The Topology of Preferential Attachment

How Random Interaction Begets Holes

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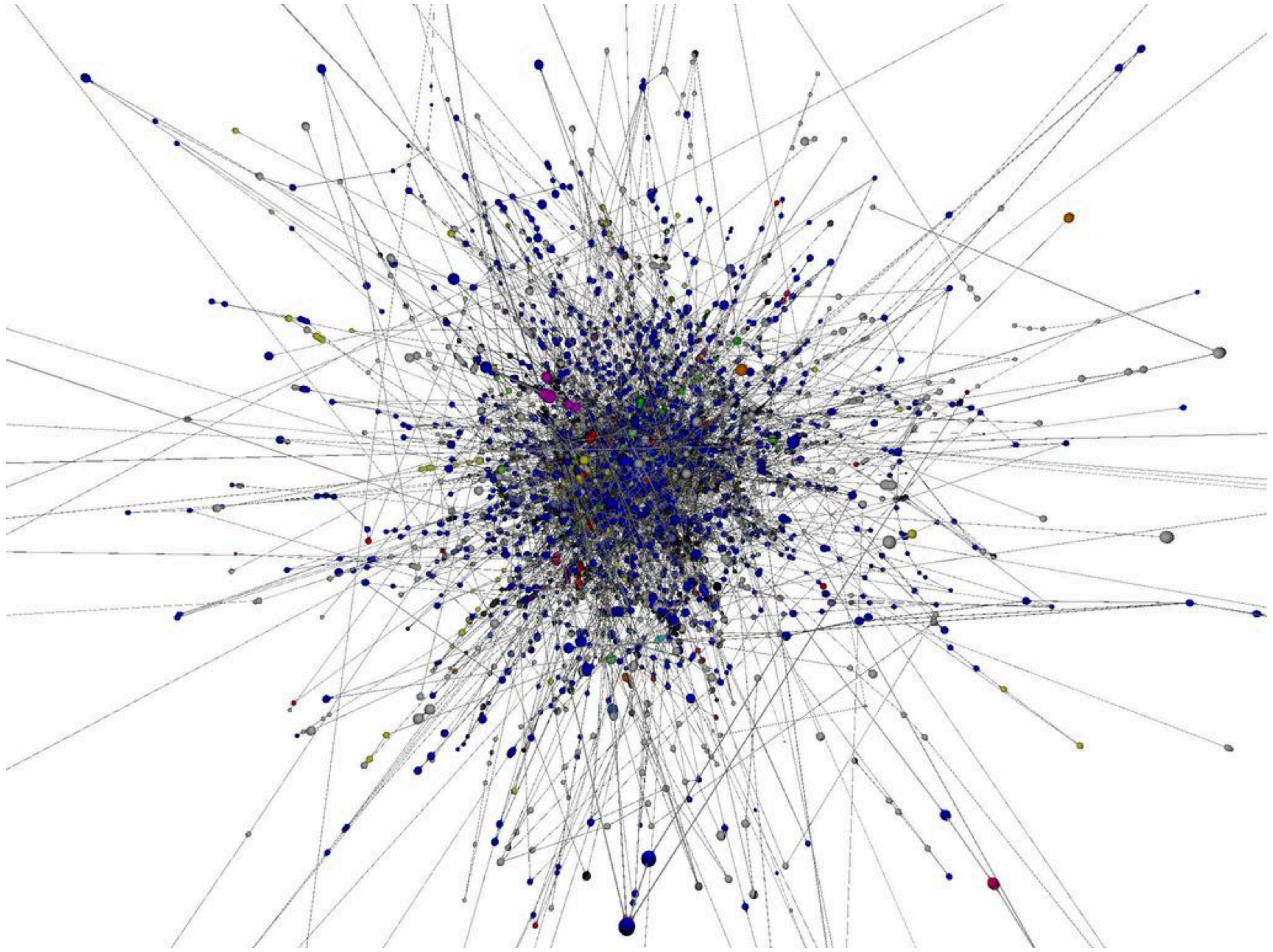
Cornell University
cs2323@cornell.edu

So, preferential attachment…

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Just a bouquet of circles?



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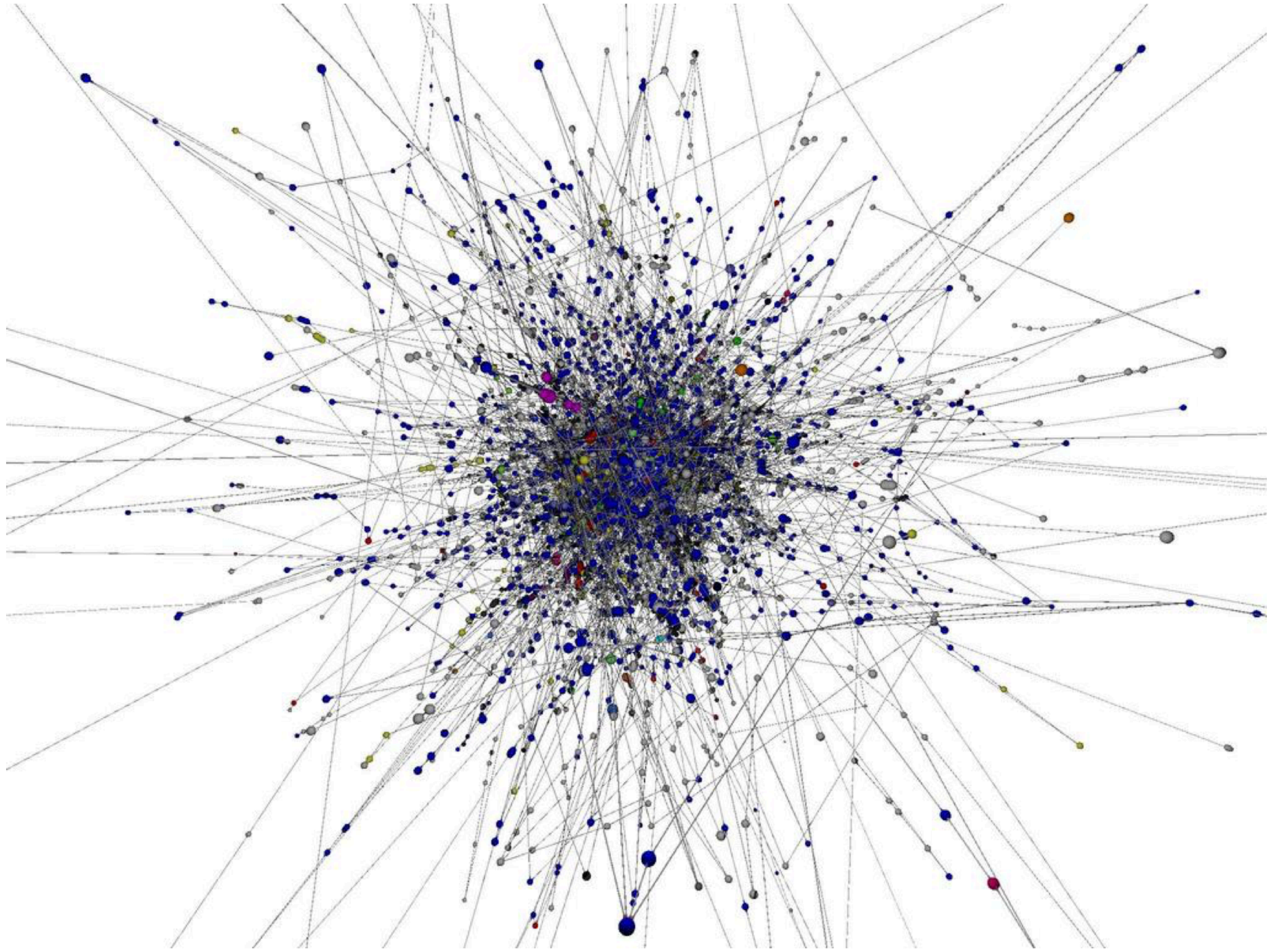

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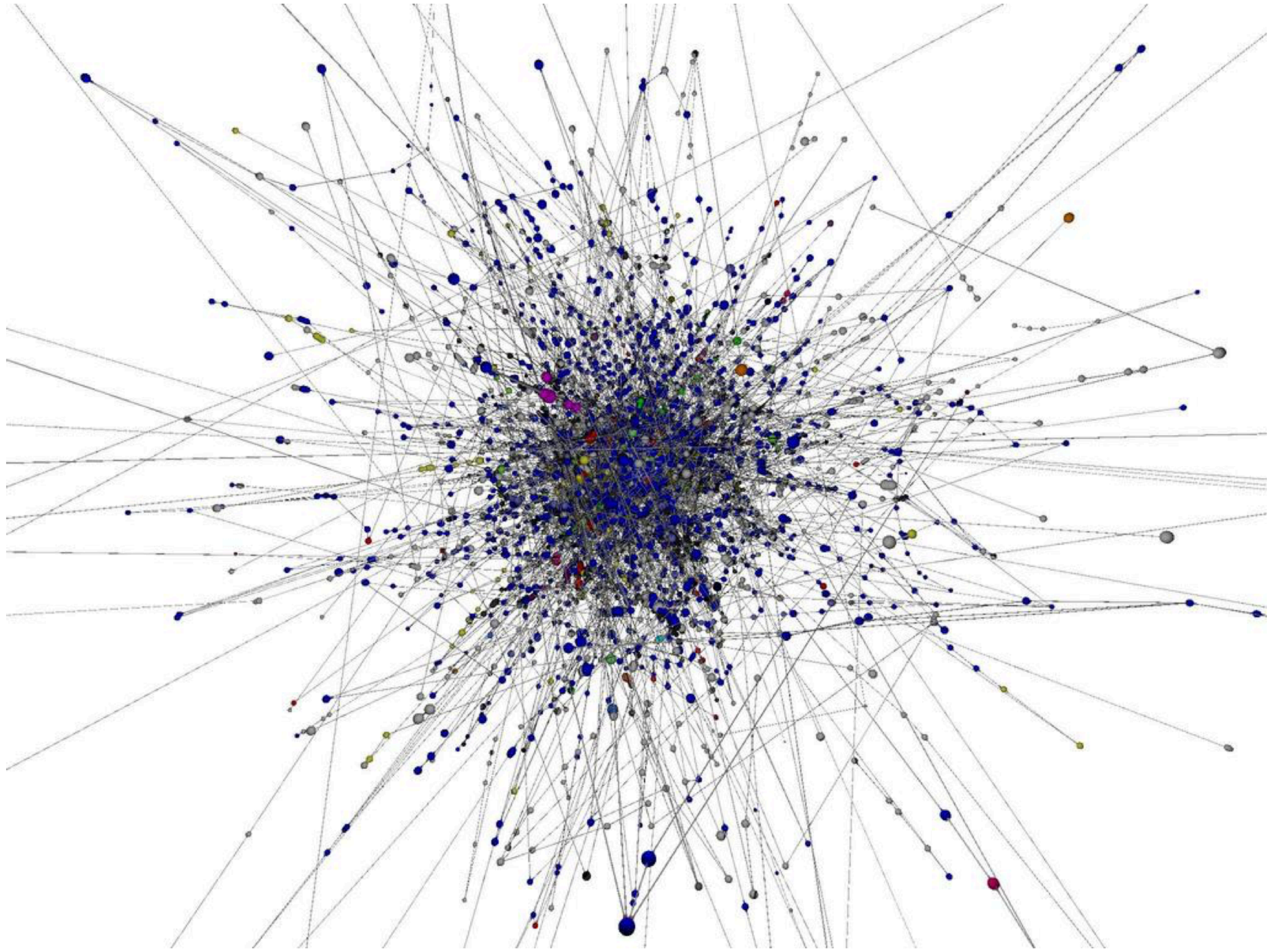

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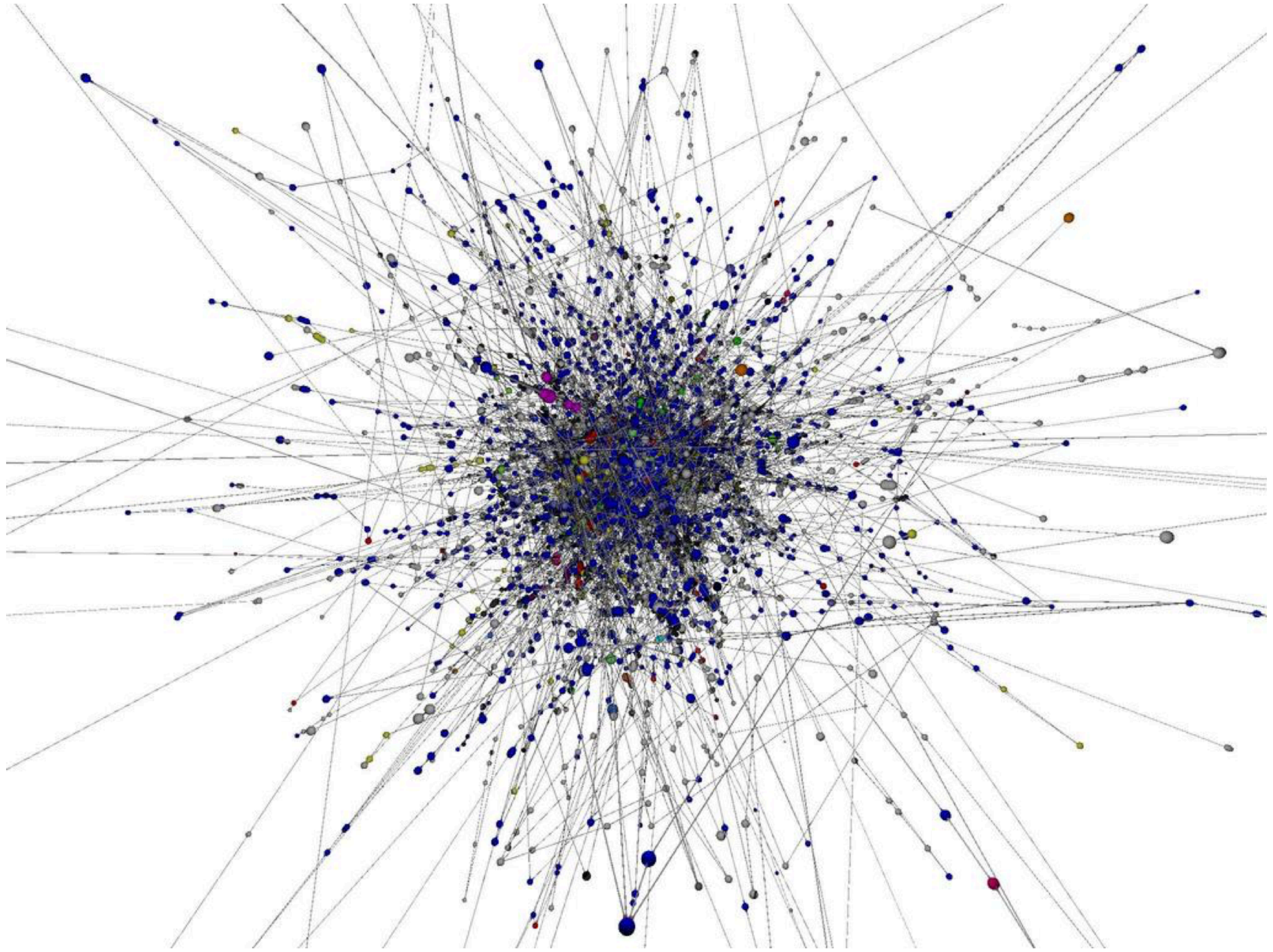
- —> random topology
  - the random process of preferential attachment



