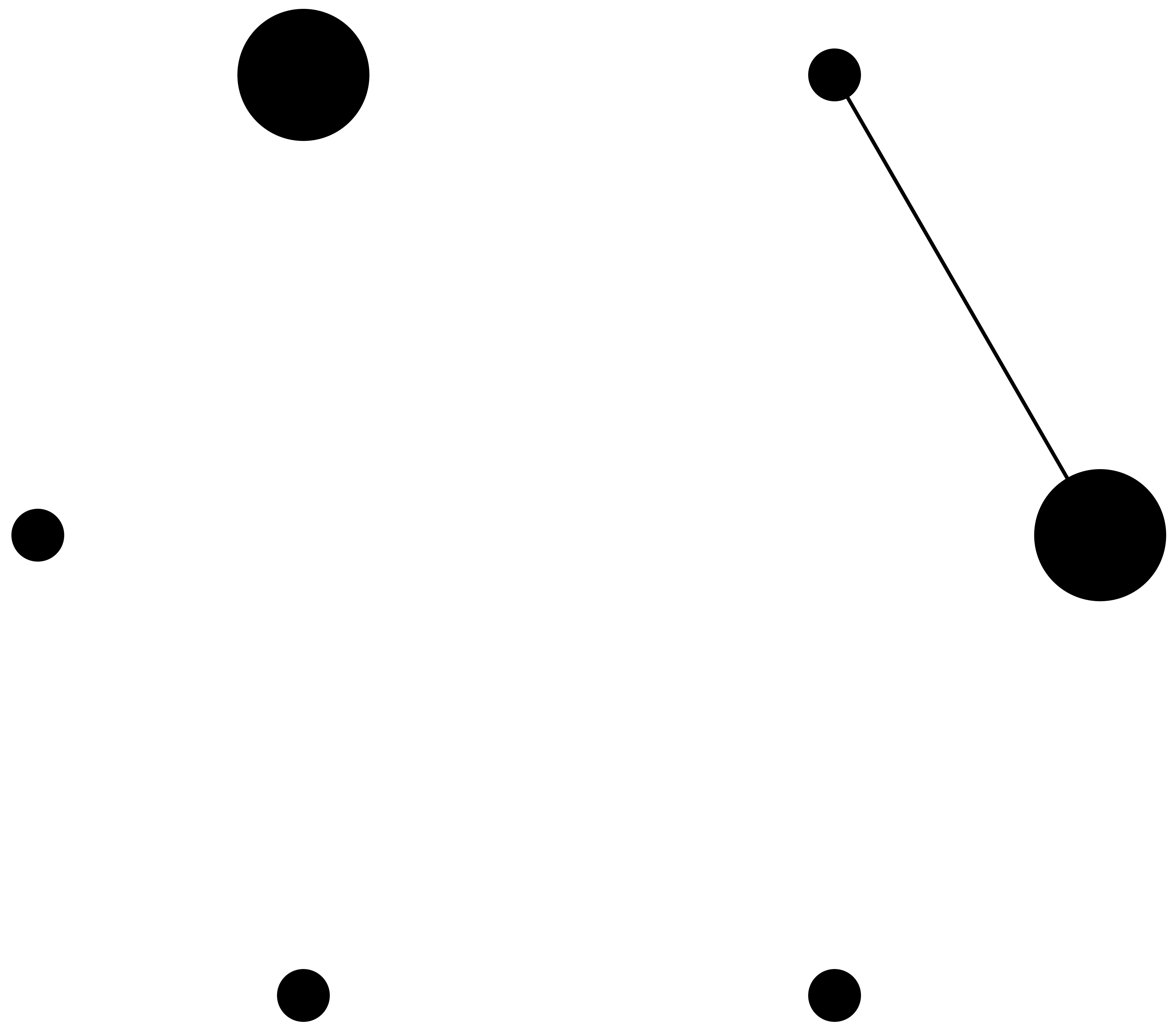

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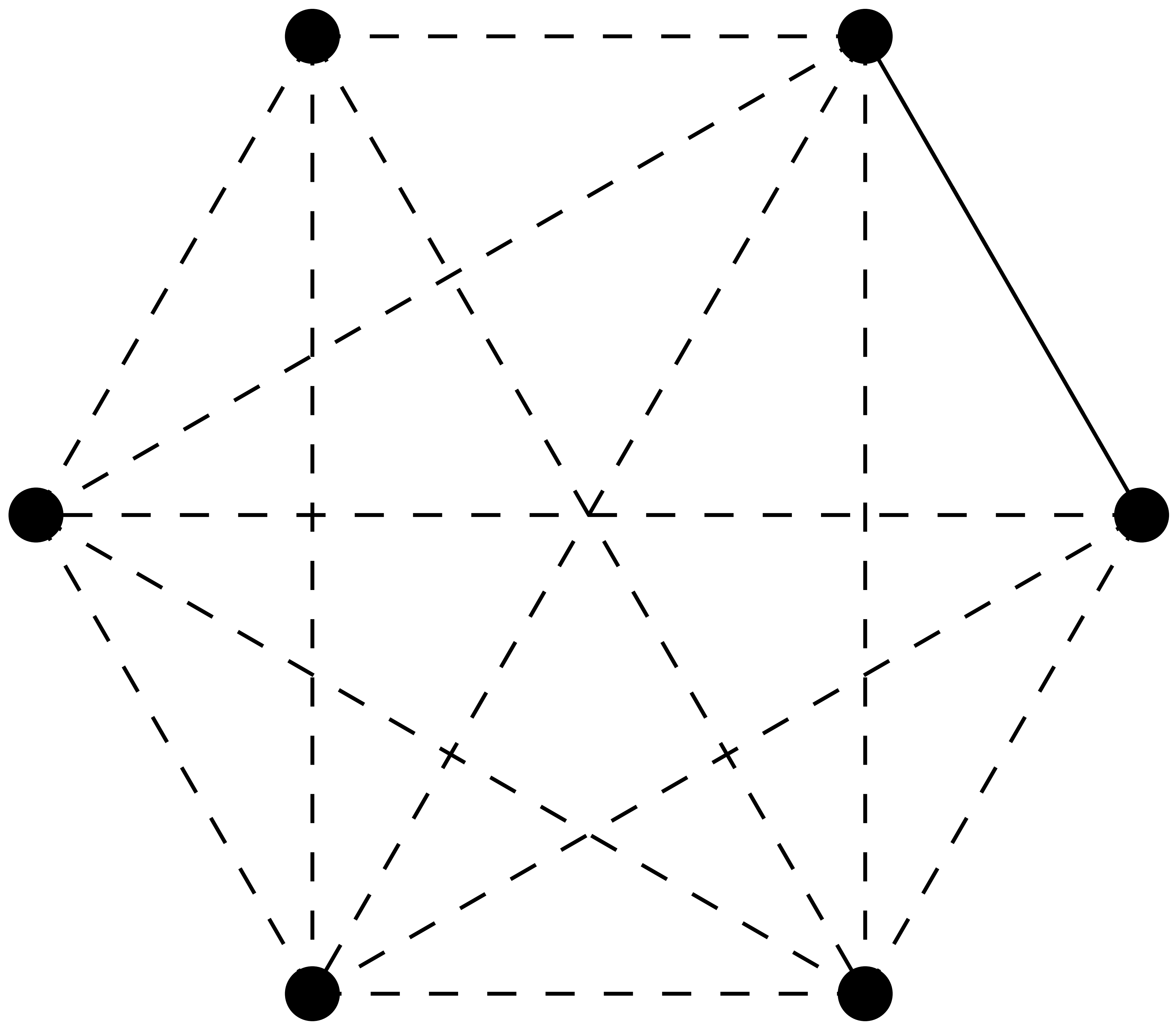

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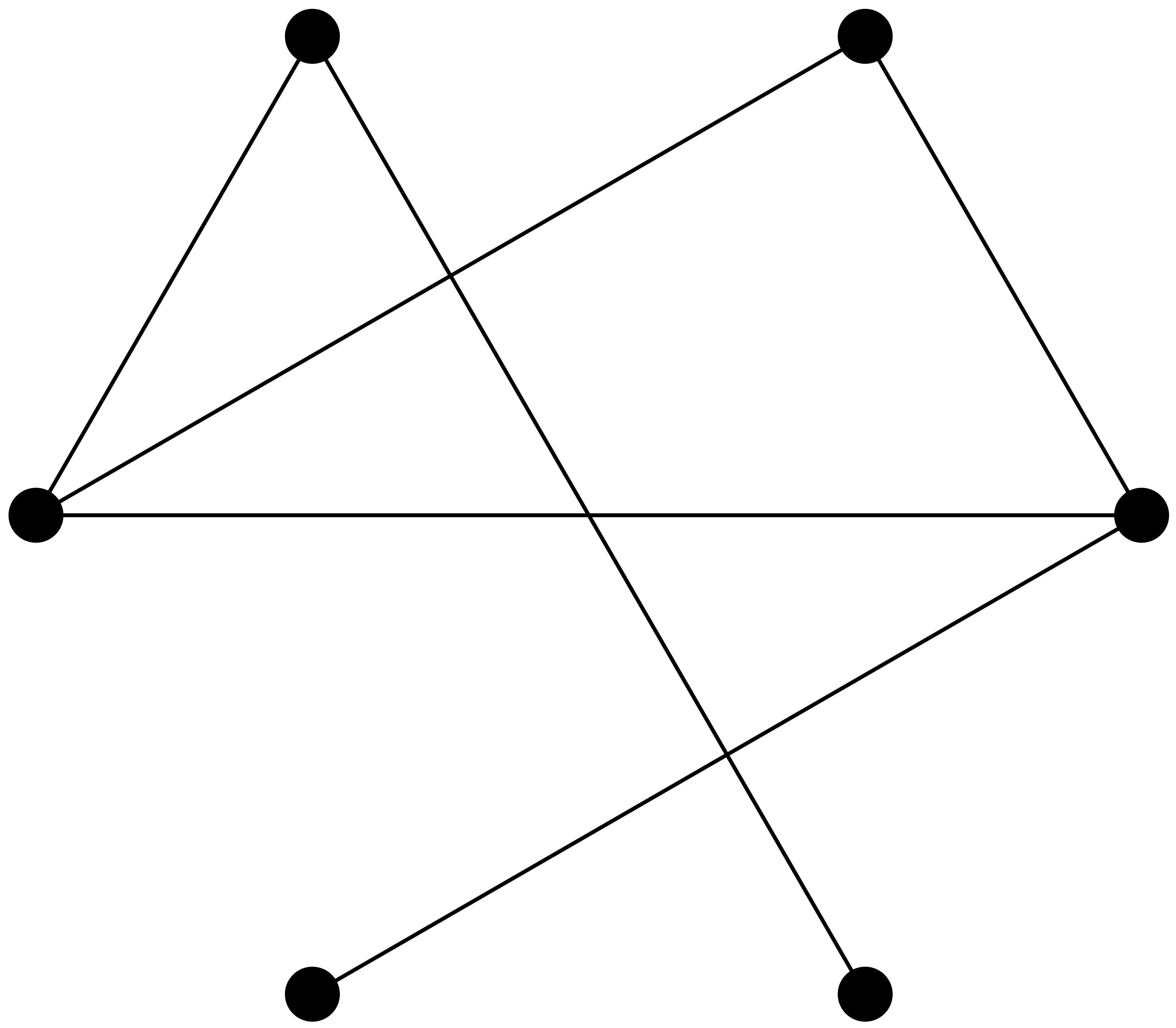

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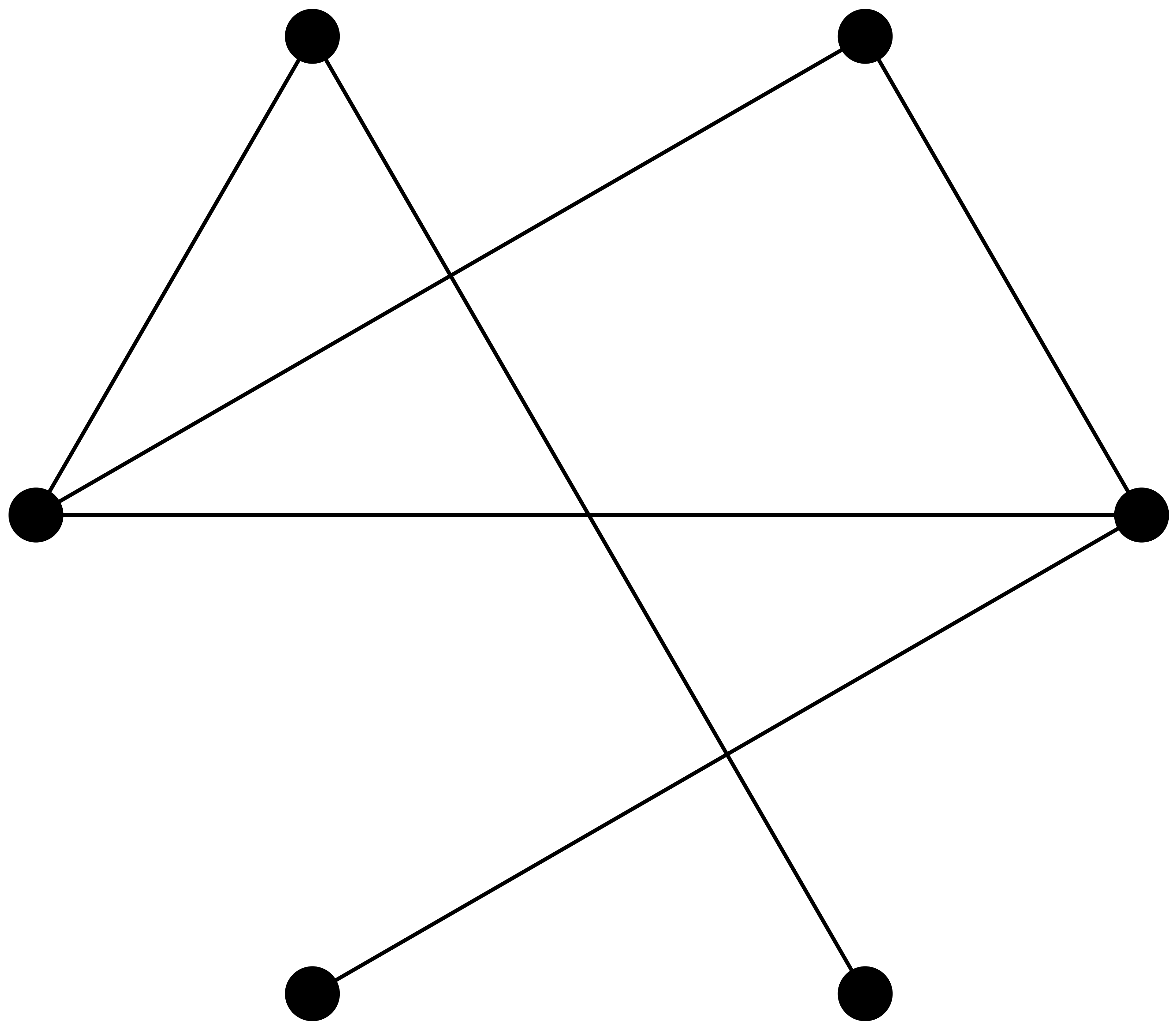
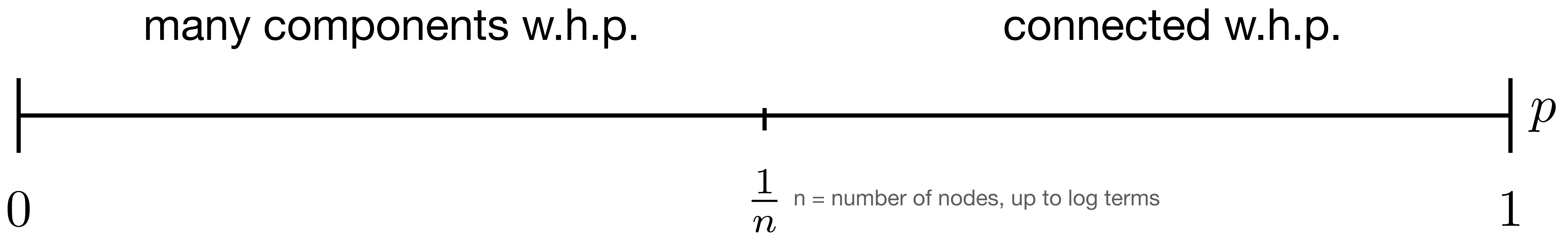


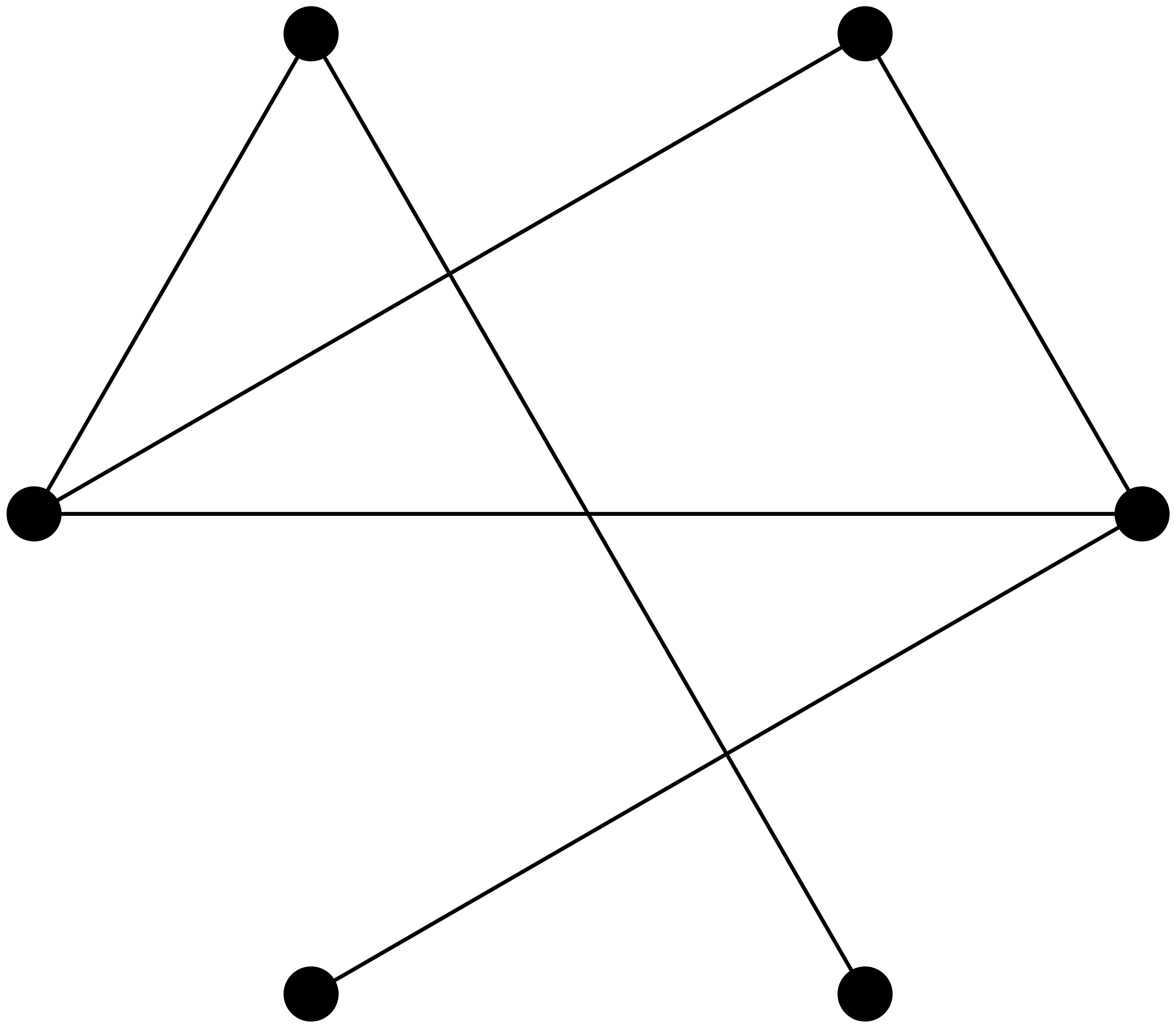
Phase Transition

[Erdos-Renyi 1960]



all log terms and constants forgone

Erdos-Renyi Clique Complex



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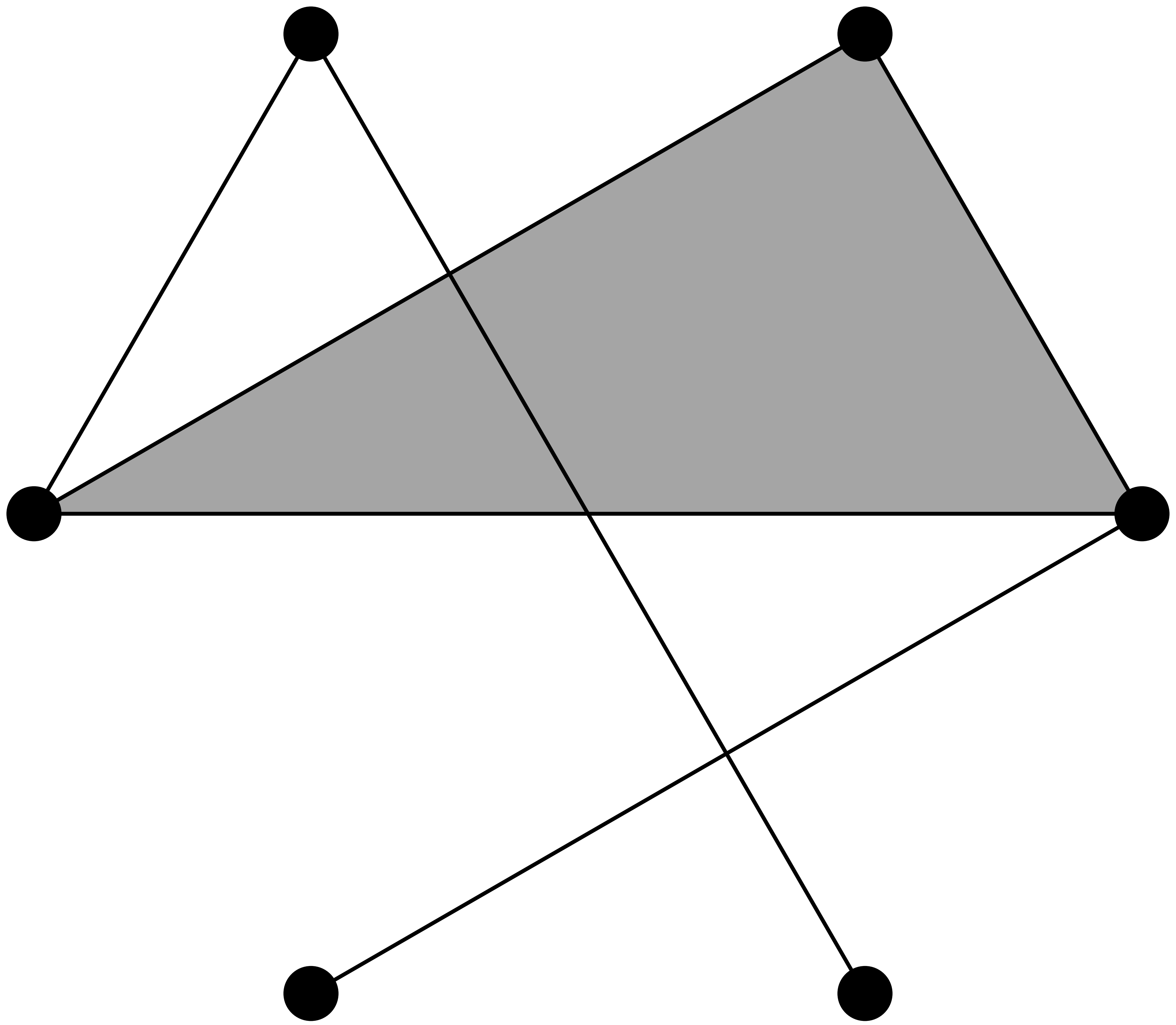
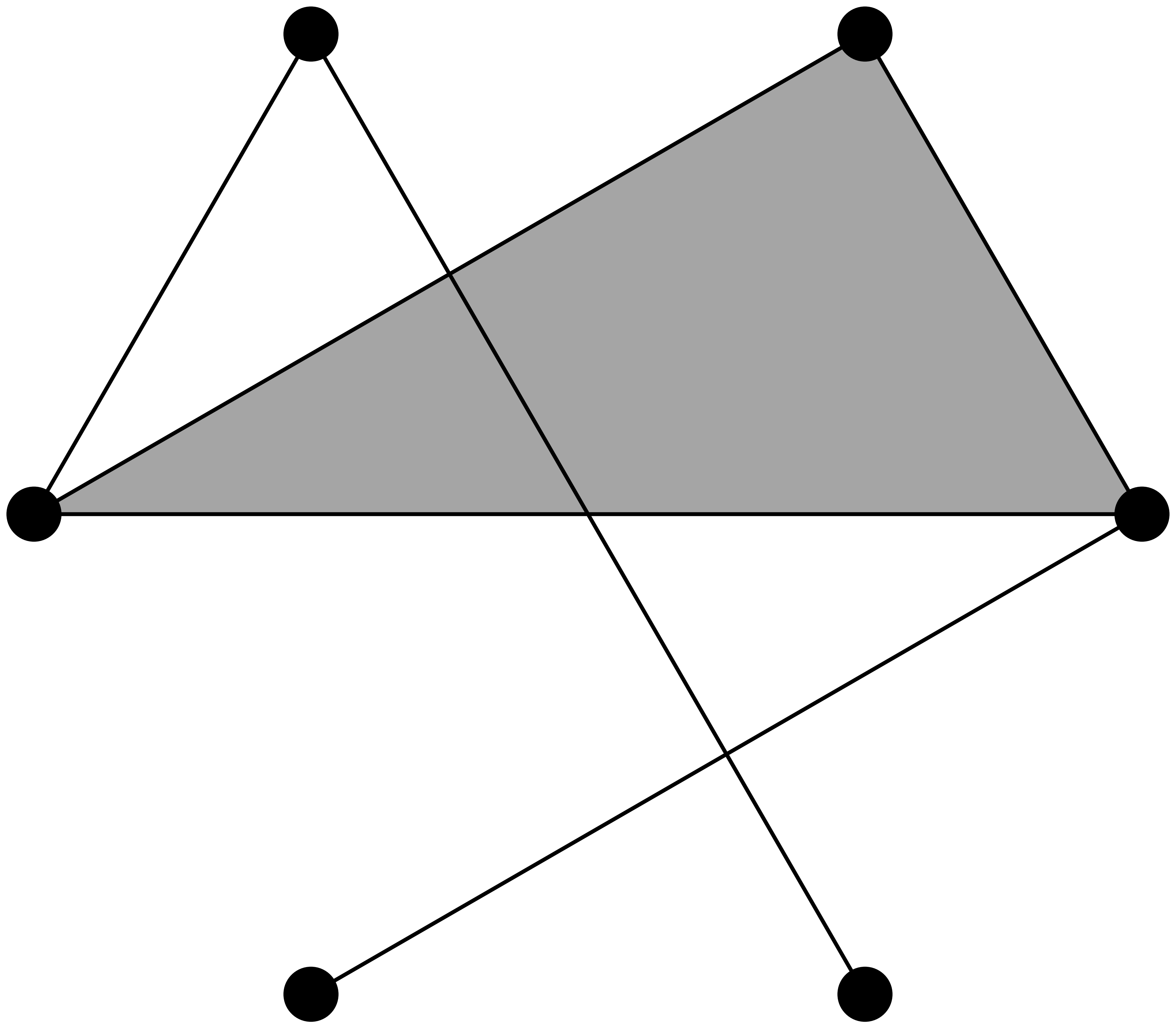
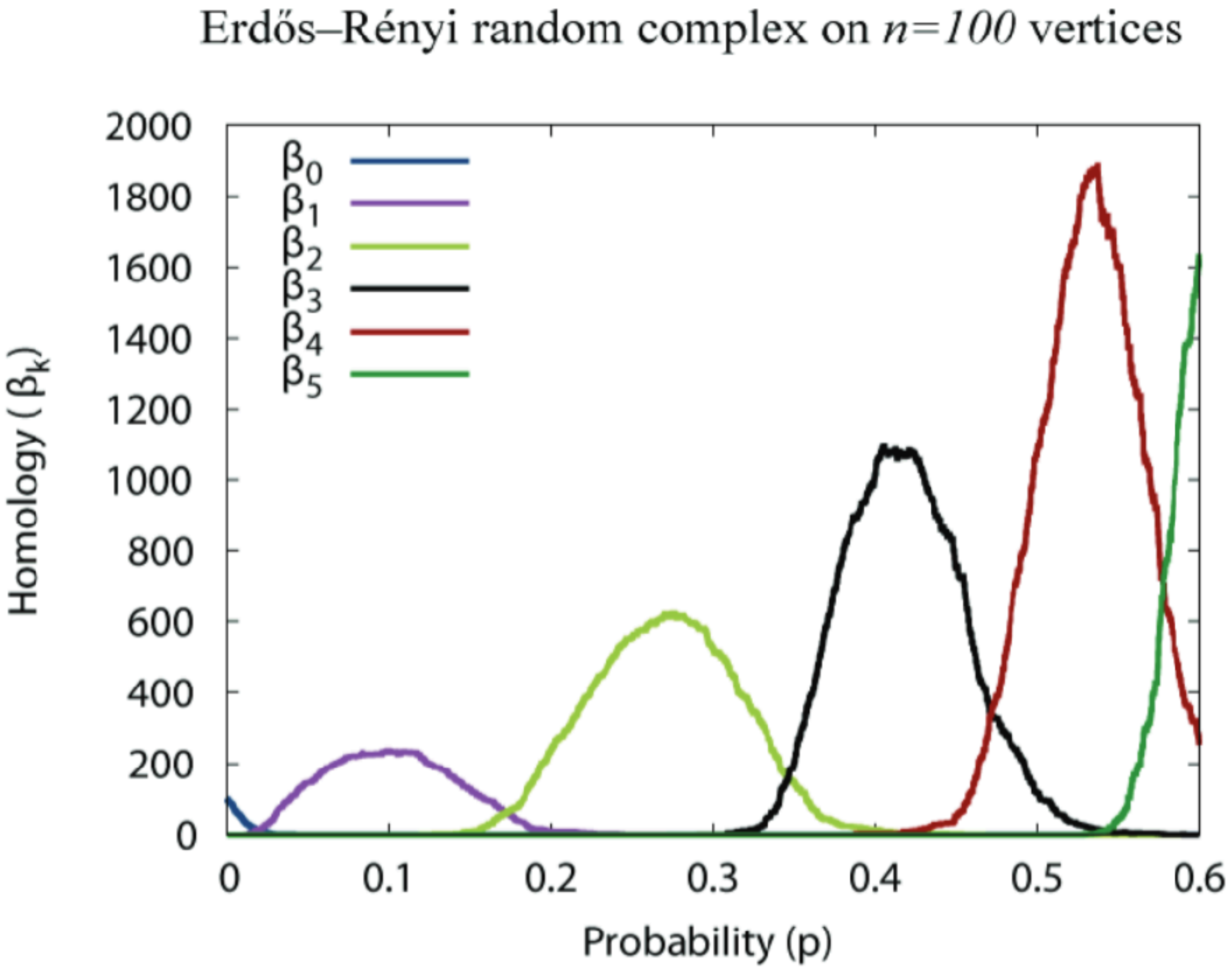


Betti Numbers

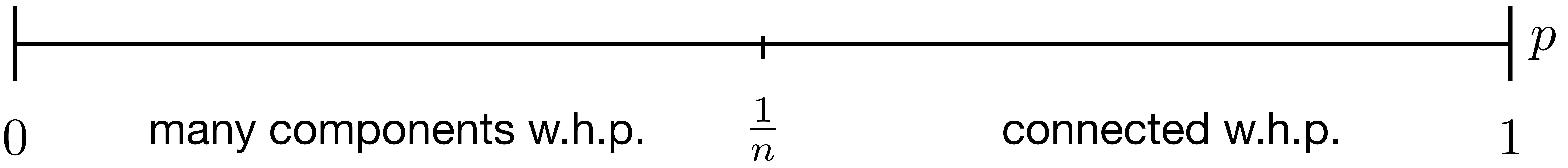




computation and plotting done by Zomorodian

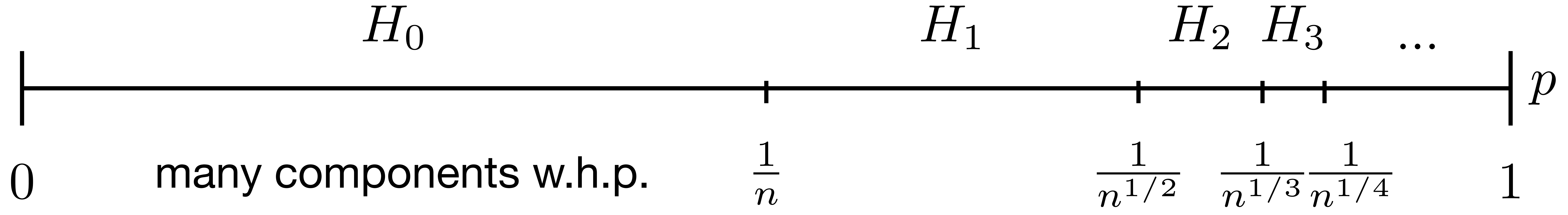
Phase Transition

[Erdos-Renyi 1960]



n = number of nodes
all log terms and constants forgone

Phase Transition
[Kahle 2009, 2014]



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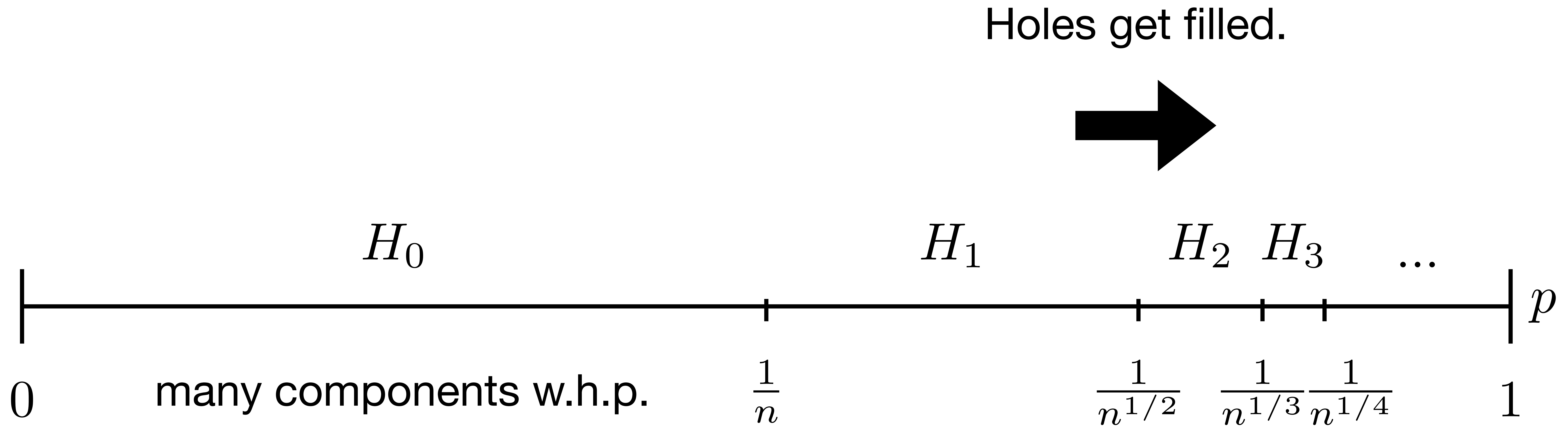

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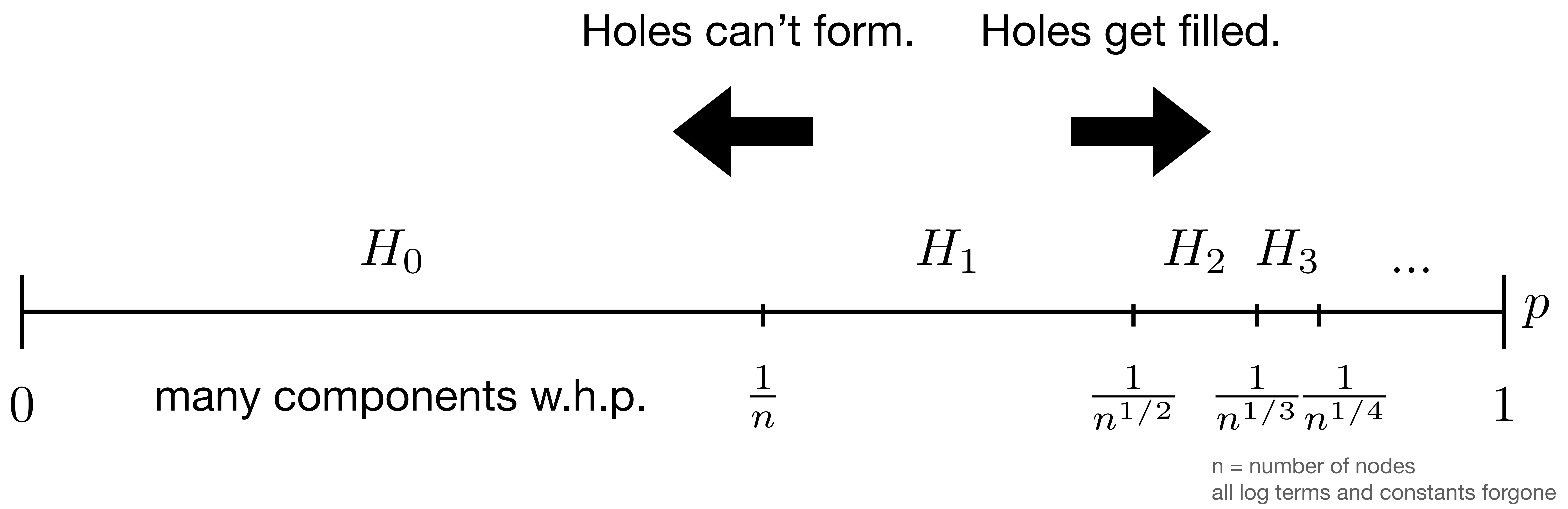
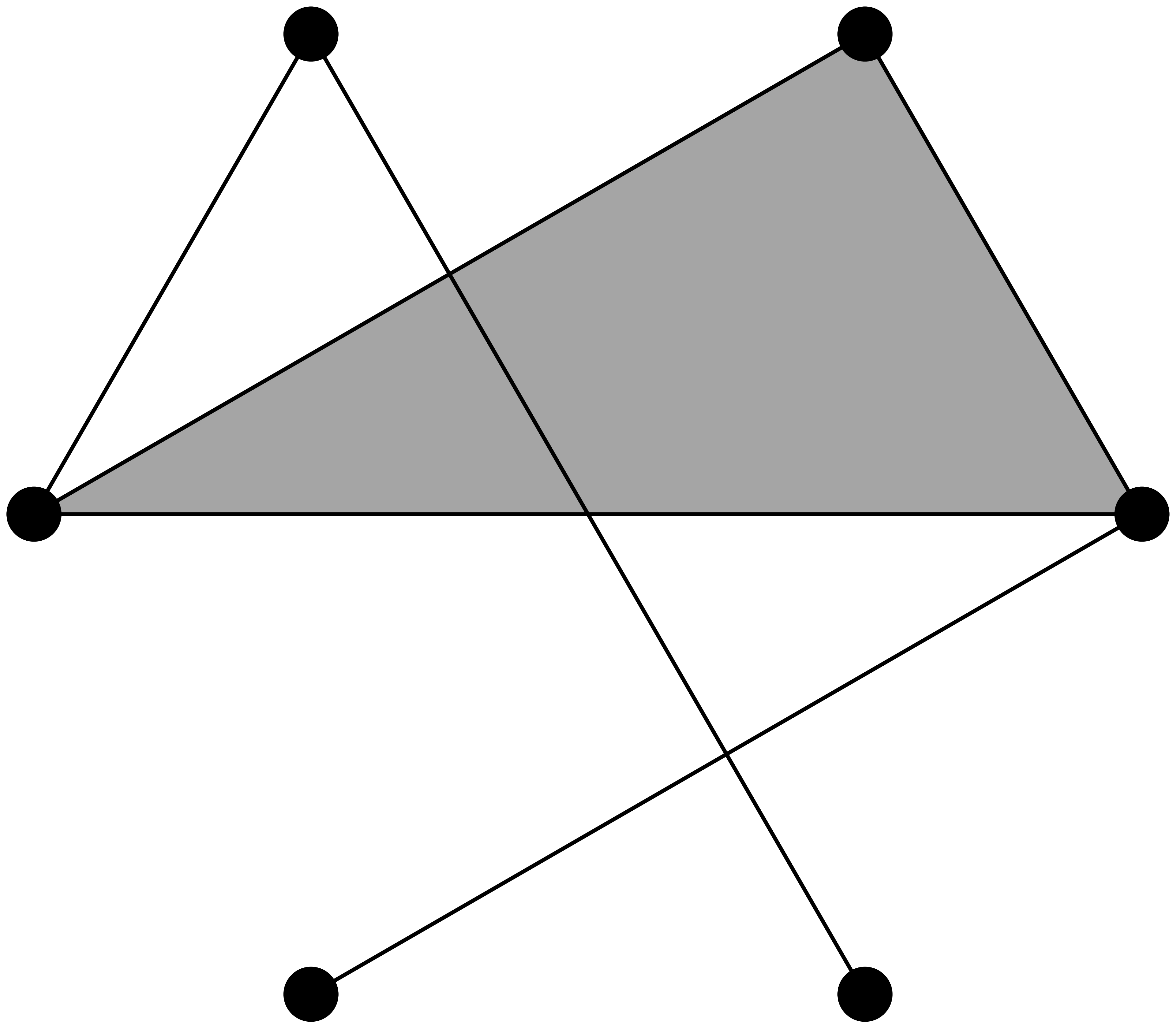