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Agenda

random topology

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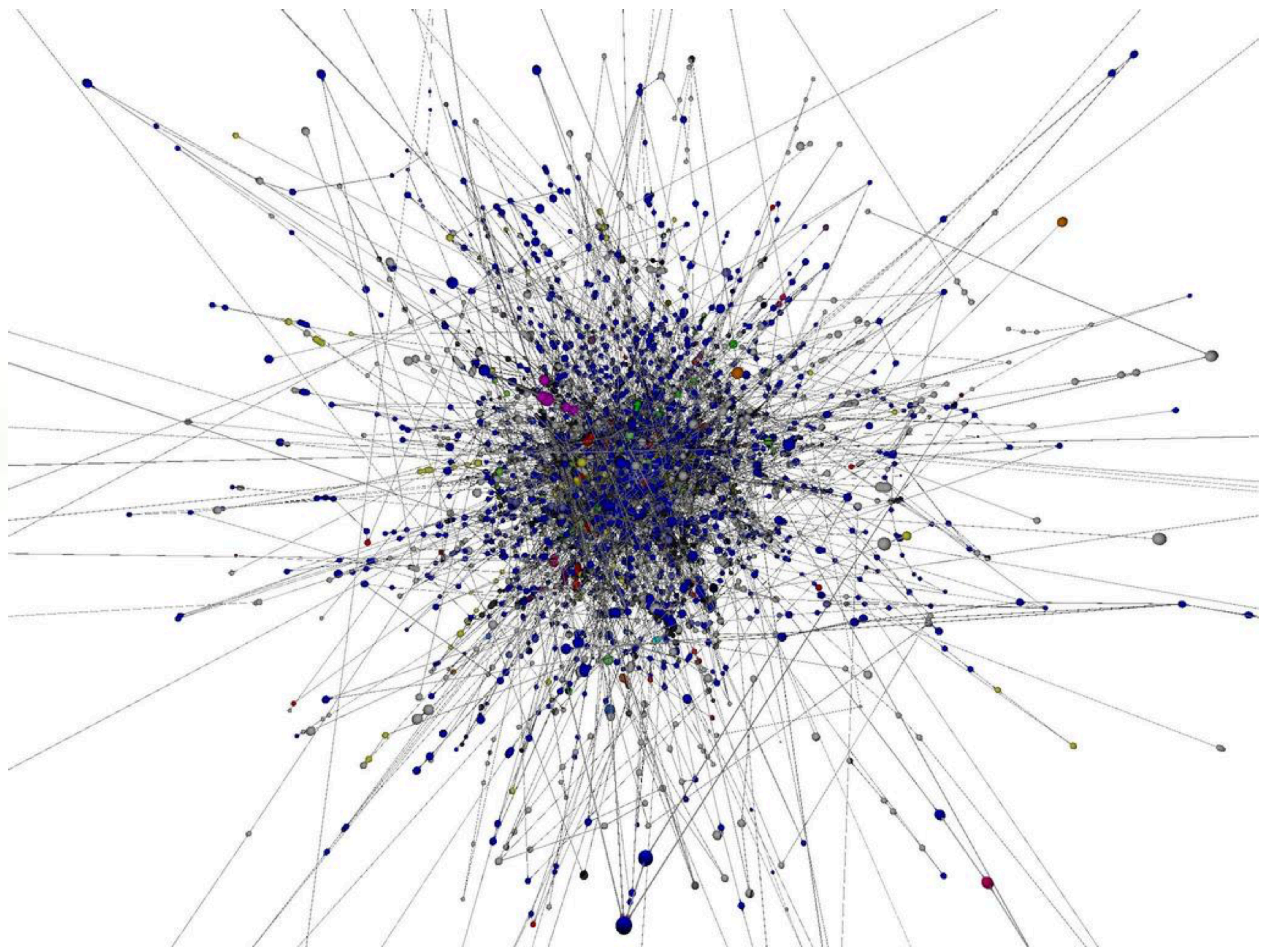
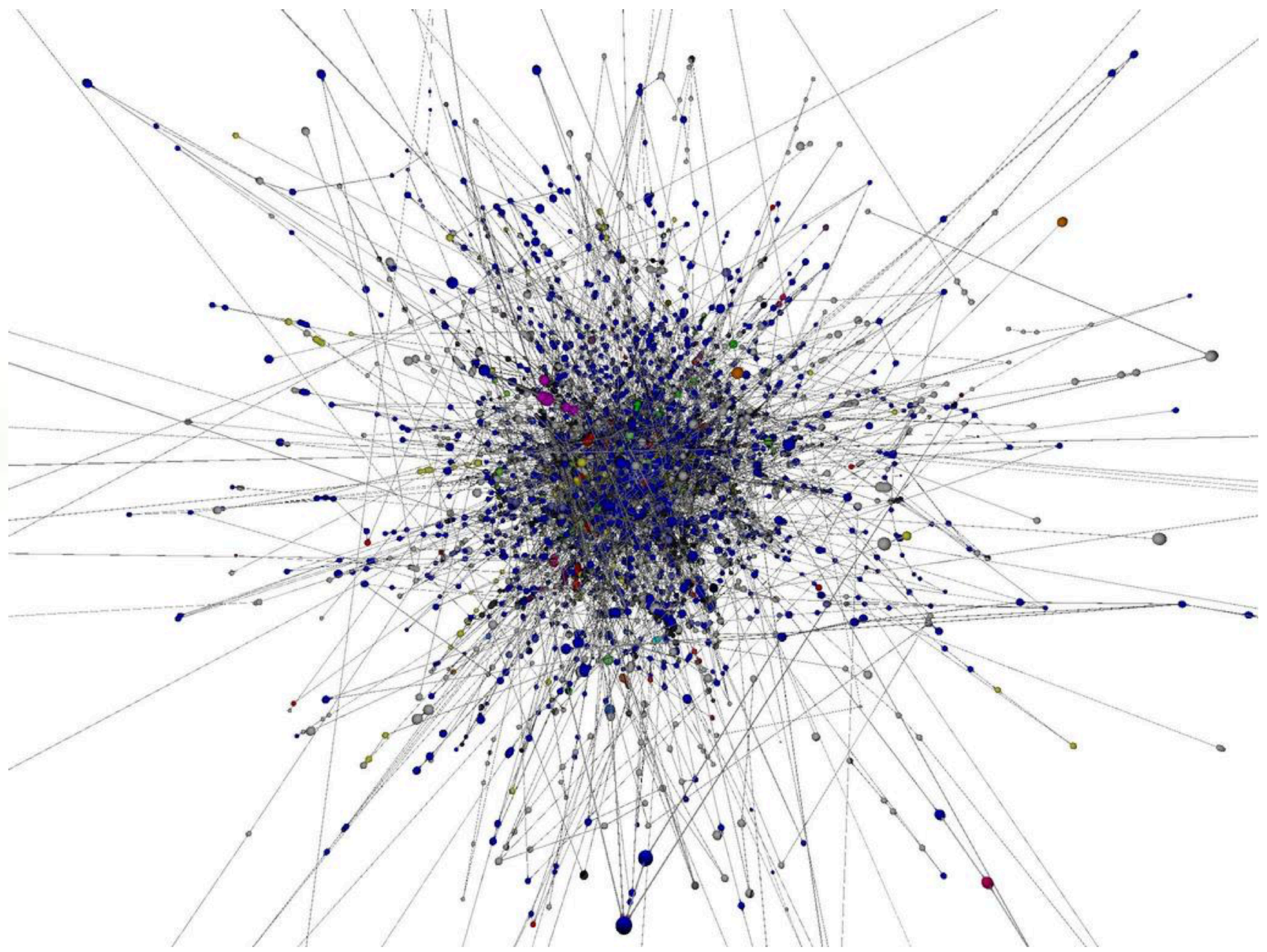


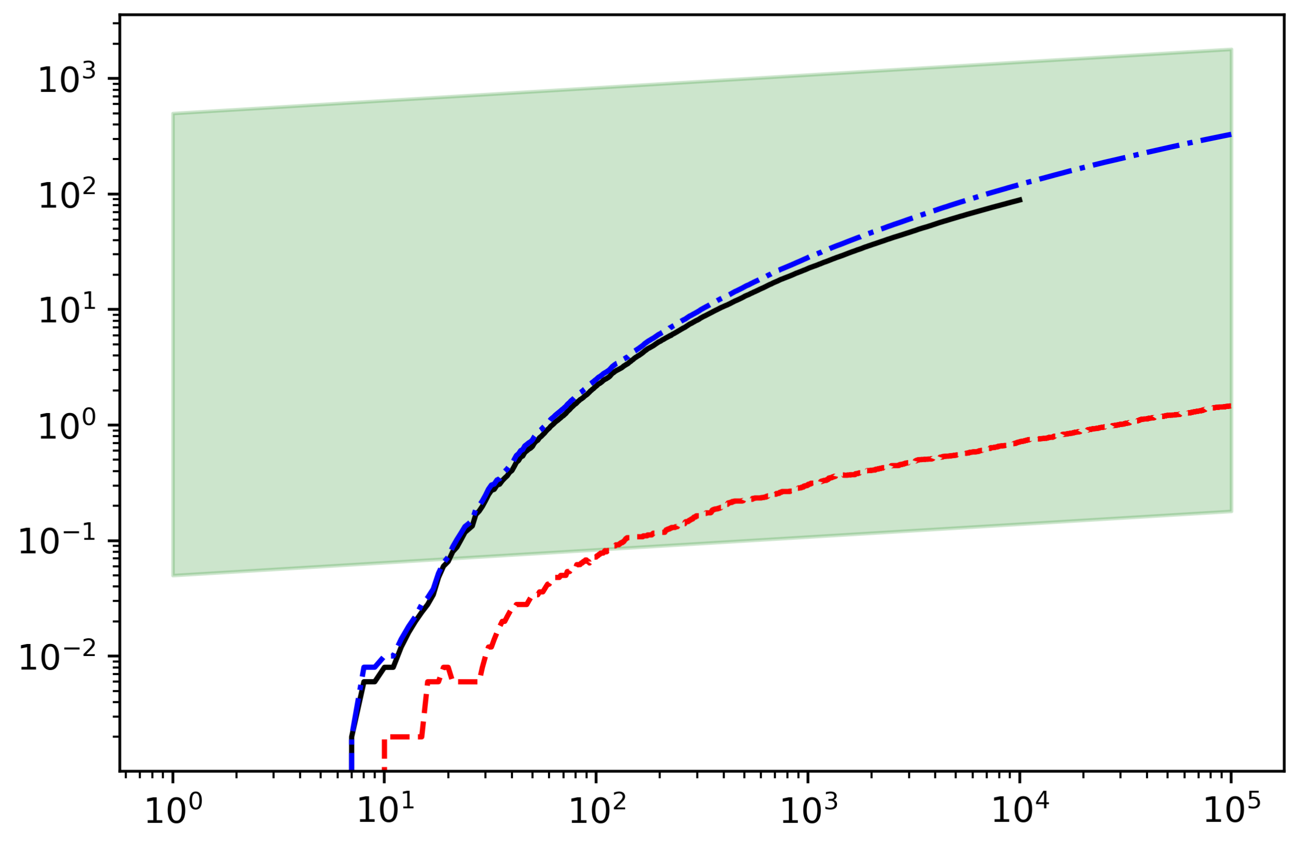
preferential attachment

Agenda

random topology



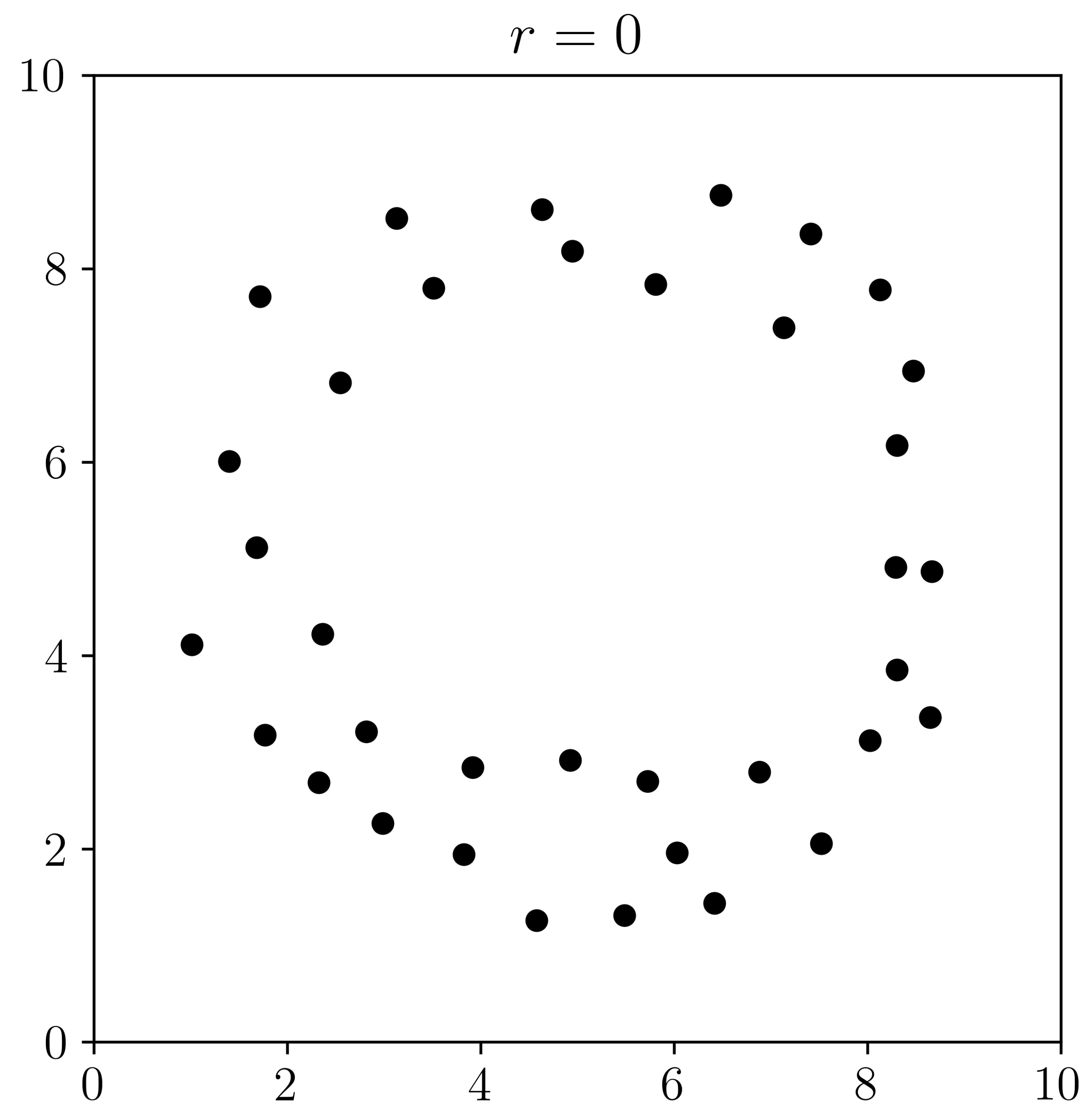
preferential attachment

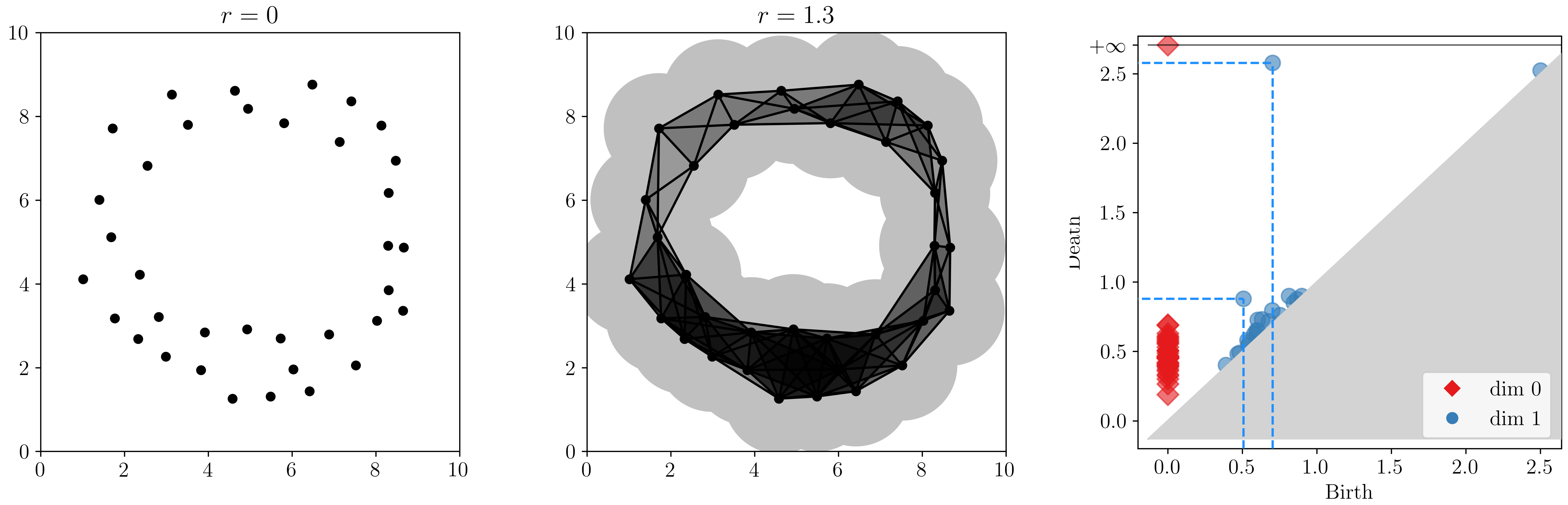


our result

I. A Probabilist’s Apology

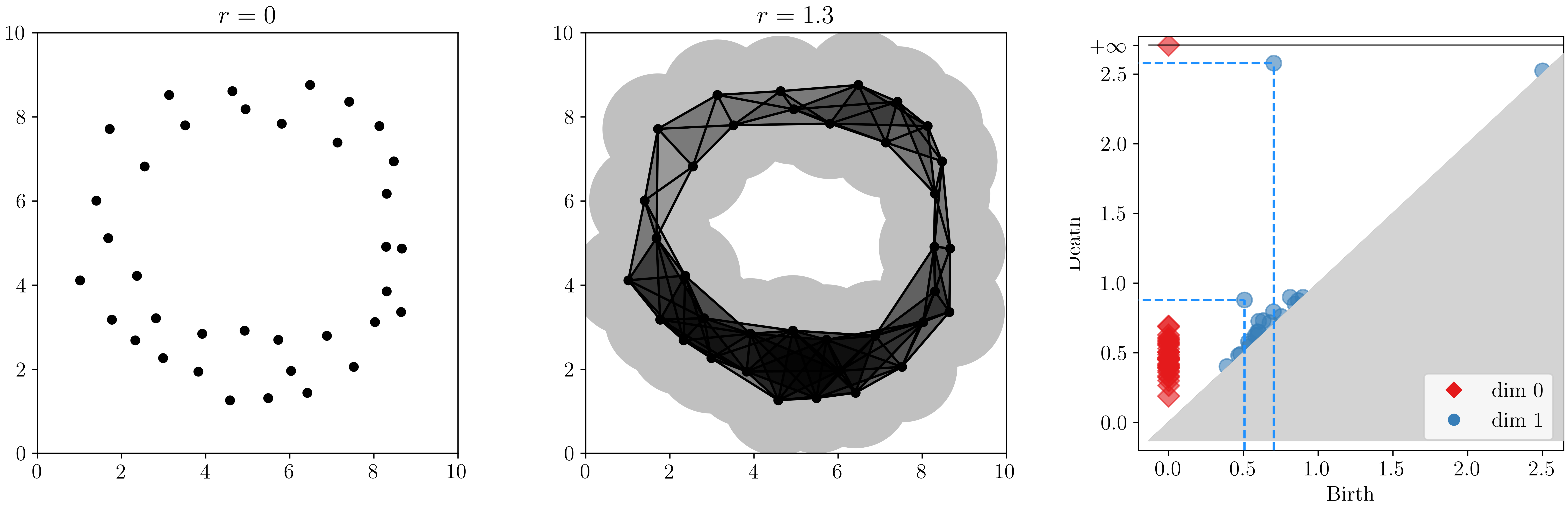
Why Random Topology


$r = 0$
10
8
6
4
2
0
0
2
4
6
8
10

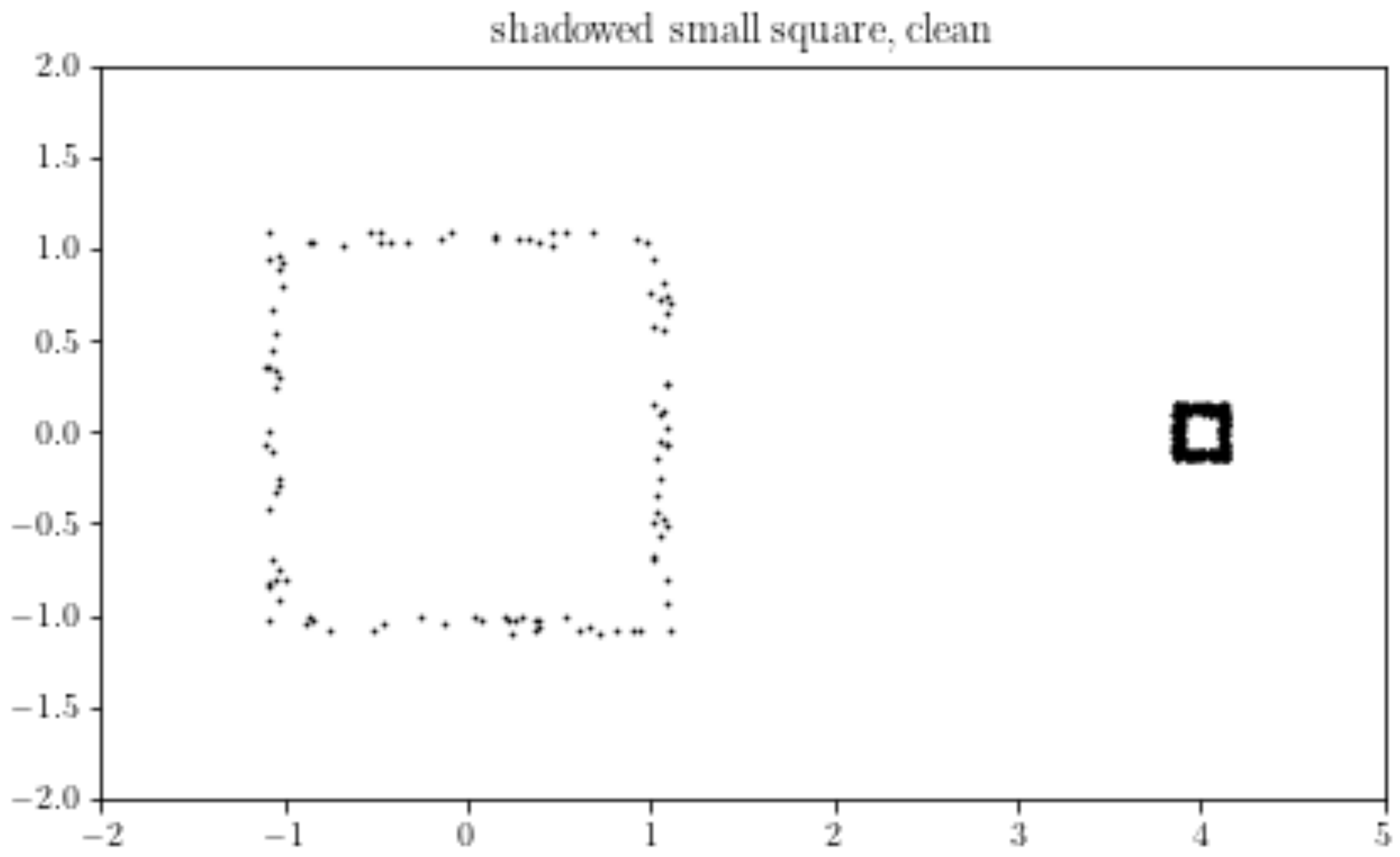


plots generated by Andrey Yao

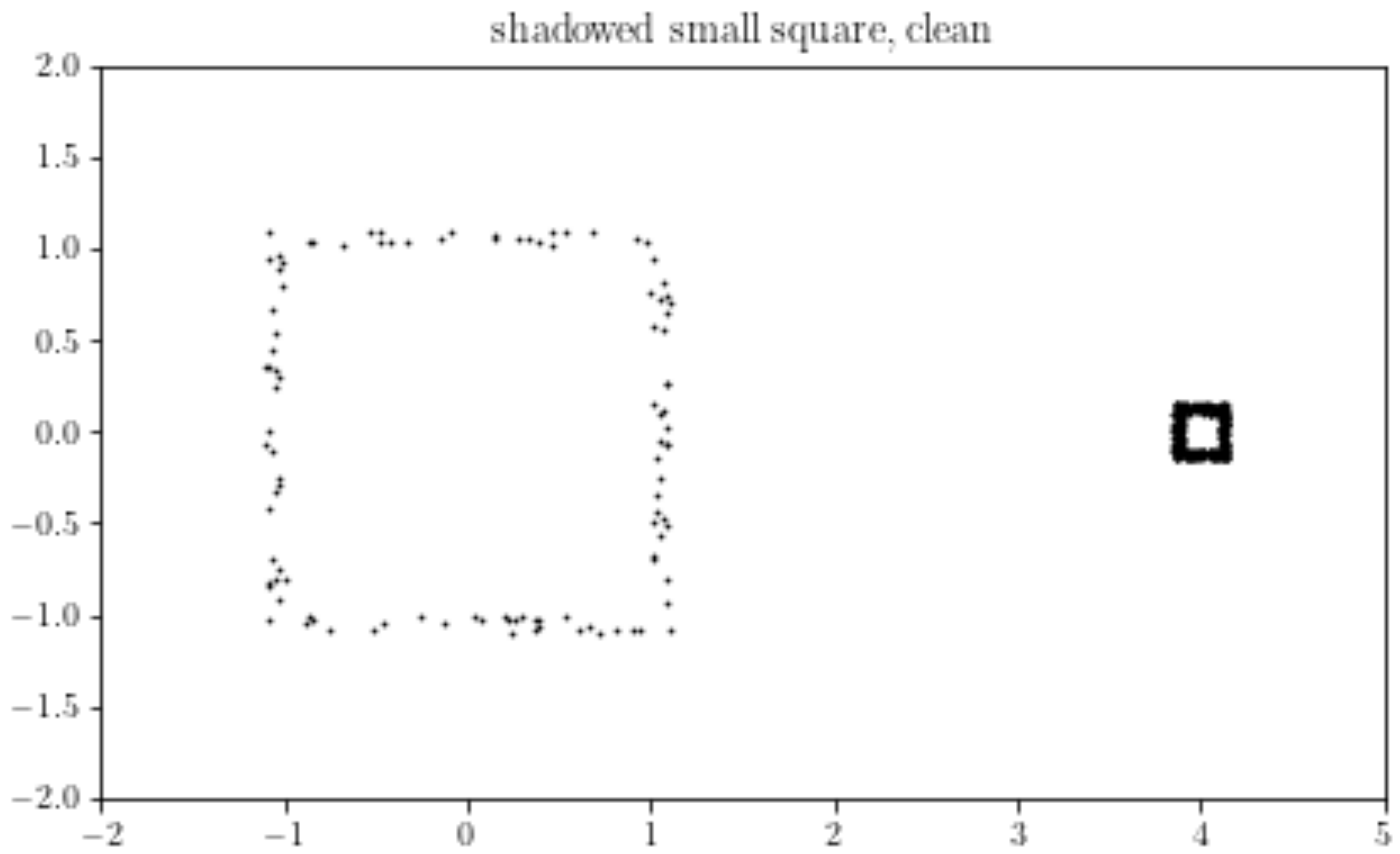
Size is Signal



Or is it?


shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5

Or is it?



shadowed small square, clean
2.0
1.5
1.0
0.5
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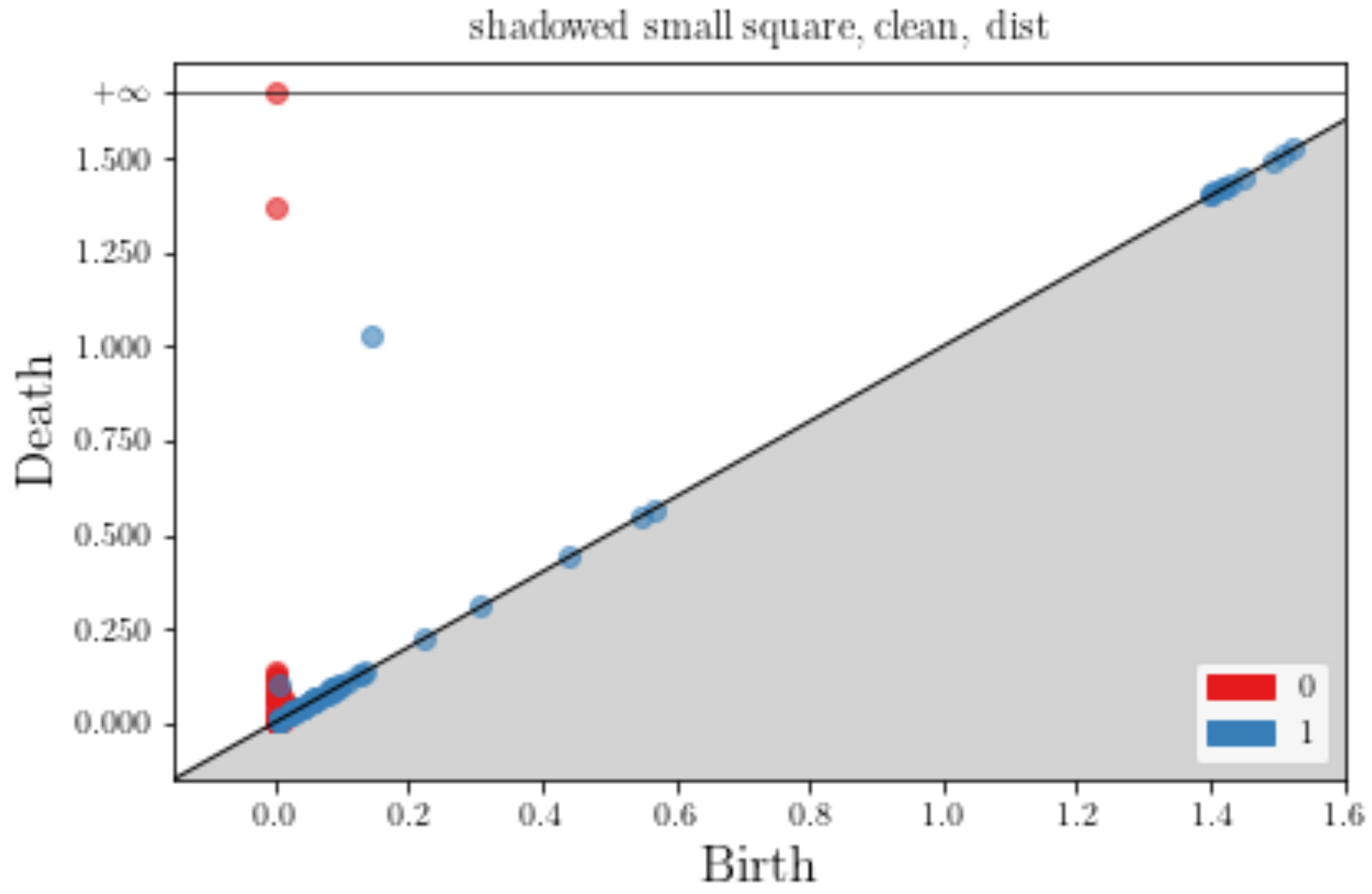
−2.0
−2
−1
0
1
2
3
4
5



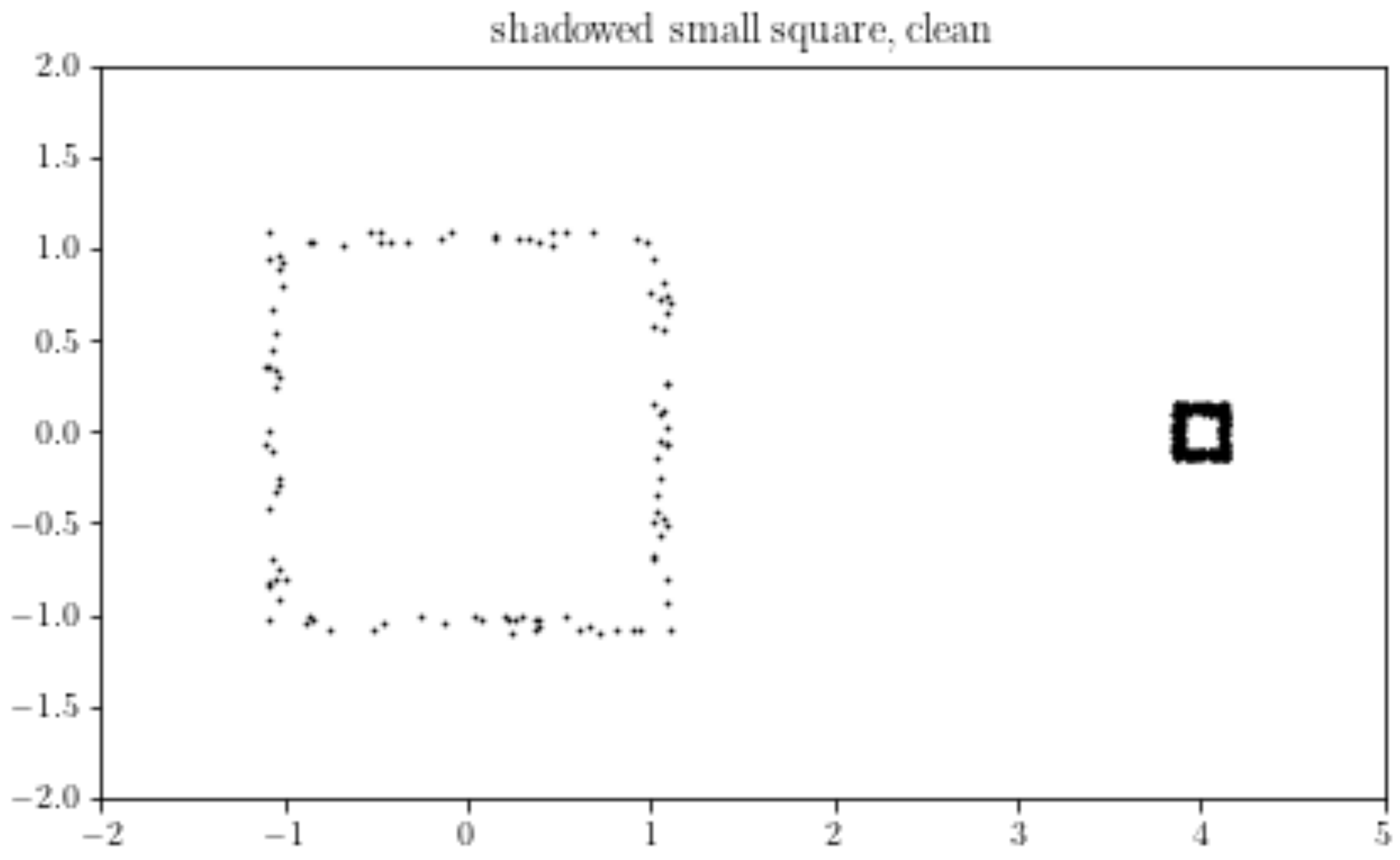
shadowed small square, clean, dist
+∞
1.500
1.250
1.000
0.750
0.500
0.250
0.000
Death
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Birth
0
1

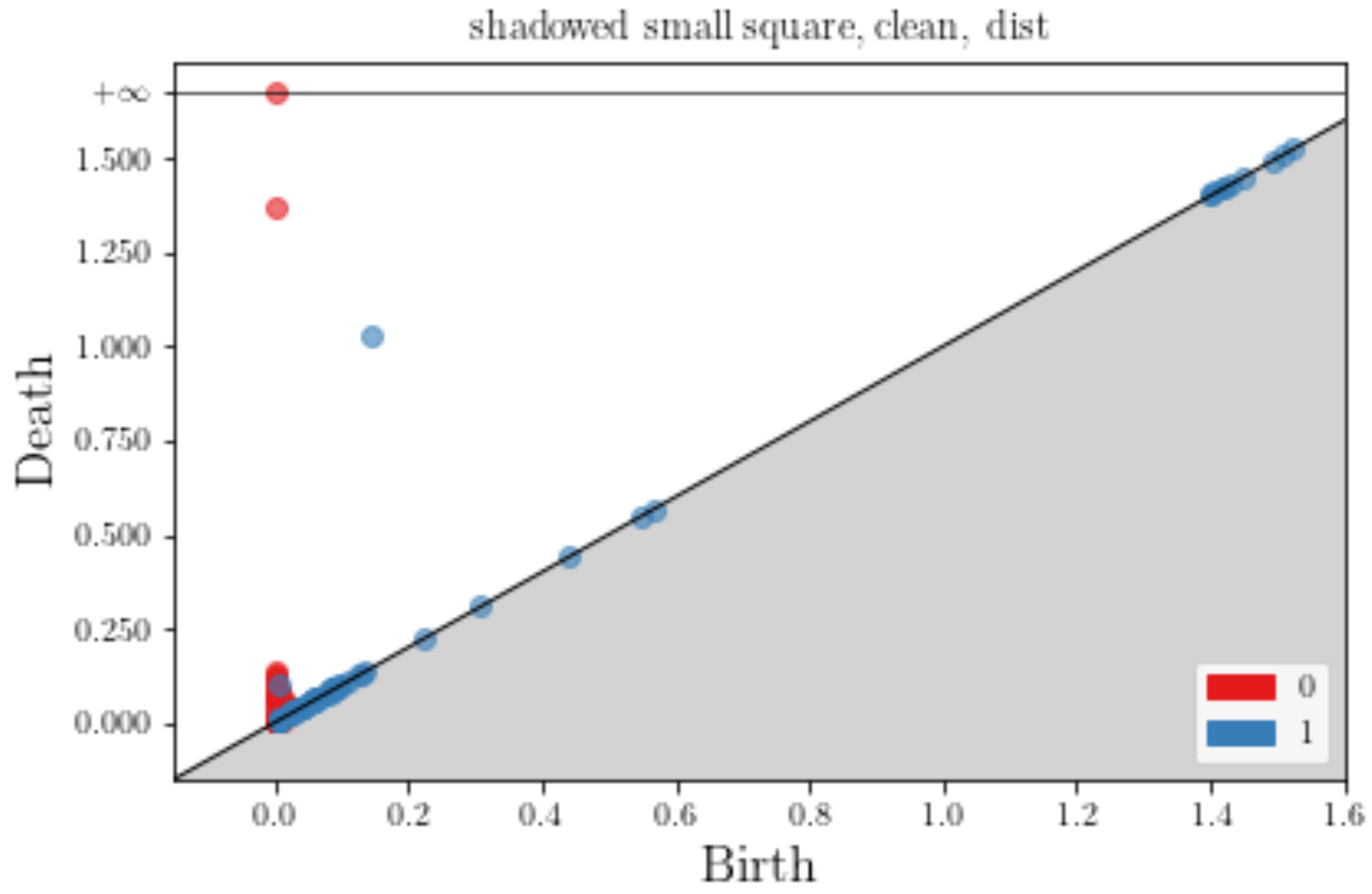
Size is Signal?