Erdos-Renyi Clique Complex



Geometric Complexes

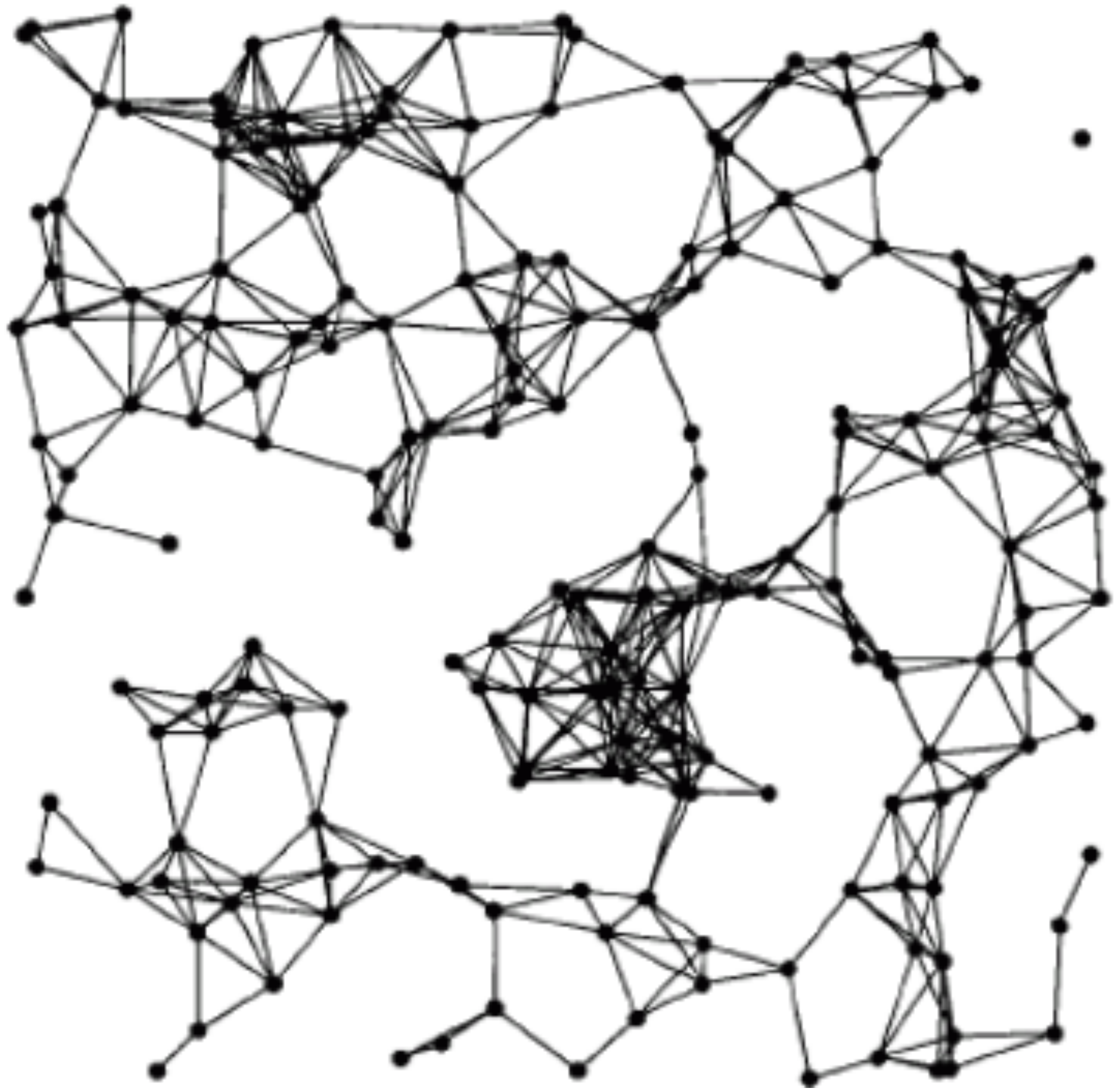


image credit: Penrose

Expected Betti numbers at dimension k

[Kahle 2011]

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- n, the number of points

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- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension

Expected Betti numbers at dimension k

[Kahle 2011]

- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension

$E\beta_k(\text{Cech}) \to 0$

$E\beta_k(\text{Cech}) \to \infty$

$O(\omega^k e^{-c\omega} n)$

under convexity assumption

r

0

very sparse

$n^{-\frac{1}{D}\left(1 - \frac{1}{k+2}\right)}$

sparse

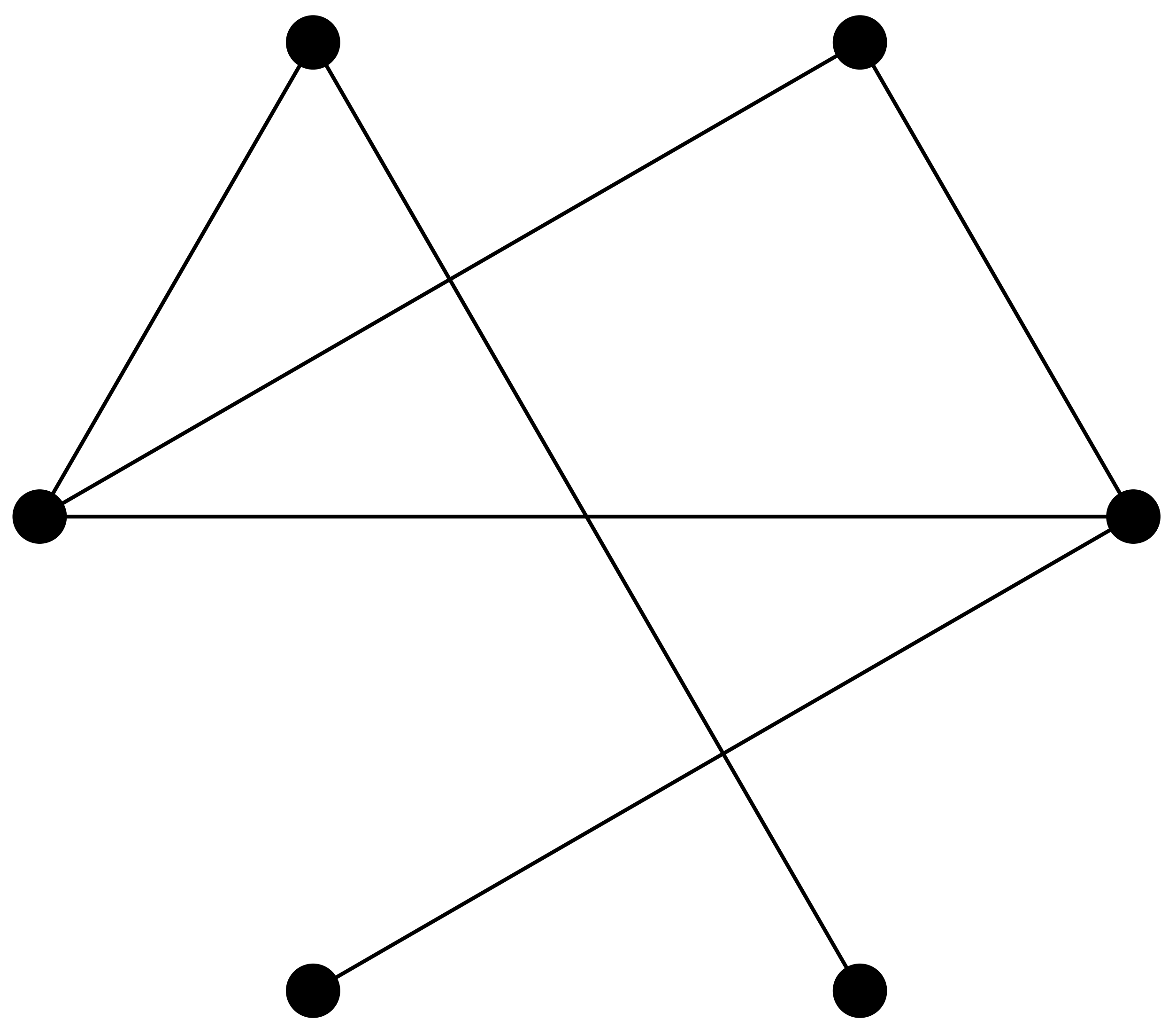
$n^{-1/D}$

dense

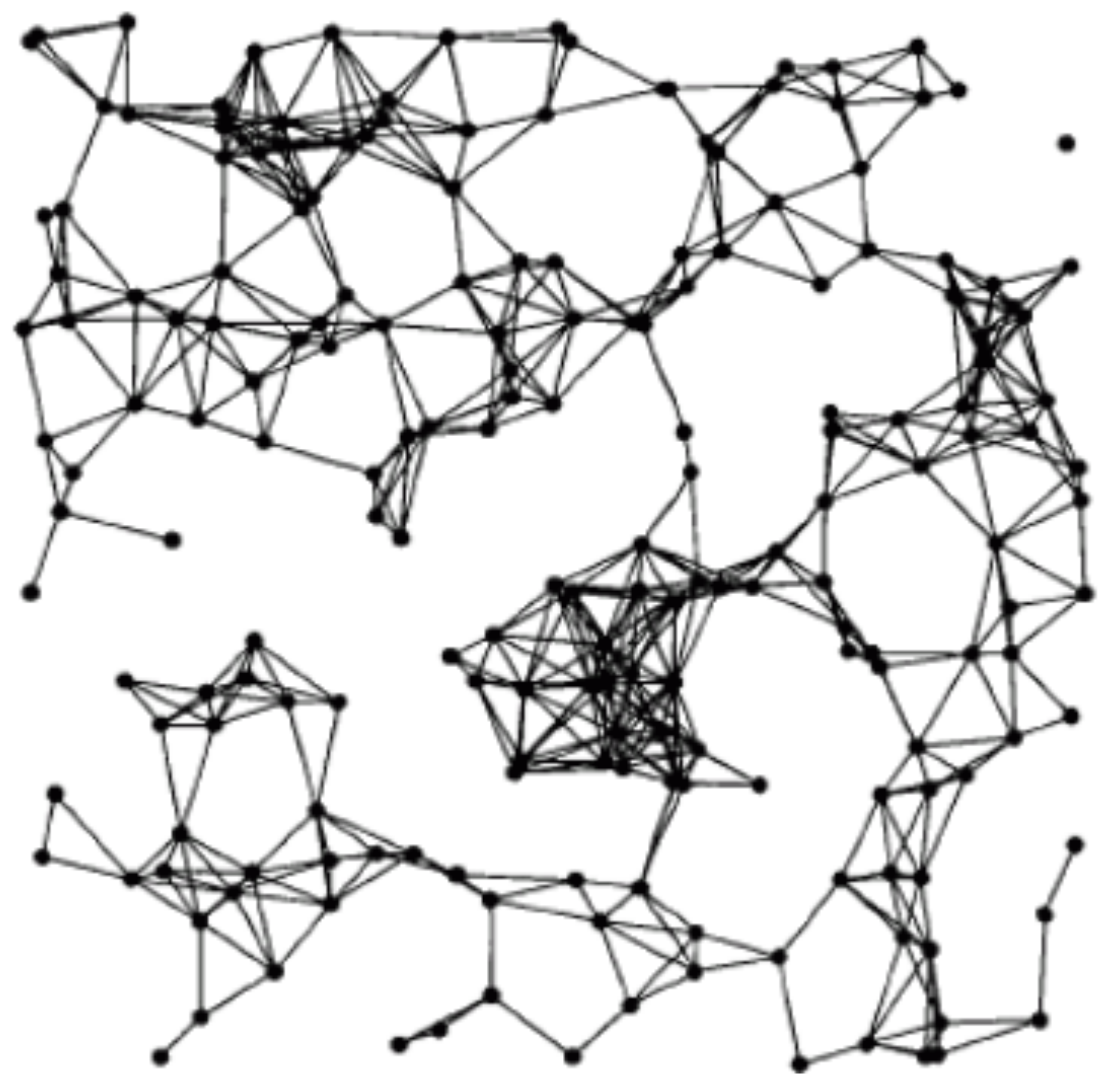
n = number of nodes
all log terms and constants forgone

$E\beta_k(\text{Cech}) \sim \omega^{2k+1} n$

Examples in Random Topology



Erdo-Renyi Complexes



Geometric Complexes

II. Preferential Attachment

Beyond independence and homogeneity

Independent and identically distributed?

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(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Preferential Attachment

[Albert and Barabasi 1999]

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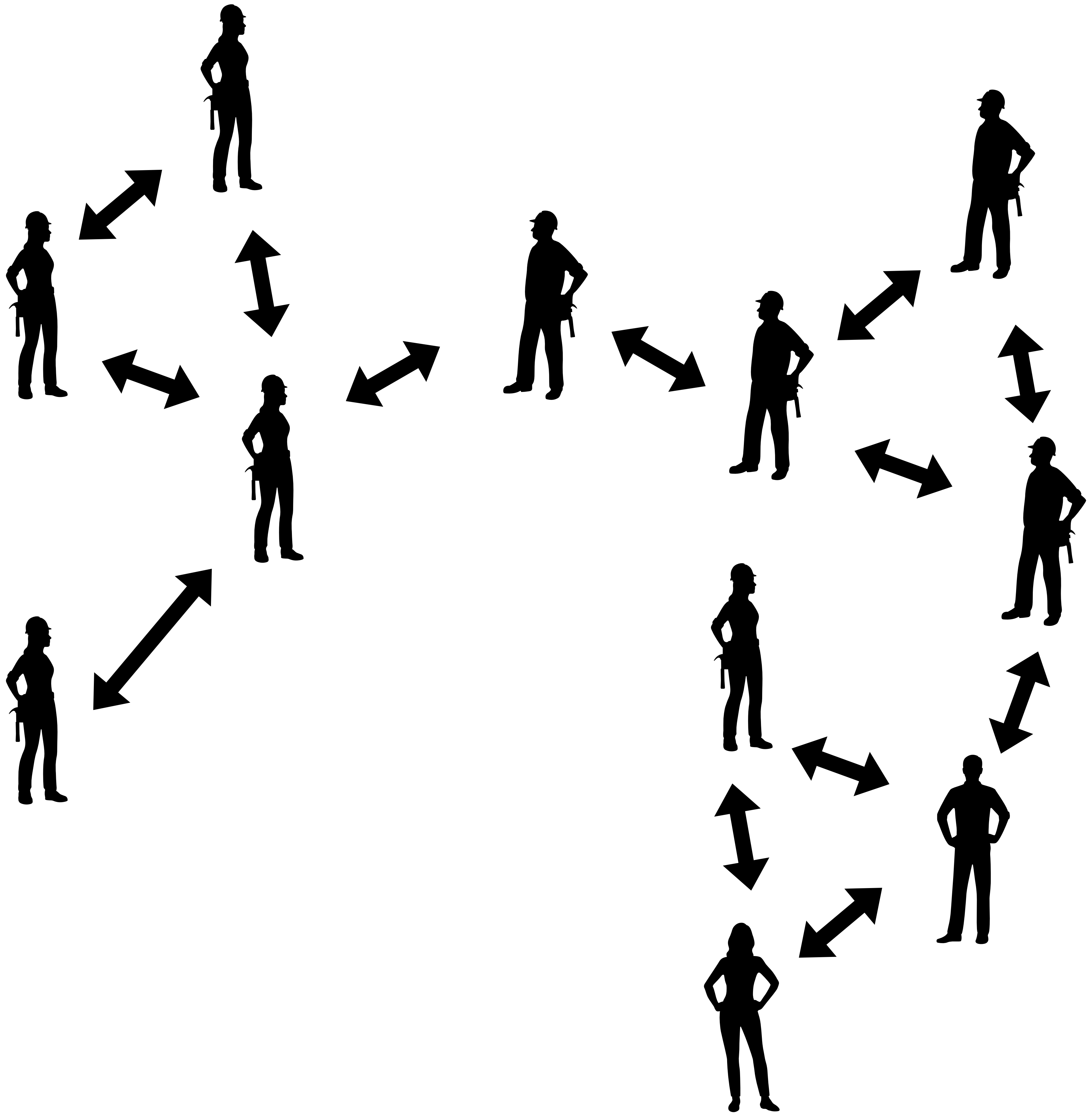
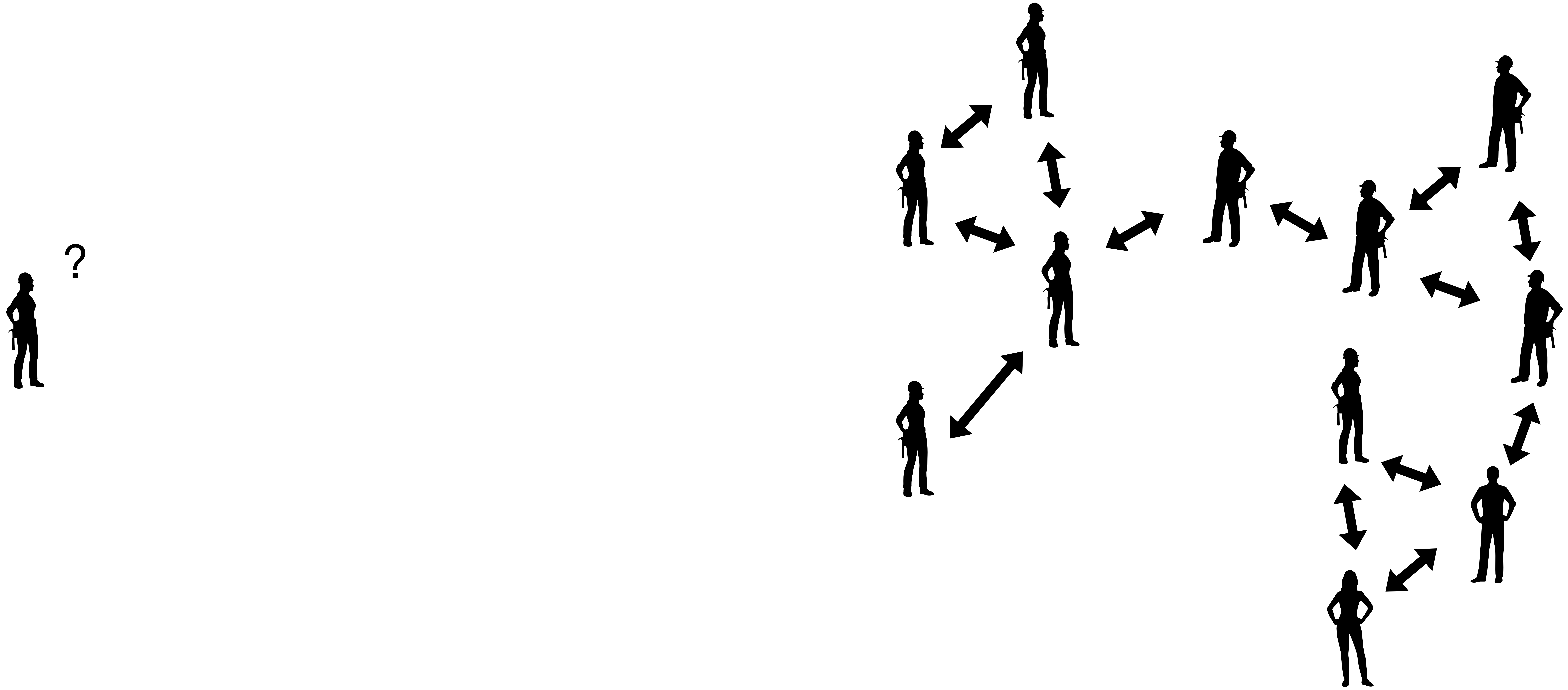


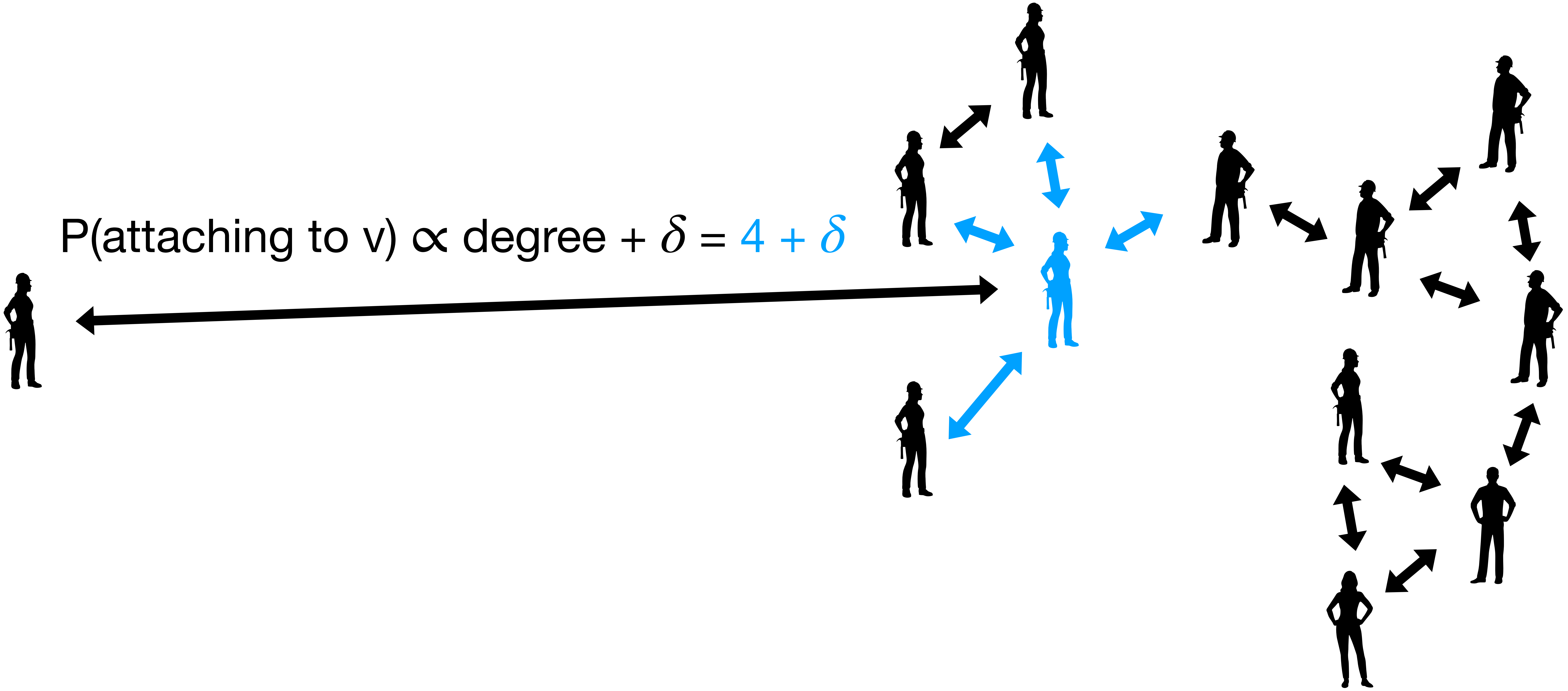
Preferential Attachment

[Albert and Barabasi 1999]



Preferential Attachment

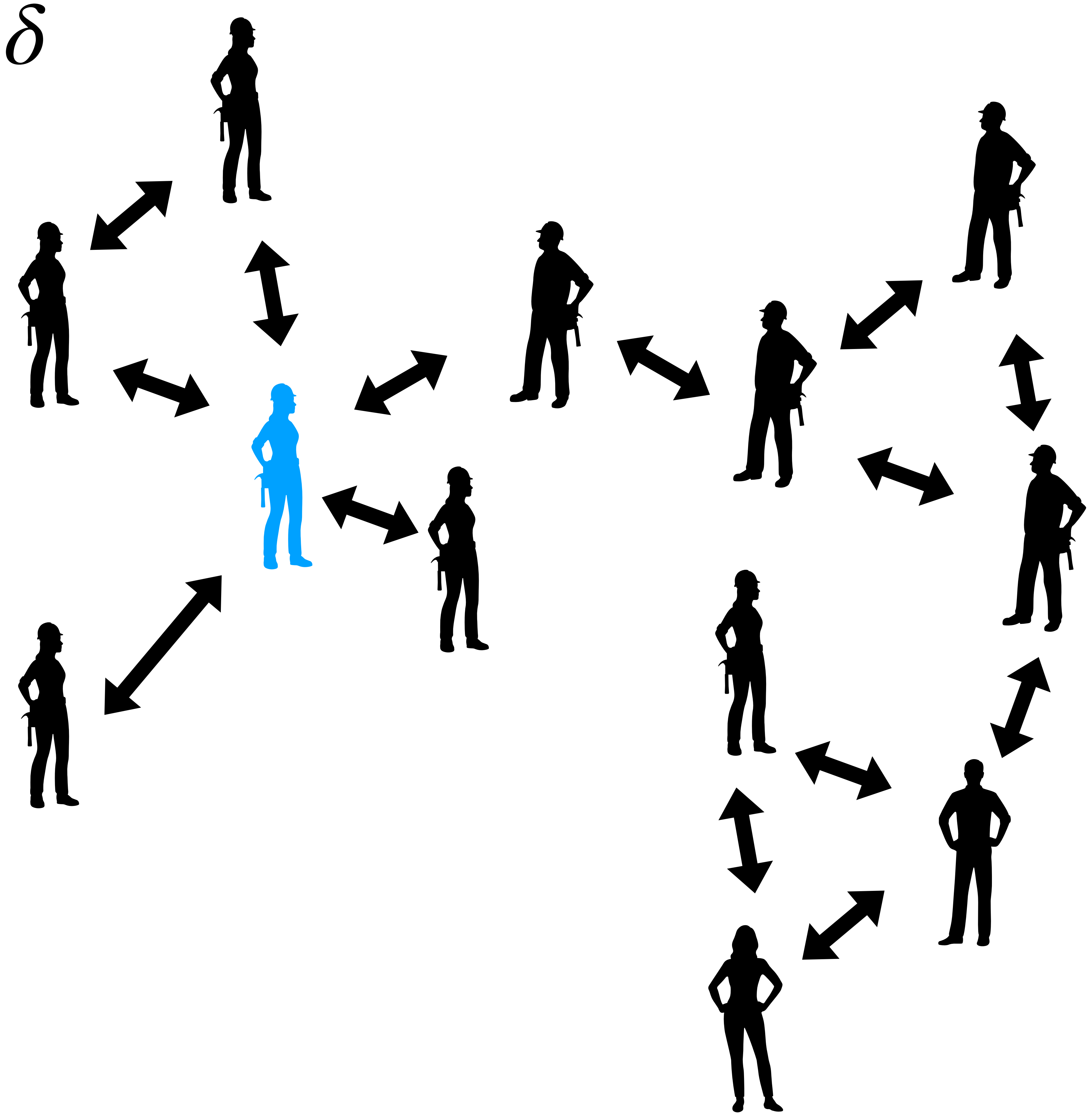
[Albert and Barabasi 1999]



Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) $\propto$ degree + a tuning parameter δ



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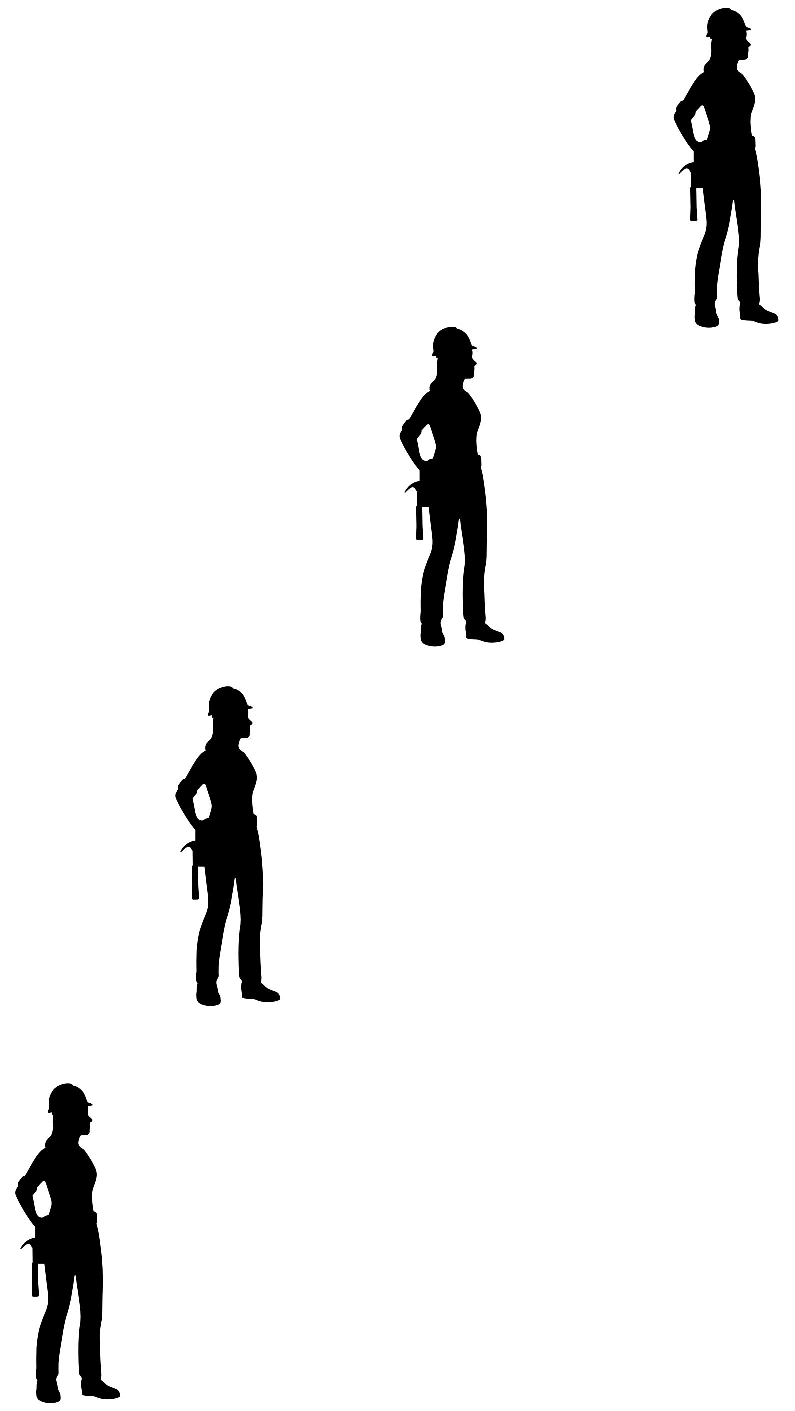

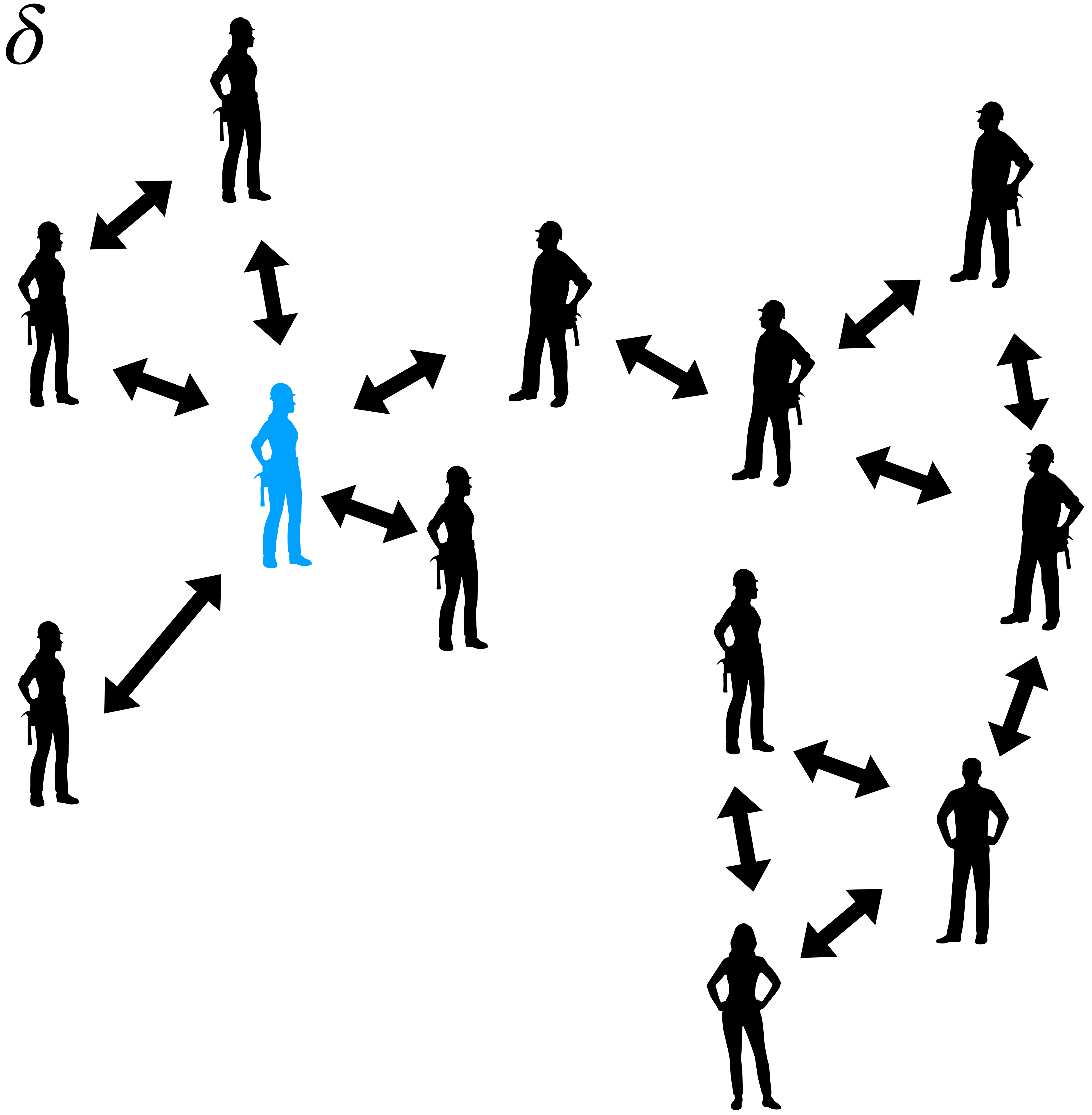

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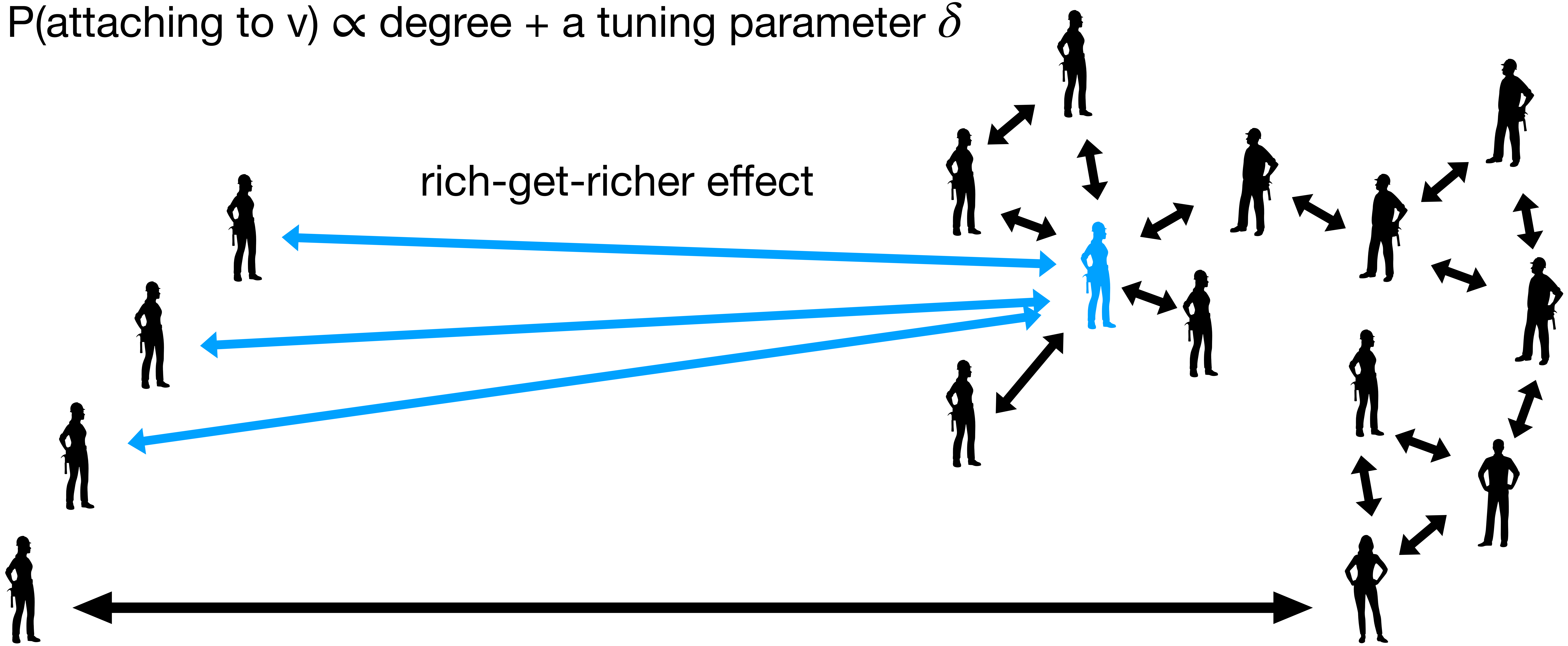

What do we know?

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- degree distribution [Albert and Barabasi 1999]

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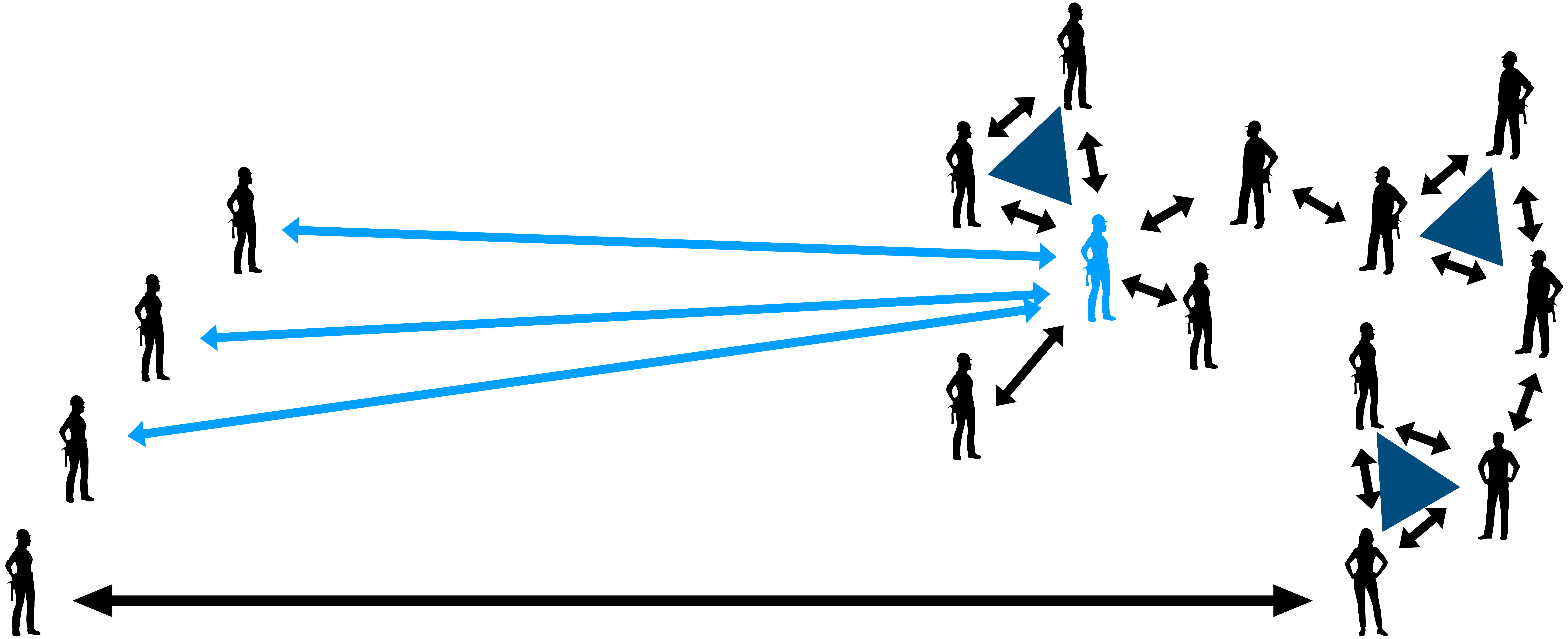
- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]

What do we know?

- degree distribution [Albert and Barabasi 1999]
- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]
- and more…

Clique Complex

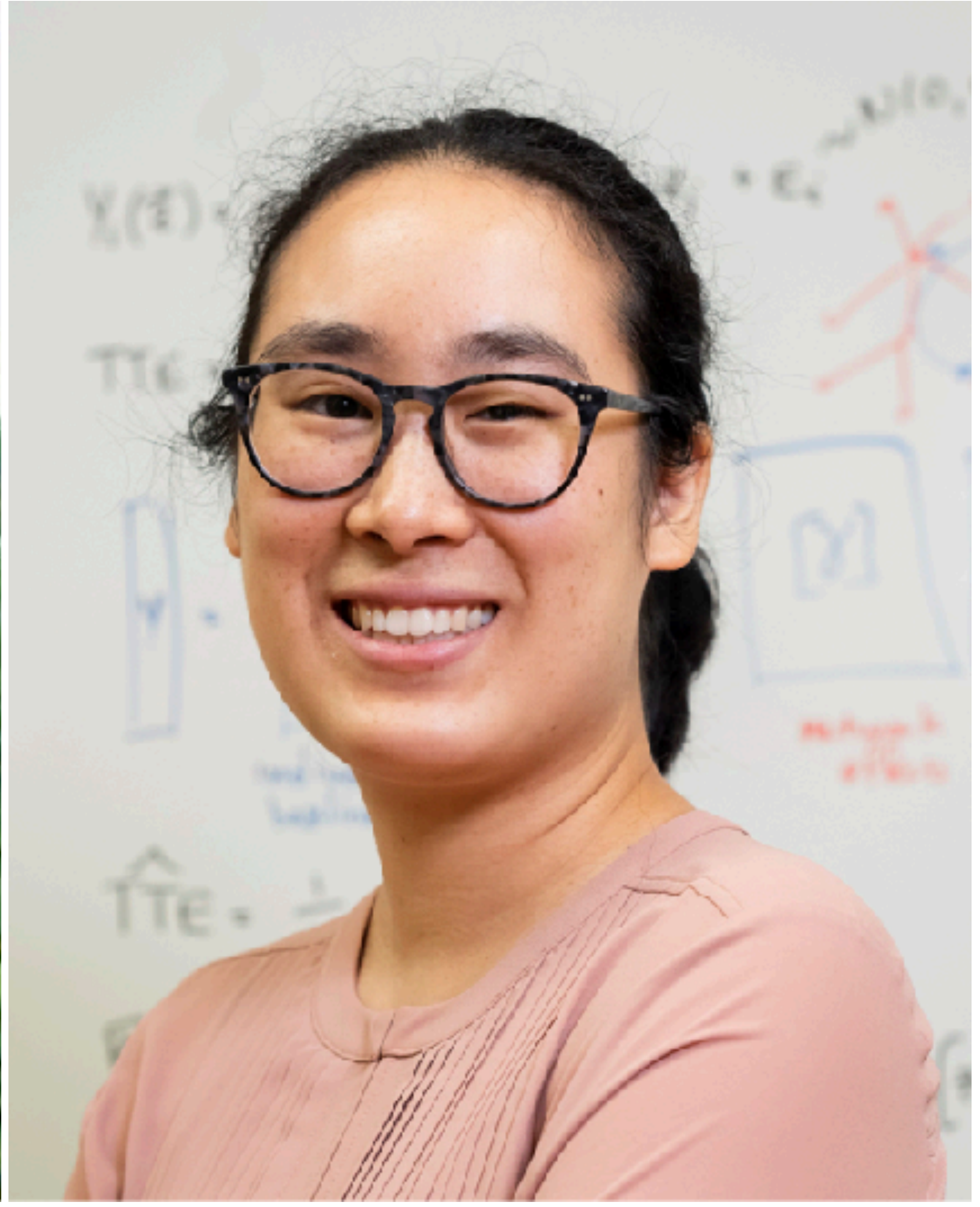
aka Flag Complex



III Topology of Preferential Attachment

My Lovely Collaborators

Avhan Misra



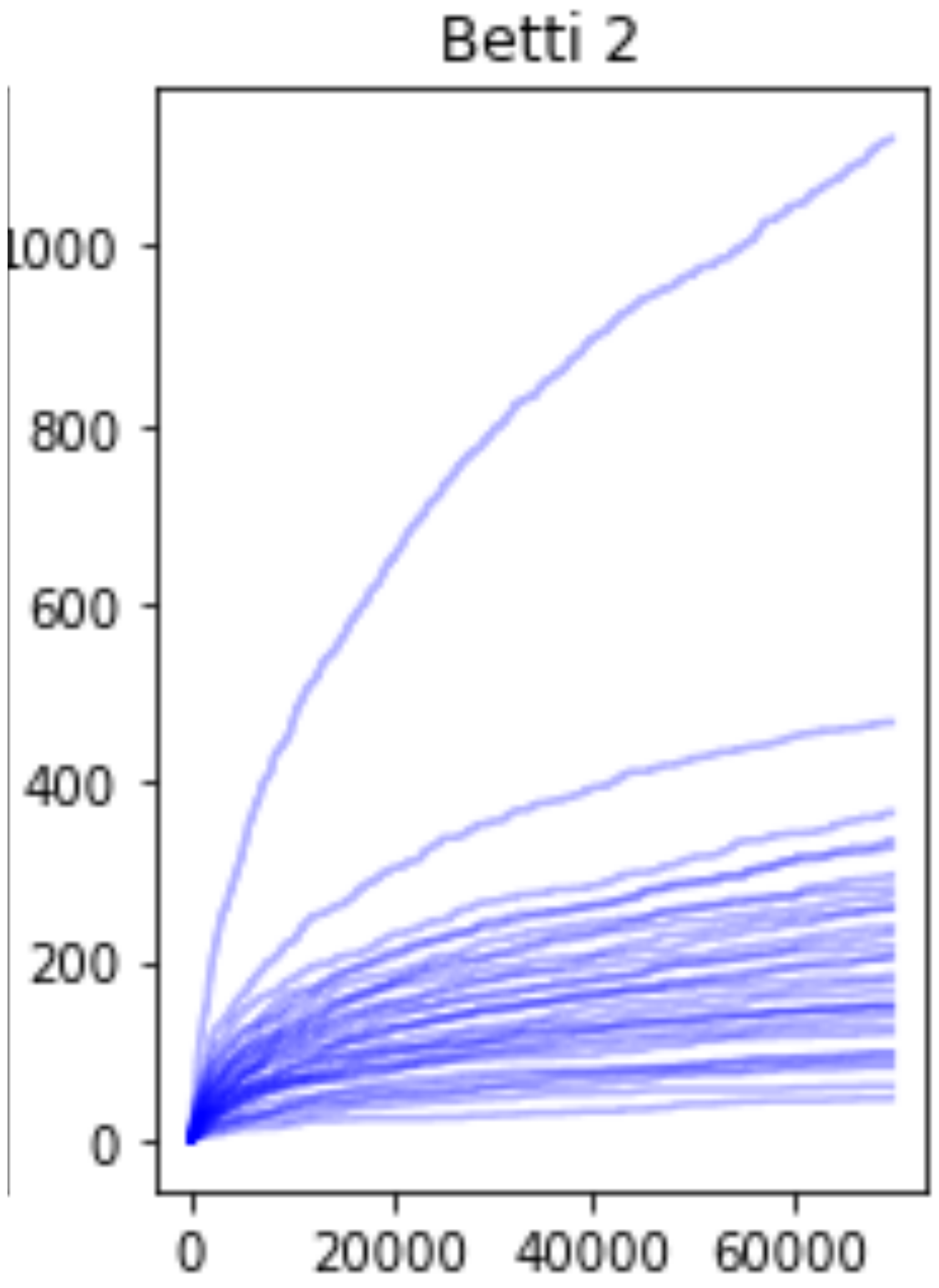
Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

Betti Number β_q

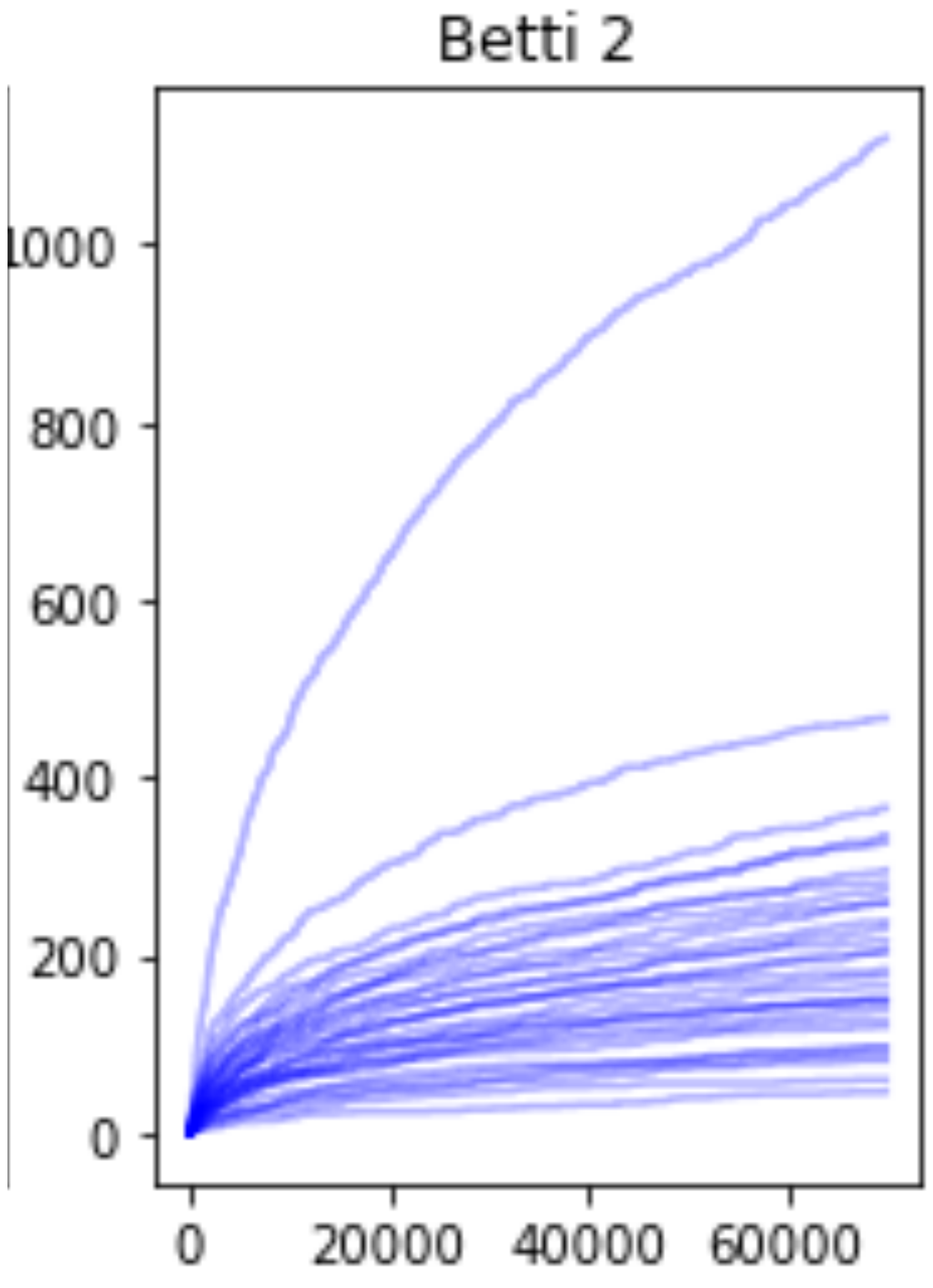
Betti Number β_q



Different curves, different random seeds.
All curves have the same model parameters.

Betti Number β_q

- increasing trend



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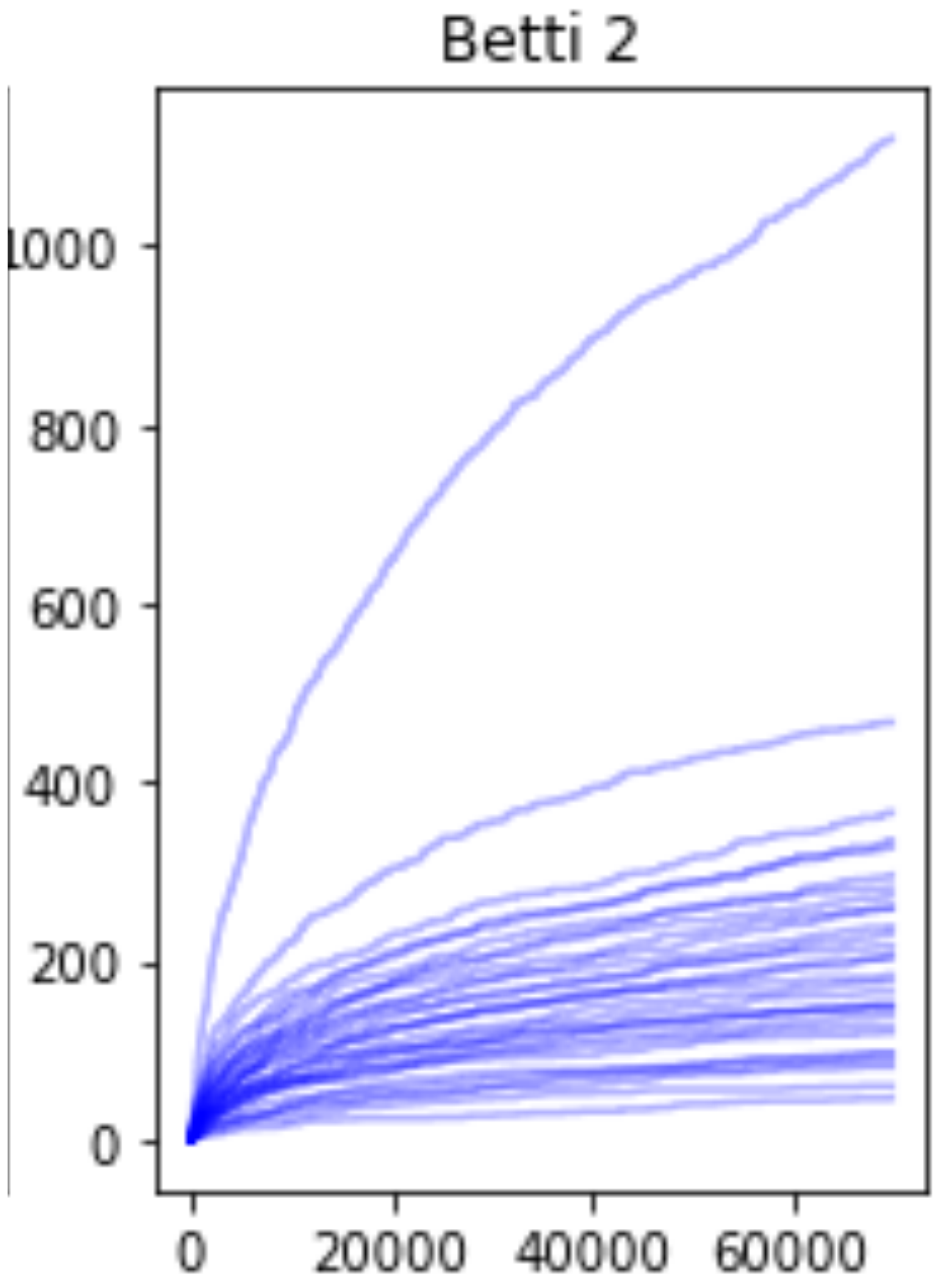
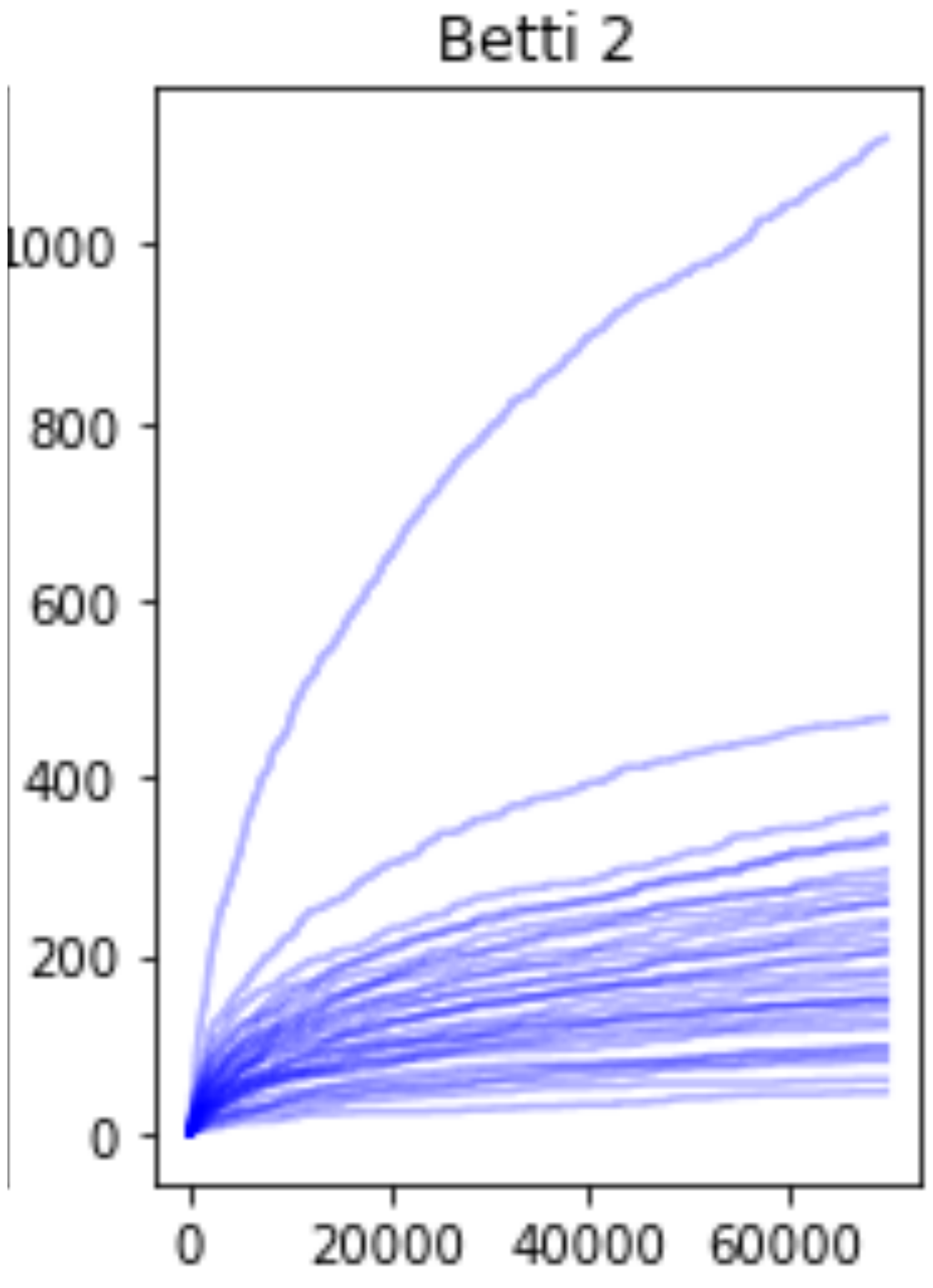


Different curves, different random seeds.
All curves have the same model parameters.

Betti Number β_q

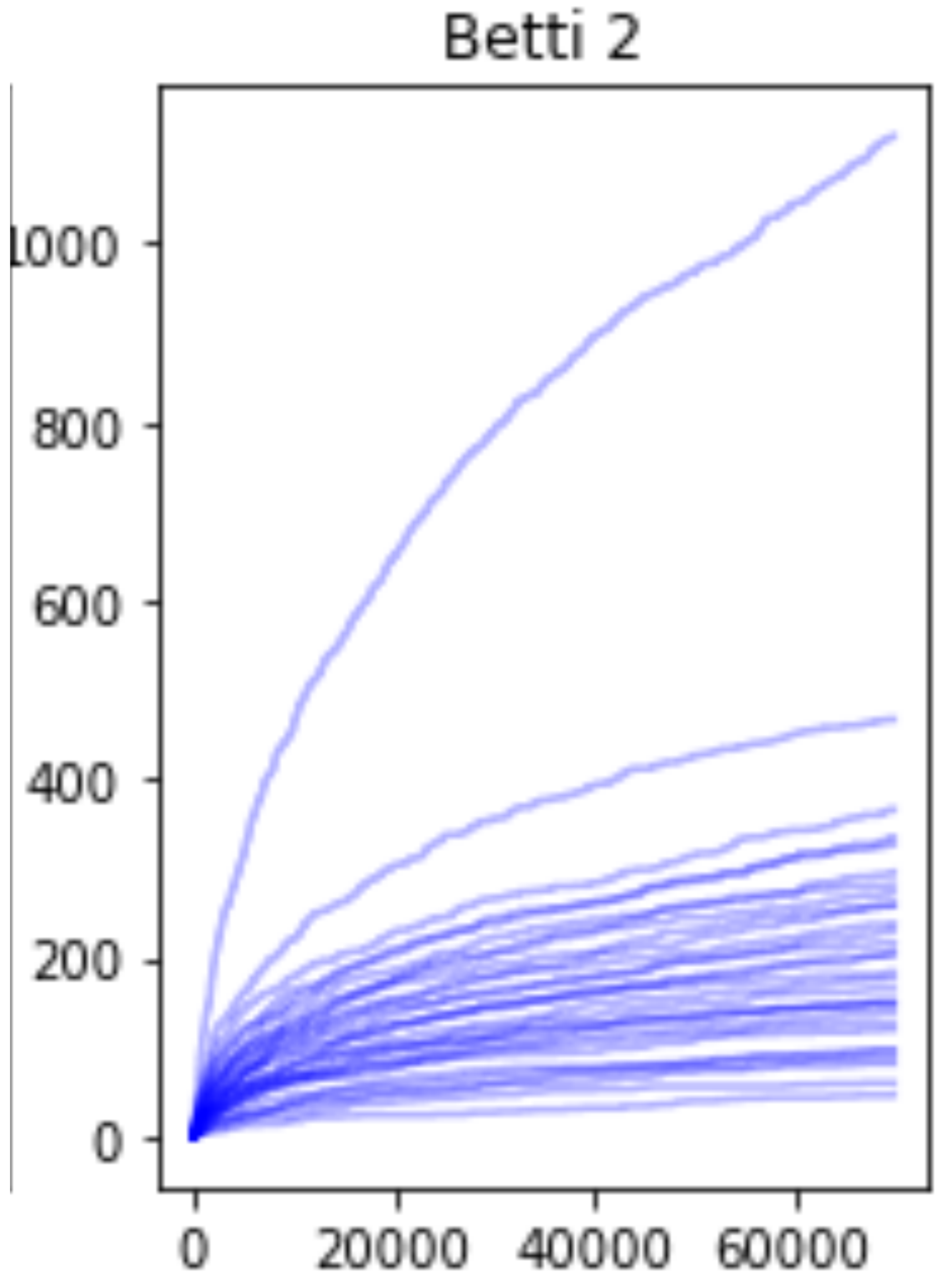
- increasing trend
- concave growth
- outlier



Different curves, different random seeds.
All curves have the same model parameters.

Betti Number β_q

- With probability at least $1 - \varepsilon$,
- c_ε(num of nodes^{1-4x}) $\leq \beta_2 \leq C_\varepsilon$(num of nodes^{1-4x})
  - $x \in (0,1/2)$ decreases with the preferential attachment strength
    - $P[T$ attaches to $i] \propto T^{-x}$



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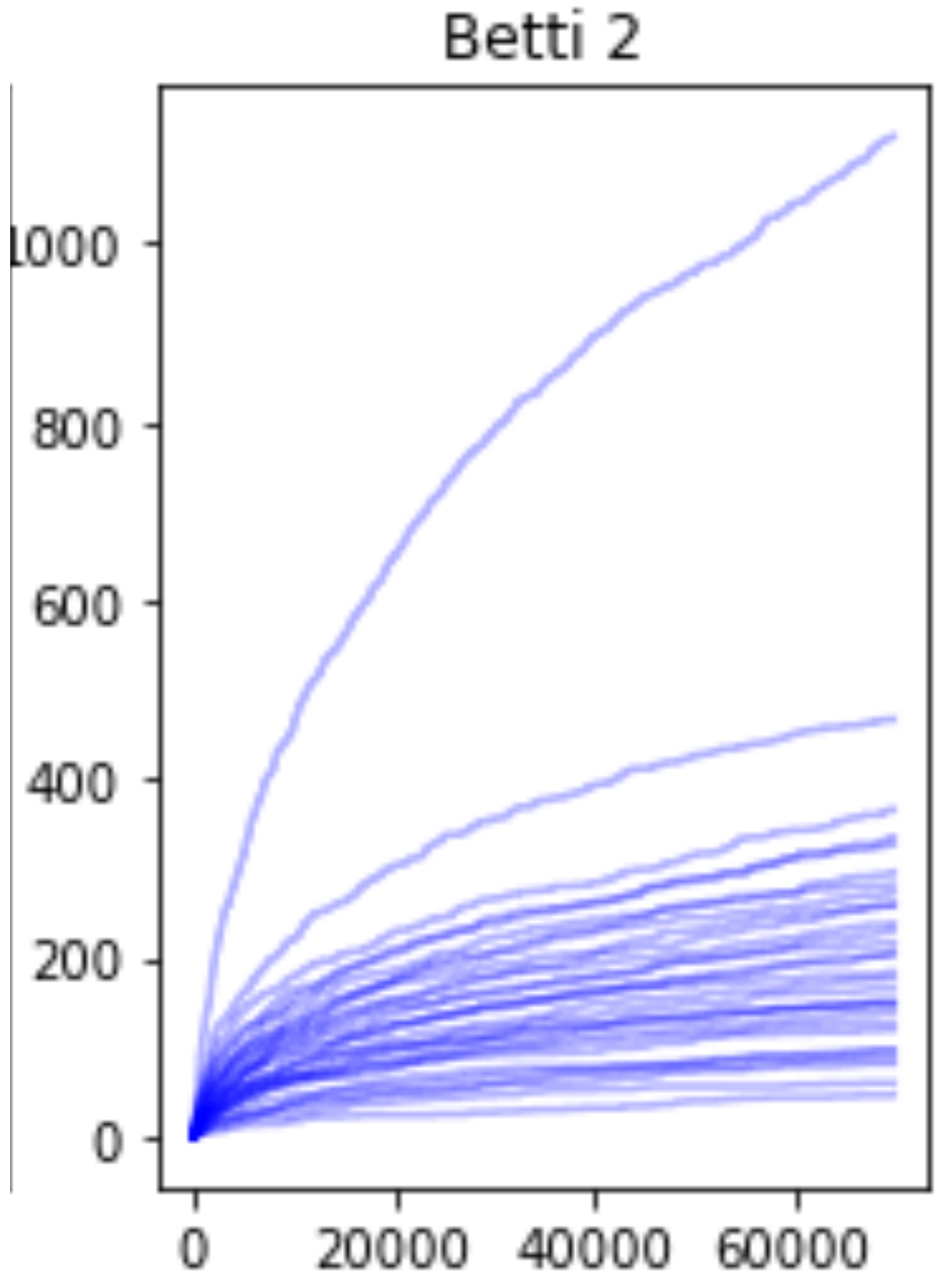
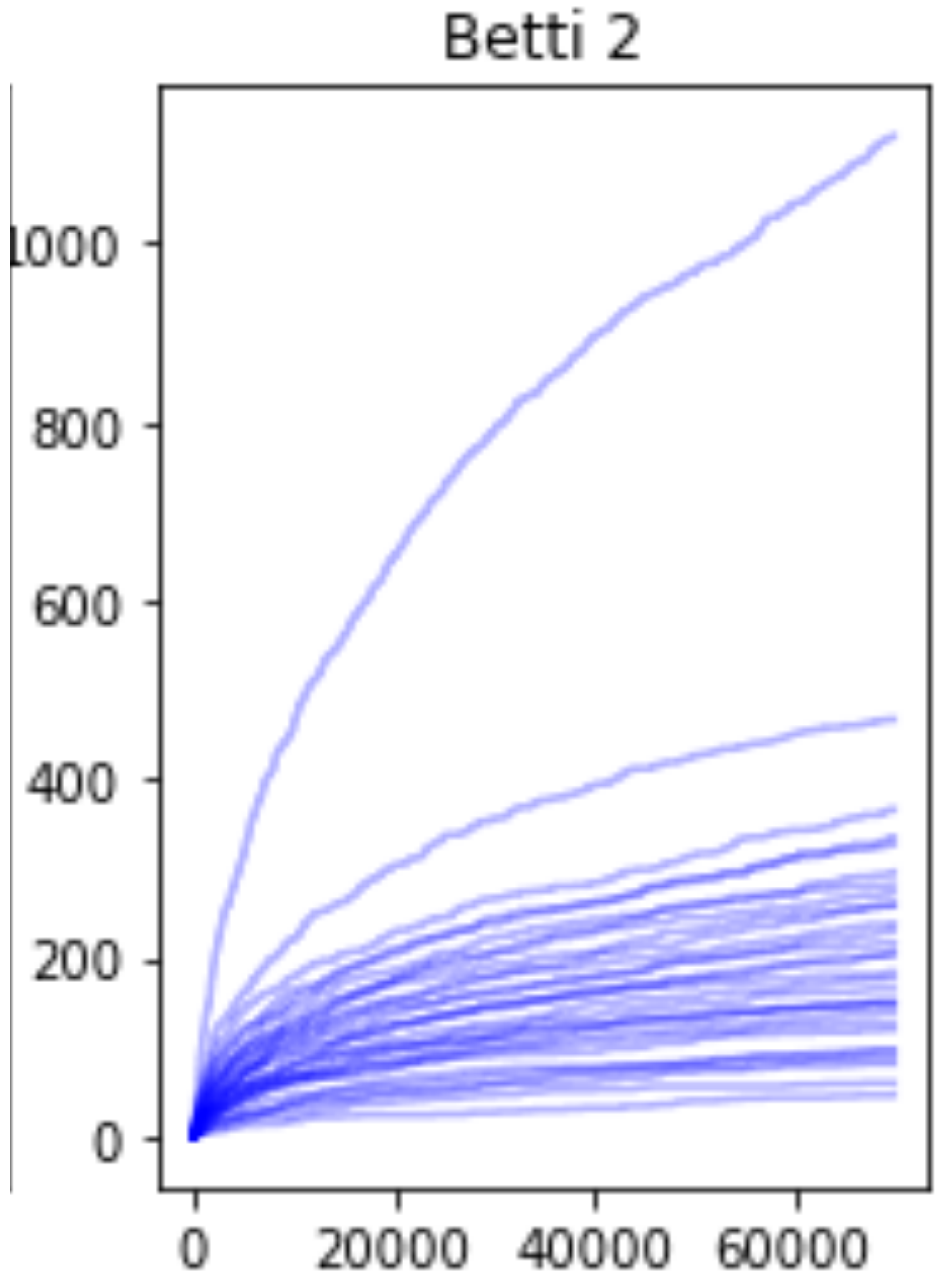
  - If $\boxed{1 - 4x} < 0$, then $\beta_2 \le C_\varepsilon$.



Betti Number β_q

- With probability at least $1 - \varepsilon$,
- $c_\varepsilon(\text{num of nodes}^{1-\boxed{4}x}) \leq \beta_2 \leq C_\varepsilon(\text{num of nodes}^{1-\boxed{4}x})$
  - $x \in (0,1/2)$ decreases with the preferential attachment strength
    - $P[T \text{ attaches to } i] \propto T^{-x}$
  - If $1 - 4x < 0$, then $\beta_2 \leq C_\varepsilon$.
- $c_\varepsilon(\text{num of nodes}^{1-\boxed{2q}x}) \leq \beta_q \leq C_\varepsilon(\text{num of nodes}^{1-\boxed{2q}x})$ for $q \geq 2$.

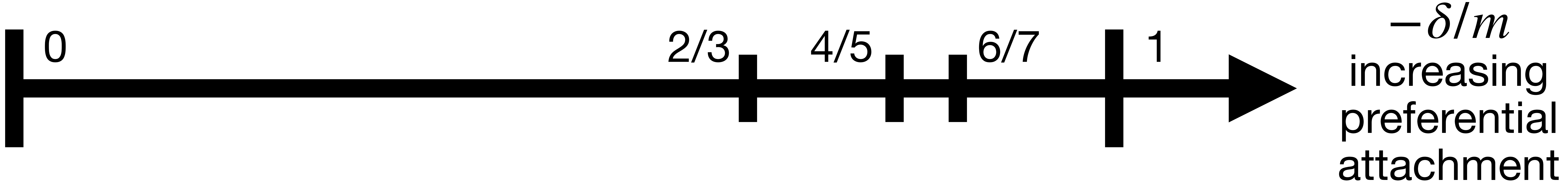


Phase transition

Recall

P(attaching to v) $\propto$ degree + δ

m = number of edges per new node



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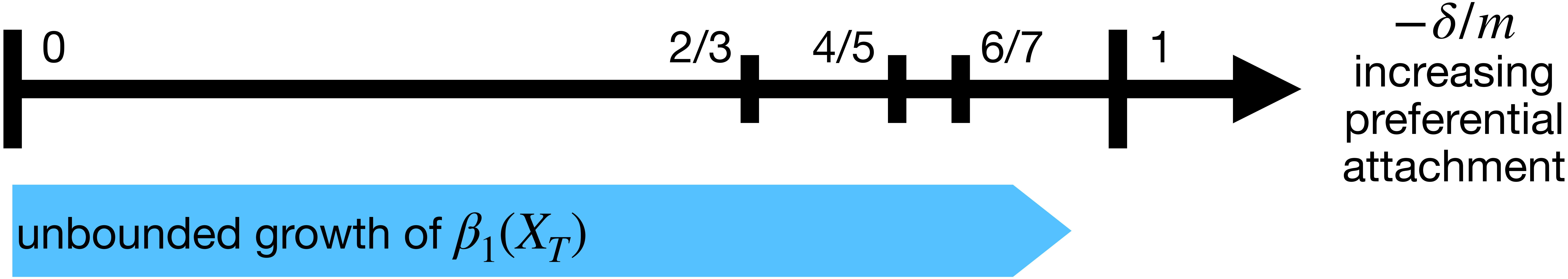

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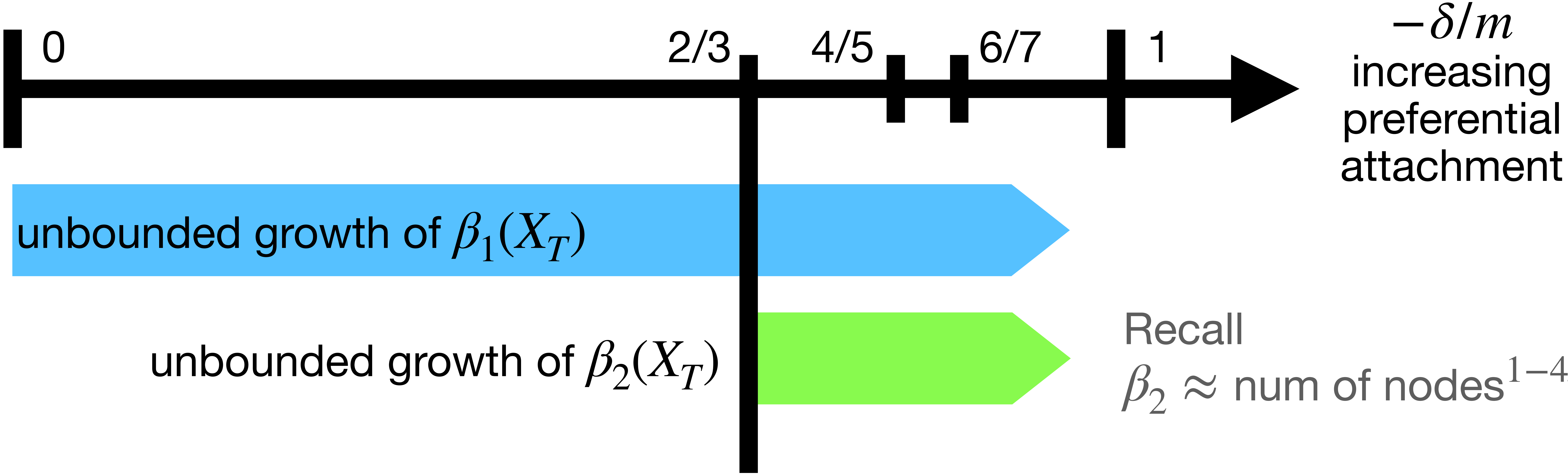

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$\beta_2 \approx$ num of nodes^{1-4x}

Phase transition

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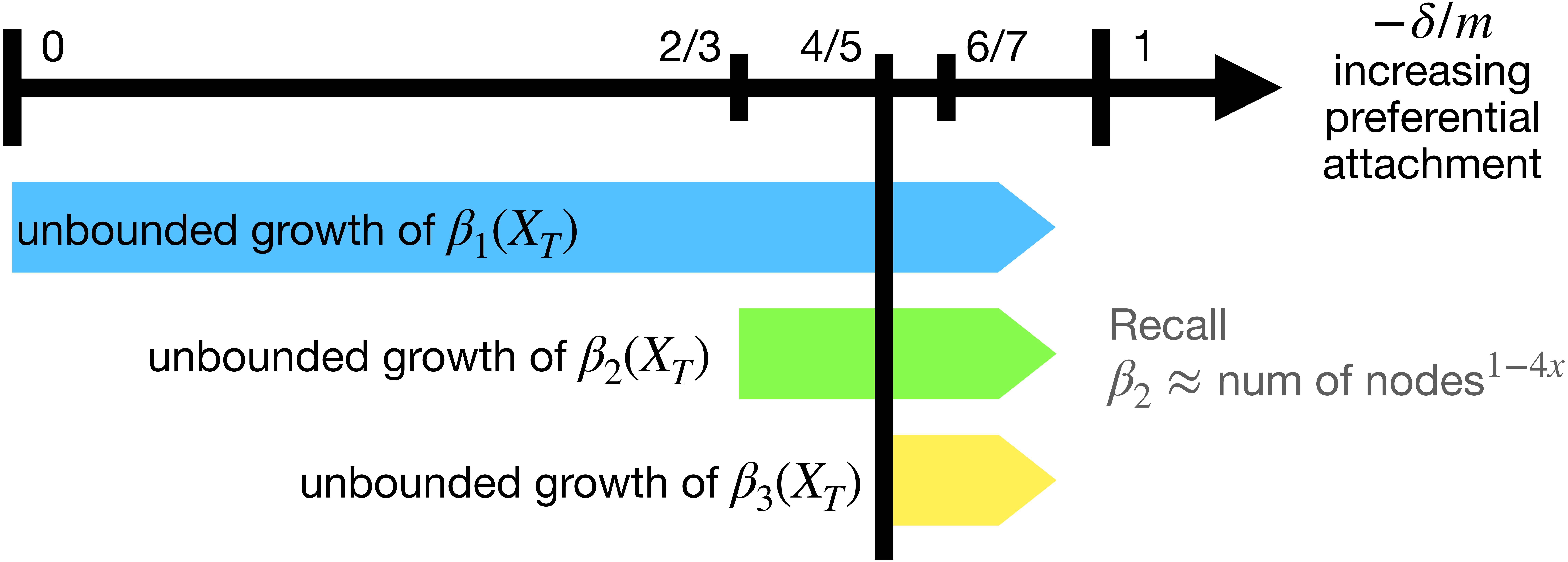

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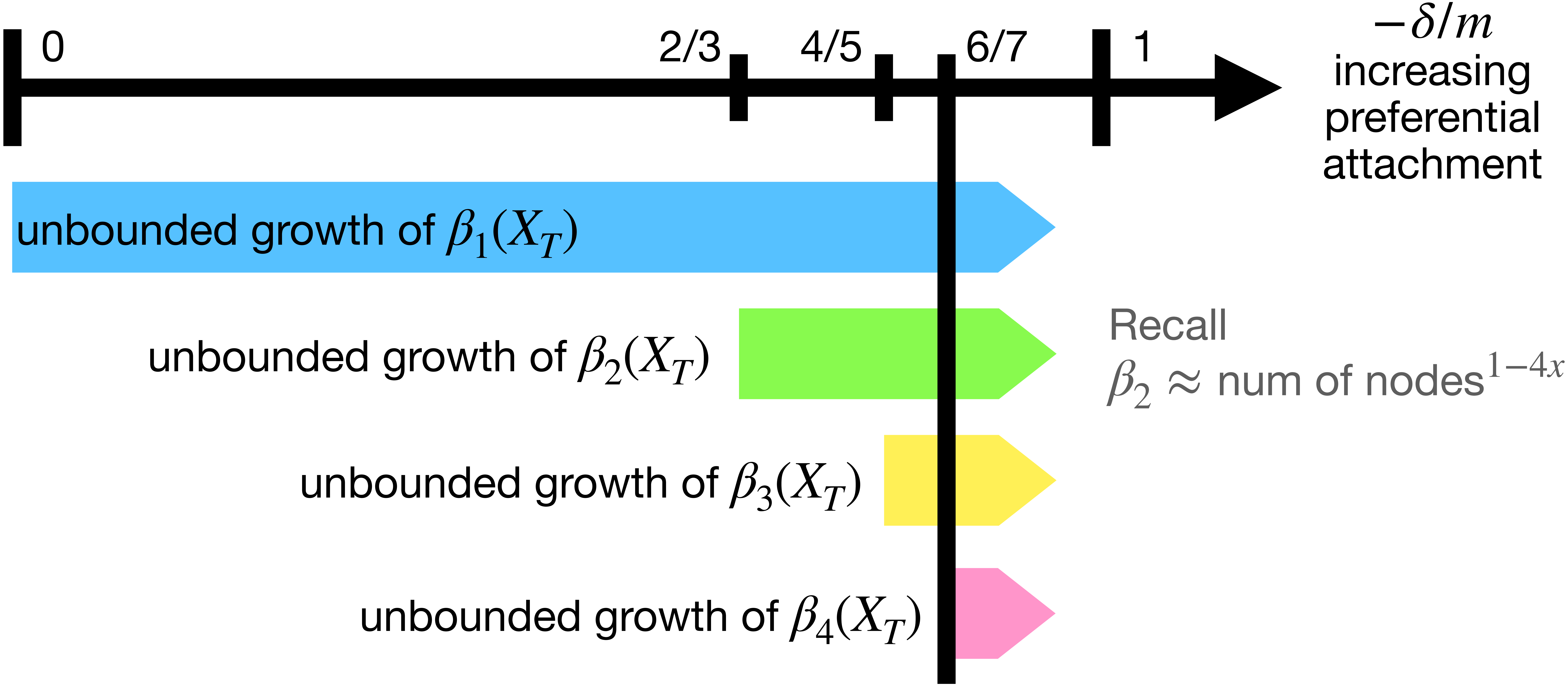