Surprise

~~Size~~ is Signal.

Random points don't do that.





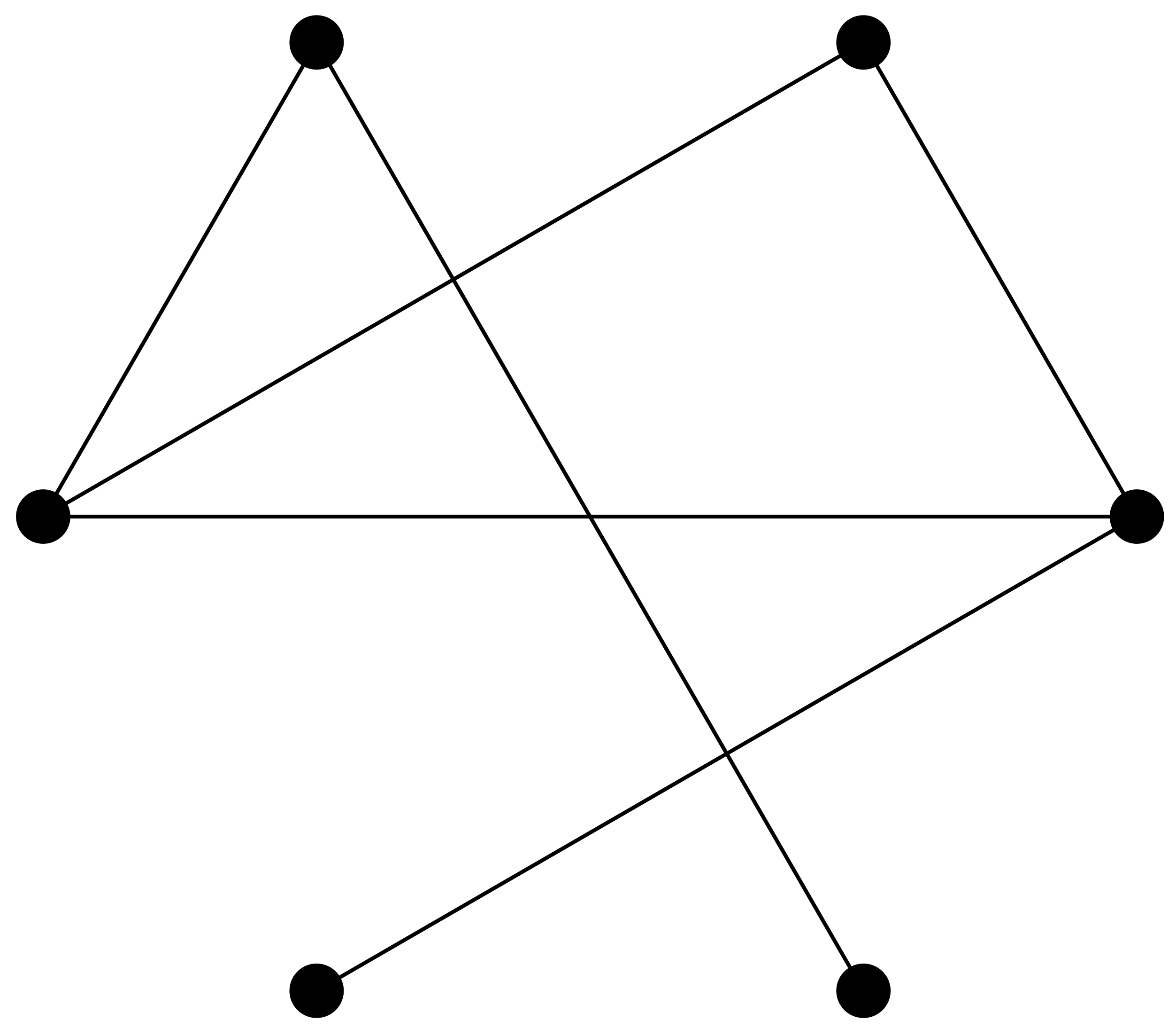
Signal is what is not random.