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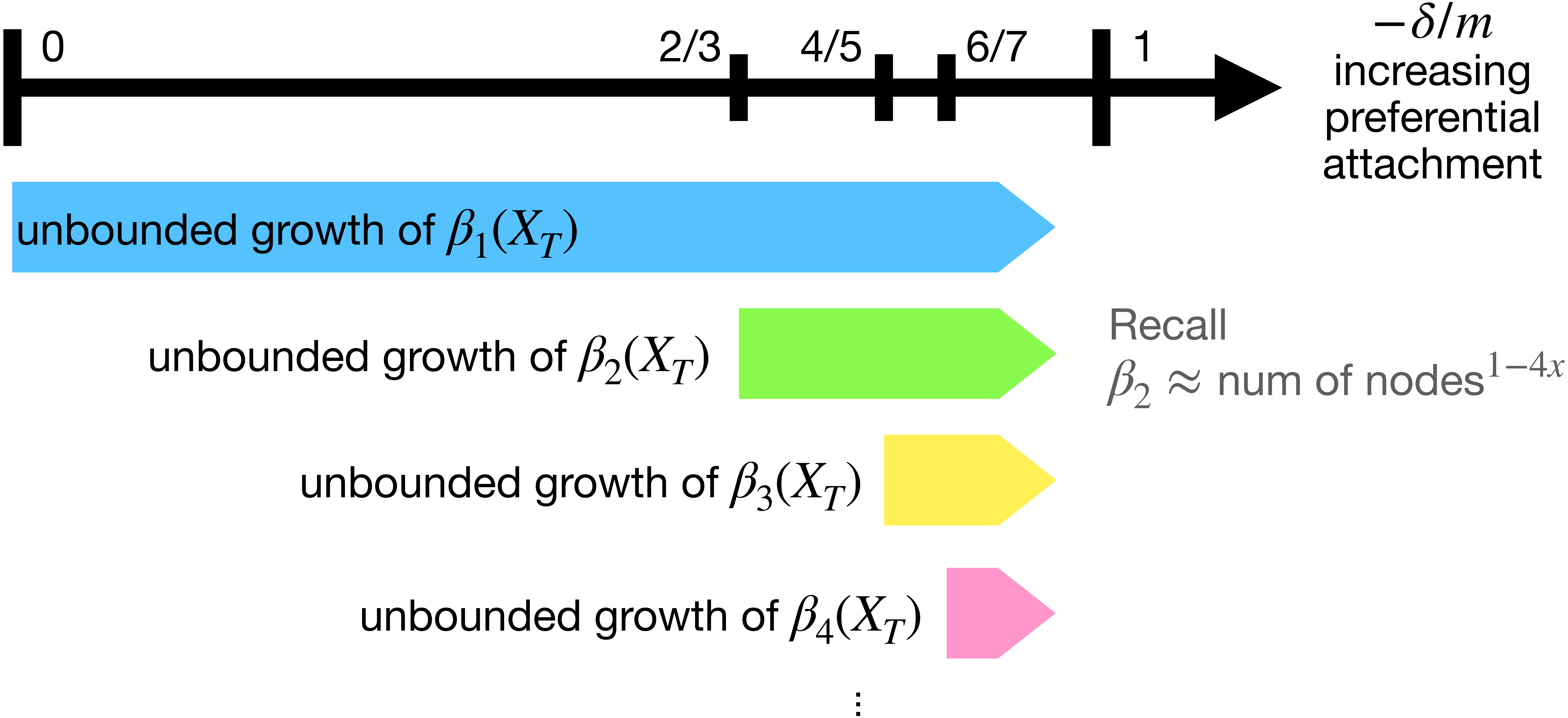
P(attaching to v) $\propto$ degree + δ

m = number of edges per new node



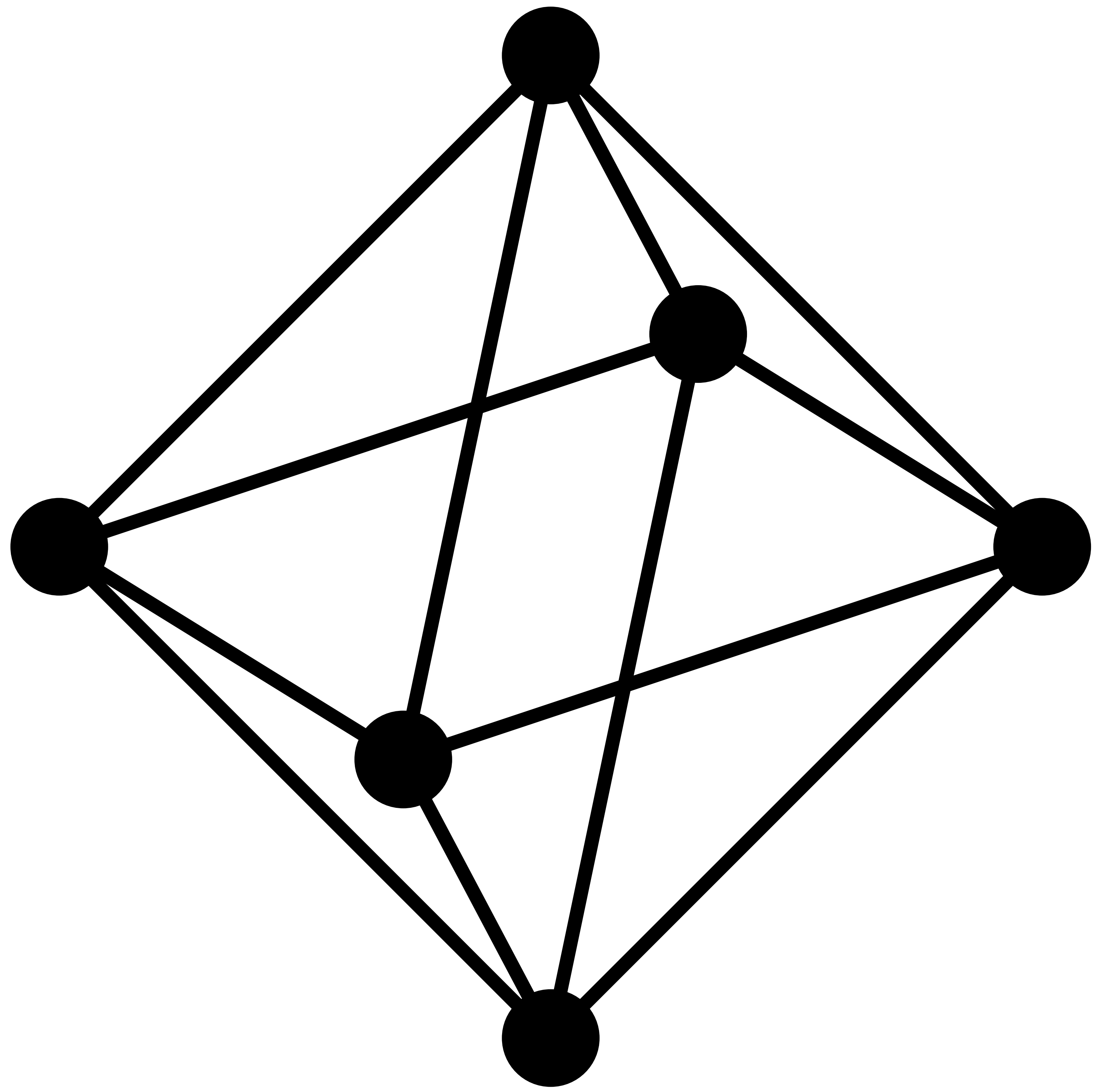
Recall

$\beta_2 \approx$ num of nodes^{1-4x}

Theorem: $\beta_2 \approx$ num of nodes^{1-4x}

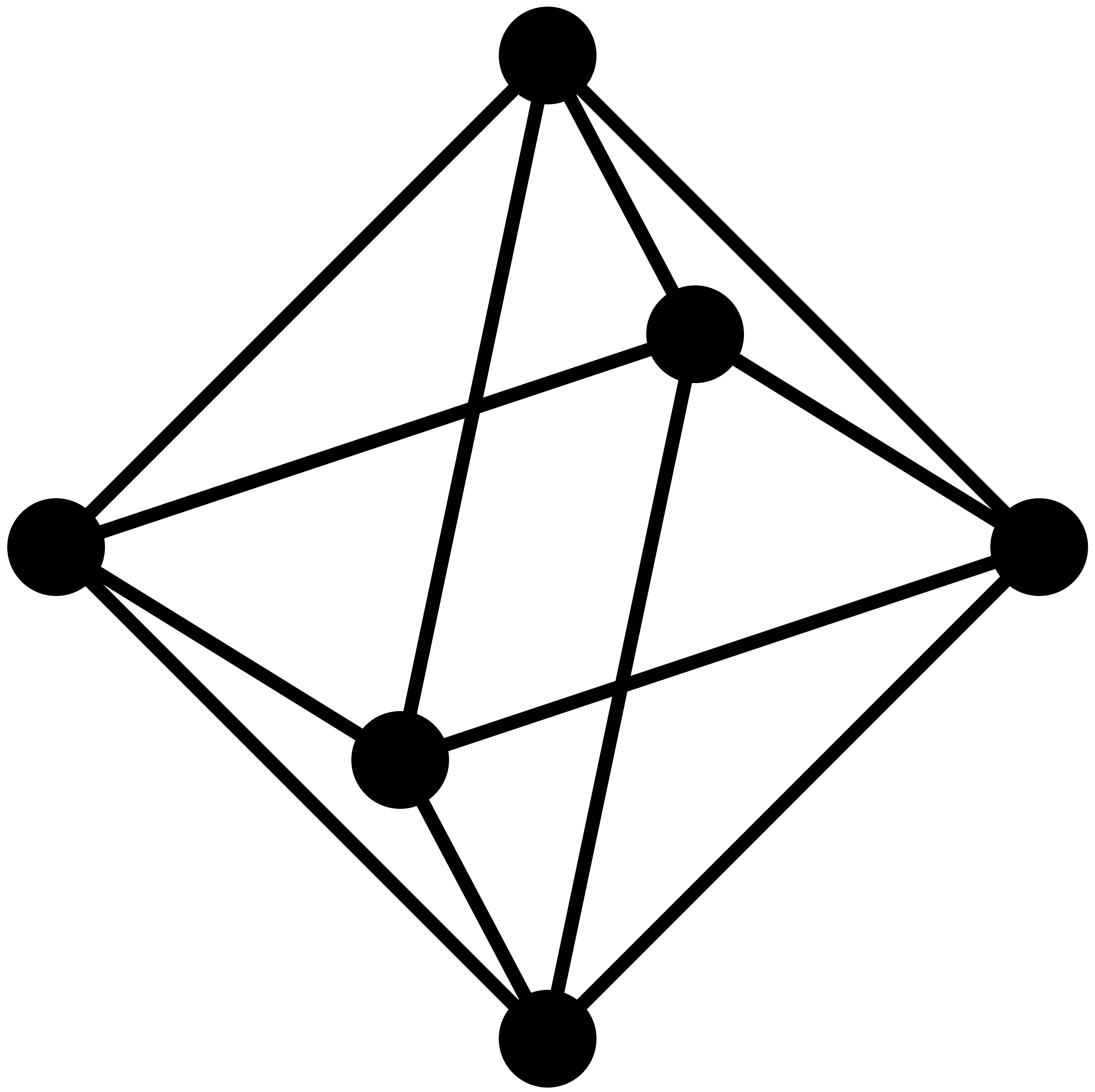
Proof?

Proof of $\beta_2 \approx$ num of nodes^{1-4x}

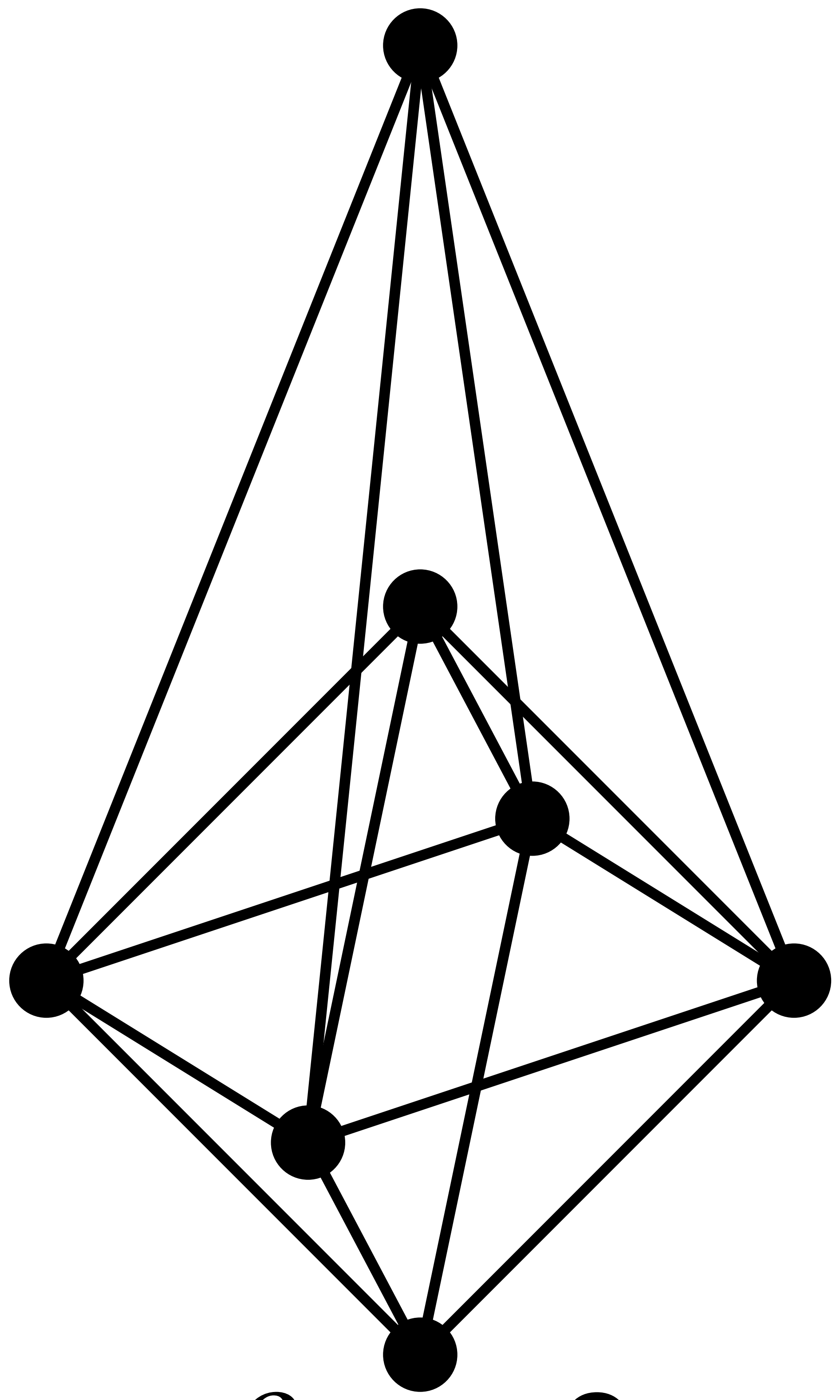


$\beta_2 = 1$

Proof of $\beta_2 \approx$ num of nodes^{1-4x}

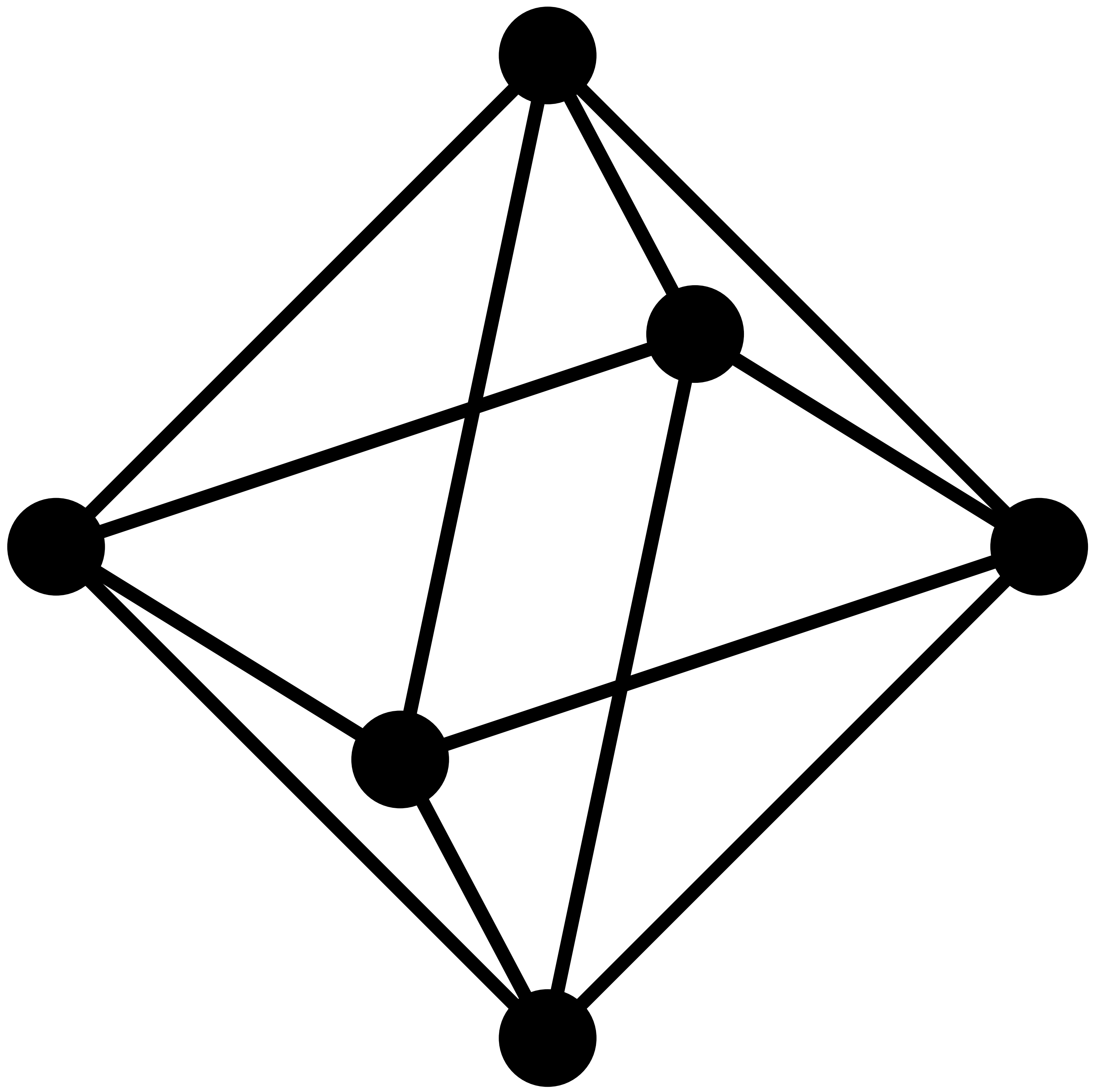


$\beta_2 = 1$

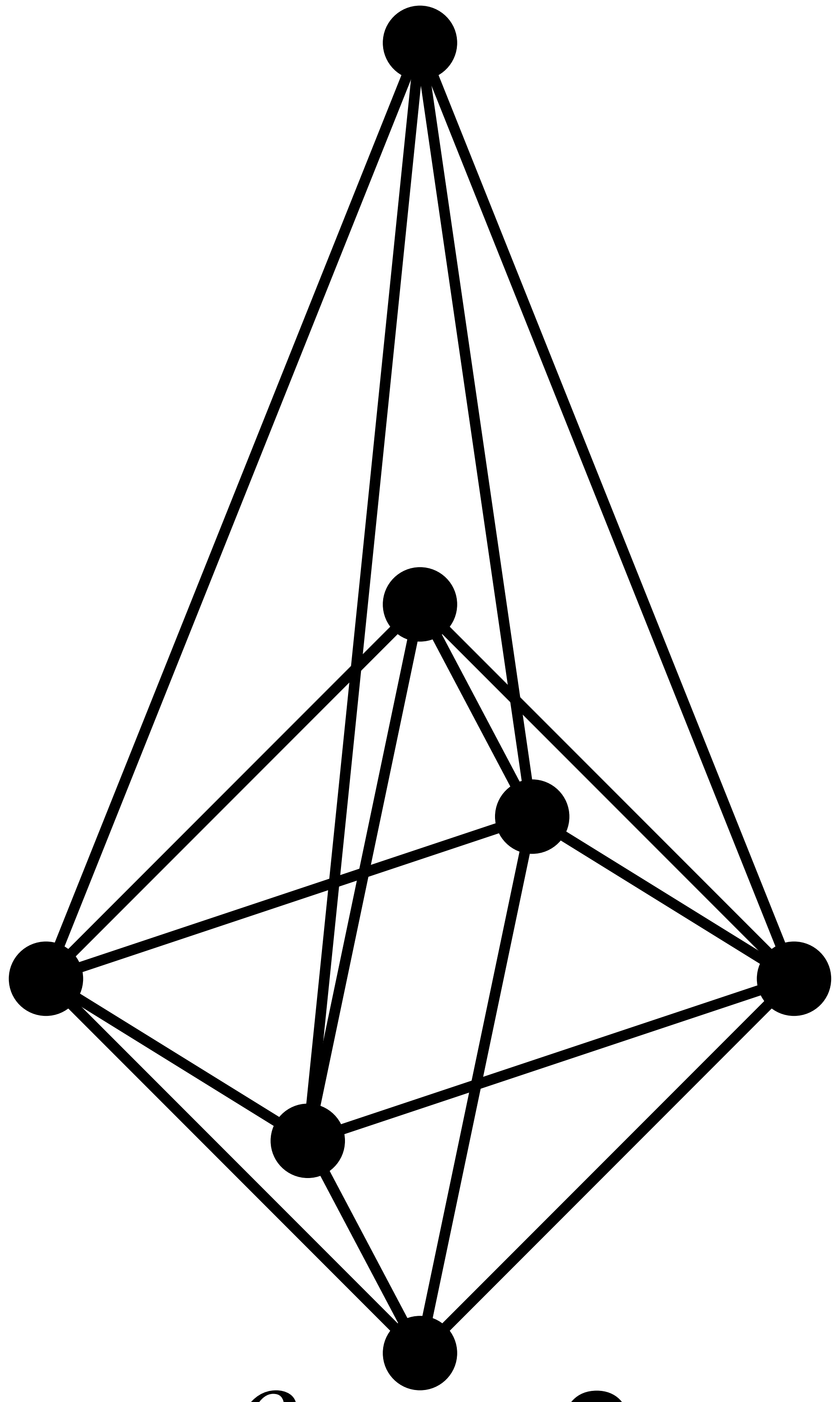


$\beta_2 = 2$

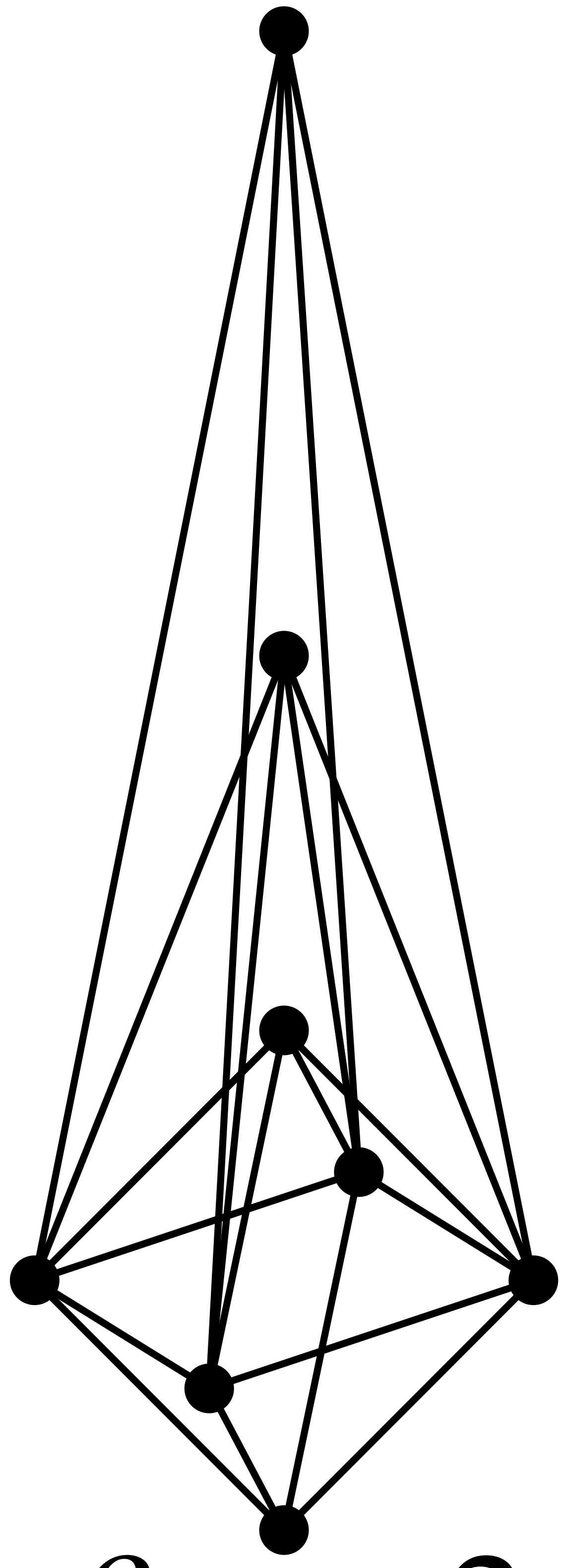
Proof of $\beta_2 \approx$ num of nodes^{1-4x}



$\beta_2 = 1$



$\beta_2 = 2$



$\beta_2 = 3$

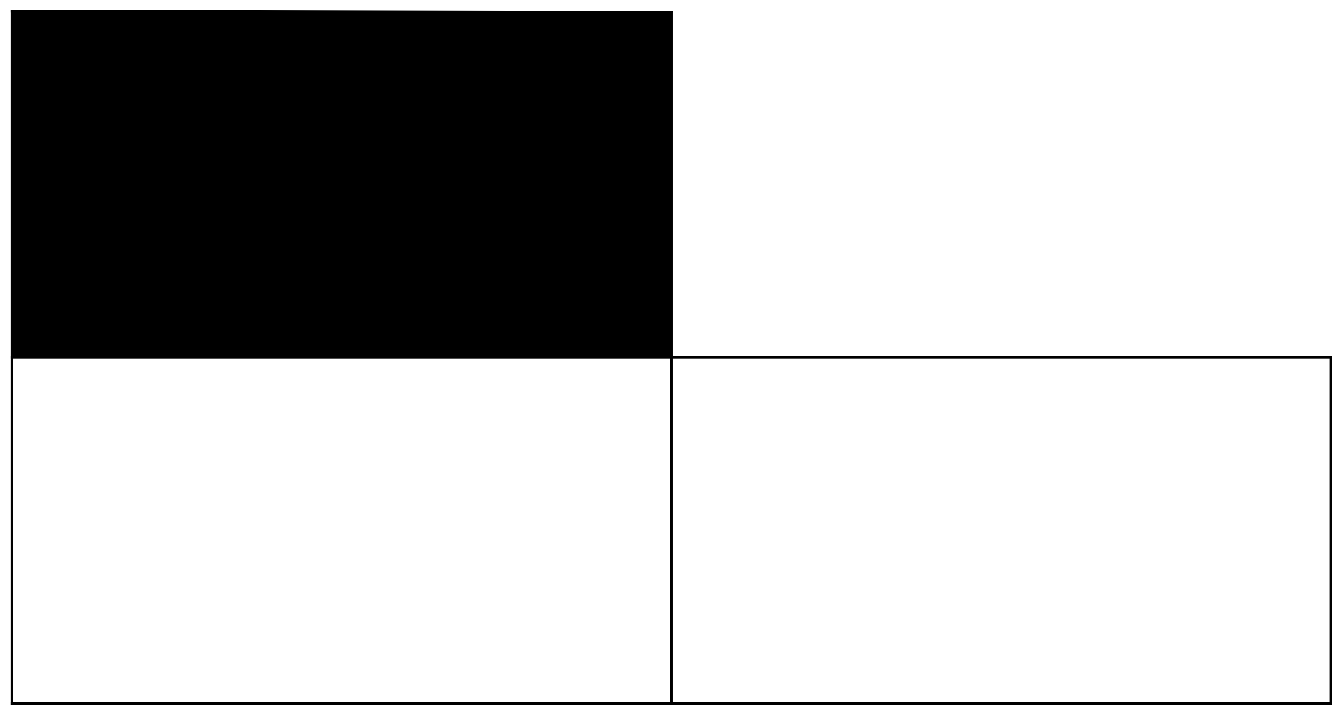
Homotopy-Connectivity?

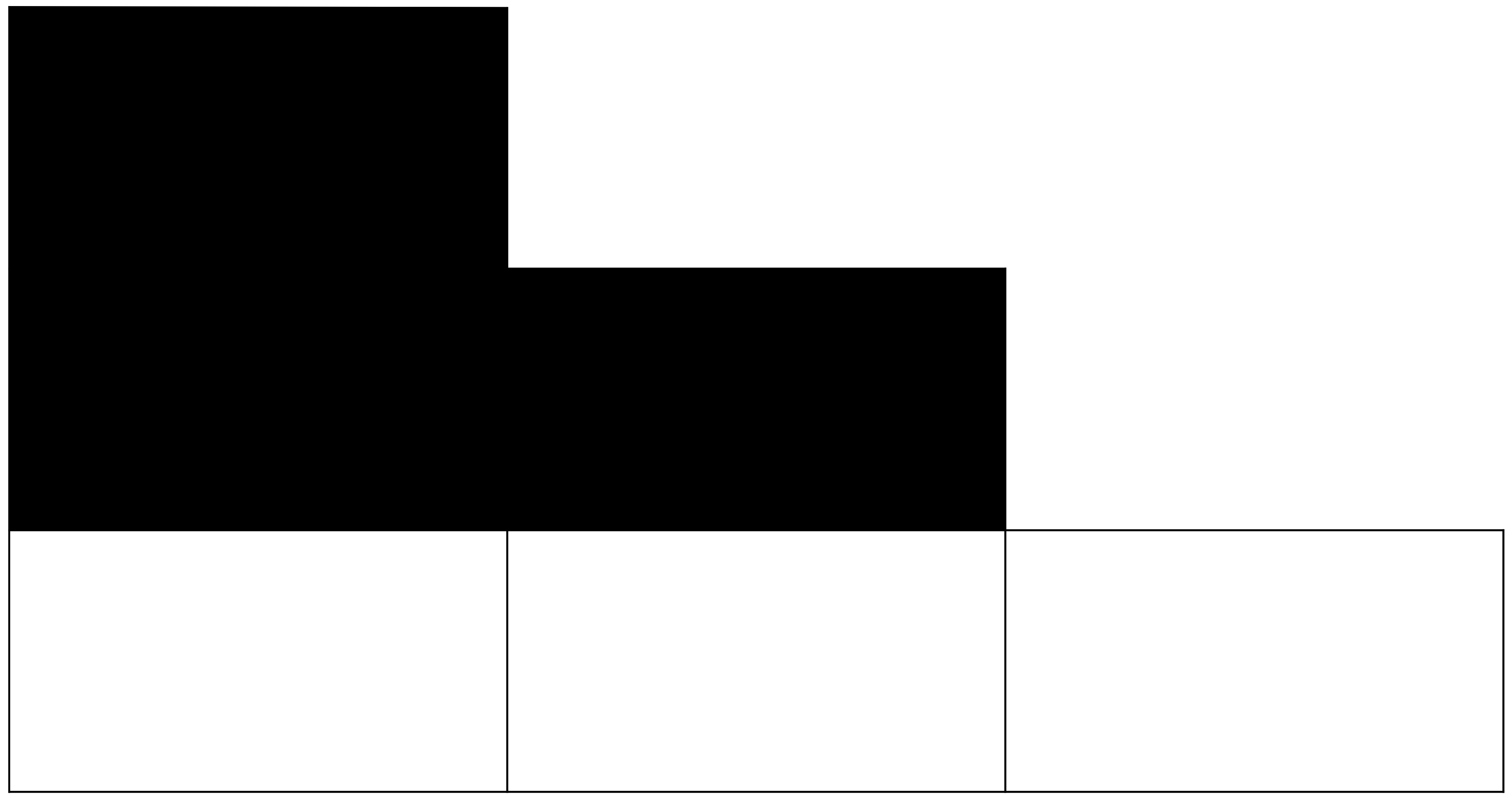
Homotopy-Connectivity?

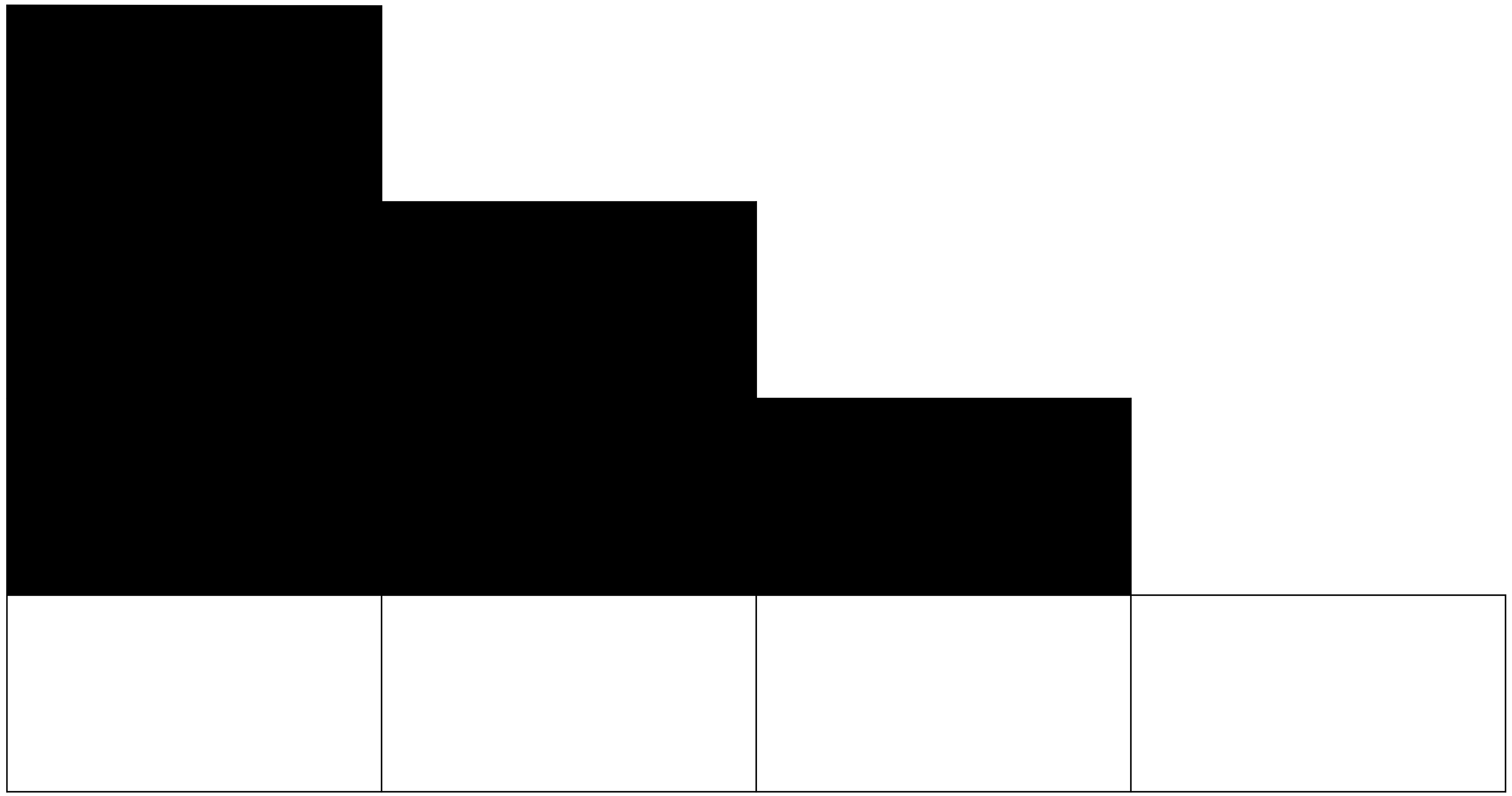
$$\beta_2 \approx \text{num of nodes}^{1-4x}$$

Pass to infinity

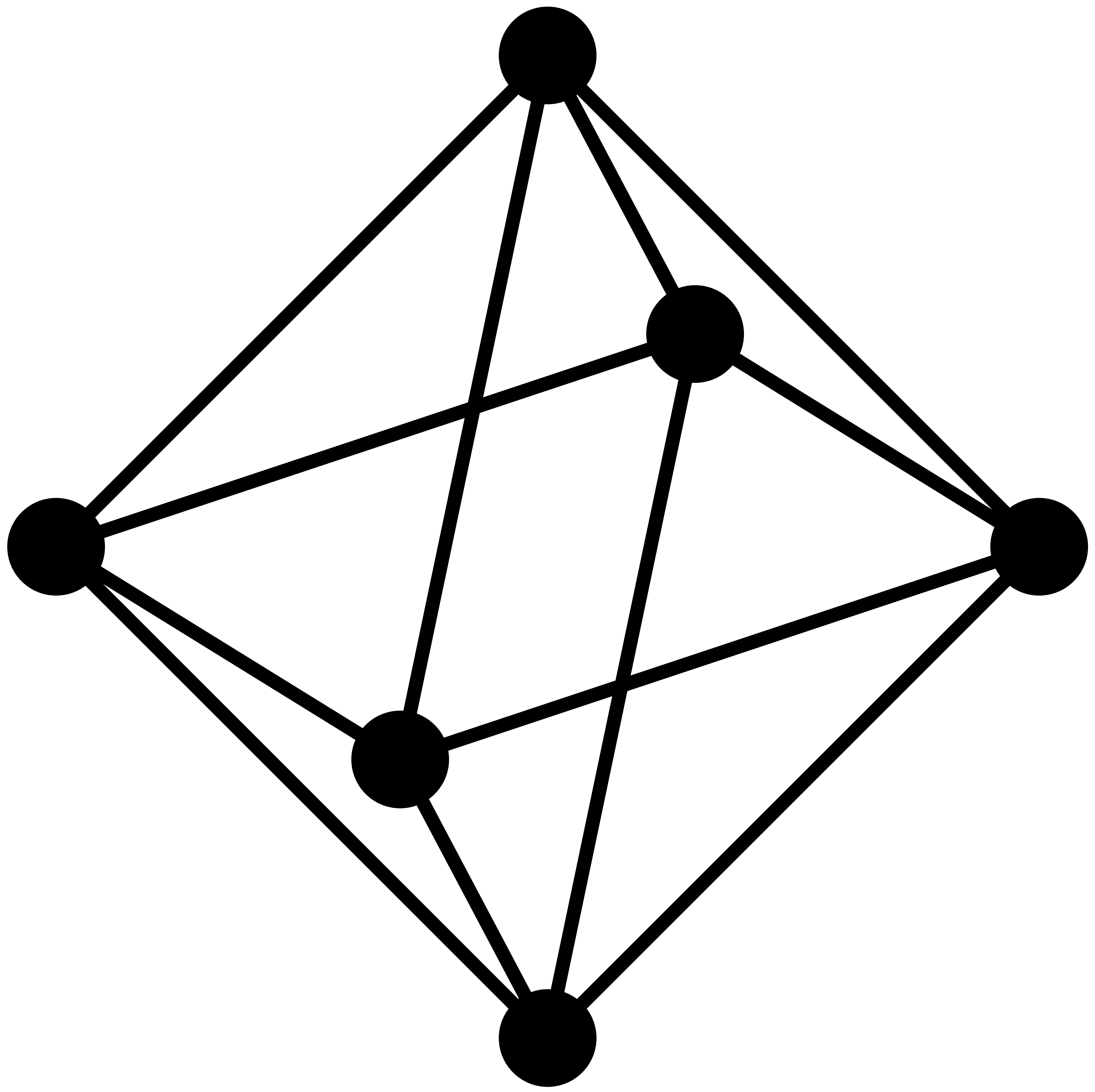
S Weinberger, 2023
private conversation



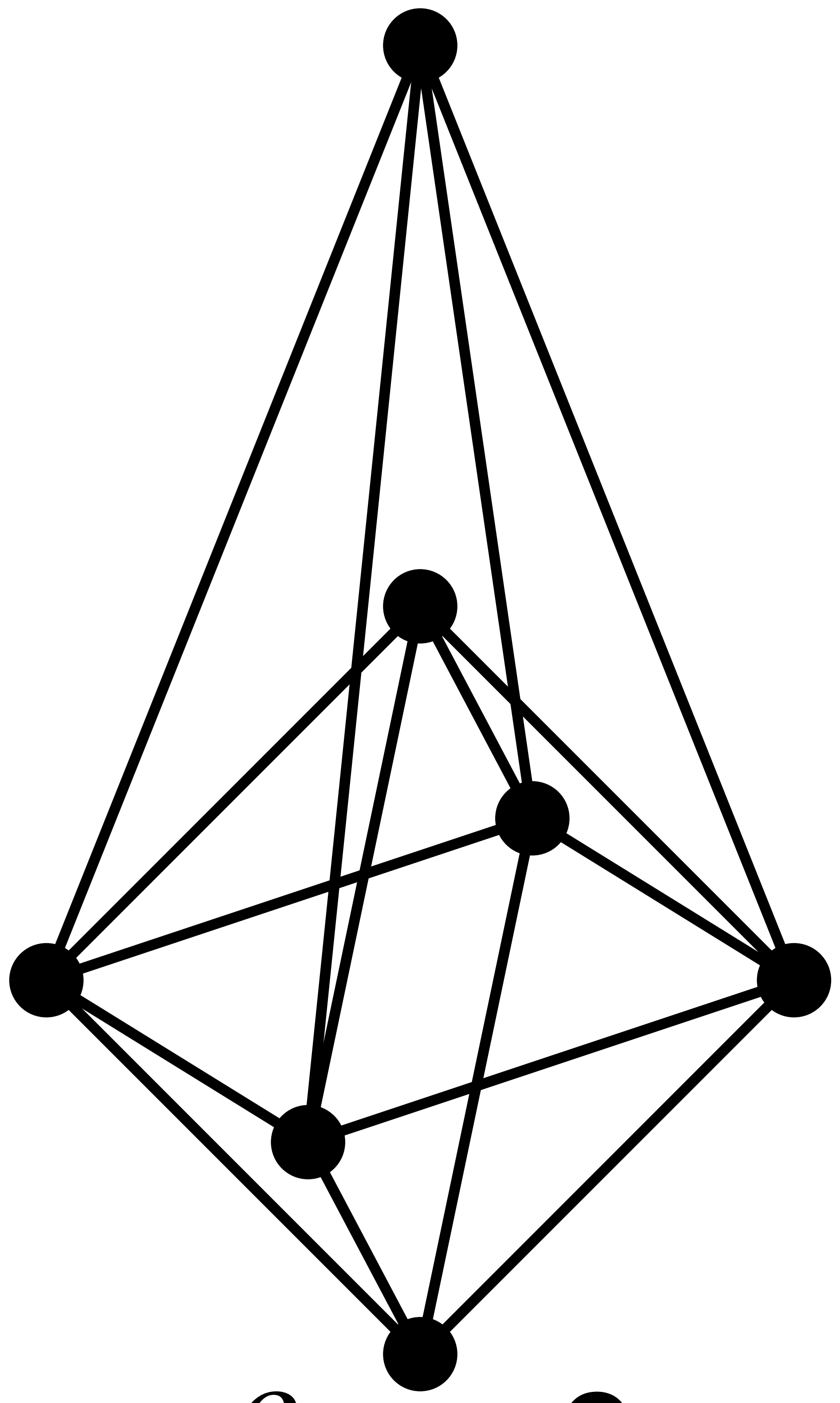




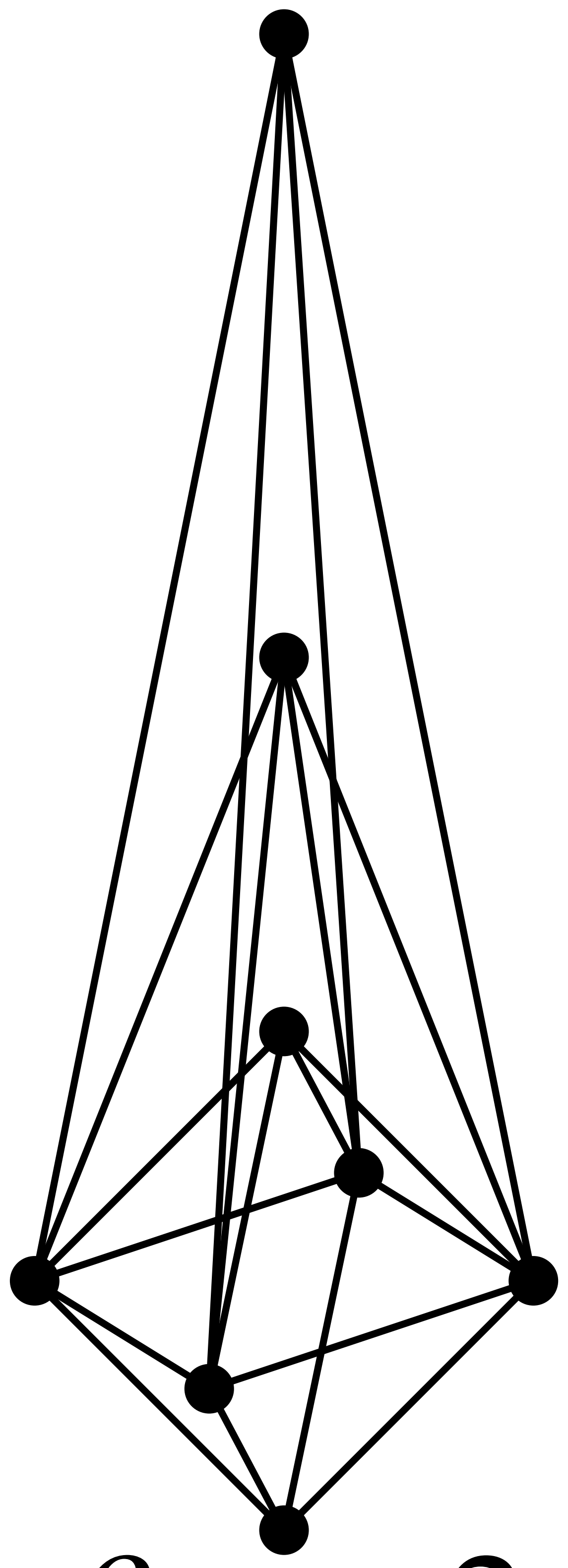
Will all of these be filled in at infinity?



$\beta_2 = 1$



$\beta_2 = 2$



$\beta_2 = 3$

[Barmak 2023]

- A clique complex is q-homotopy-connected
- if every collection of $2(q + 1)$ nodes has a common neighbor.

Homotopy-Connected

- Almost surely, the infinite preferential attachment complex
- is q-homotopy-connected if $x \leq \dfrac{1}{2(q+1)}$

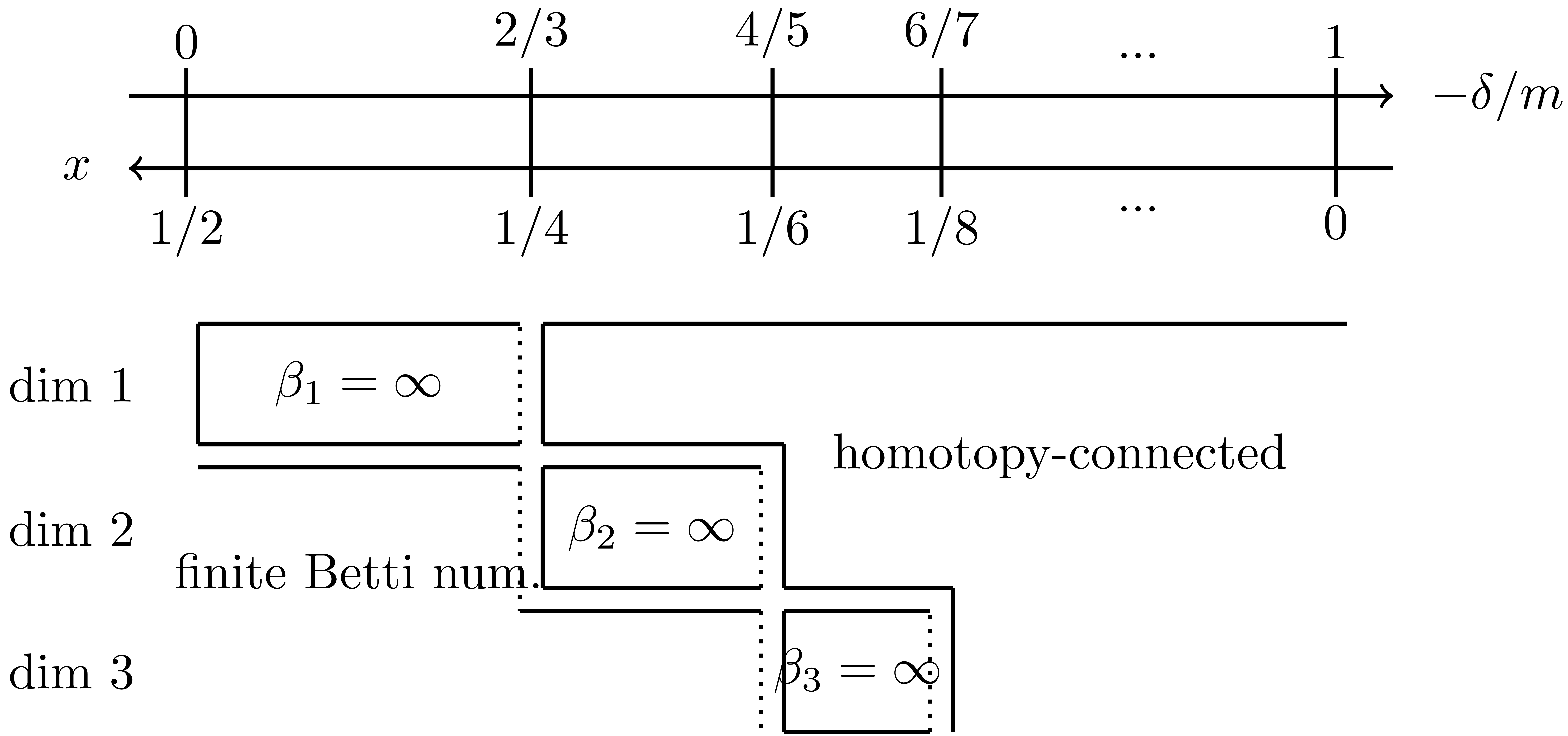
Recall:
$x \in (0,1/2)$ decreases with
the preferential attachment strength
$P[T$ attaches to $i] \propto T^{-x}$

Homotopy-Connected

- Almost surely, the infinite preferential attachment complex
- is q-homotopy-connected if $x \leq \dfrac{1}{2(q+1)}$
- has infinite Betti number at dimension q if $\dfrac{1}{2(q+1)} < x \leq \dfrac{1}{2q}$

Recall:
$x \in (0,1/2)$ decreases with
the preferential attachment strength
$P[T \text{ attaches to } i] \propto T^{-x}$

Phase Transition

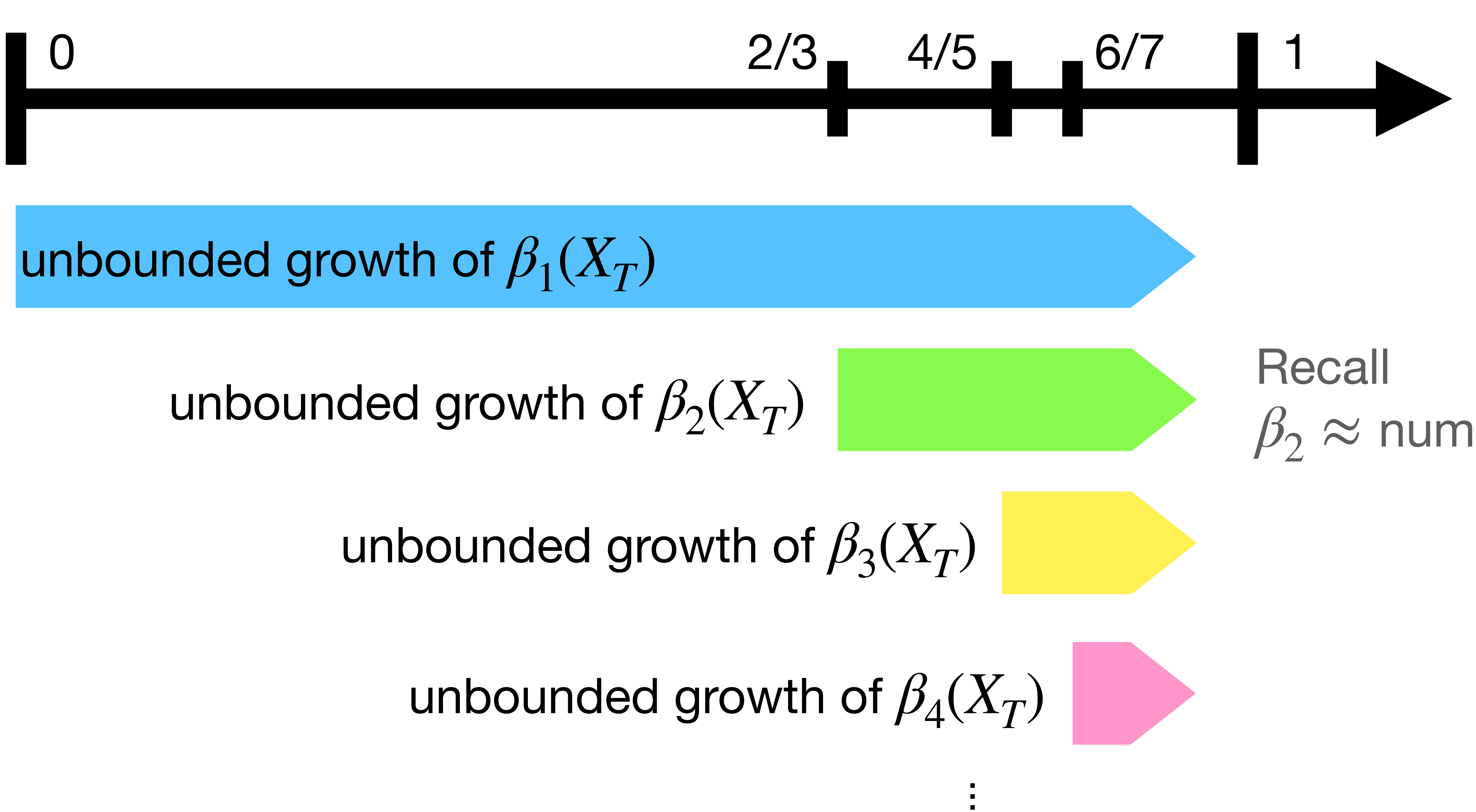


Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



$-\delta/m$

increasing preferential attachment

Recall

$\beta_2 \approx$ num of nodes^{1-4x}

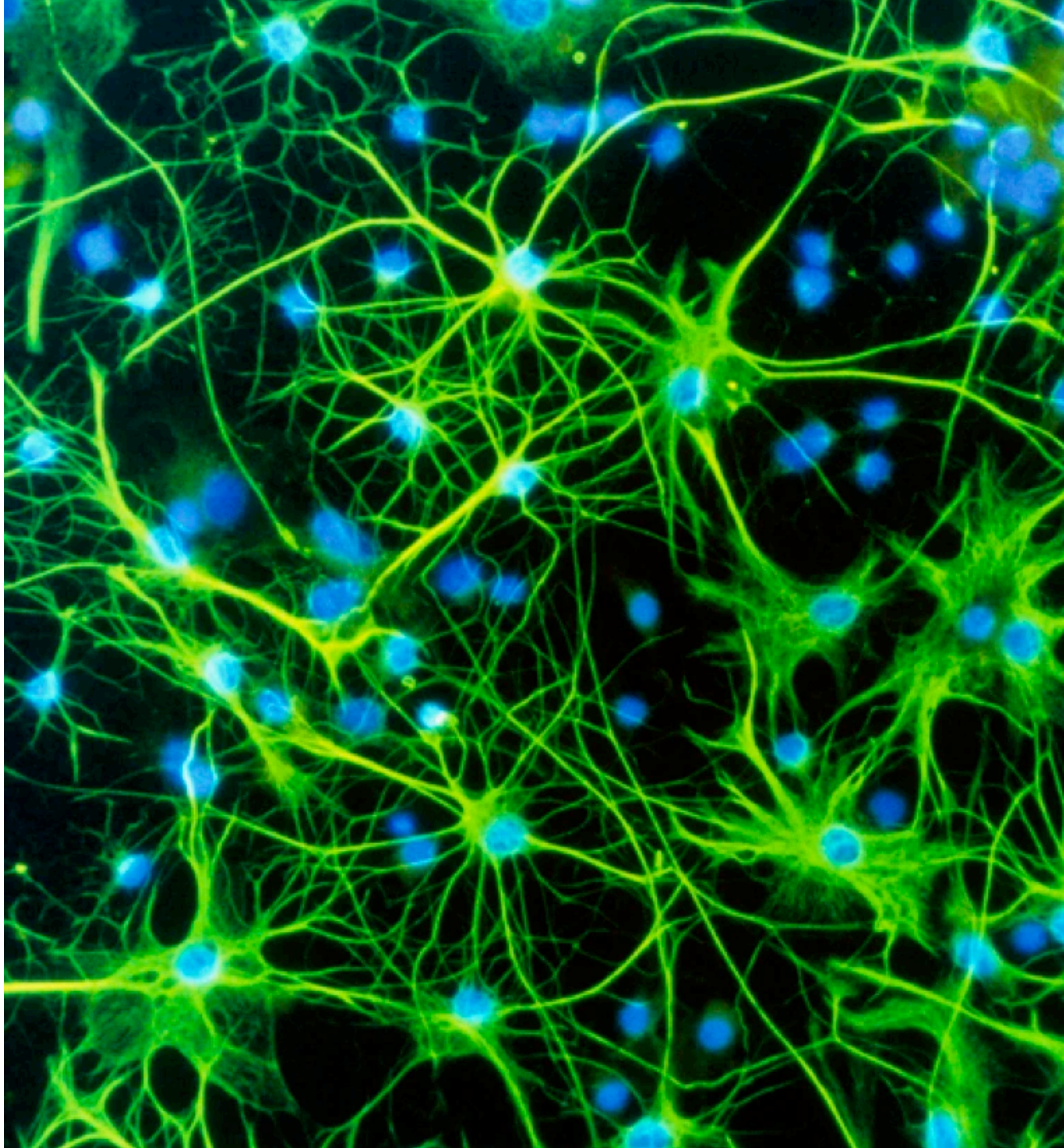
- If the preferential attachment effect is strong enough,
- $\beta_q(X_T)$ grows sublinearly with high probability
- $\pi_q(X_\infty) \cong 0$ almost surely

IV. Rumination on Neuroscience

What I am doing now

- depression, ADHD and other psychiatric conditions
- whole-brain functional MRI

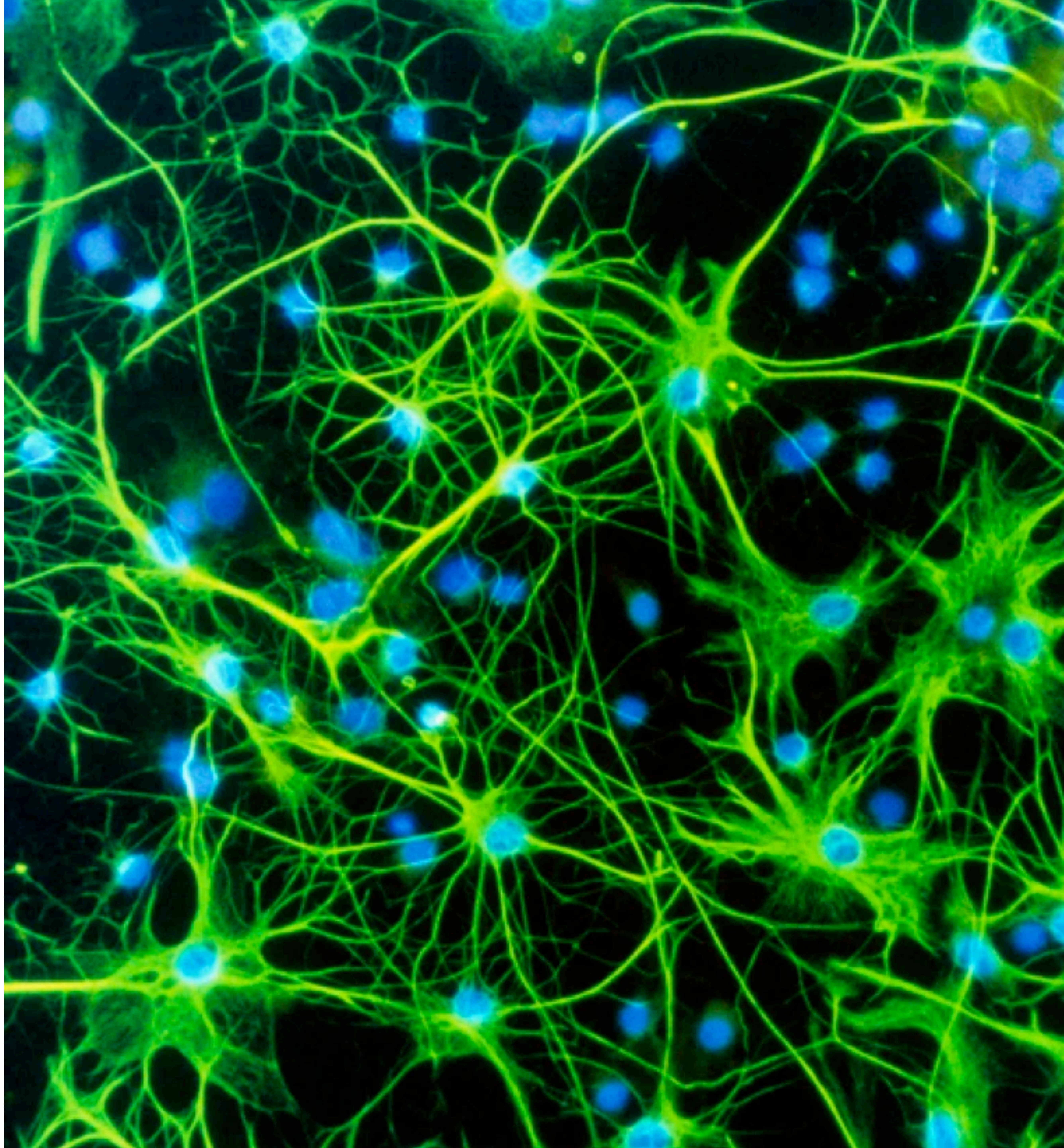
Question



Credit: Nancy Kedersha/SPL
https://www.nature.com/articles/d41586-019-02153-y

Question

- Not completely random (?)



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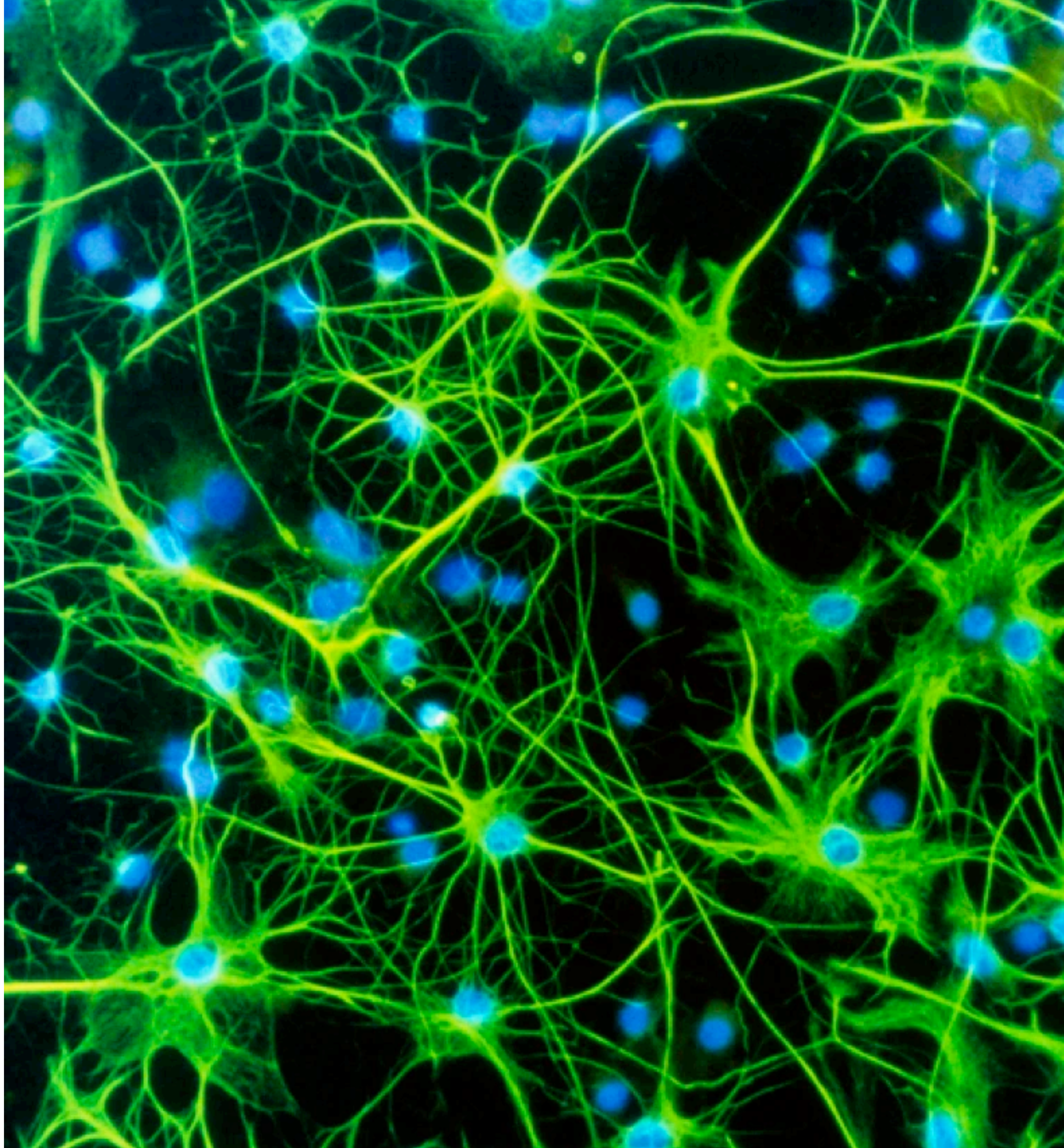
Question

- Not completely random (?)
  - local connections more likely (?)



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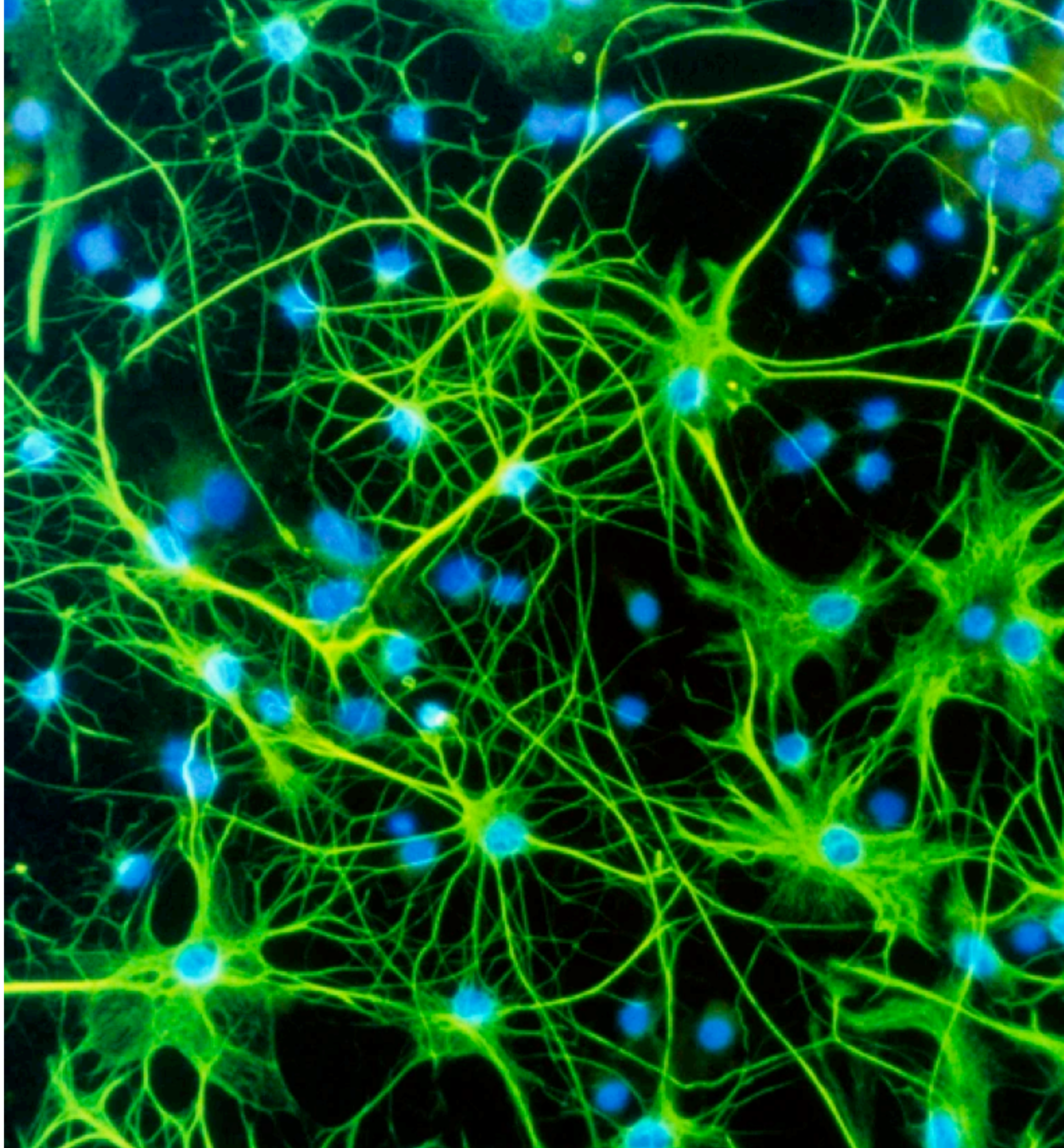
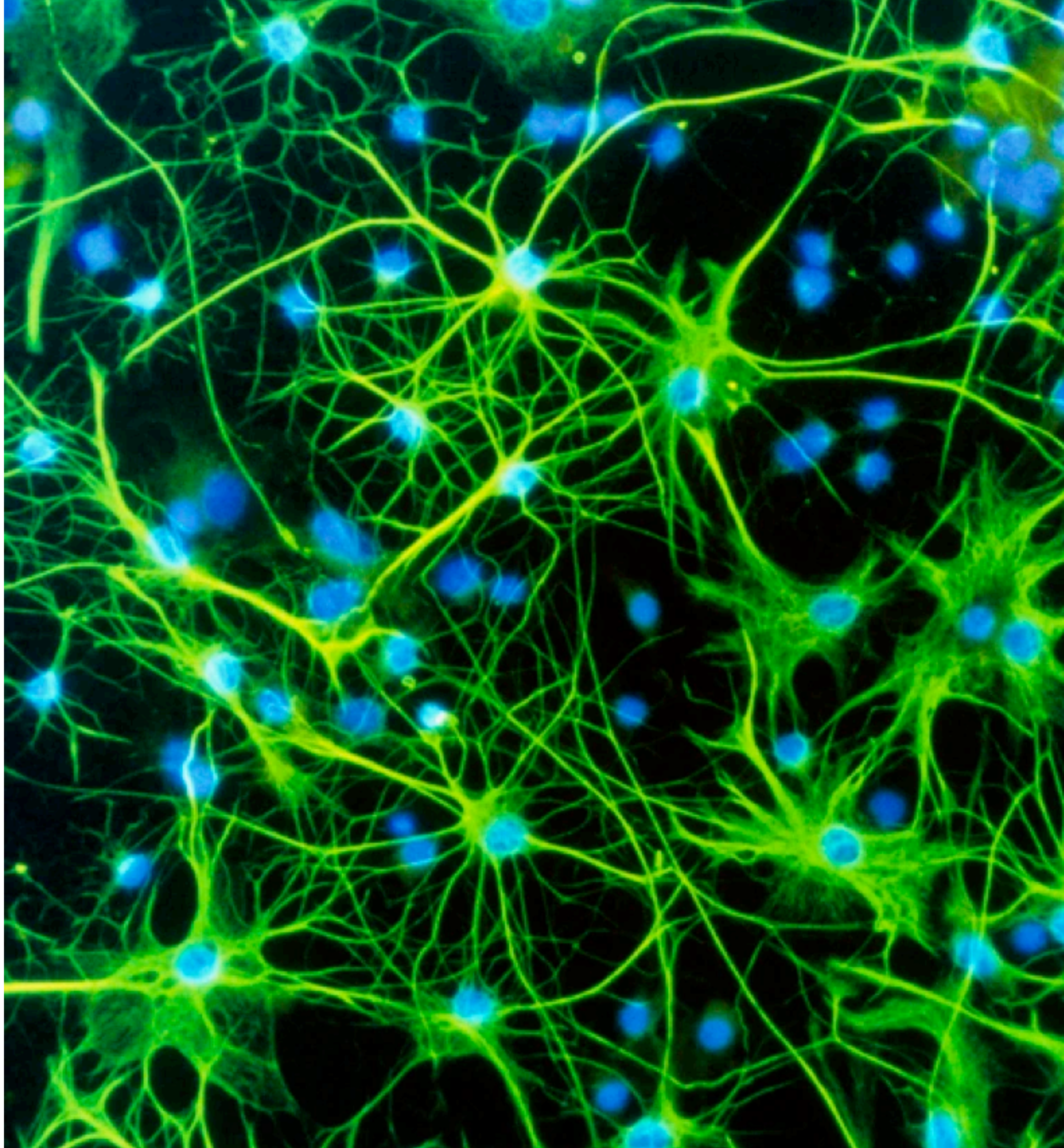


Credit: Nancy Kedersha/SPL
https://www.nature.com/articles/d41586-019-02153-y

Question

- Not completely random (?)
  - local connections more likely (?)
- Not preferential attachment (?)
  - e.g. neurons do not “arrive” one by one (?)

Credit: Nancy Kedersha/SPL
https://www.nature.com/articles/d41586-019-02153-y



What is the right model?

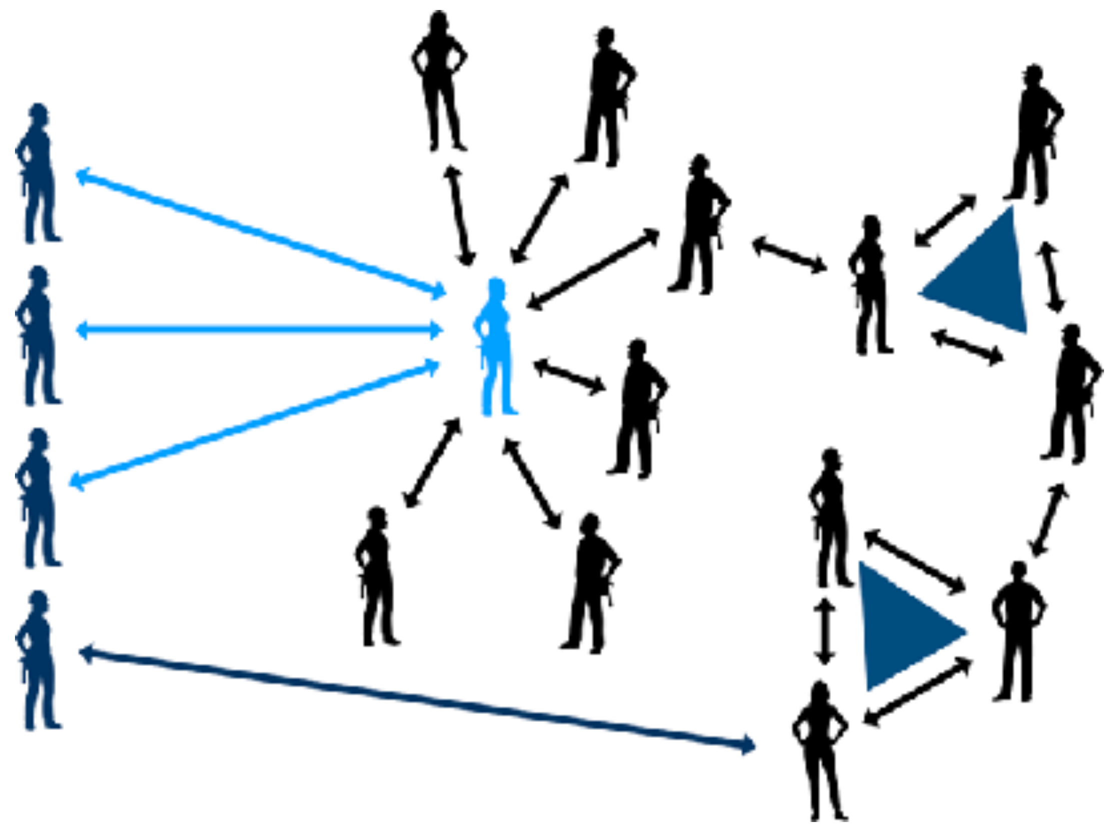
What is the right model?

- Are neuronal distribution (locally) spatially uniform?

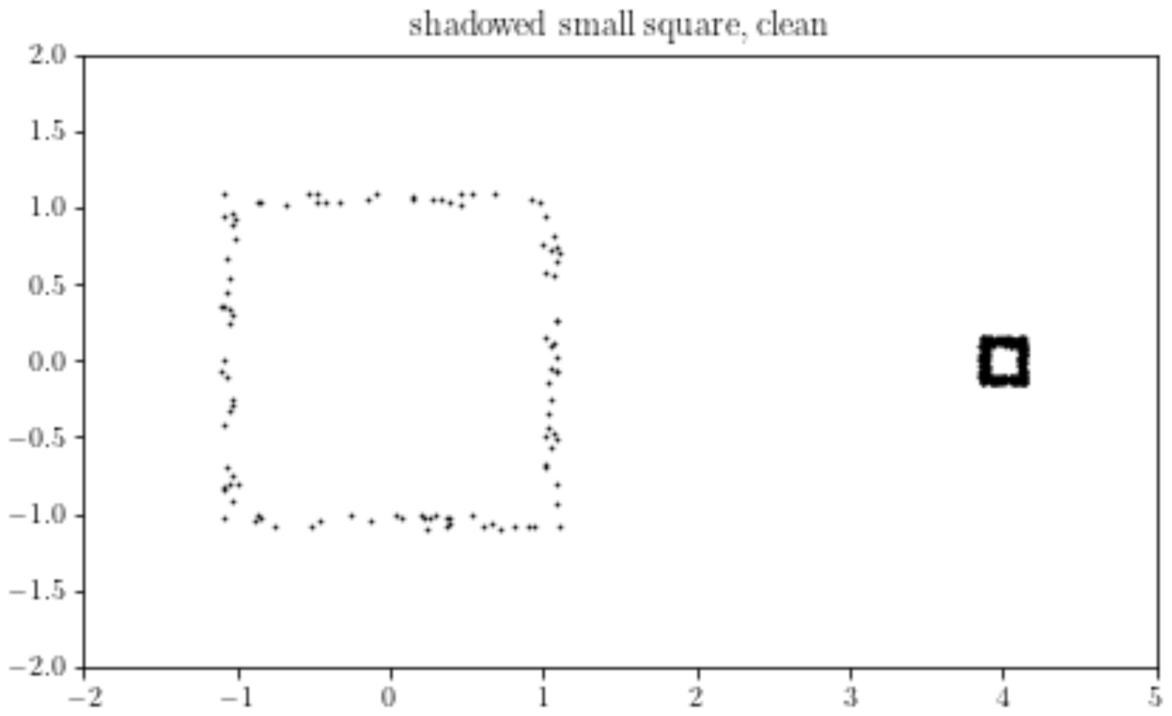
What is the right model?

- Are neuronal distribution (locally) spatially uniform?
- Does neuron connection probability depend only on distance?

Chunyin Siu **Stanford University, School of Medicine**
siuc@stanford.edu



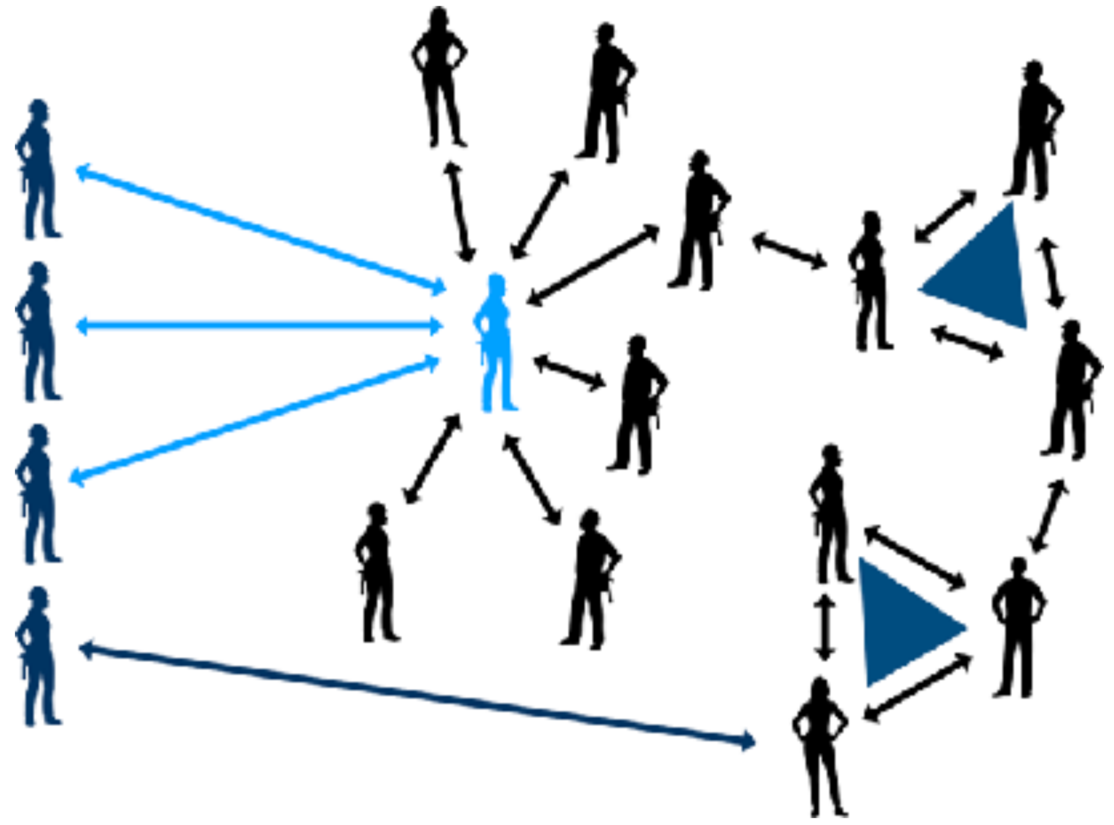
arxiv paper



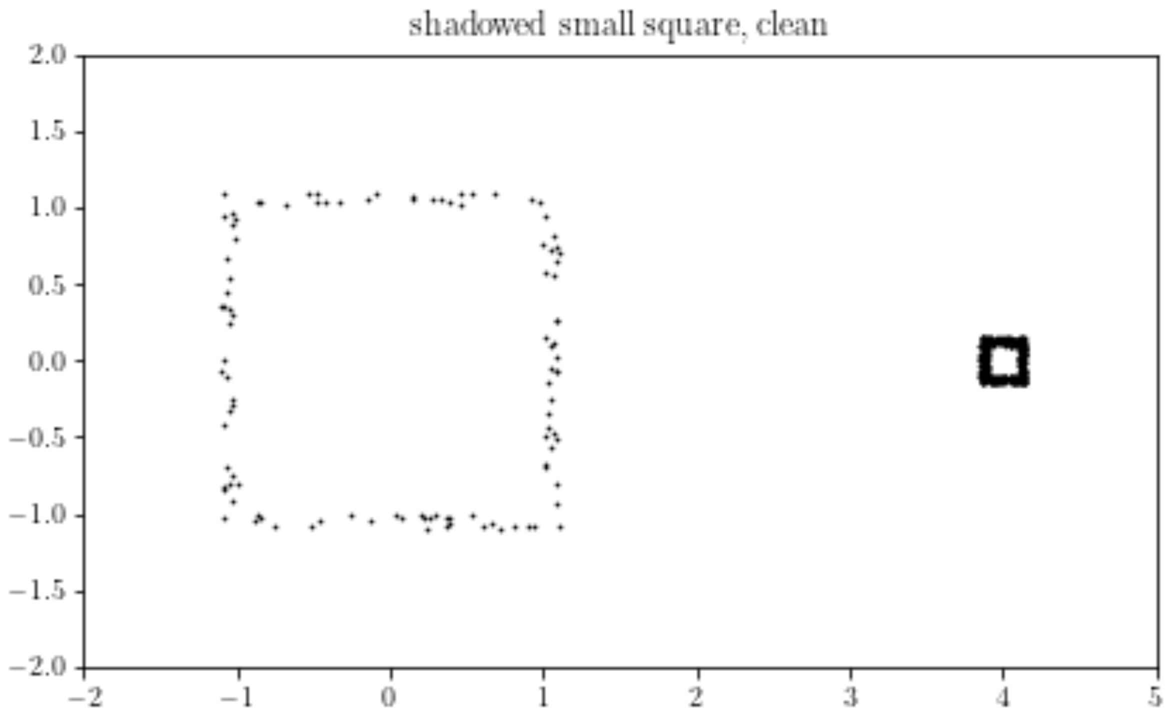
my video about small holes

Thank you!