Signal is what is not **random.**
So what is **random?**

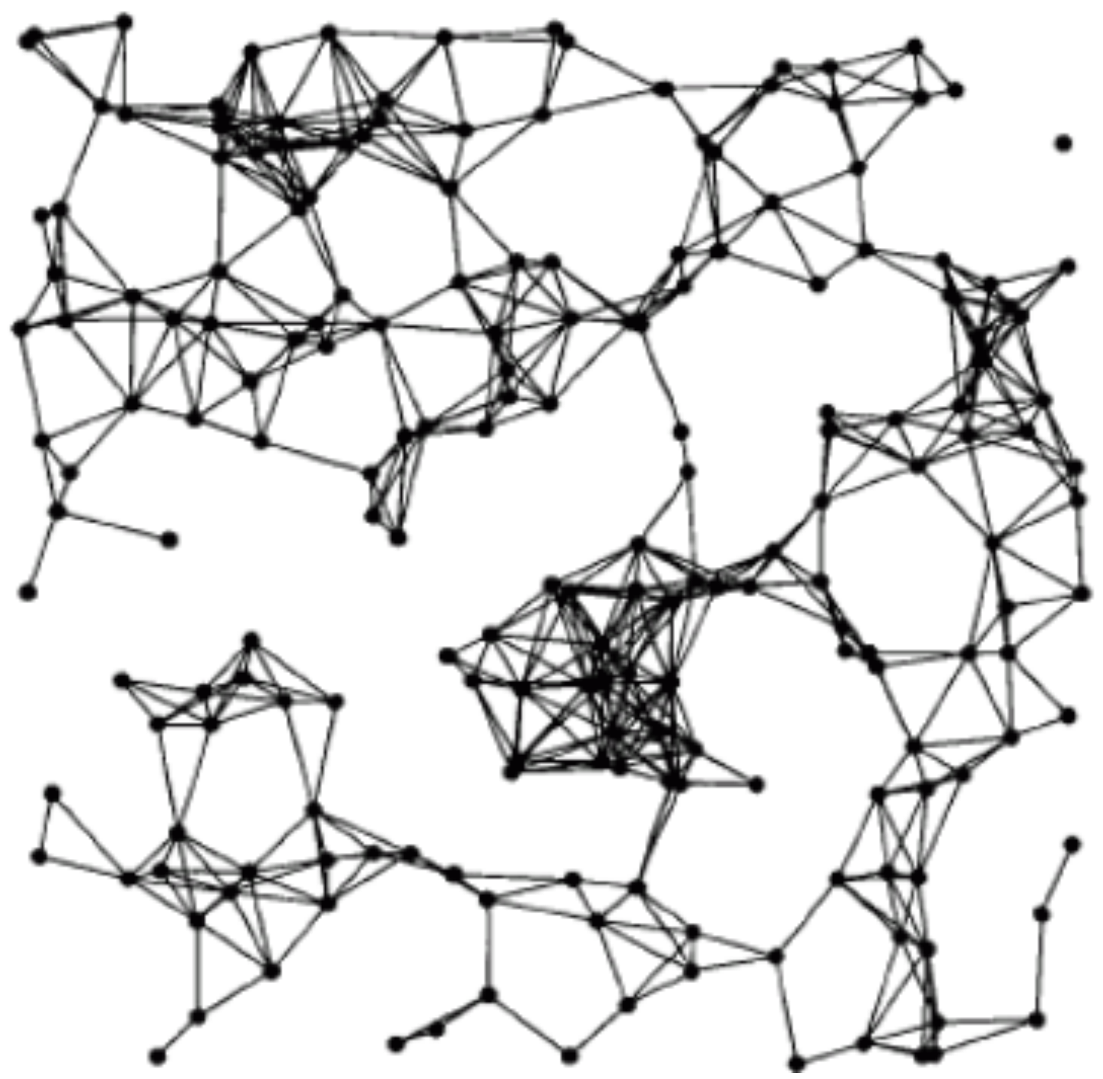
II. Random Walk in the Literature

What Random Topologists Already Know

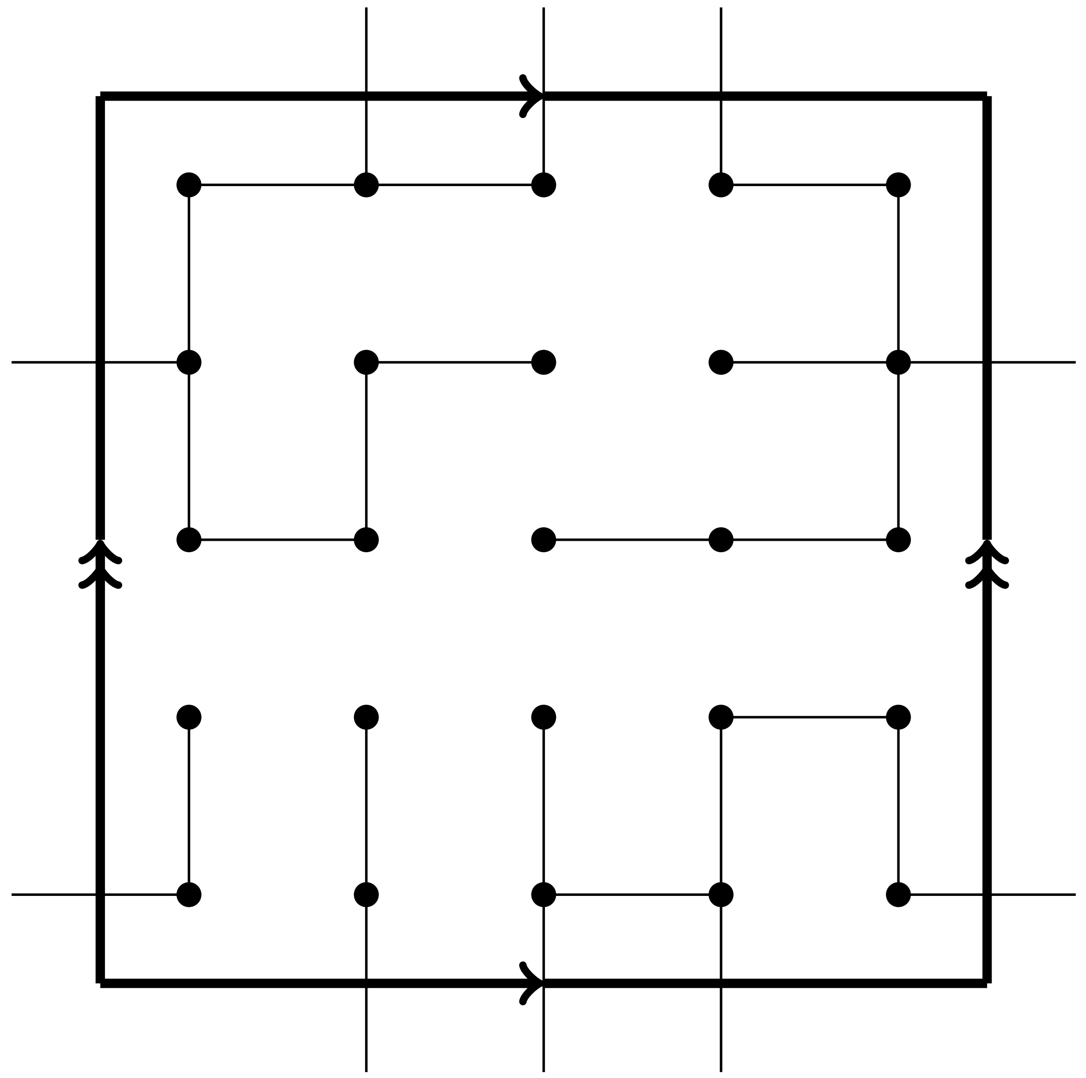
Tapas de Random Topology



Erdo-Renyi Complexes

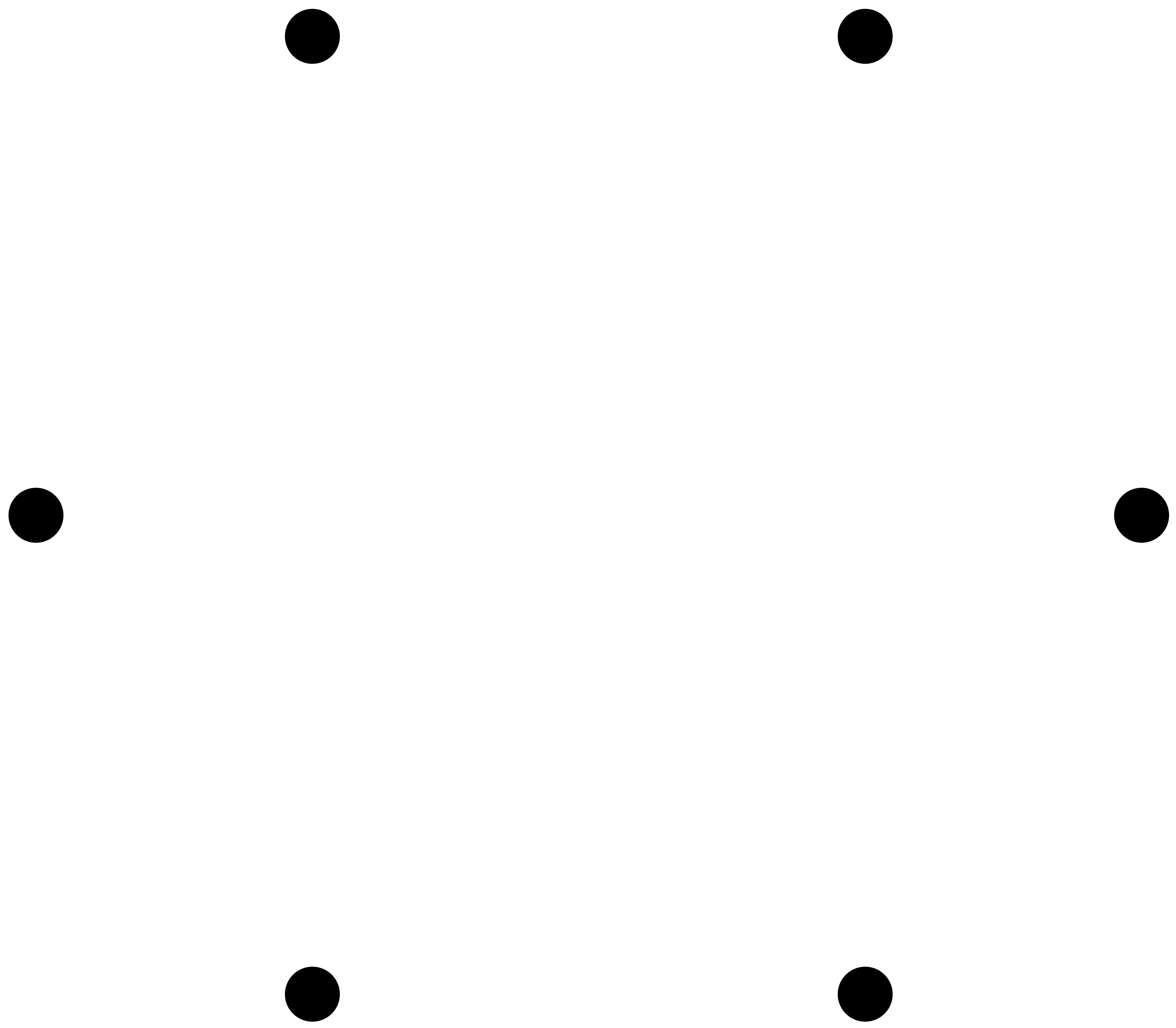


Geometric Complexes

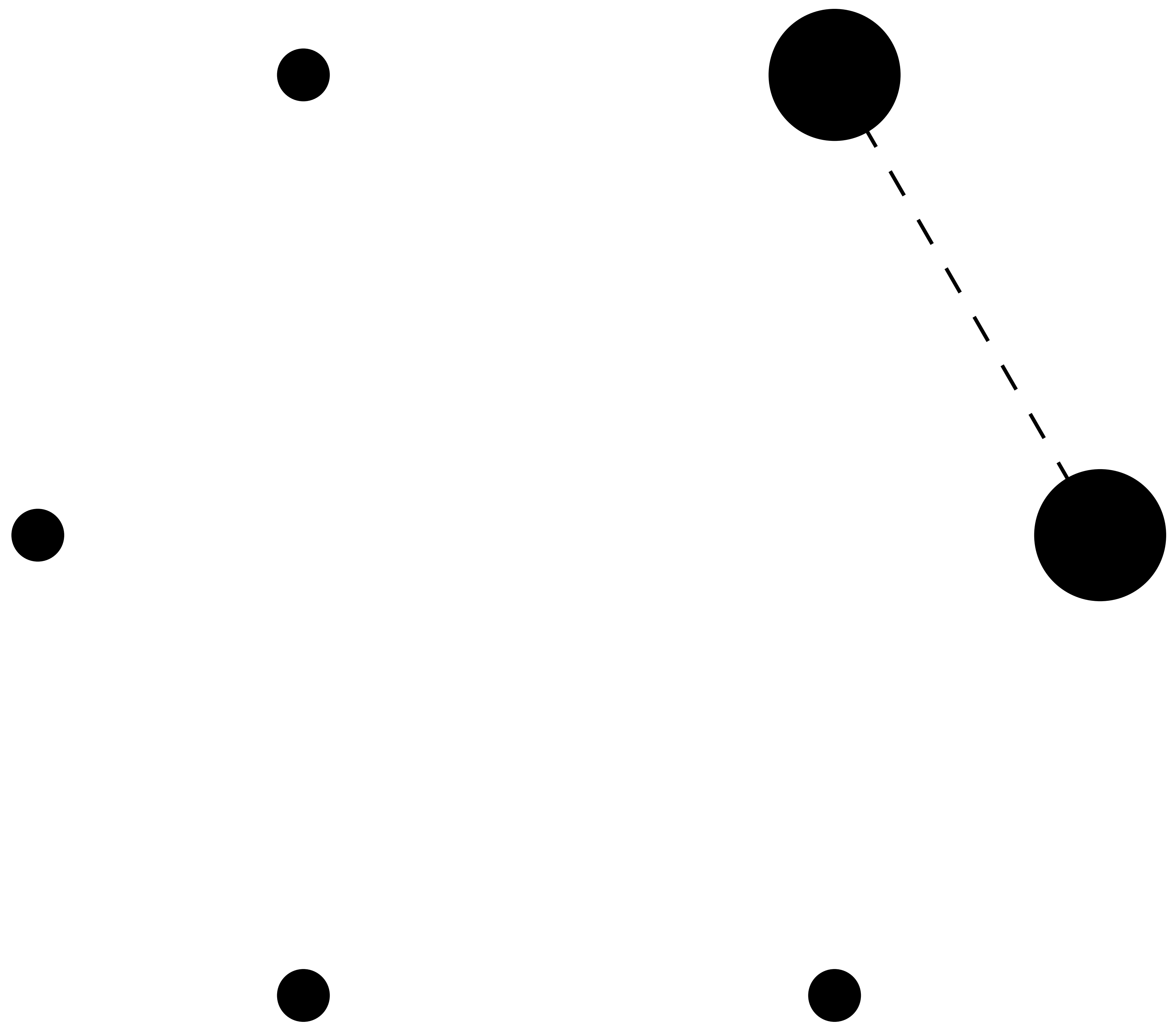


Topological Percolation

Erdos-Renyi graphs



Erdos-Renyi graphs



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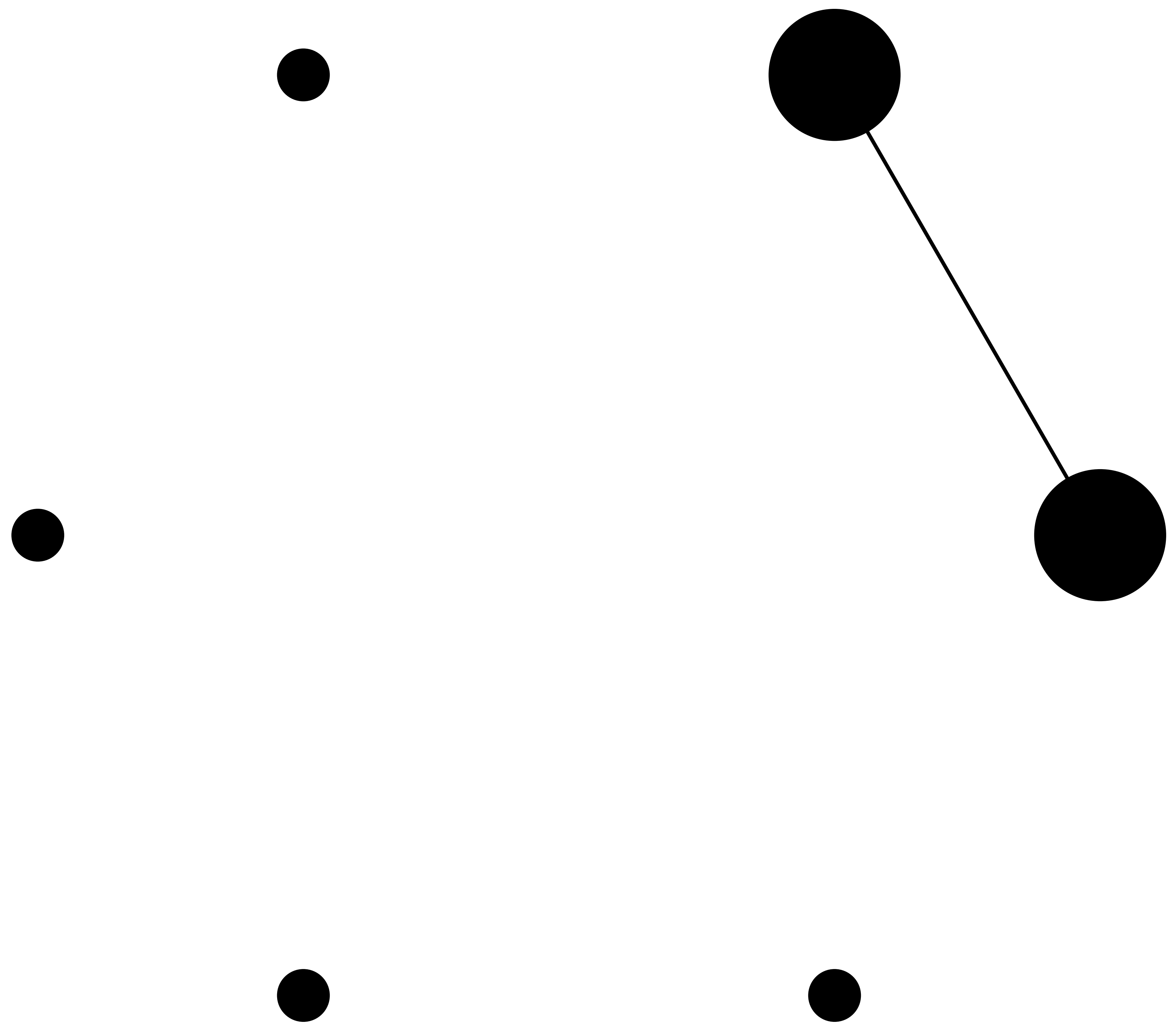

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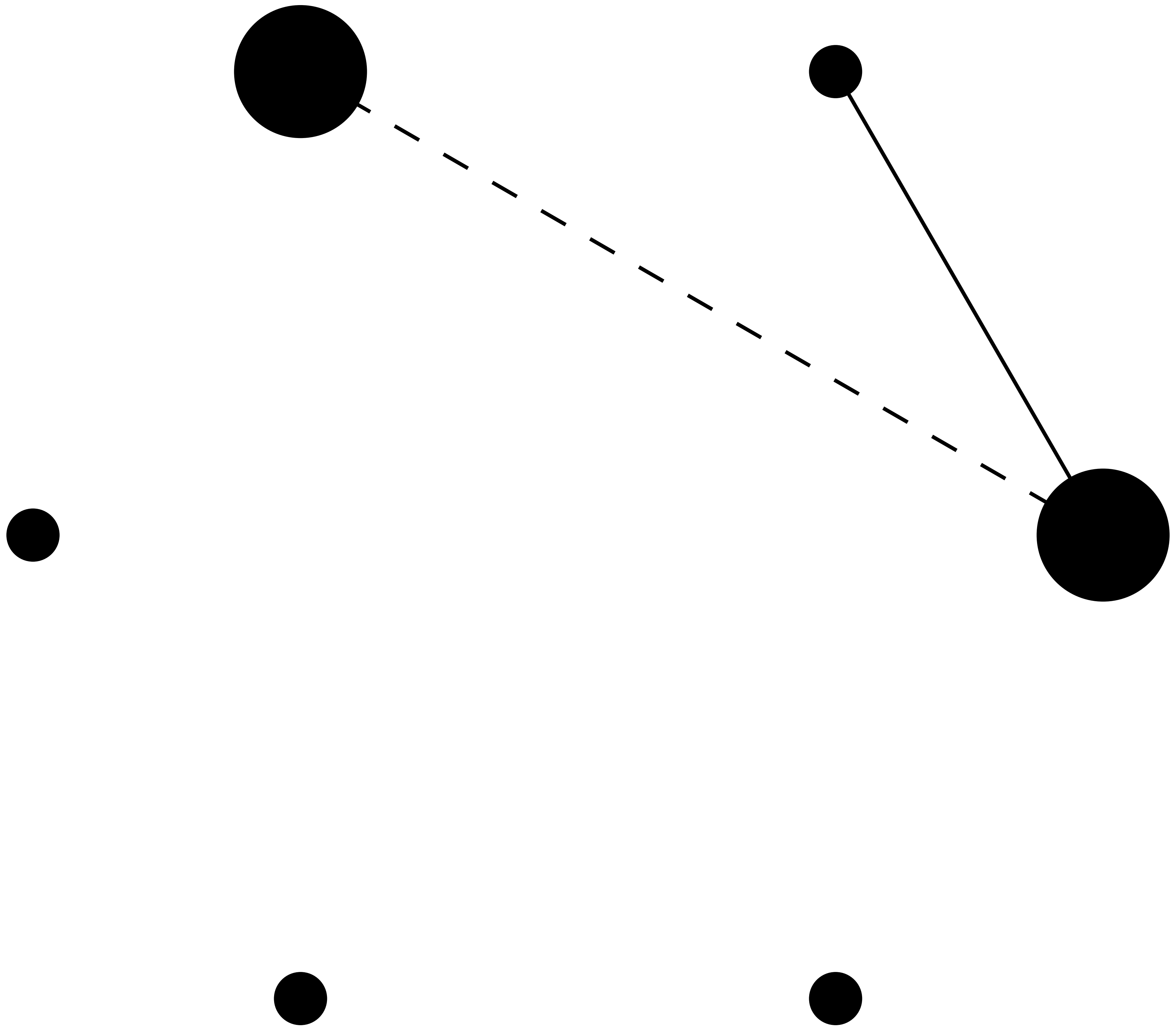

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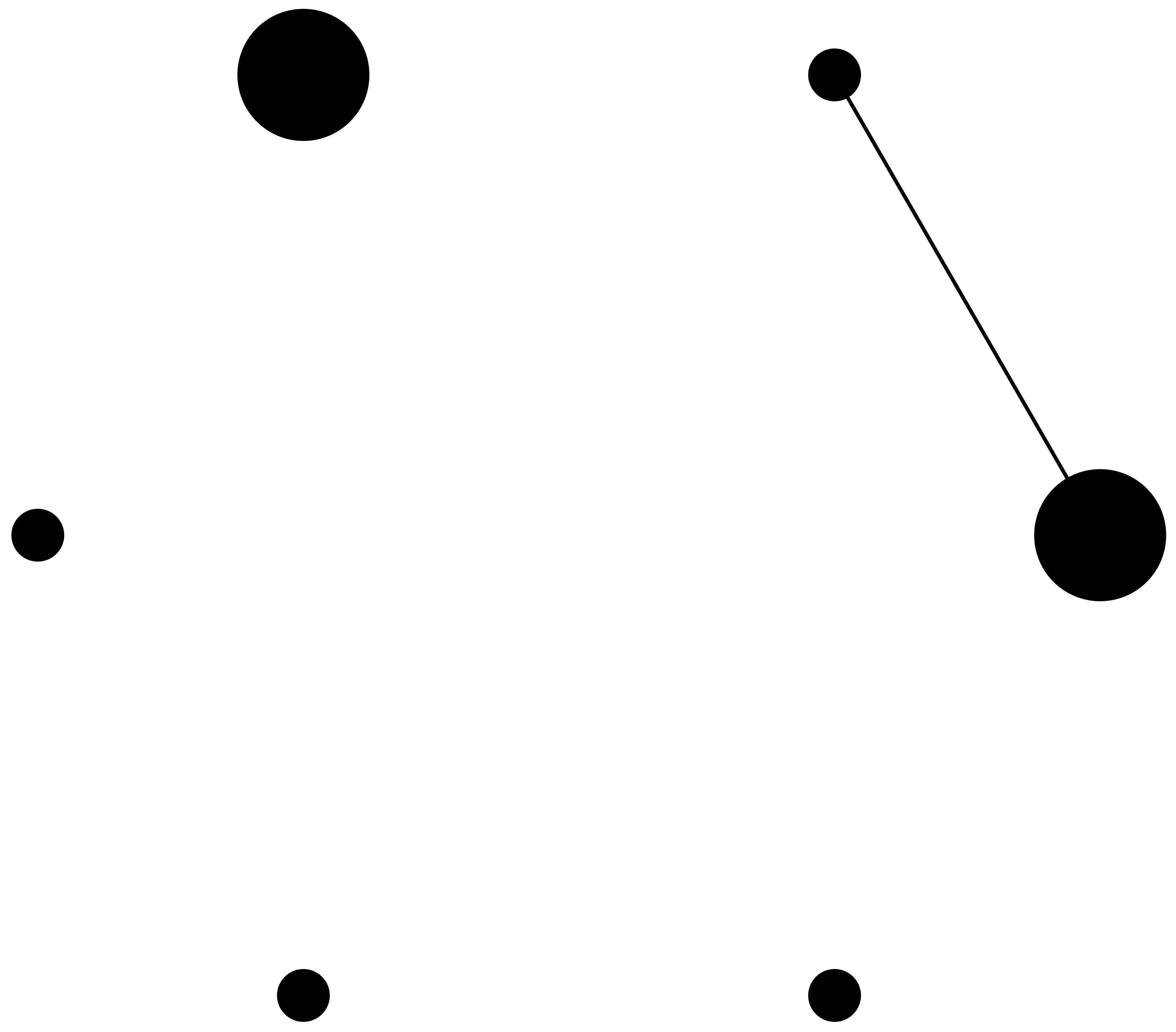