Chunyin Siu **Stanford University, School of Medicine**
siuc@stanford.edu



arxiv paper



my video about small holes

What we know
[not meant to be complete]

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- Erdos-Renyi clique complexes

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- Erdos-Renyi clique complexes
  - Kahle 2009, 2014
  - Kahle and Meckes 2013
  - Costa et al 2015
  - Malen 2023
  - etc

What we know
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  - Kahle 2011
  - Kahle and Meckes 2013
  - Yogeshwaran and Adler 2015
  - Bobrowski et al 2017
  - Hiraoka et al 2018
  - Thomas and Owada 2021a, b
  - Owada and Wei 2022
  - etc

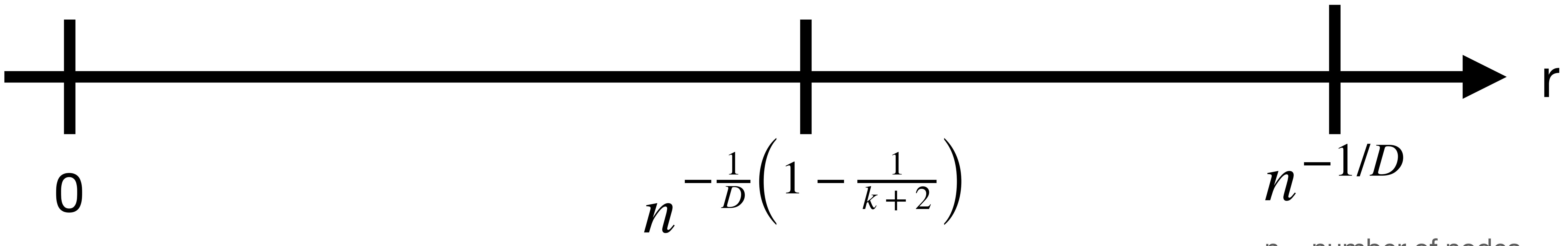
Functional Convergence at dimension k?

[Thomas and Owada 2020]

n = number of nodes
all log terms and constants forgone

Functional Convergence at dimension k?
[Thomas and Owada 2020]

- Cech: weak convergence in finite-dimensional sense



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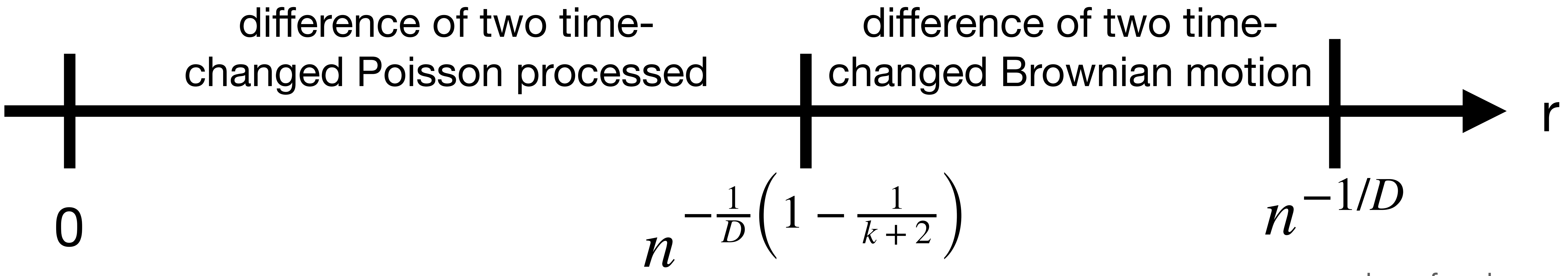

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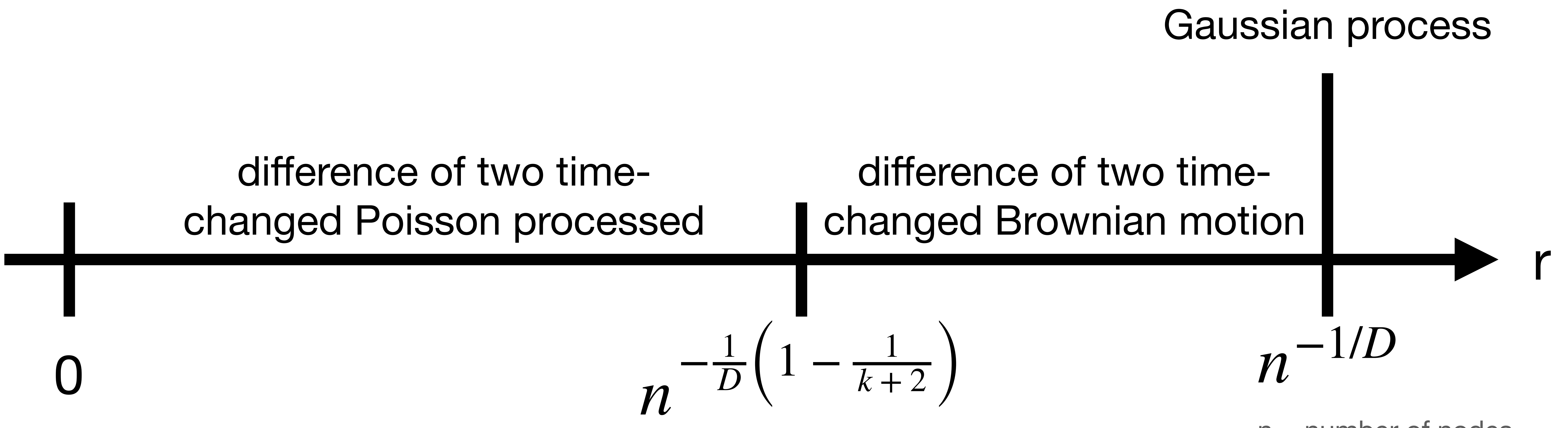

n = number of nodes
all log terms and constants forgone

- 4 CPU cores
- 40 minutes for the Betti numbers
- 7.5 hours for bounds
- memory issues for larger graphs

Subtleties

- Need homological algebra to relate Betti numbers with counts

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  - adding a vertex = construct mapping cone

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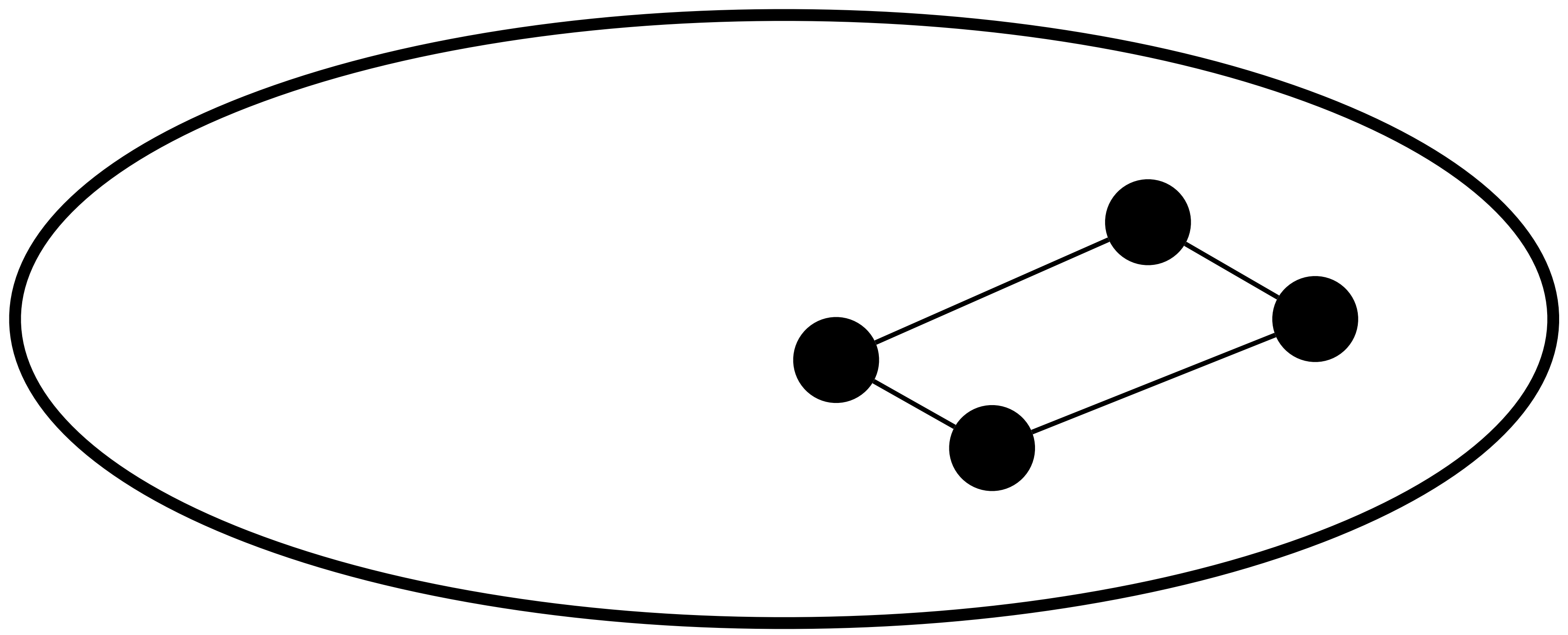
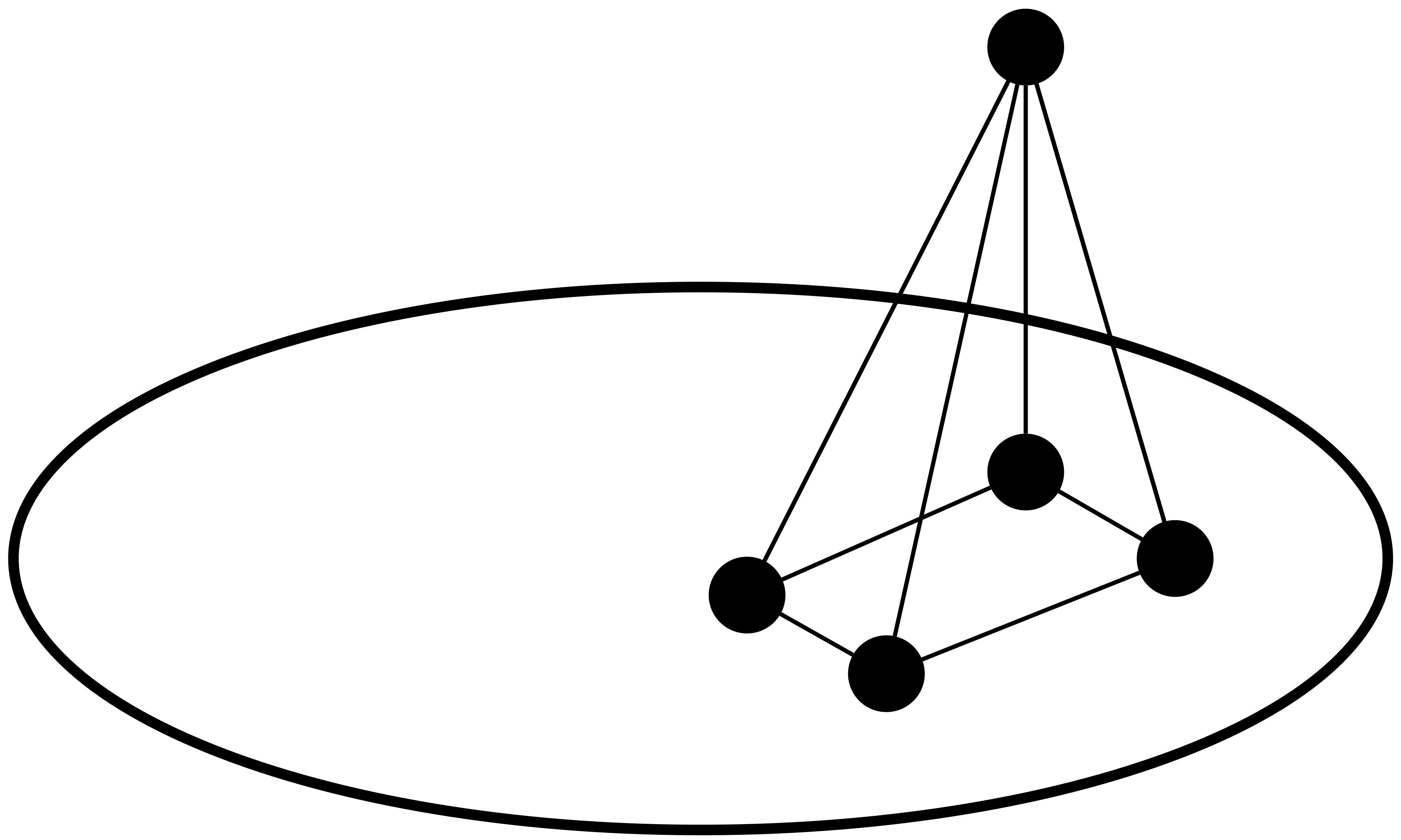
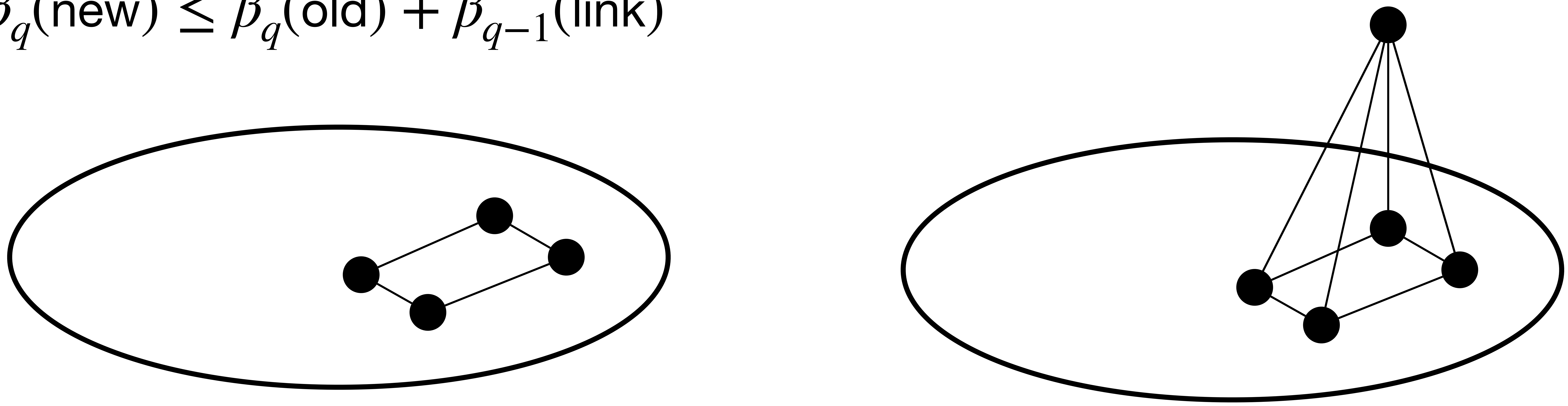




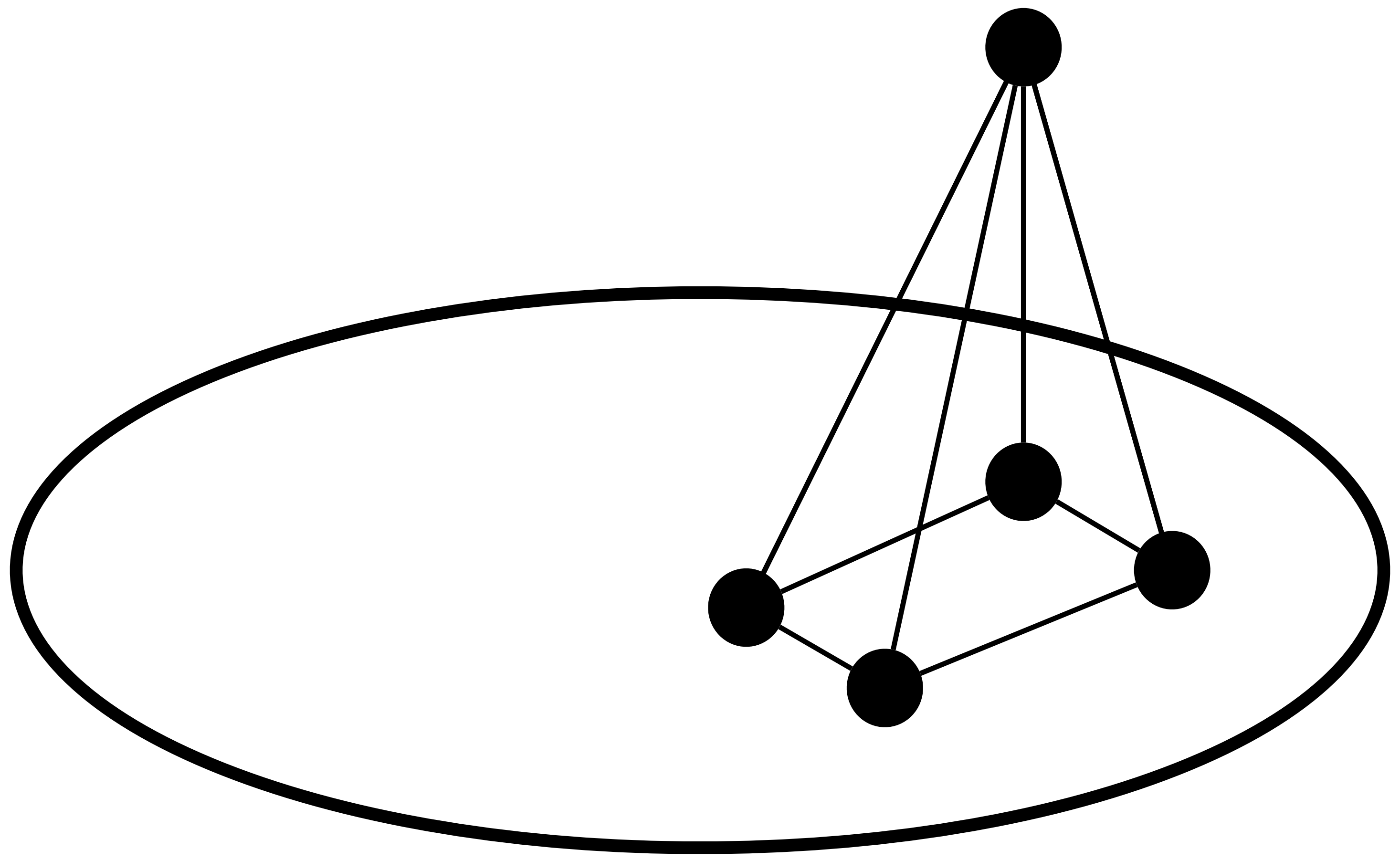
Subtleties

- Need homological algebra to relate Betti numbers with counts
  - adding a vertex = construct mapping cone
  - $\beta_q(\text{new}) \leq \beta_q(\text{old}) + \beta_{q-1}(\text{link})$



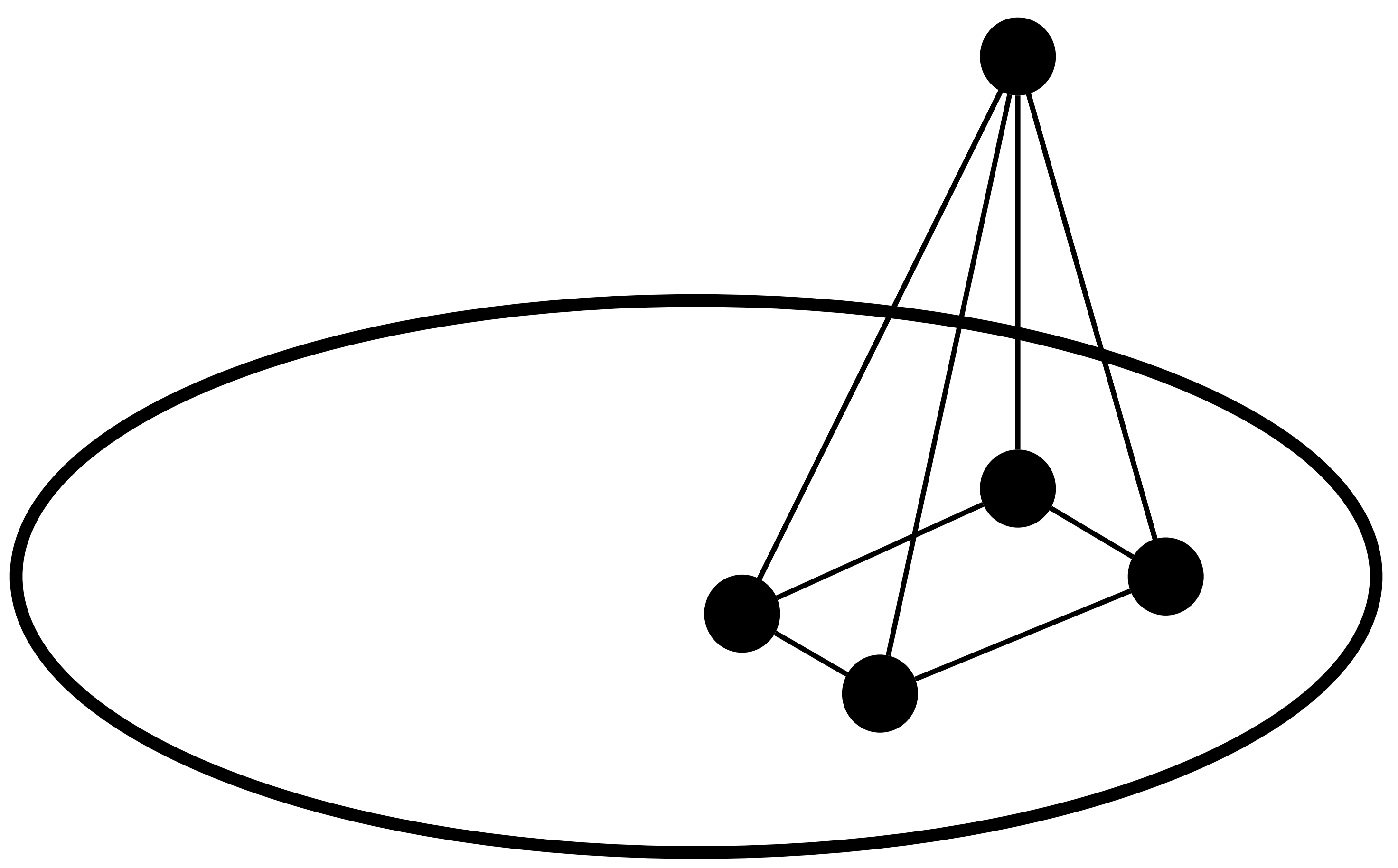
Subtleties

- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$



Subtleties

- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]



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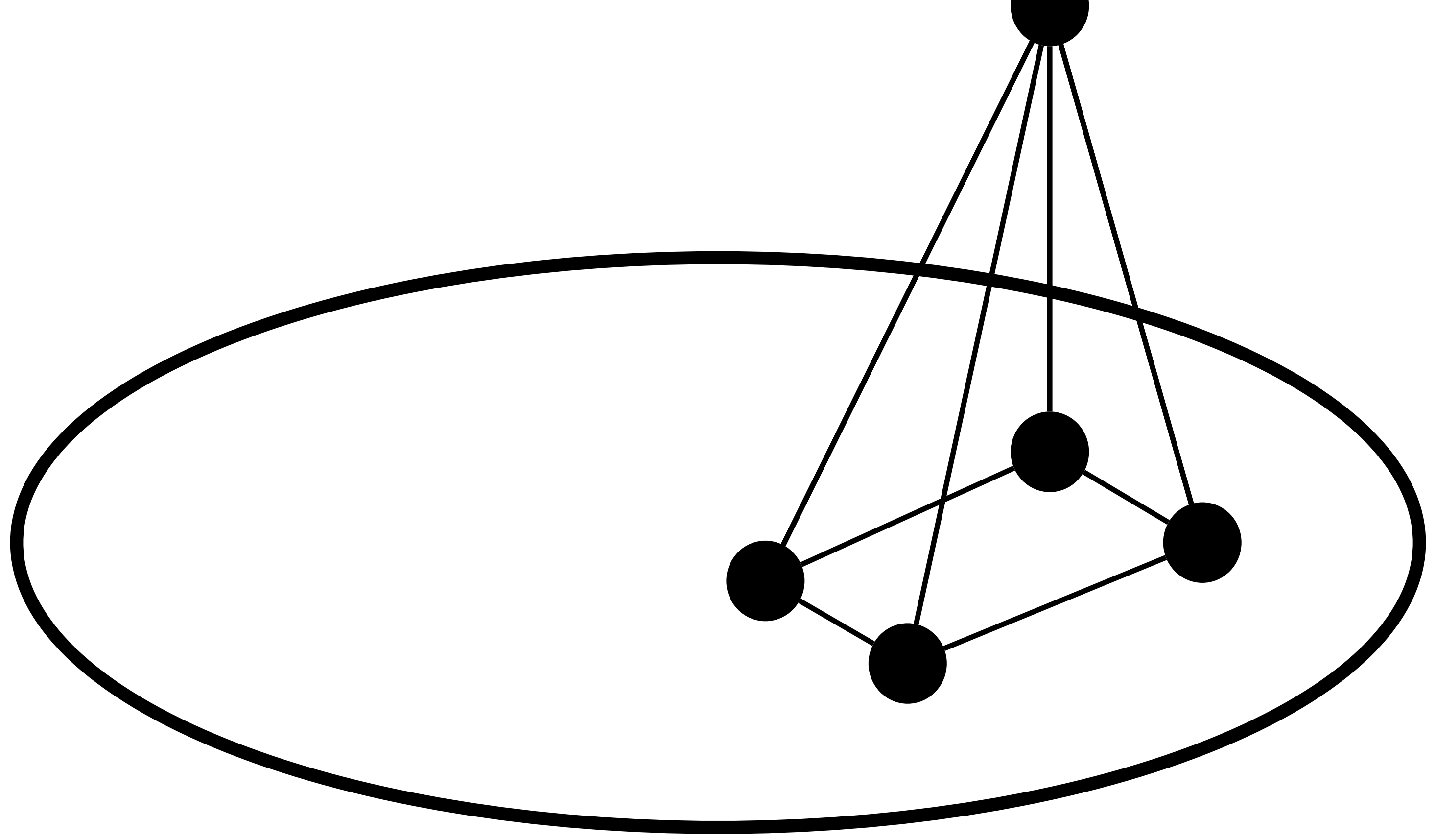
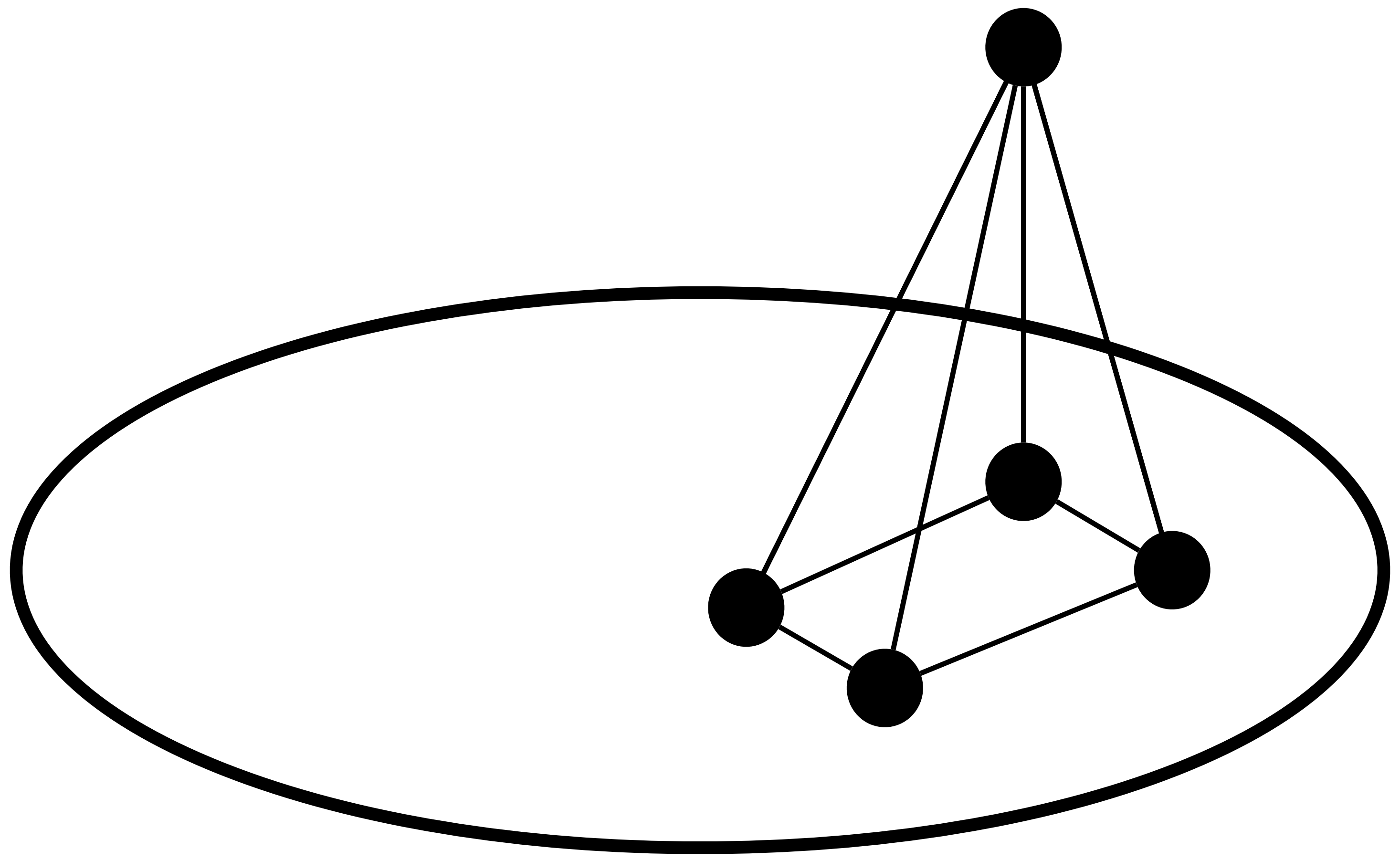
- Generalize minimal cycle results with homological algebra



Subtleties



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  - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$

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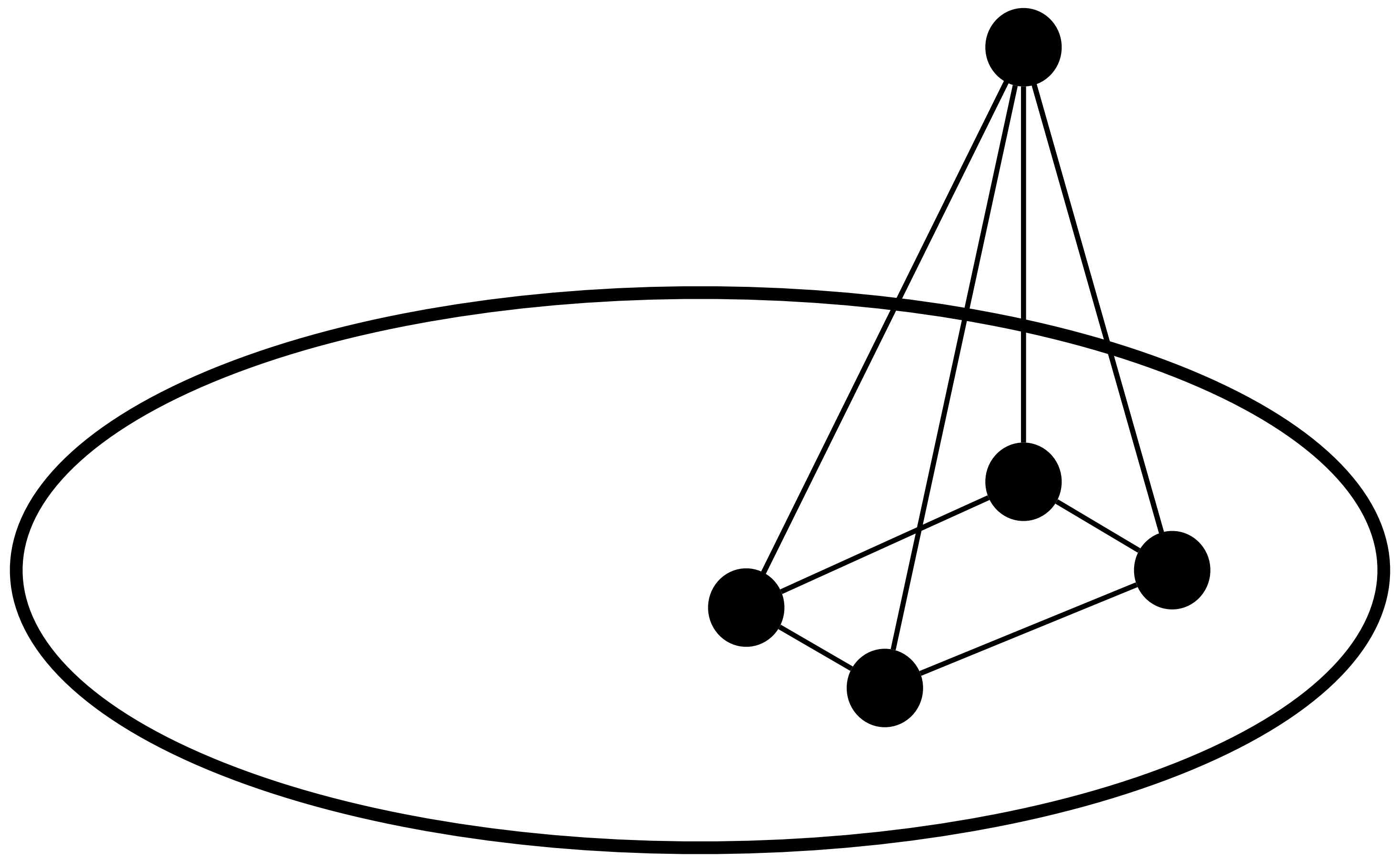

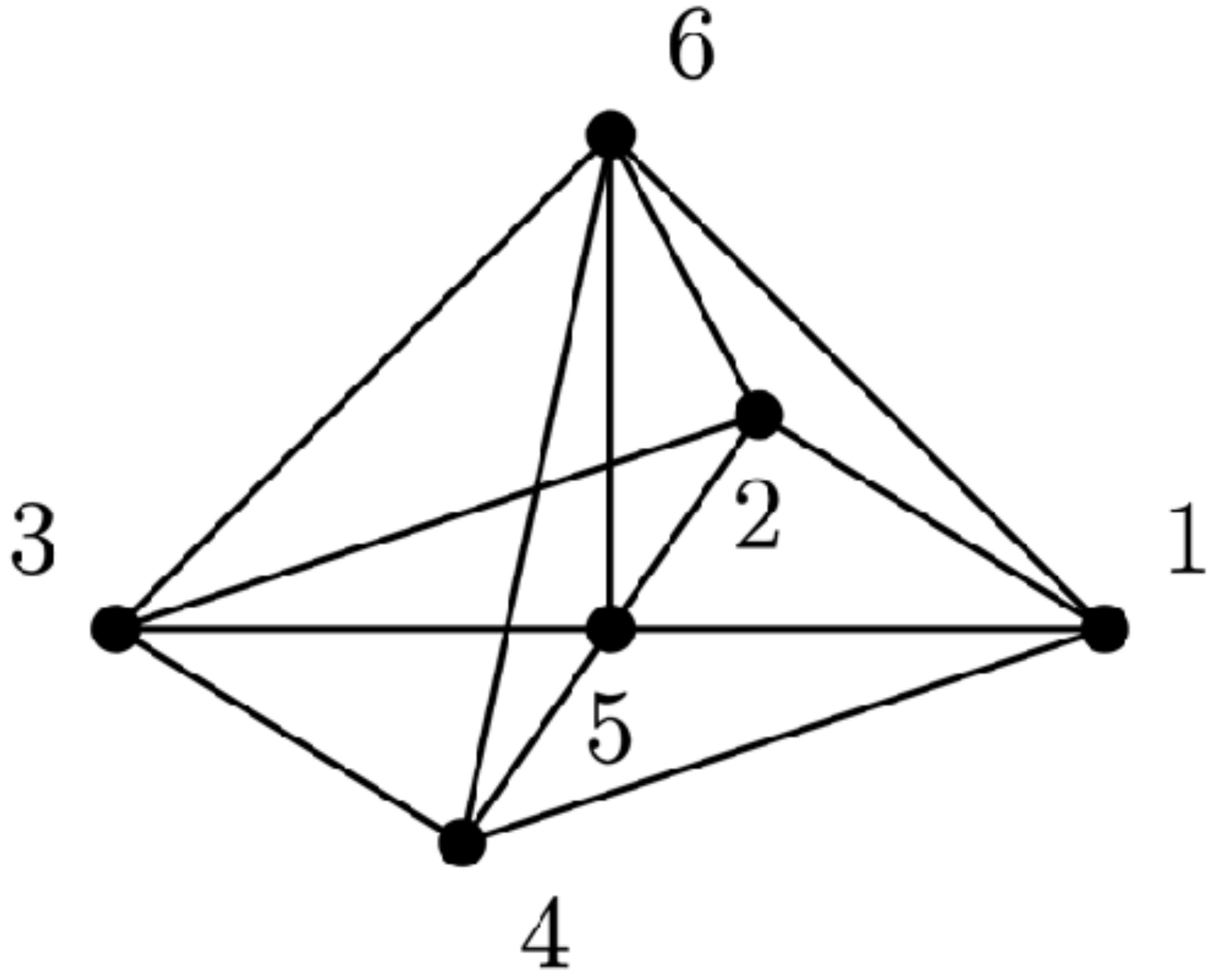

Subtleties

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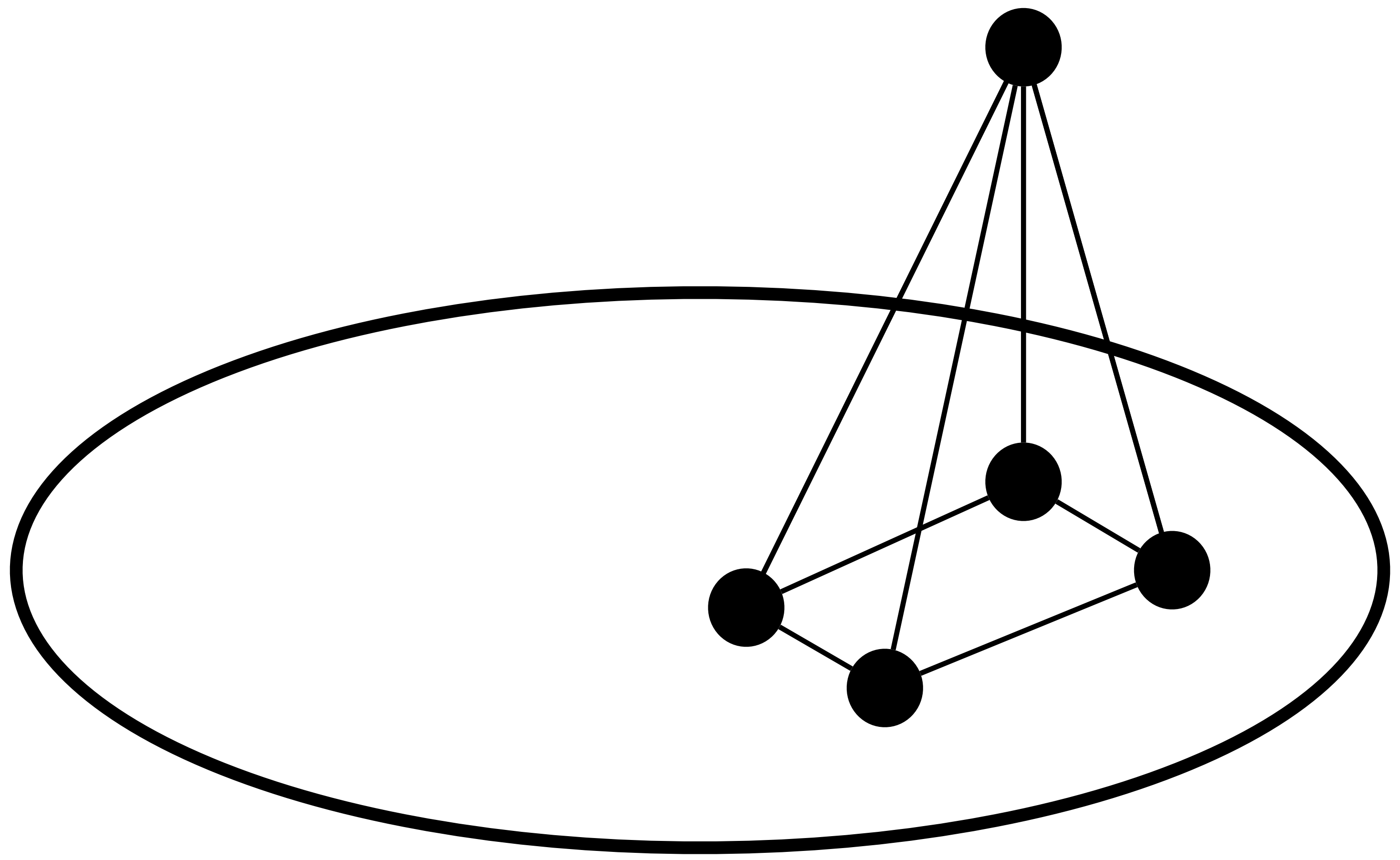
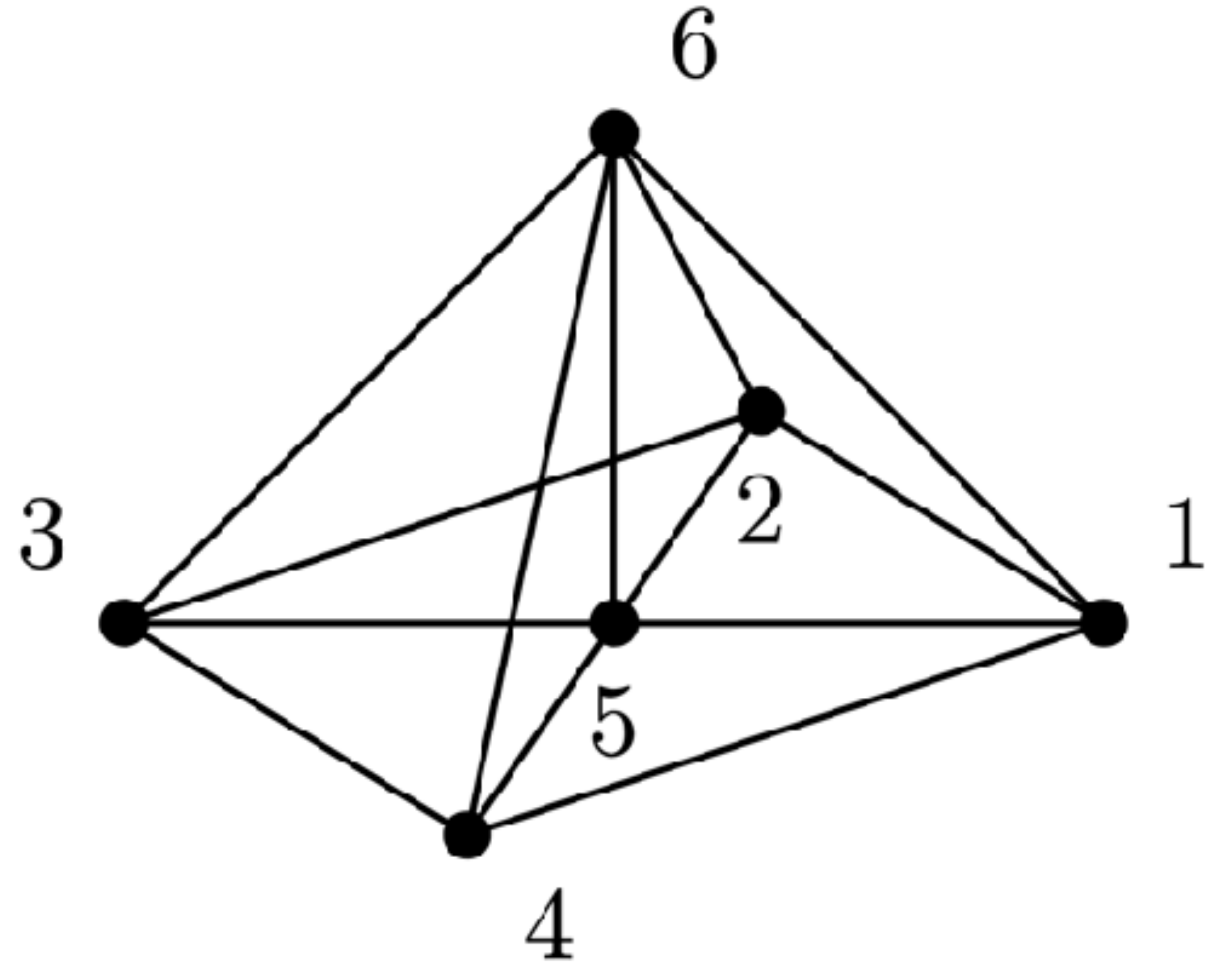
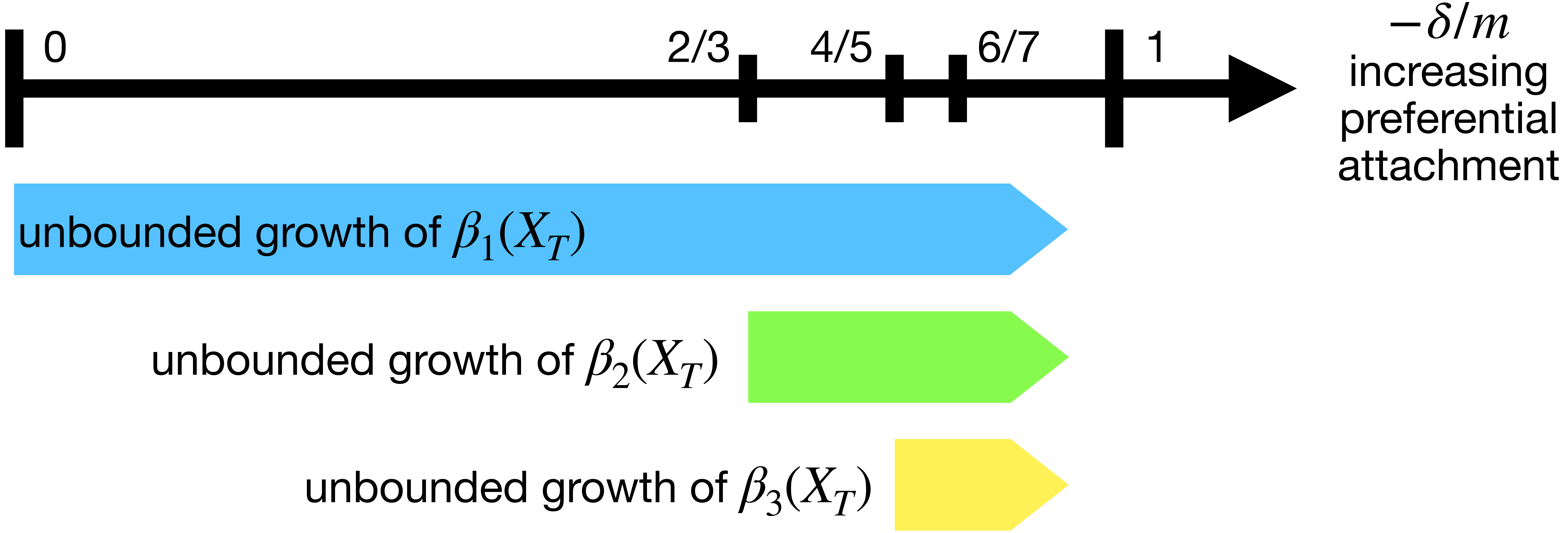
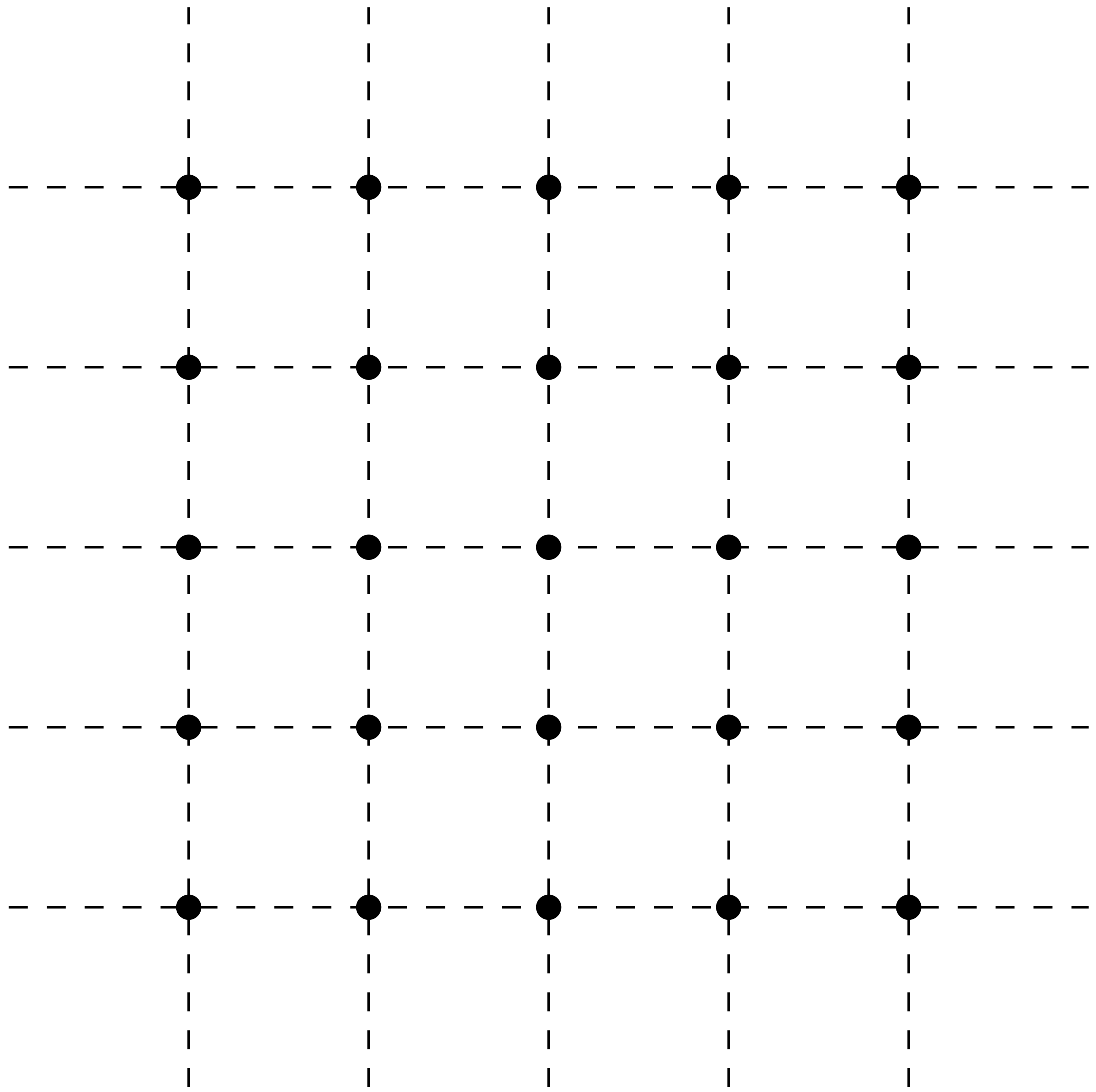
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs






0
2/3
4/5
6/7
1
$-\delta/m$
increasing
preferential
attachment
unbounded growth of $\beta_1(X_T)$
unbounded growth of $\beta_2(X_T)$
unbounded growth of $\beta_3(X_T)$

Bernoulli Bond Percolation



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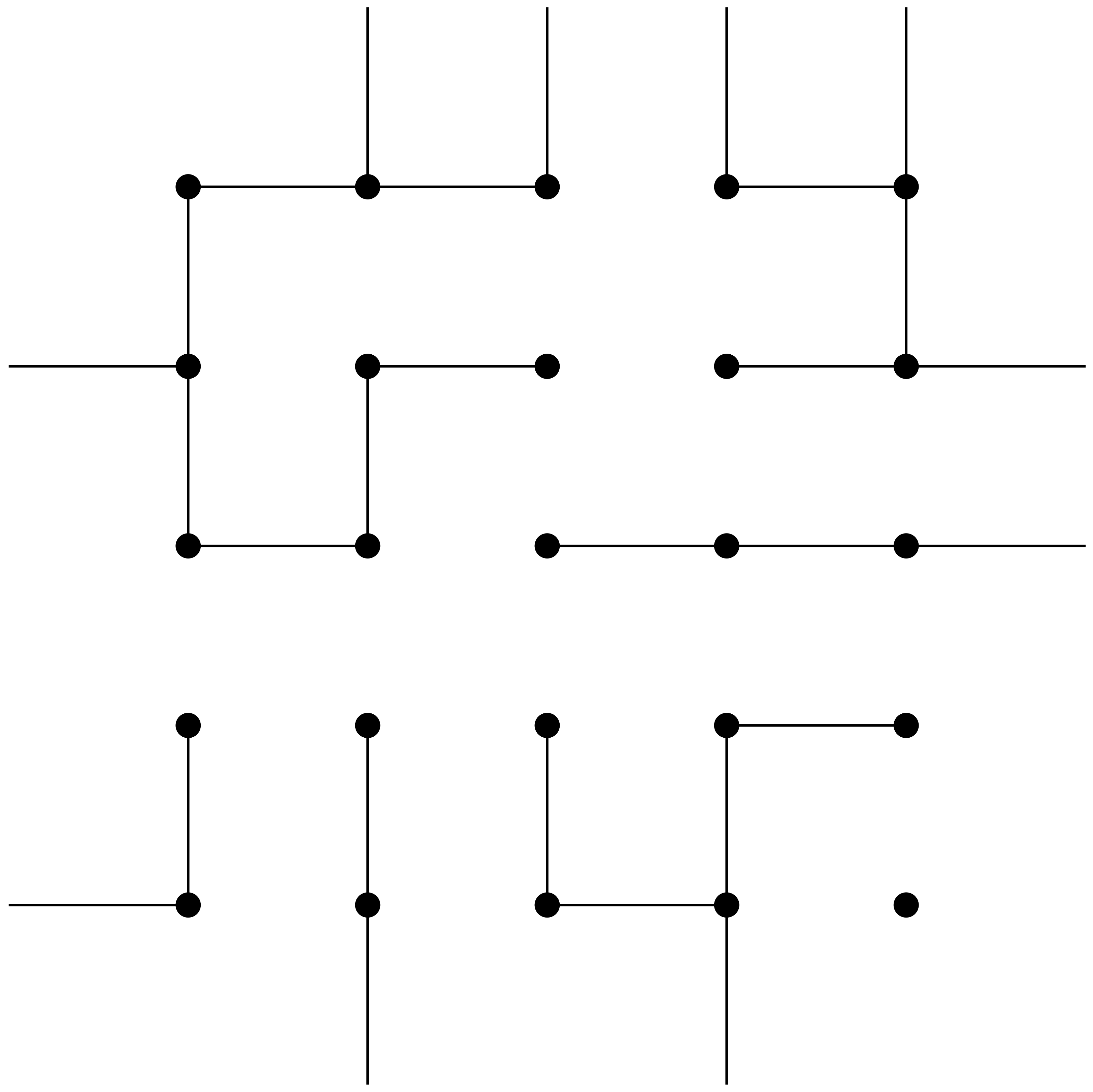
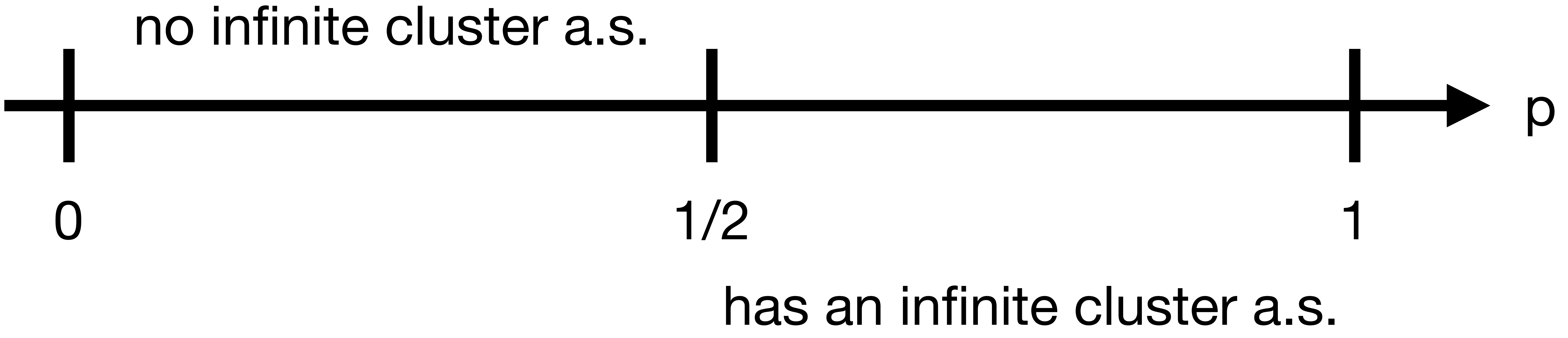
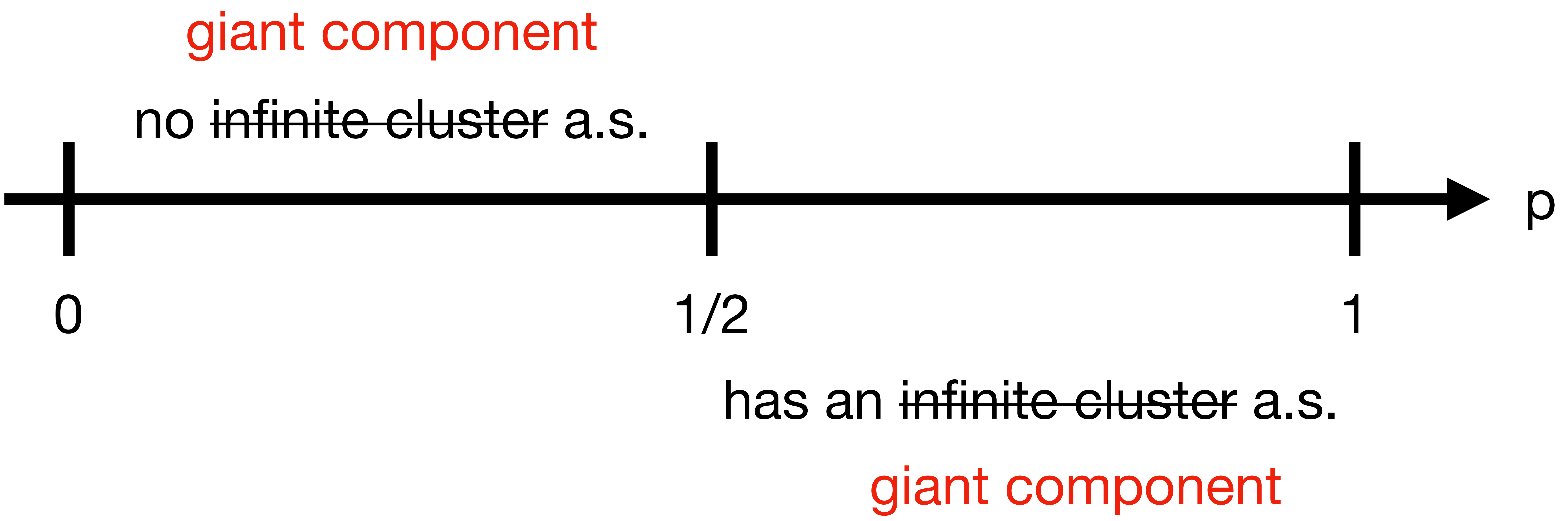


Phase Transition
[Harris 1960, Kesten 1980]



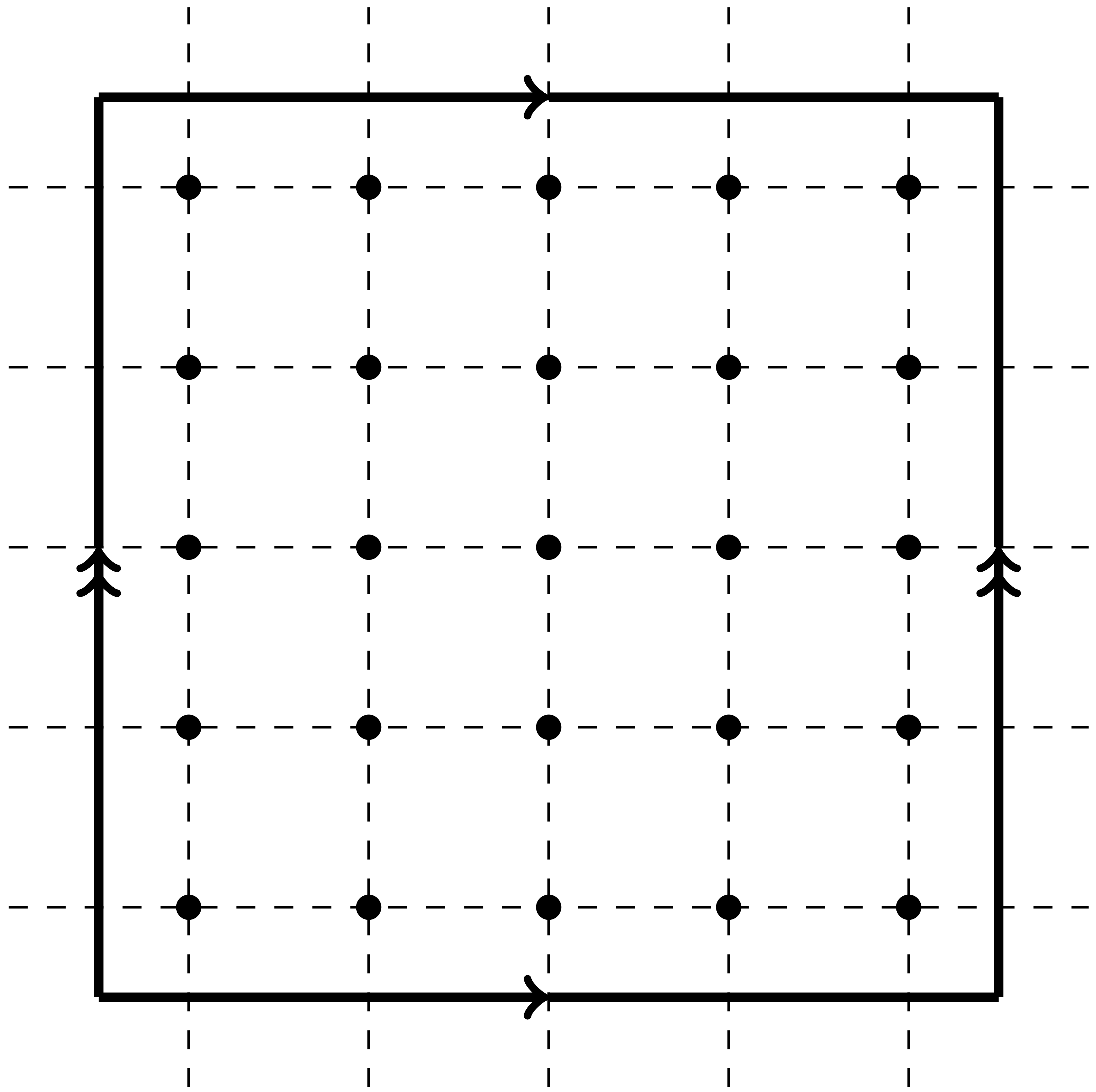
Phase Transition

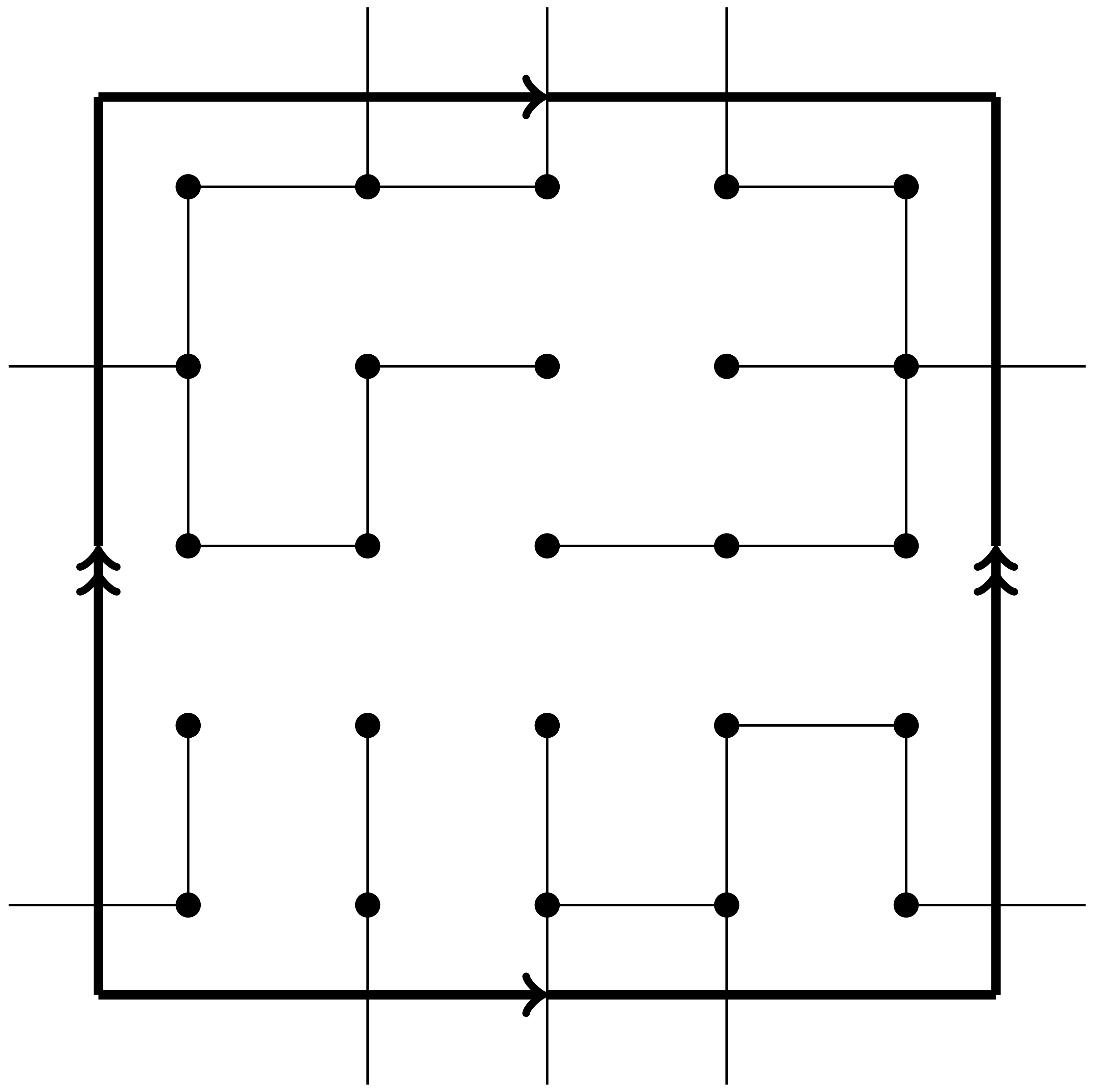
[Harris 1960, Kesten 1980]



Giant Cycles?

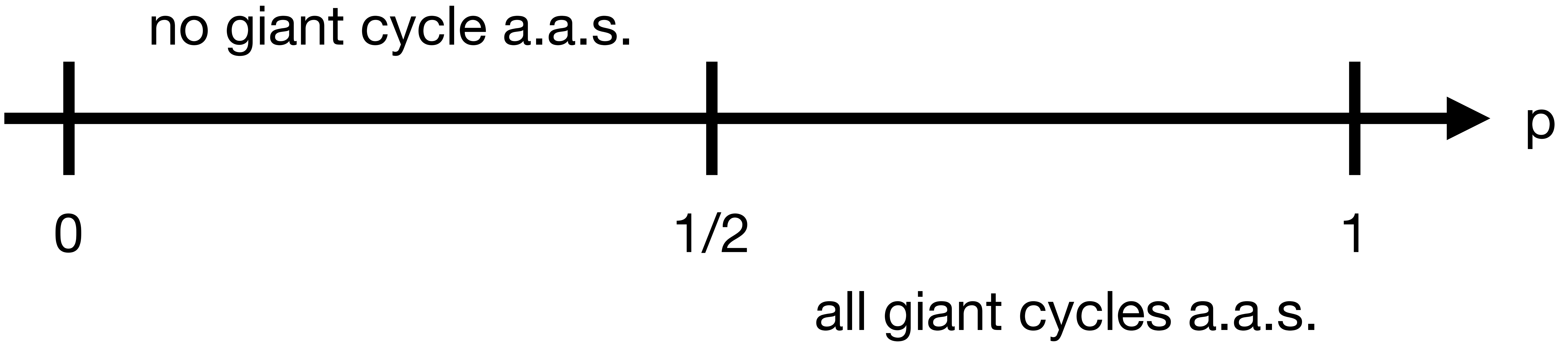
Bernoulli Bond Percolation



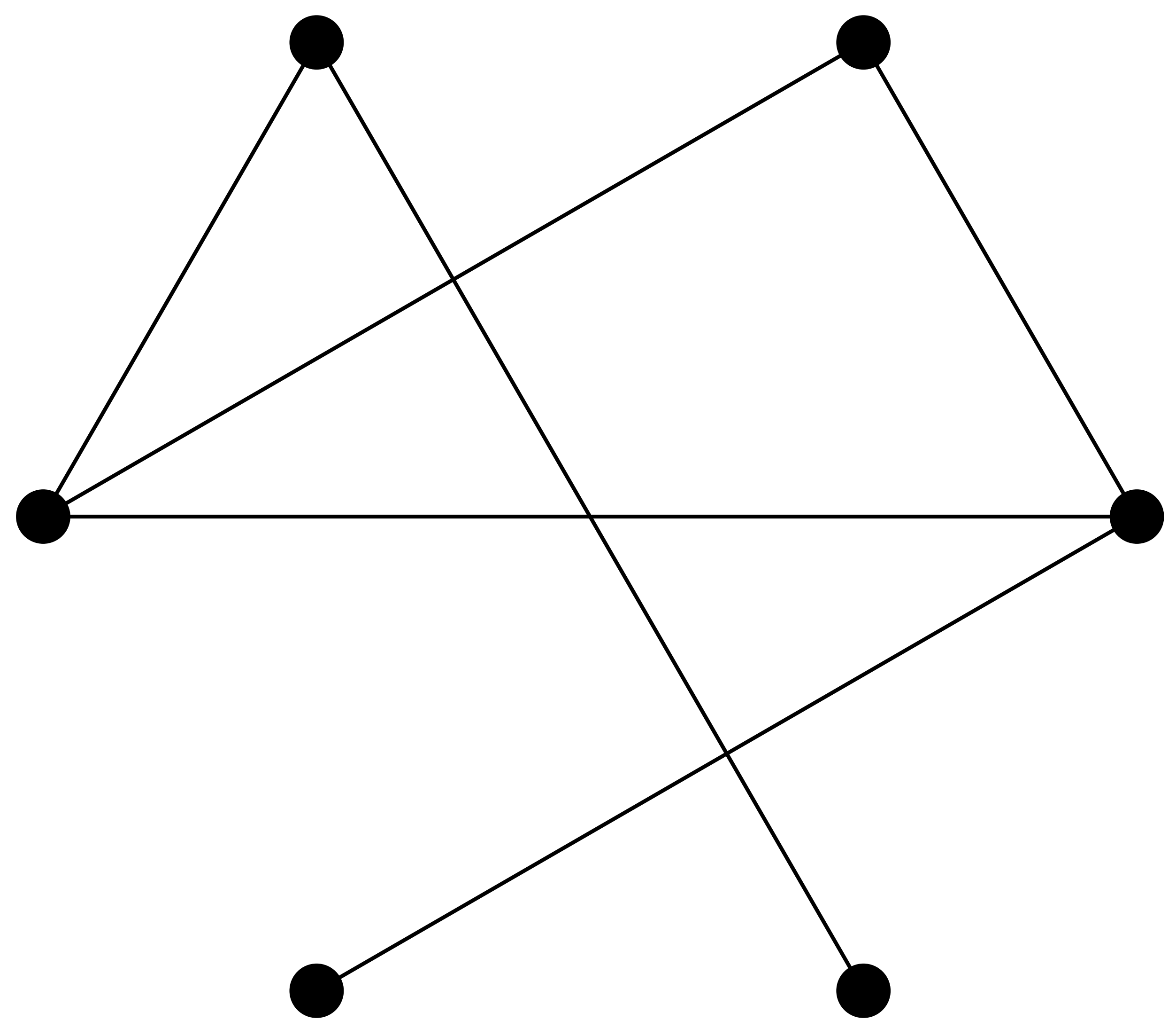


Phase Transition

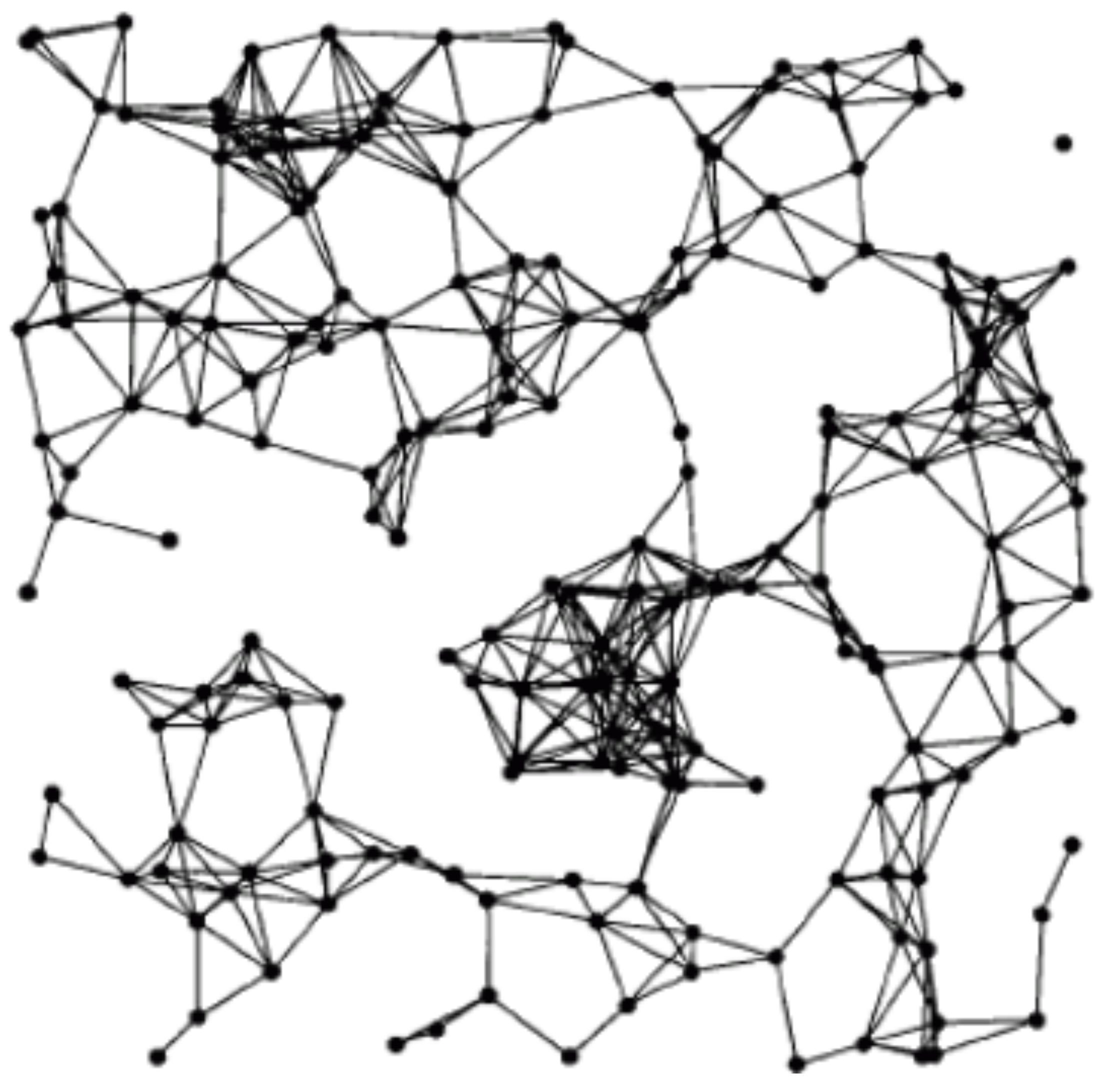
[Duncan-Kahle-Schweinhart, 2021]



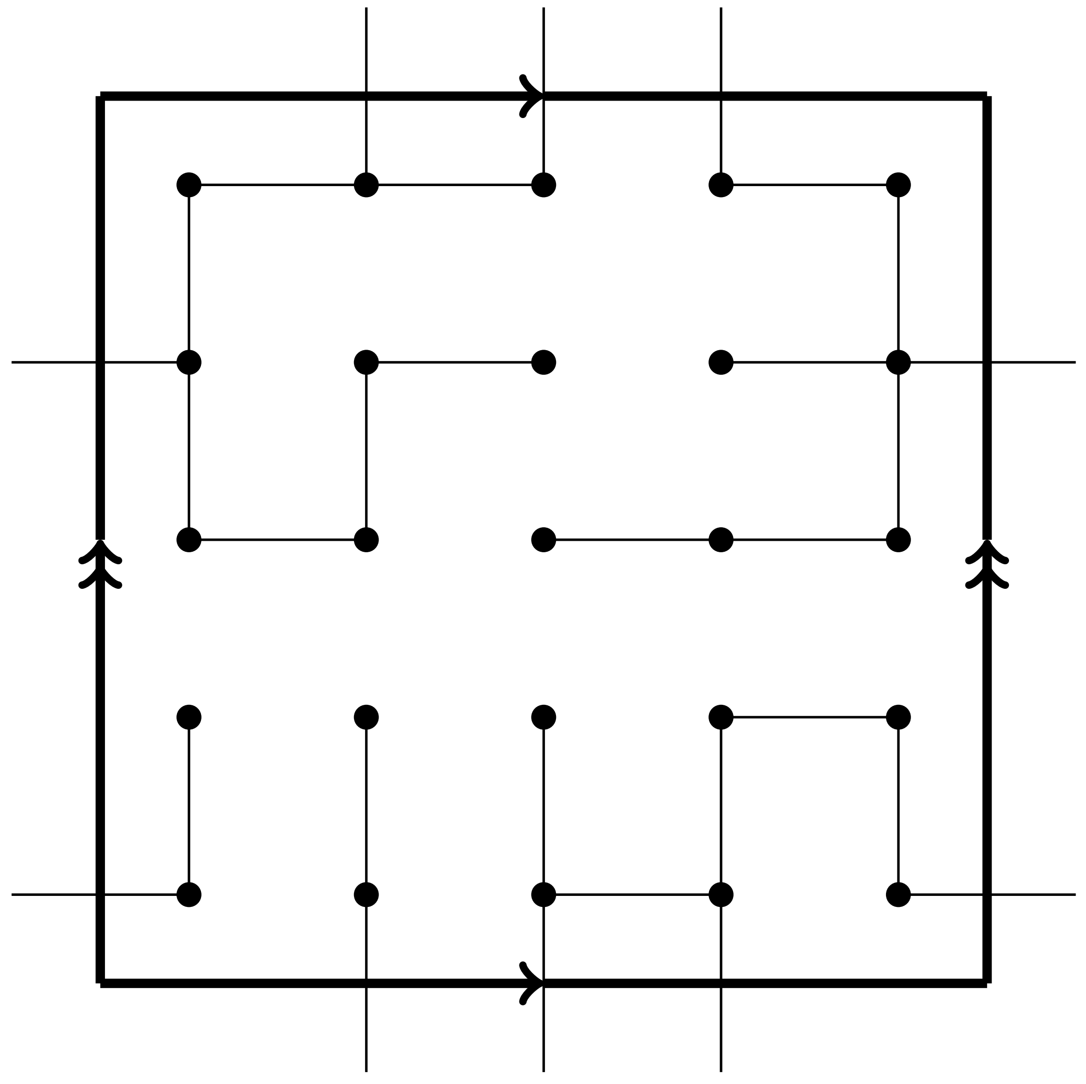
Tea with Random Topology



Erdo-Renyi Complexes



Geometric Complexes



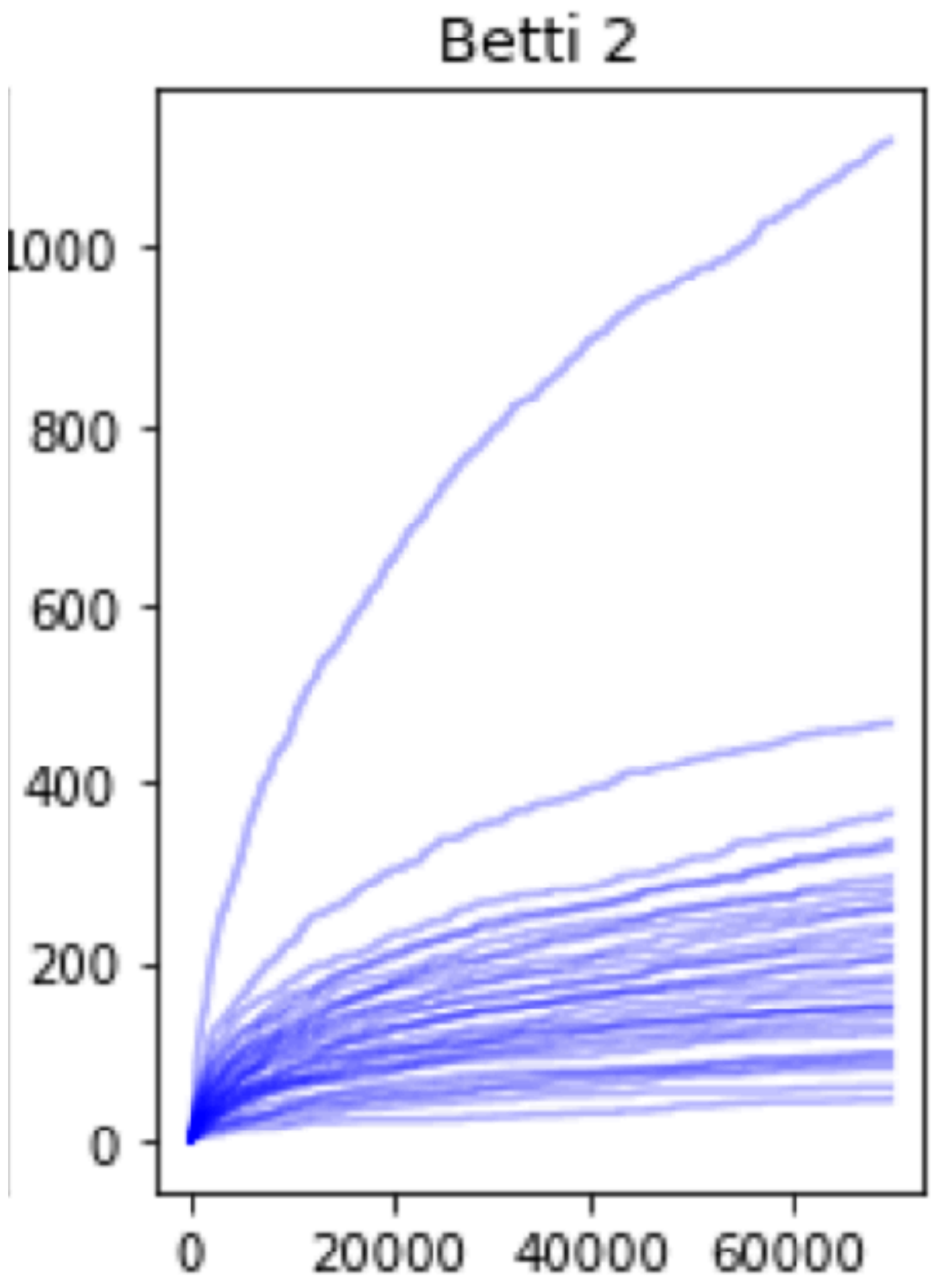
Topological Percolation

Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

In practice???