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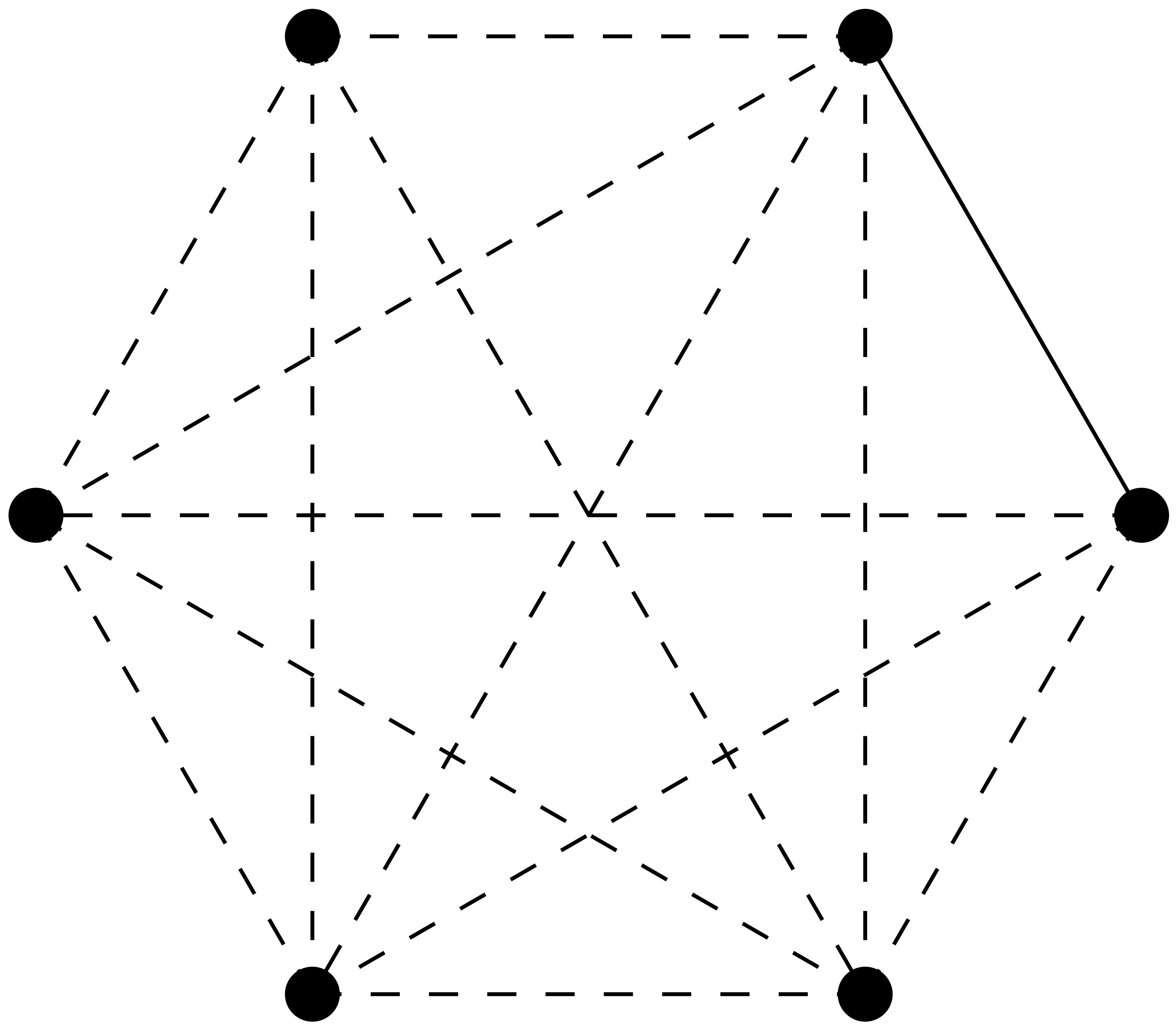

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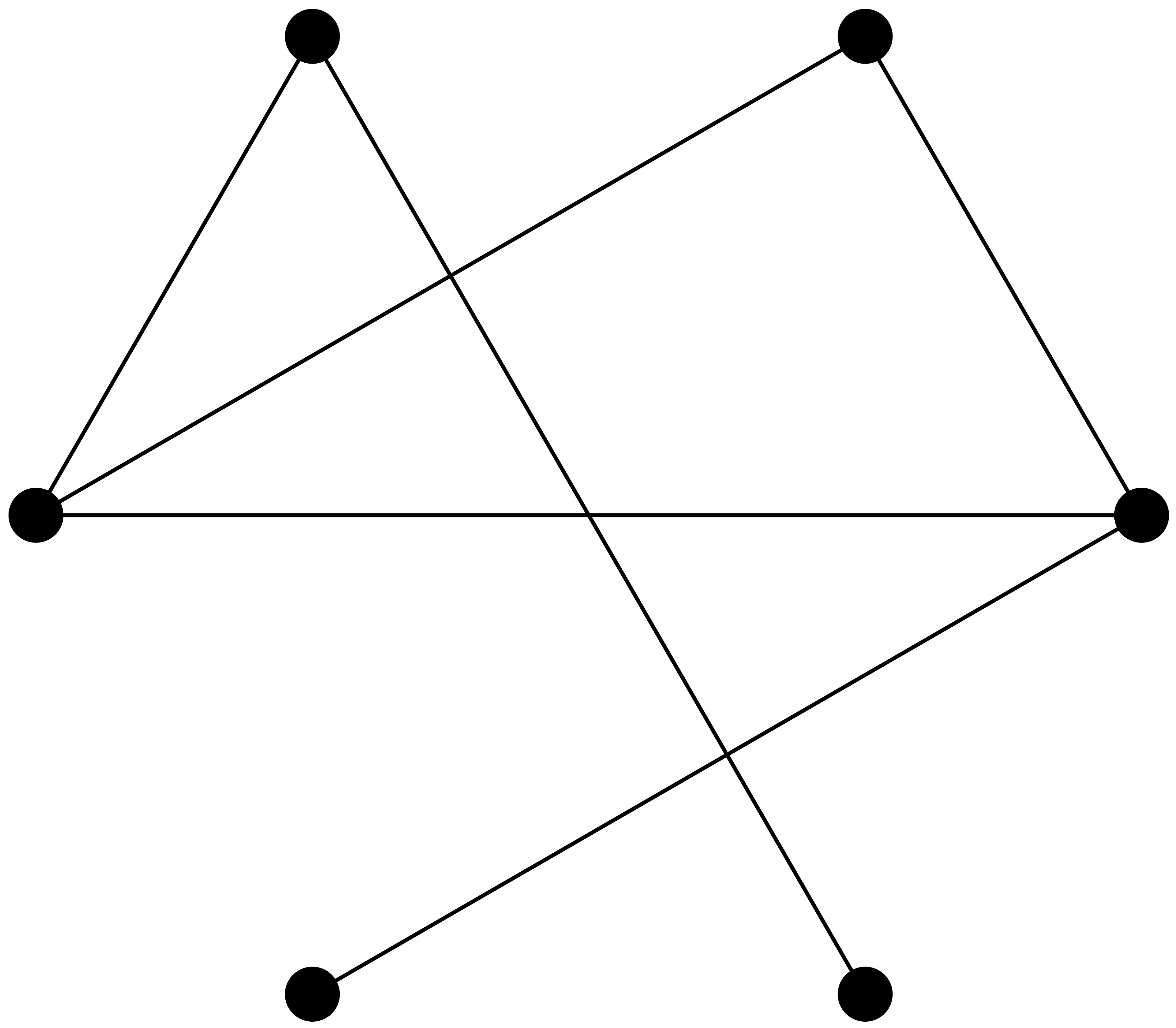
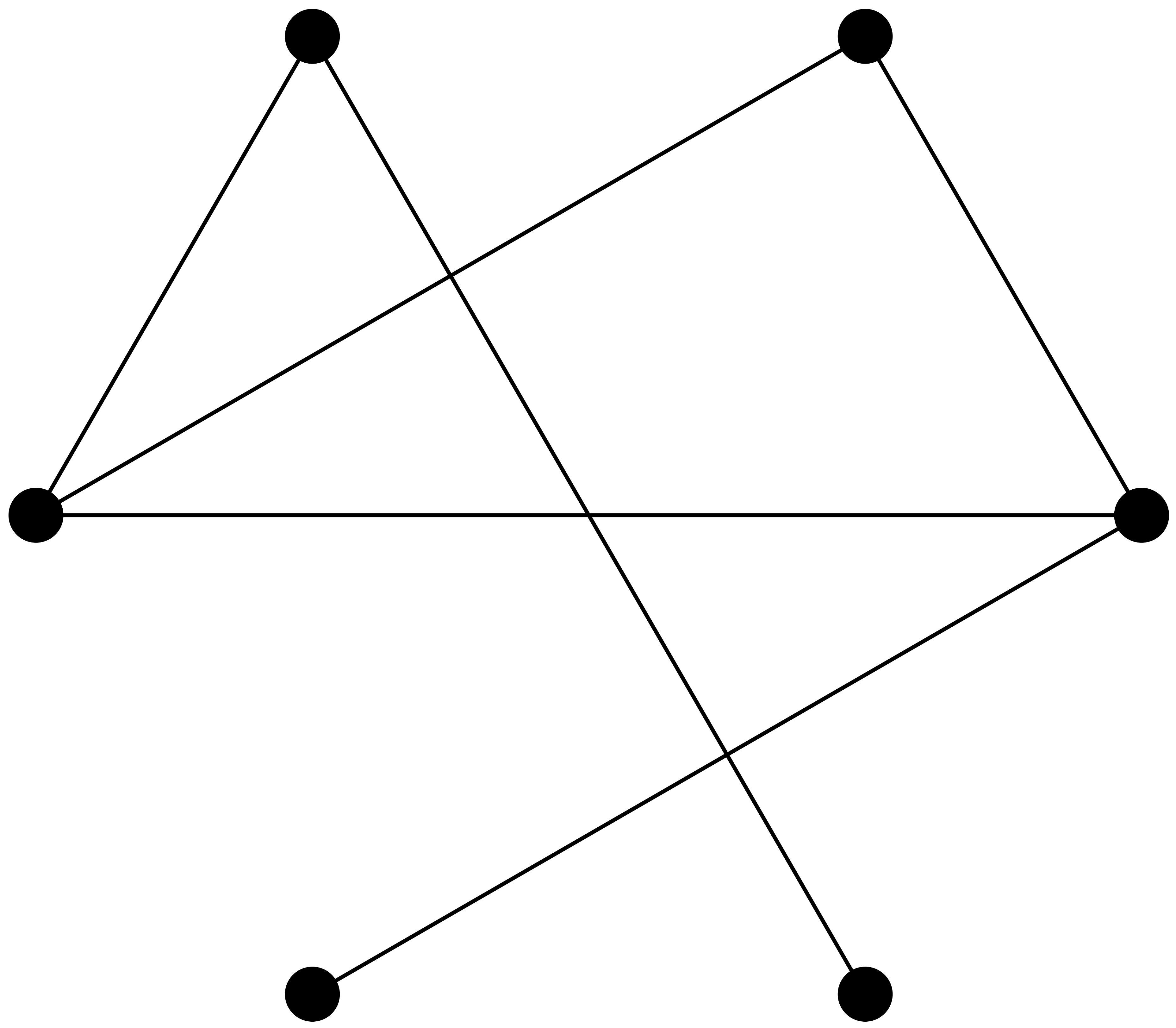
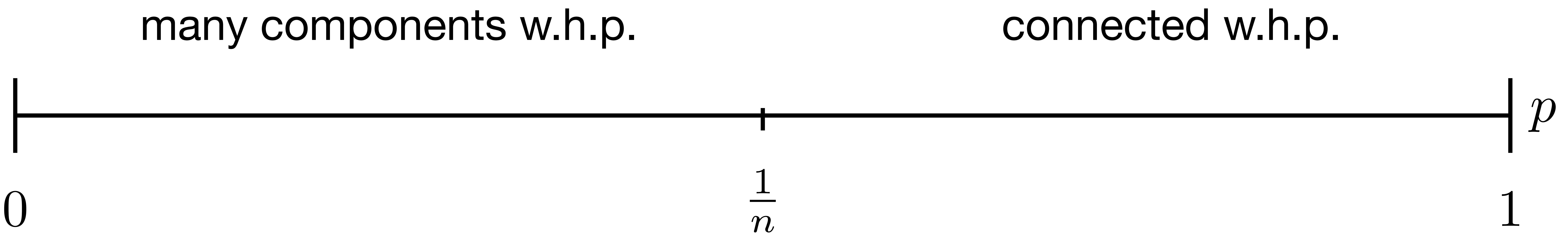


Erdos-Renyi graphs

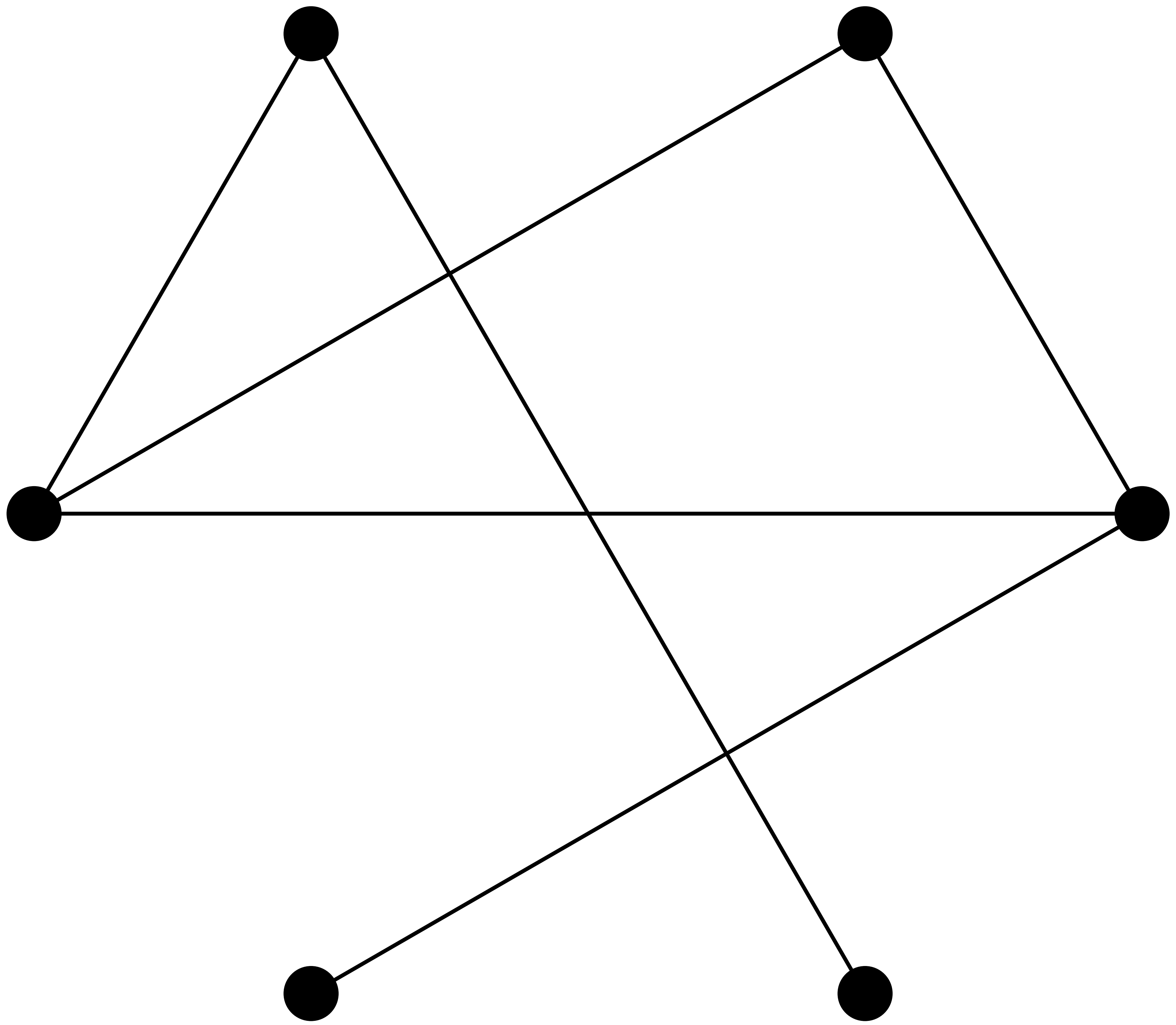


Phase Transition
[Erdos-Renyi 1960]



all log terms and constants forgone

Erdos-Renyi Clique Complex



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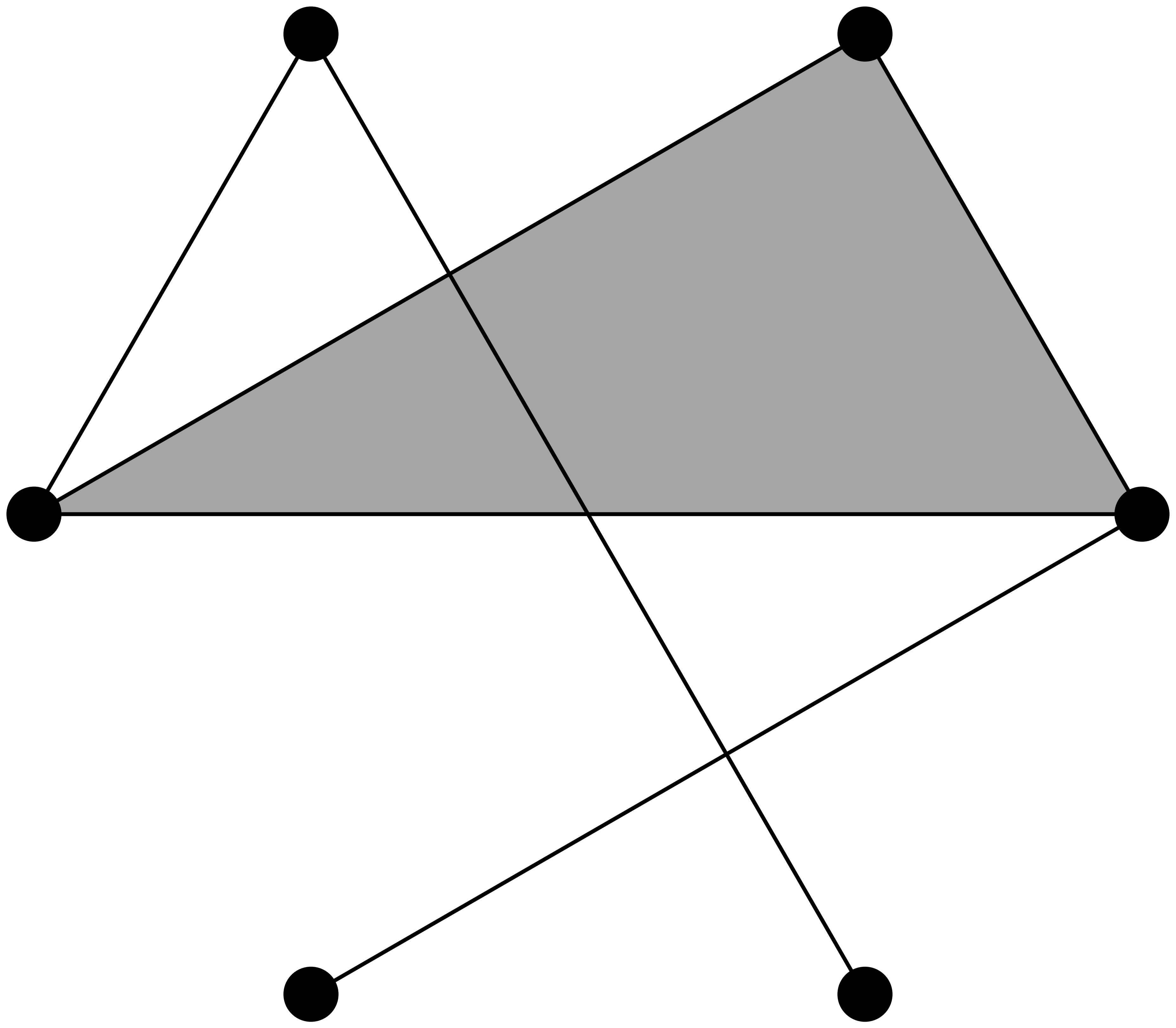
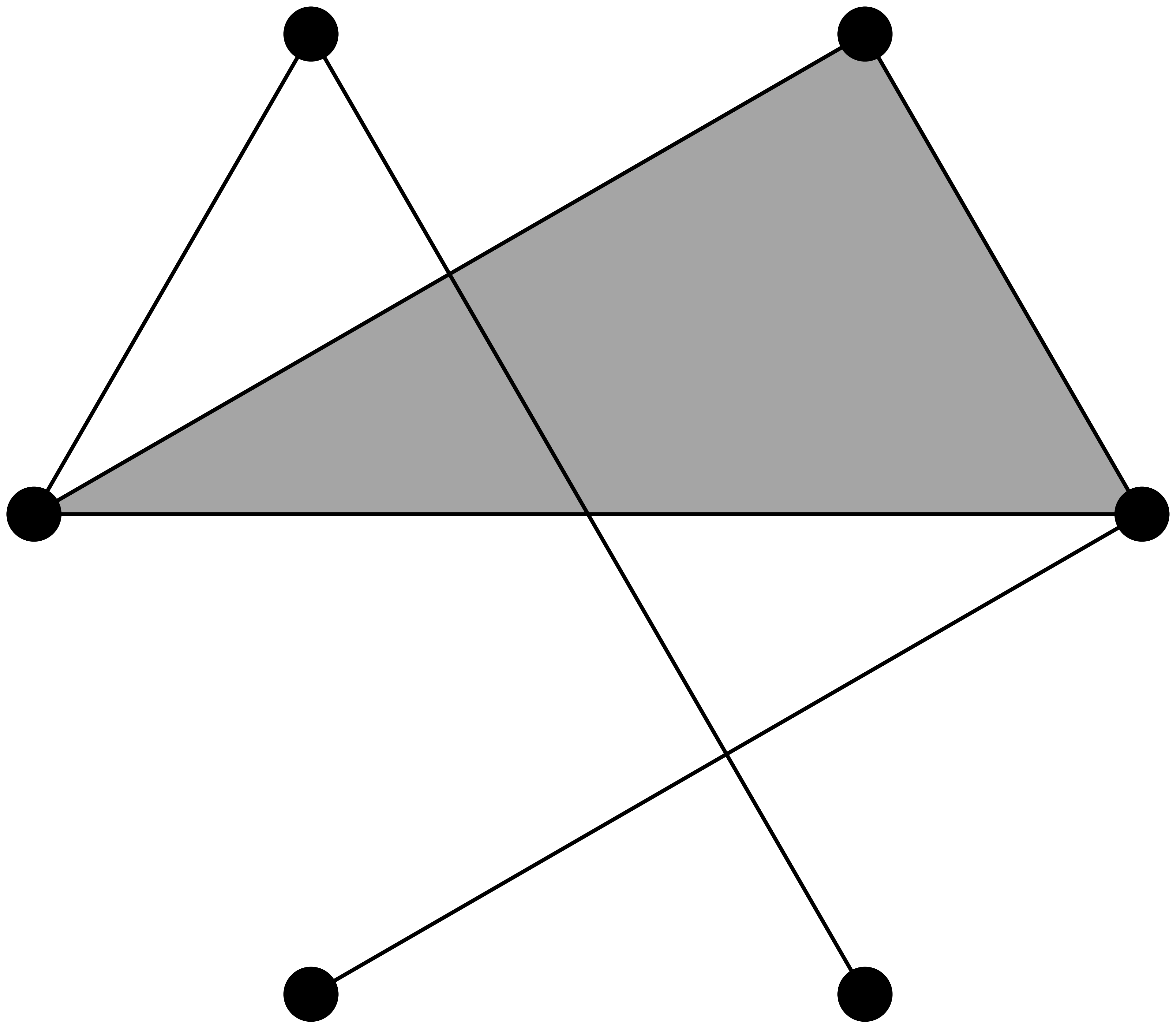
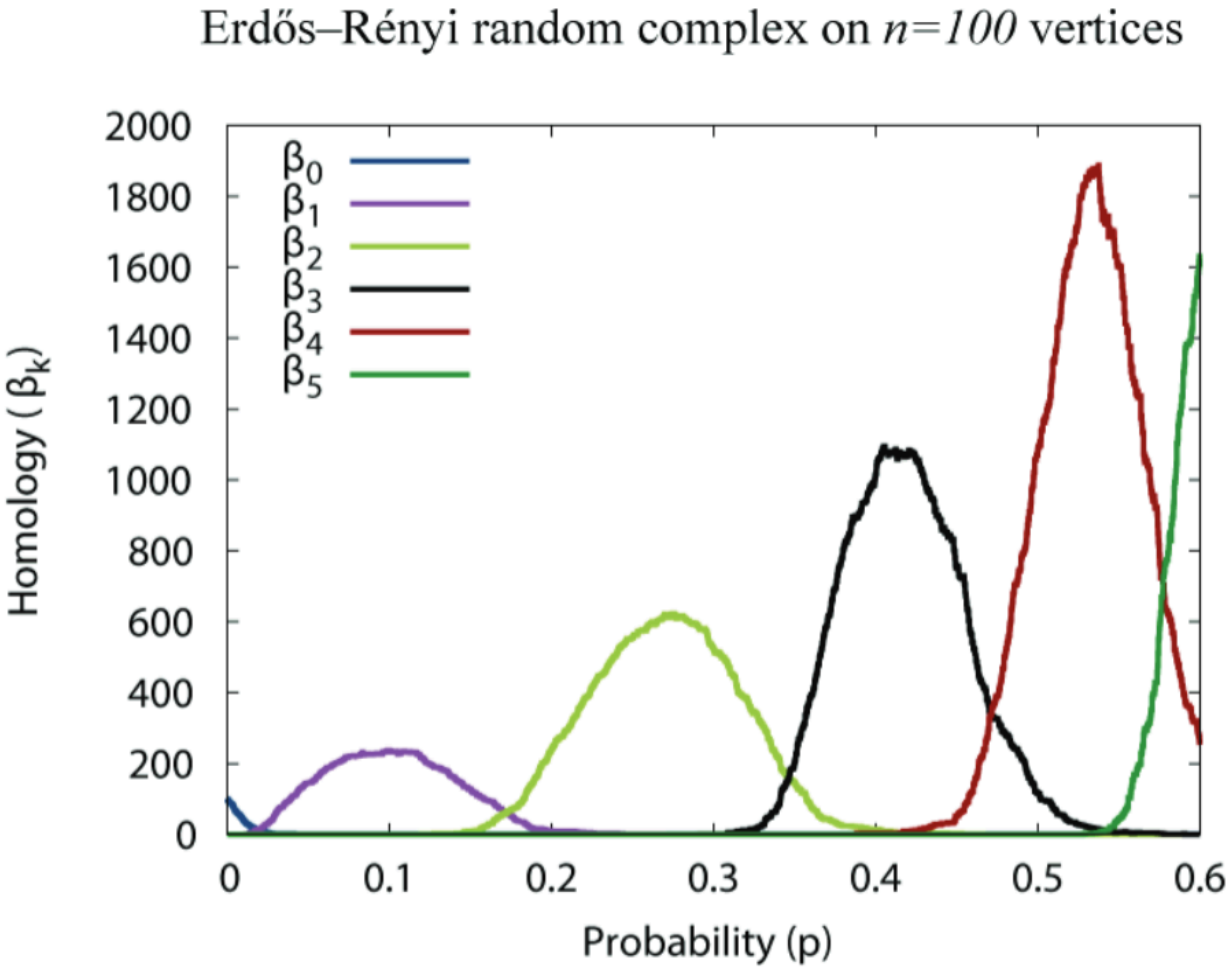


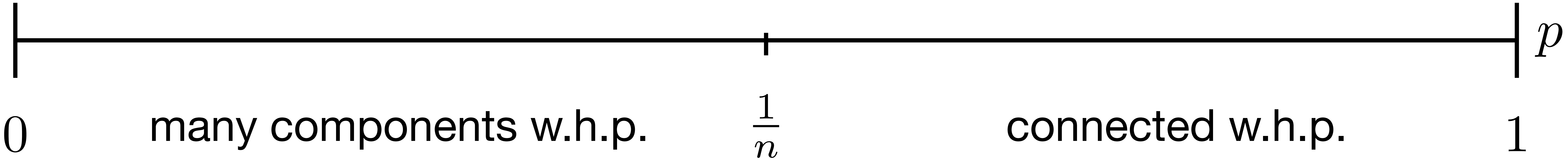
Betti Numbers





computation and plotting done by Zomorodian

Phase Transition
[Erdos-Renyi 1960]



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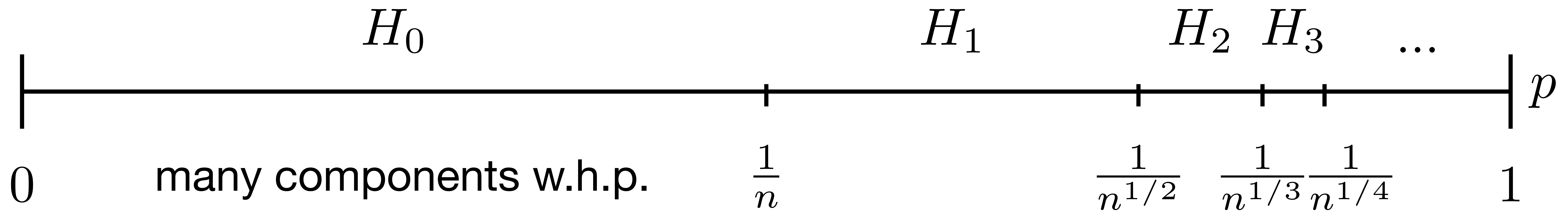
Phase Transition

[Kahle 2009, 2014]



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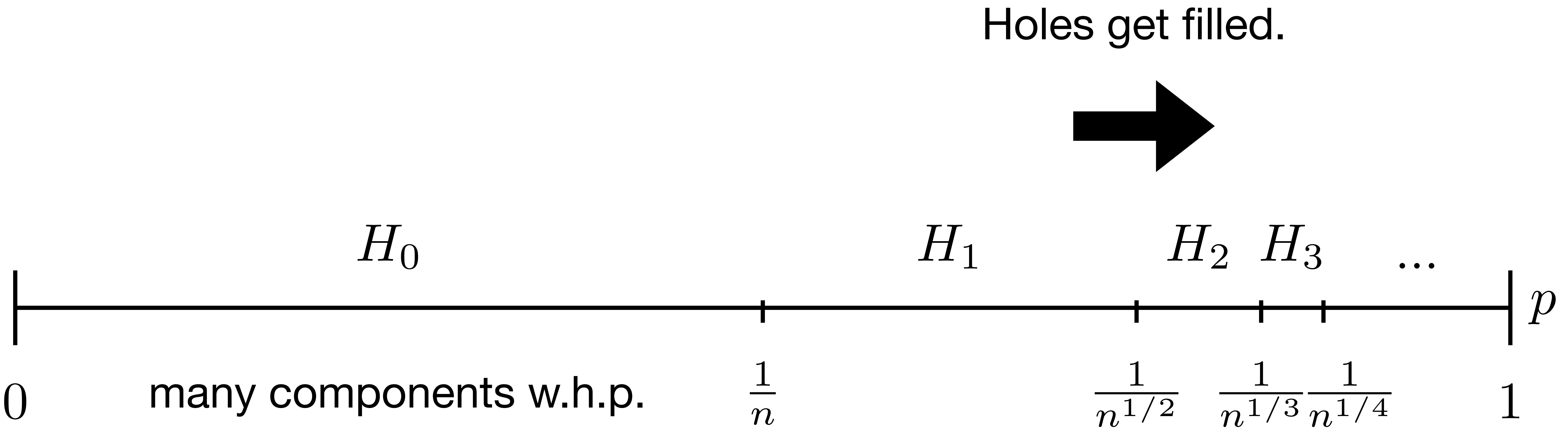

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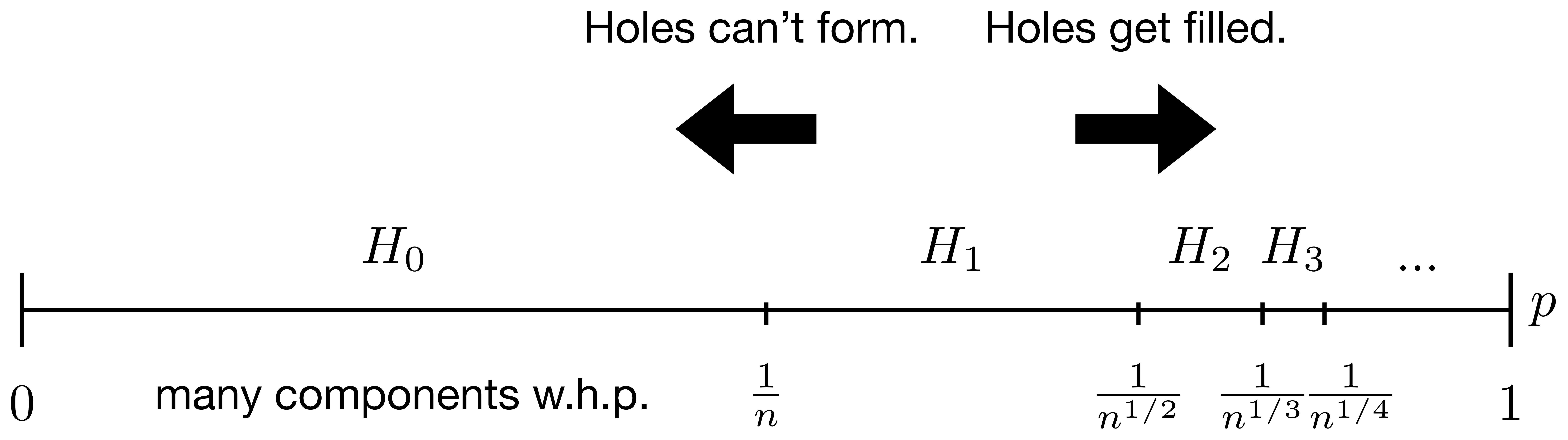
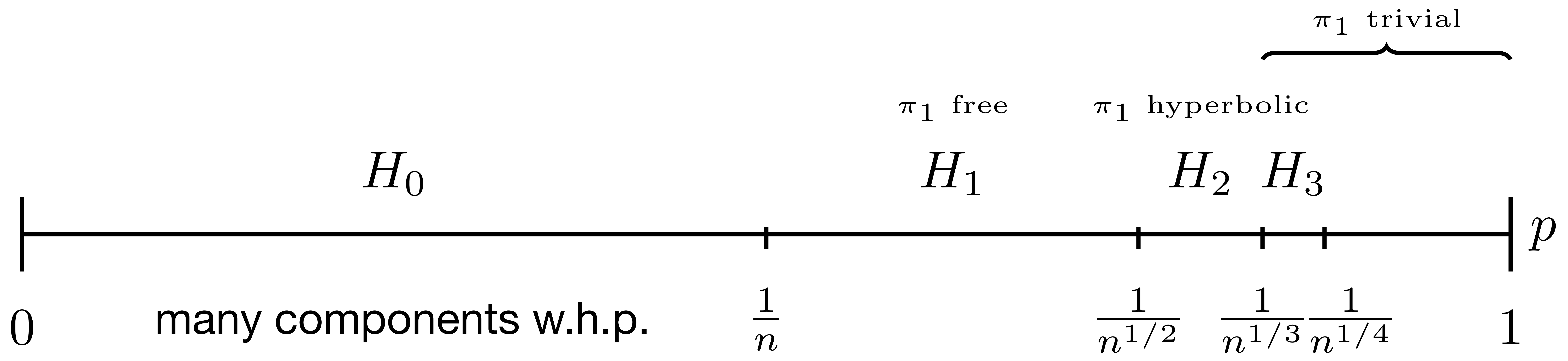


all log terms and constants forgone

Fundamental Group

[Kahle 2009, Babson 2012, Costa-Farber-Horak 2015]