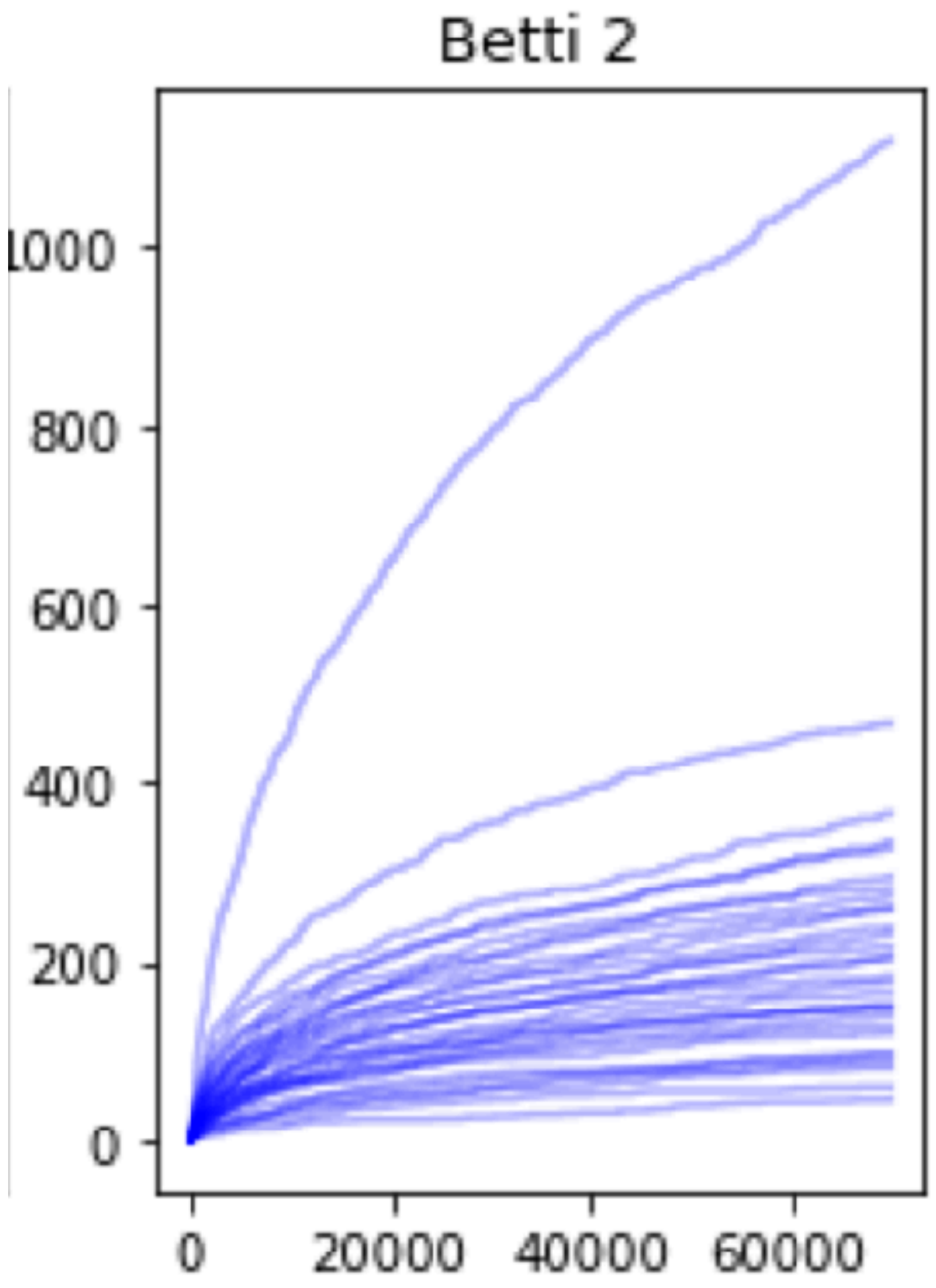
$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

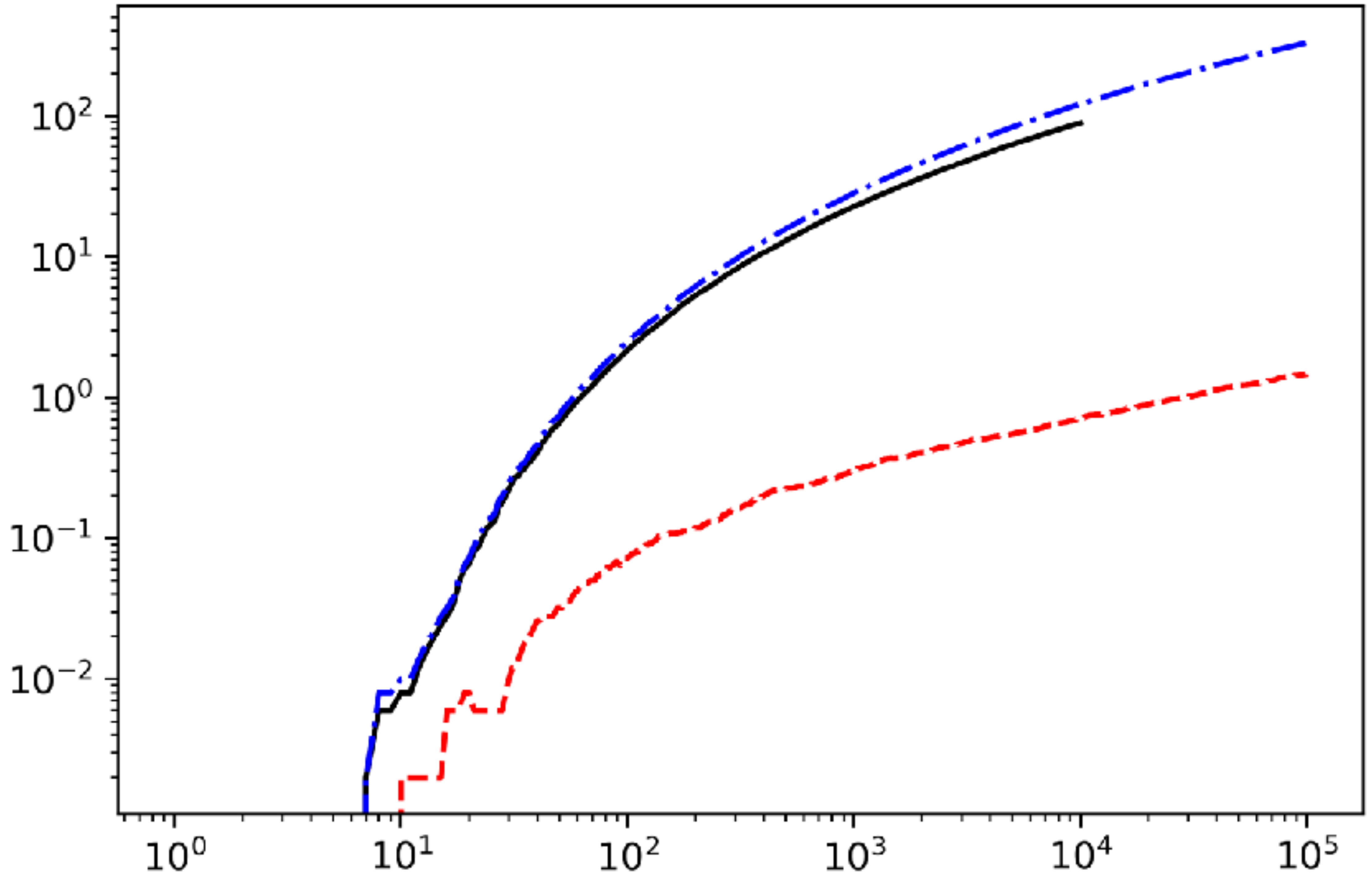
$$\log E[\beta_2] \approx (1-4x)\log(\textbf{num of nodes})$$



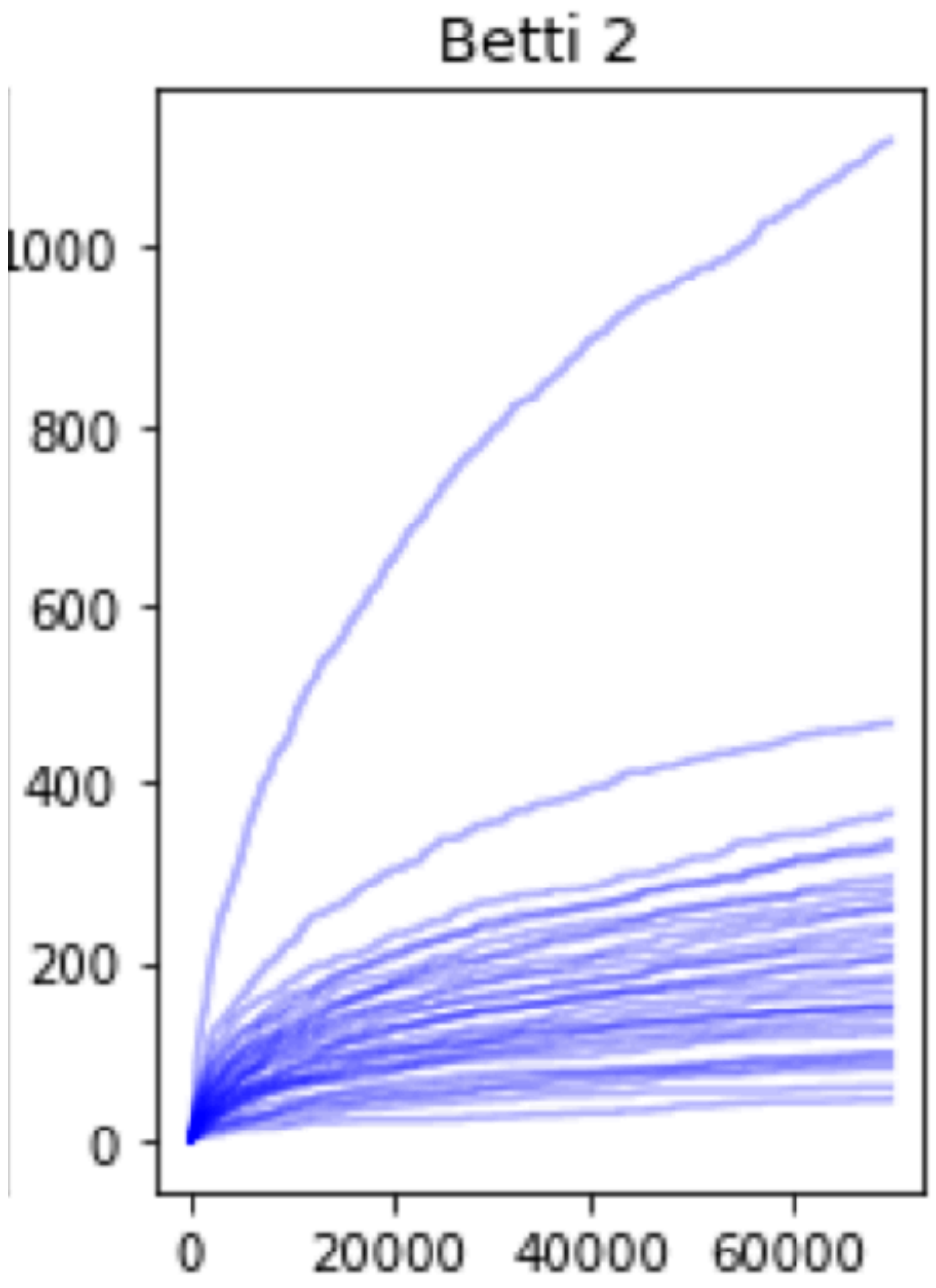
$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

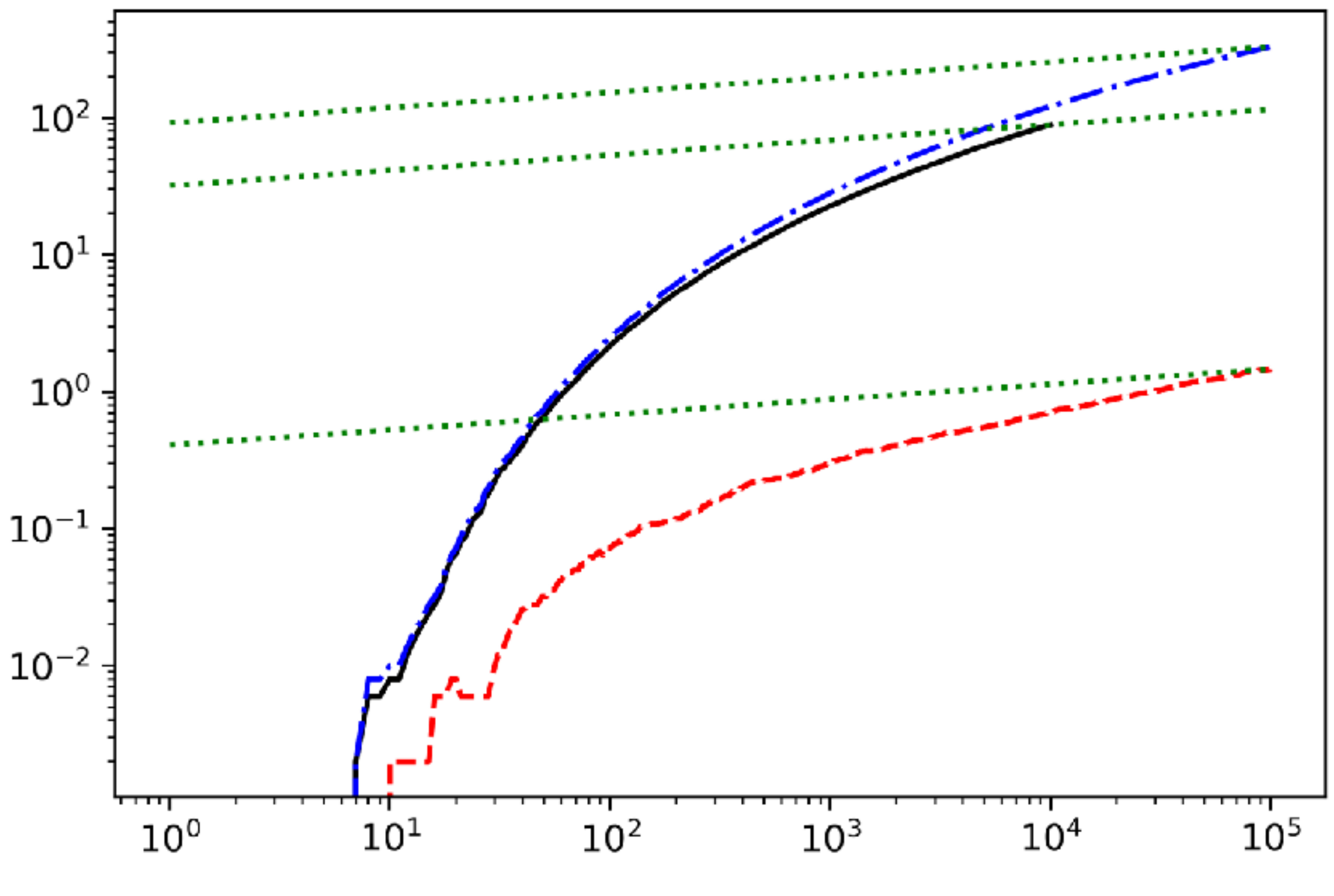
$$\log E[\beta_2] \approx (1 - 4x)\log(\textbf{num of nodes})$$



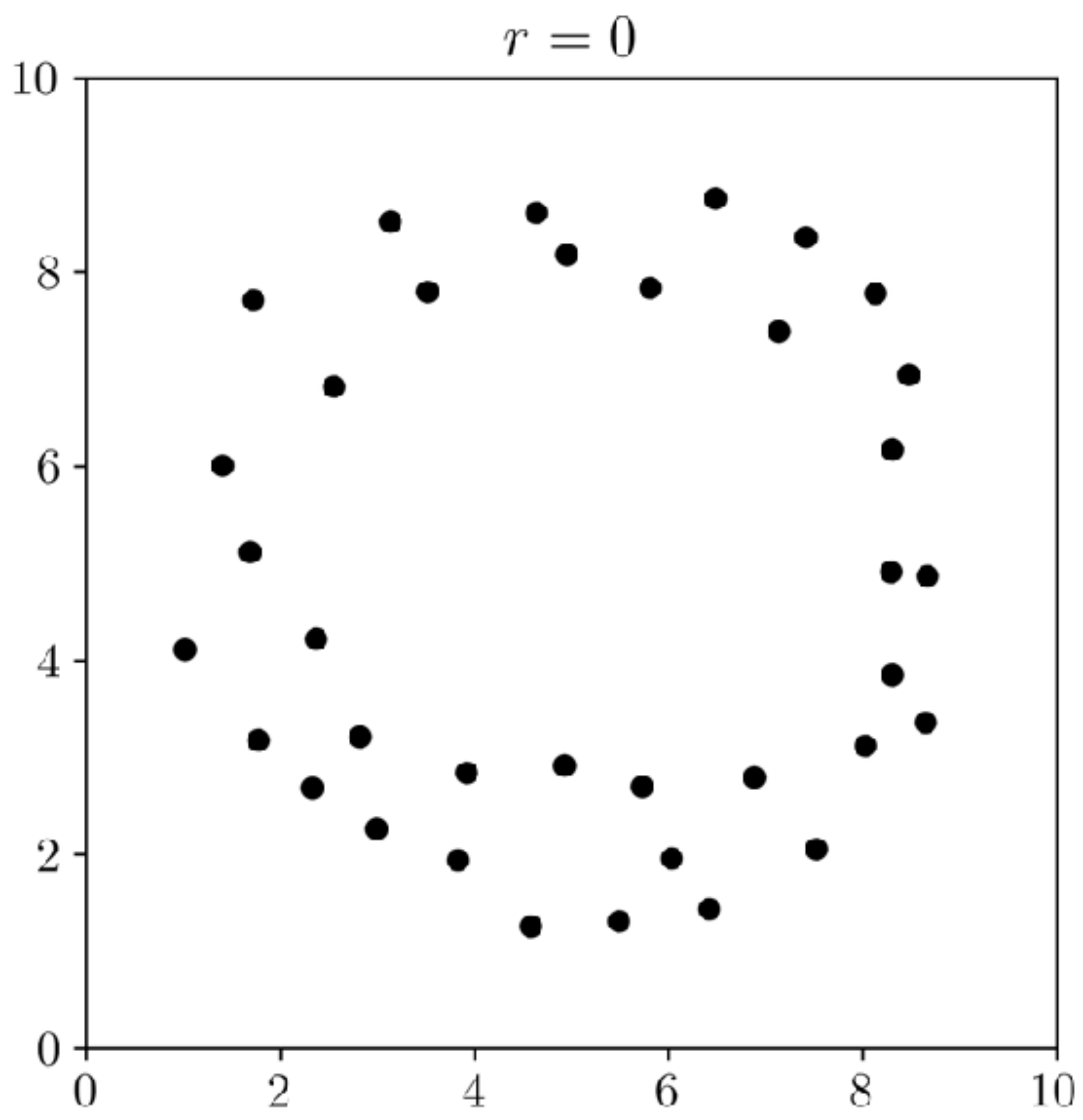


$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$





Maximally Persistent Cycles



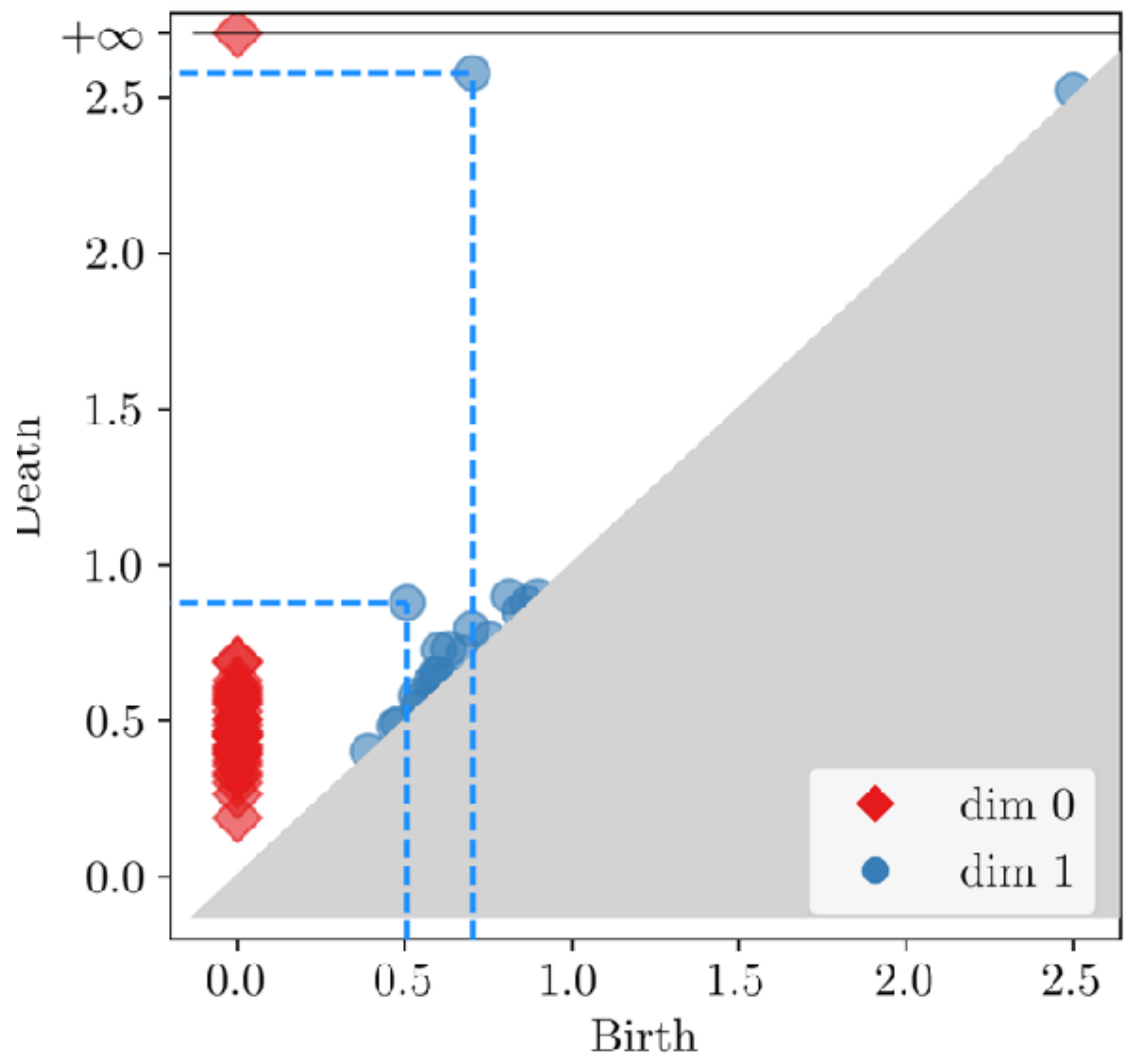


image credit: Andrey Yao

Maximally Persistent Cycles

n points in expectation

k-cycle

Maximally Persistent Cycles
[Bobrowski-Kahle-Skraba 2017]

n points in expectation

k-cycle

$$c\left(\frac{\log n}{\log\log n}\right)^{1/k} \leq \text{max persistence} \leq C\left(\frac{\log n}{\log\log n}\right)^{1/k}$$

a.a.s.

V. What lies ahead

orders of magnitude of Betti numbers

homotopy connectedness

parameter estimation?

orders of magnitude of Betti numbers

simplicial preferential attachment?

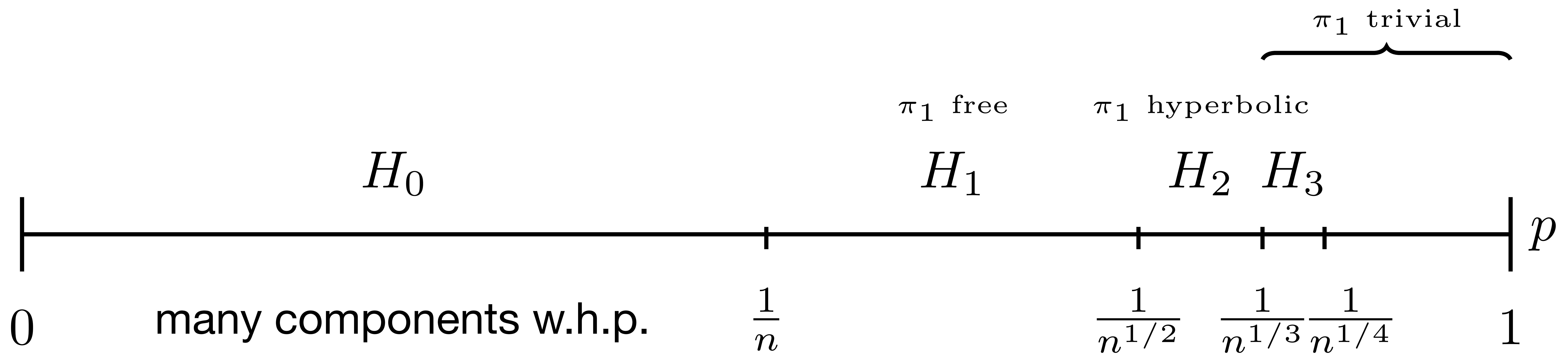
homotopy connectedness

other non-homogeneous complexes?

What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

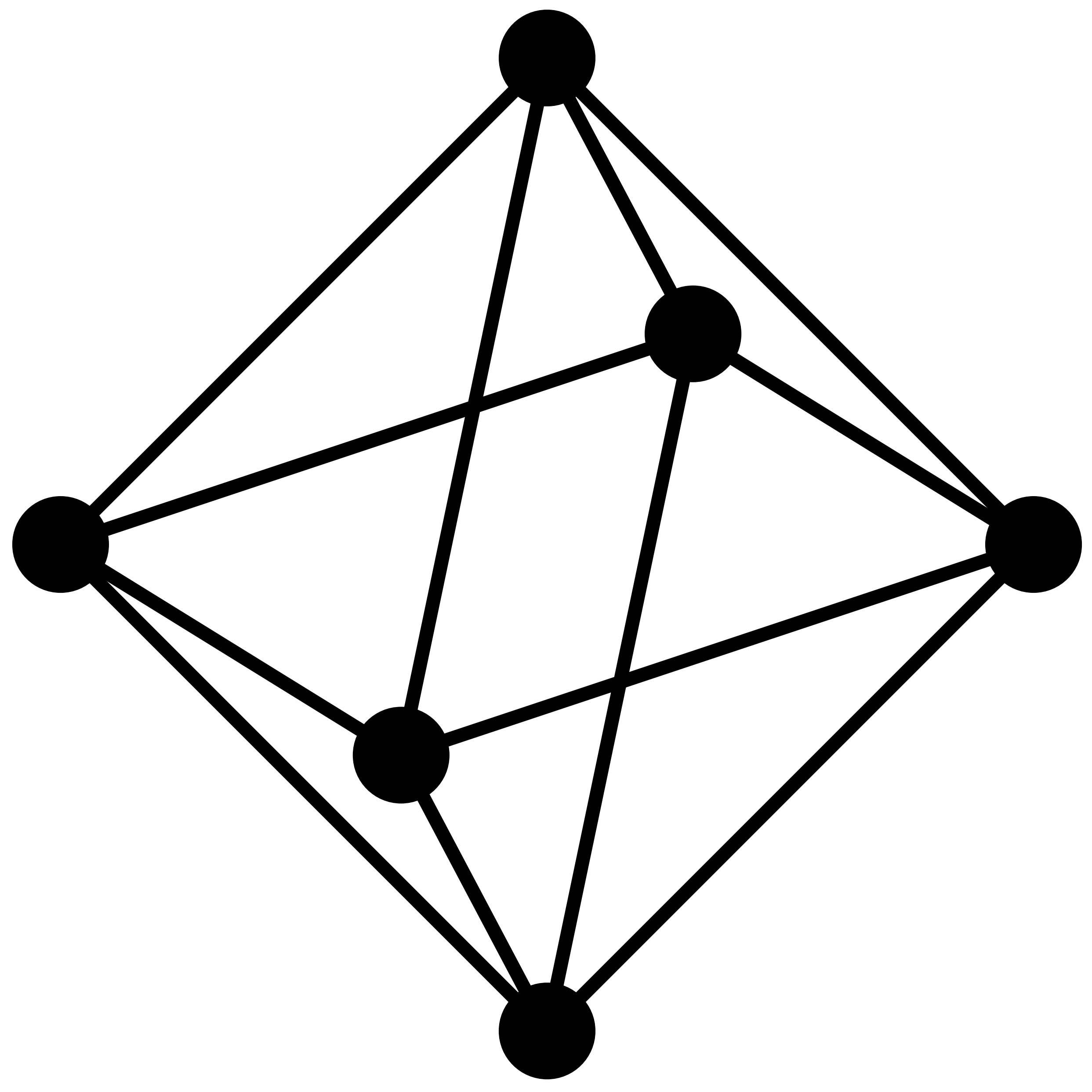
Fundamental Group
[Kahle 2009, Babson 2012, Costa-Farber-Horak 2015]



all log terms and constants forgone

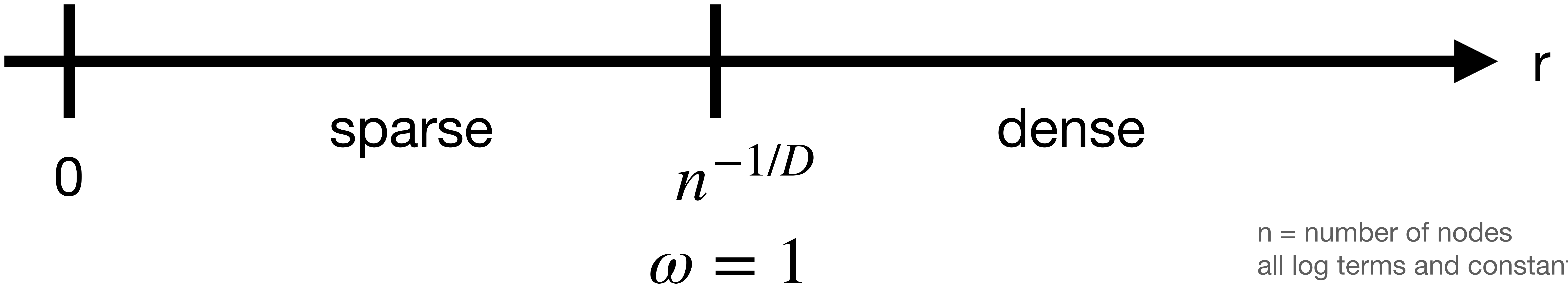
[Barmak 2023]

- A clique complex is q-homotopy-connected
- if every collection of 2(q + 1) nodes has a common neighbor.



Expected Betti numbers at dimension k
[Kahle 2011]

- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension



n = number of nodes
all log terms and constants forgone

Expected Betti numbers at dimension k
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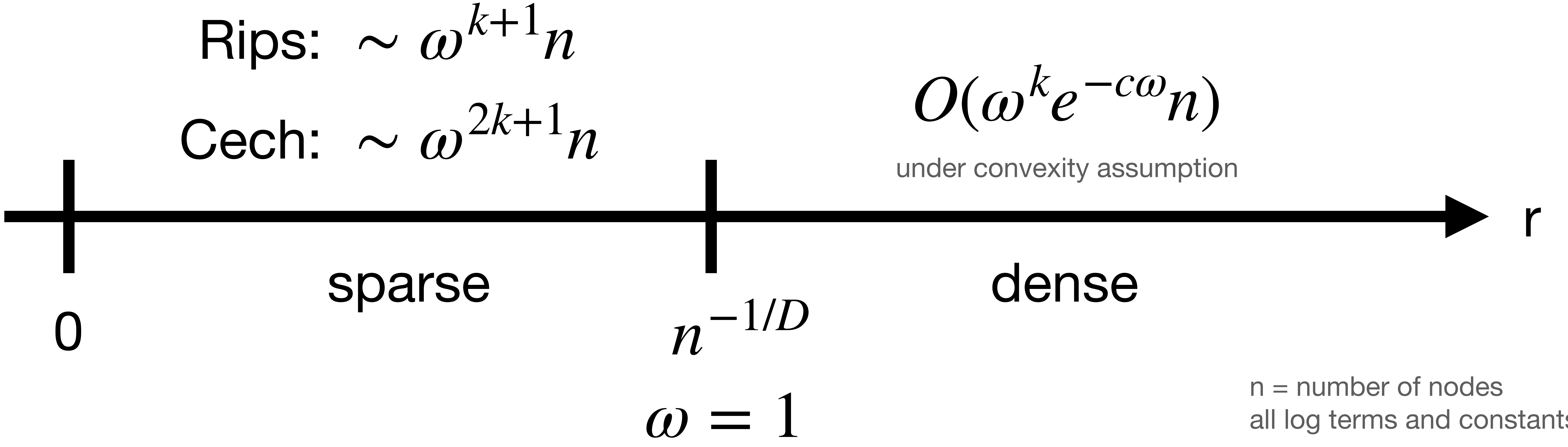

n = number of nodes
all log terms and constants forgone

Expected Betti numbers at dimension k
[Kahle 2011]

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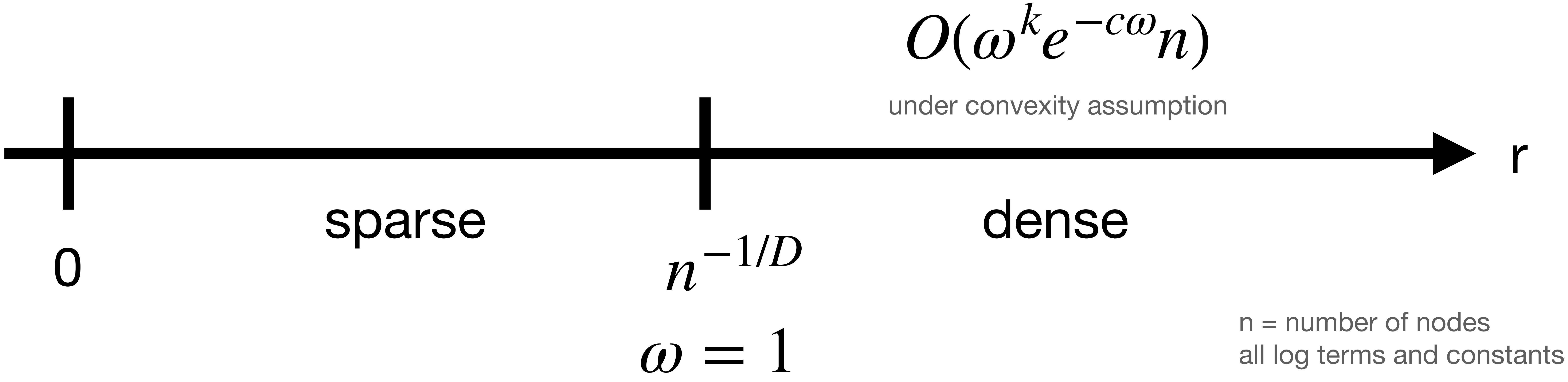
- $\omega = nr^D$, where D is the ambient dimension
- $E\beta_k(\text{Cech}) \sim \omega^{2k+1} n$



n = number of nodes
all log terms and constants forgone

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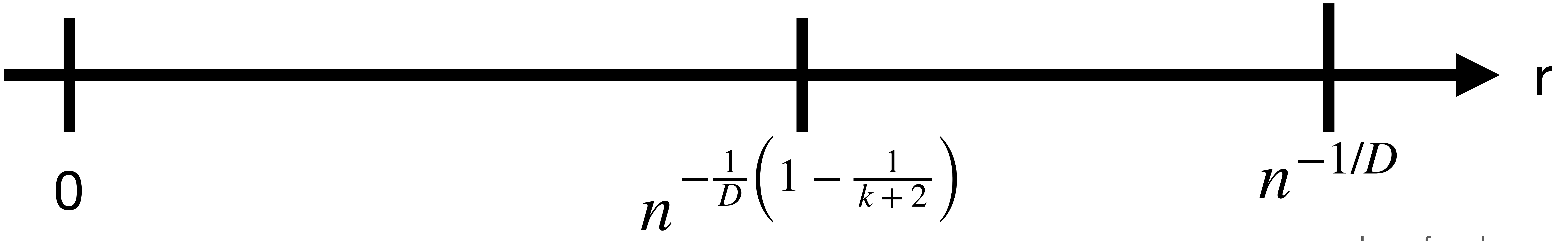

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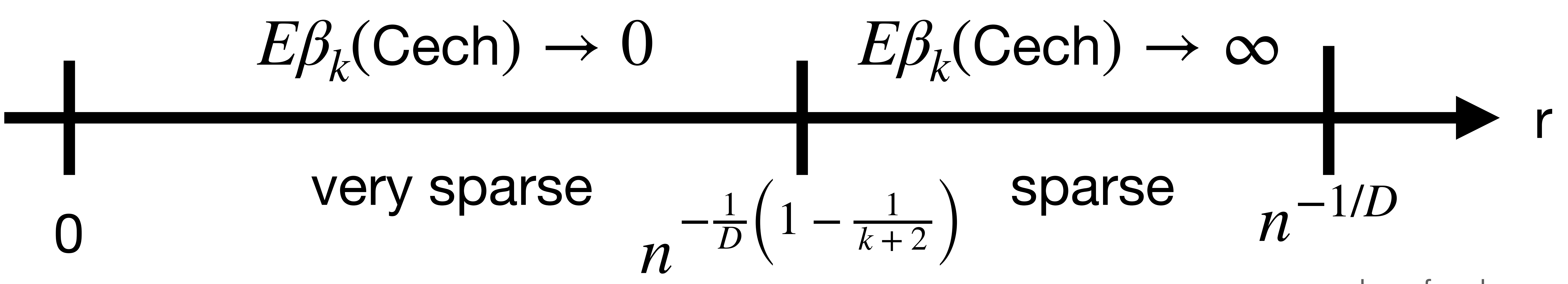

n = number of nodes
all log terms and constants forgone