all log terms and constants forgone

Geometric Complexes

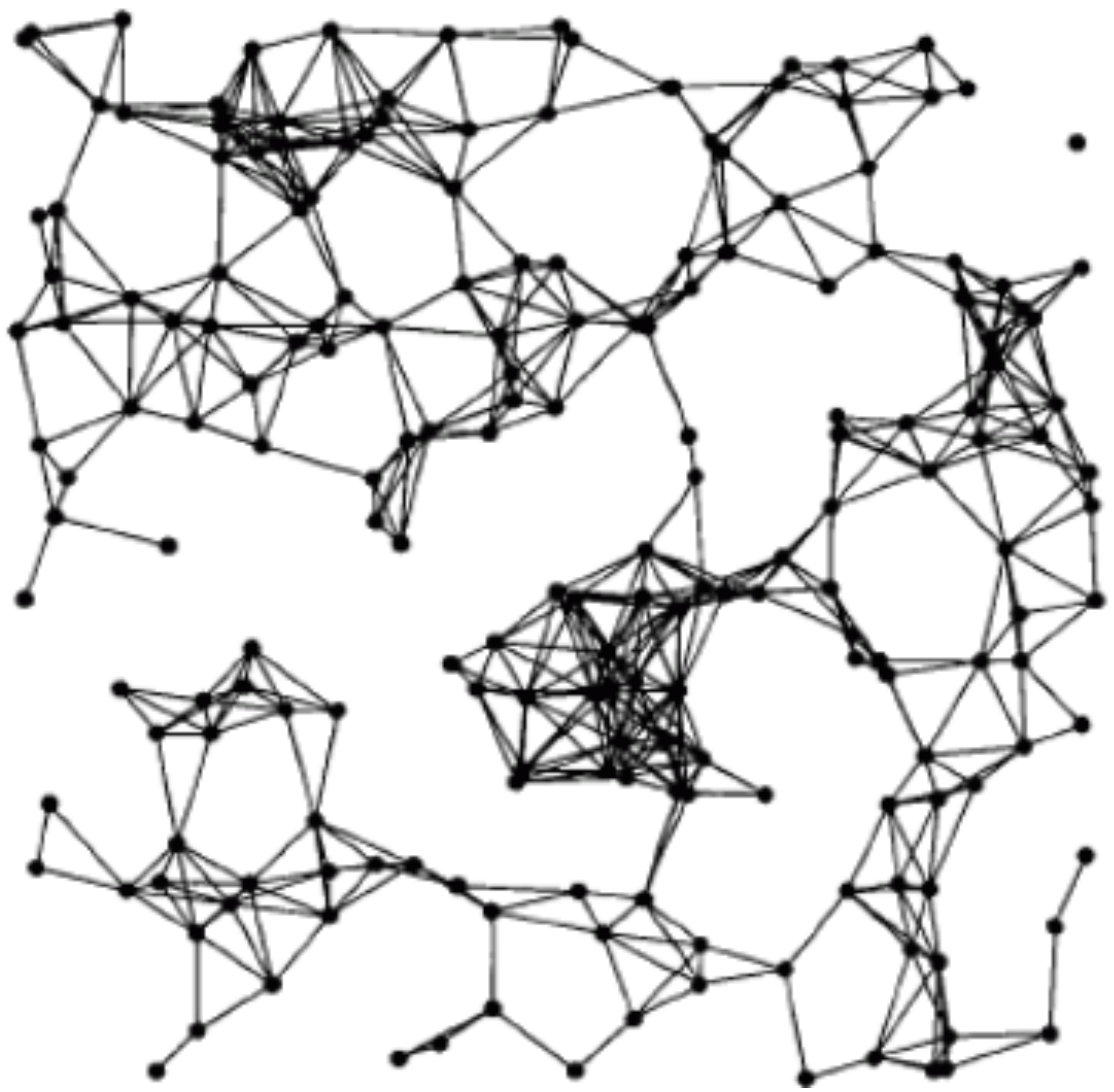


image credit: Penrose

Expected Betti numbers at dimension k

- Let $\omega = nr^D$, where D is the ambient dimension

Expected Betti numbers at dimension k
[Kahle 2011]

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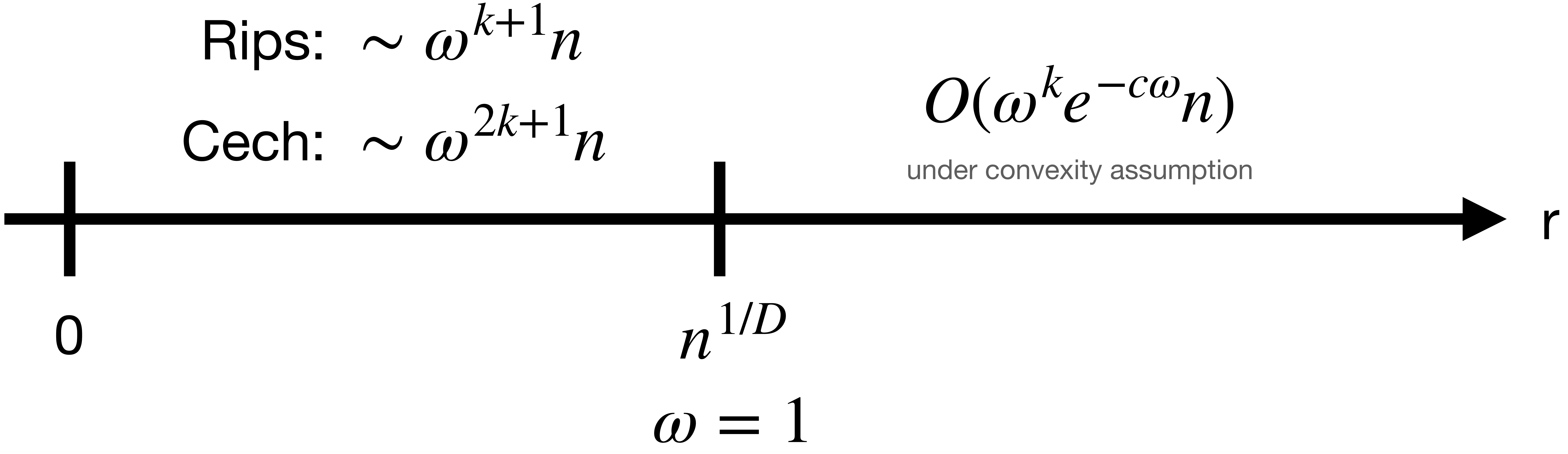
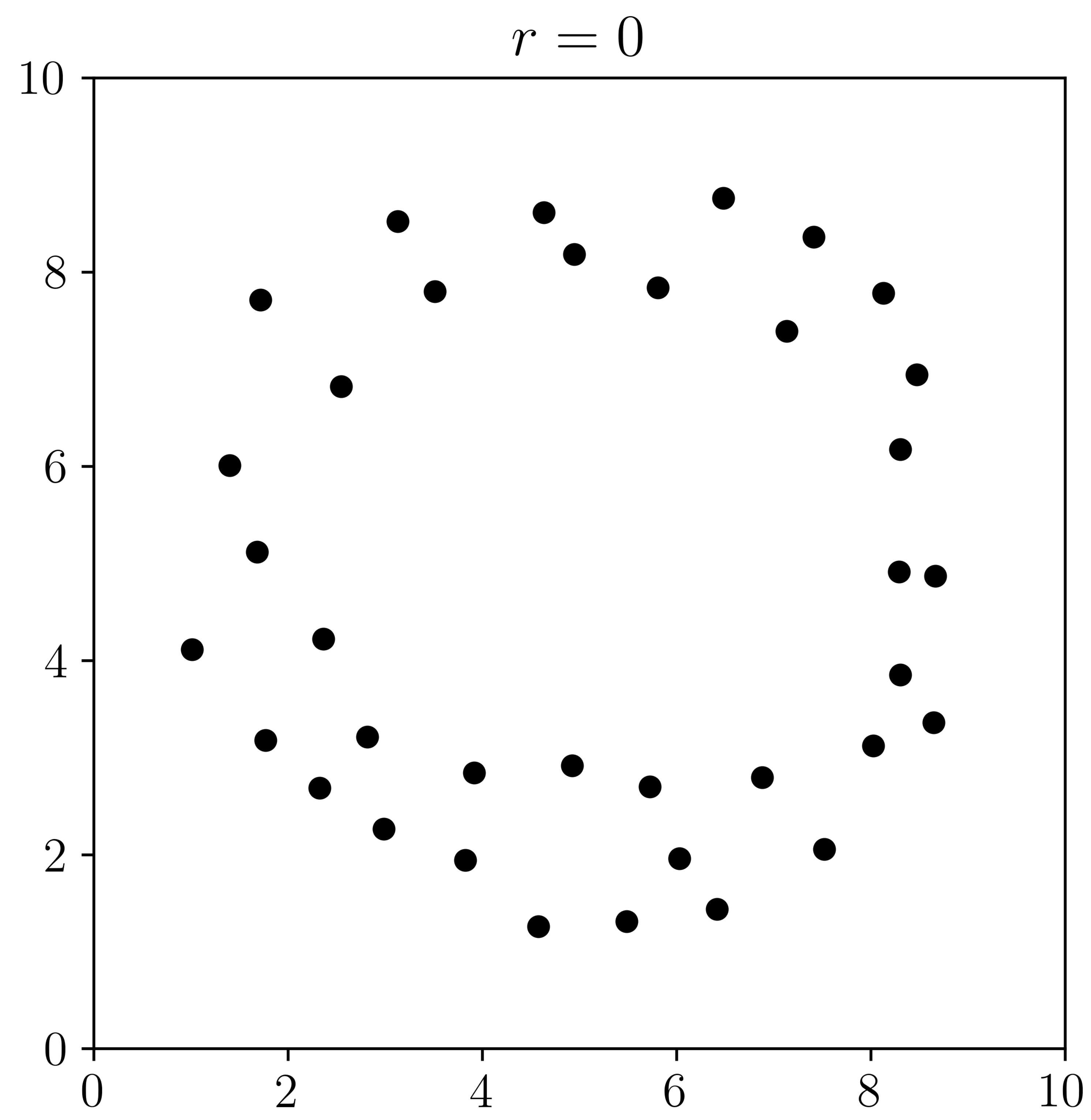


Maximally Persistent Cycles



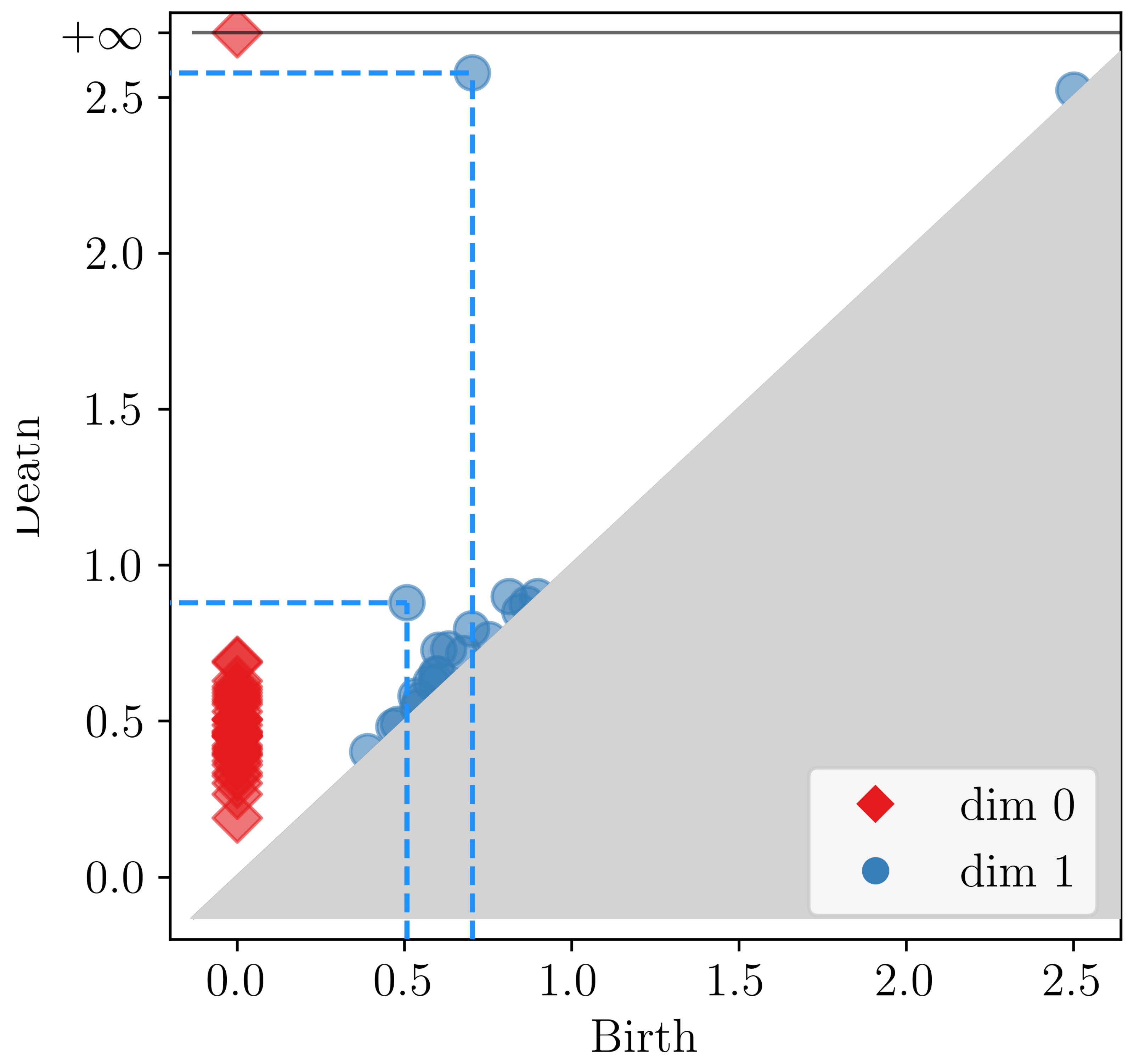


image credit: Andrey Yao

Maximally Persistent Cycles

n points in expectation

k-cycle

Maximally Persistent Cycles
[Bobrowski-Kahle-Skraba 2017]

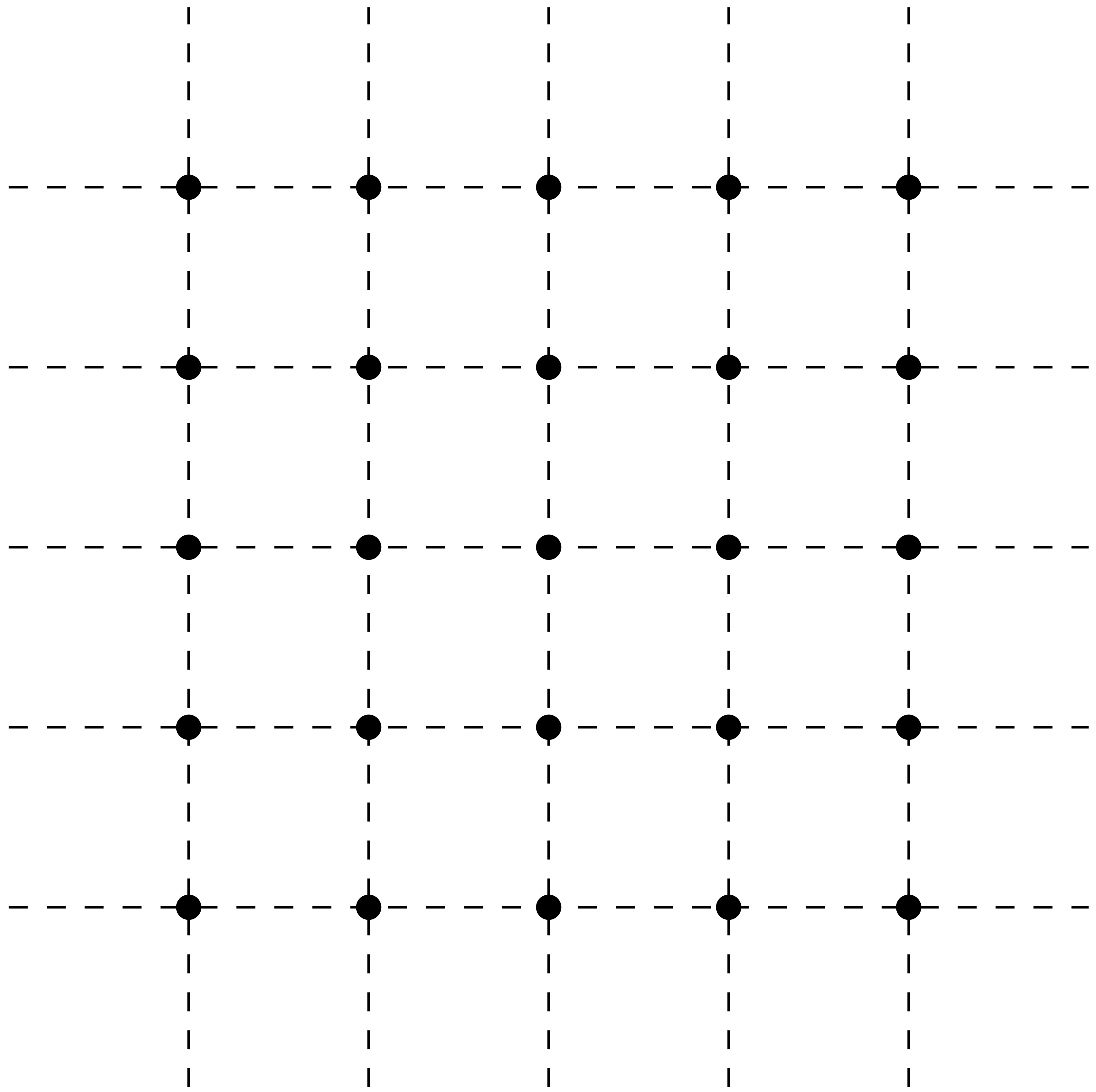
n points in expectation

k-cycle

$$c\left(\frac{\log n}{\log\log n}\right)^{1/k} \leq \text{max persistence} \leq C\left(\frac{\log n}{\log\log n}\right)^{1/k}$$

a.a.s.

Bernoulli Bond Percolation



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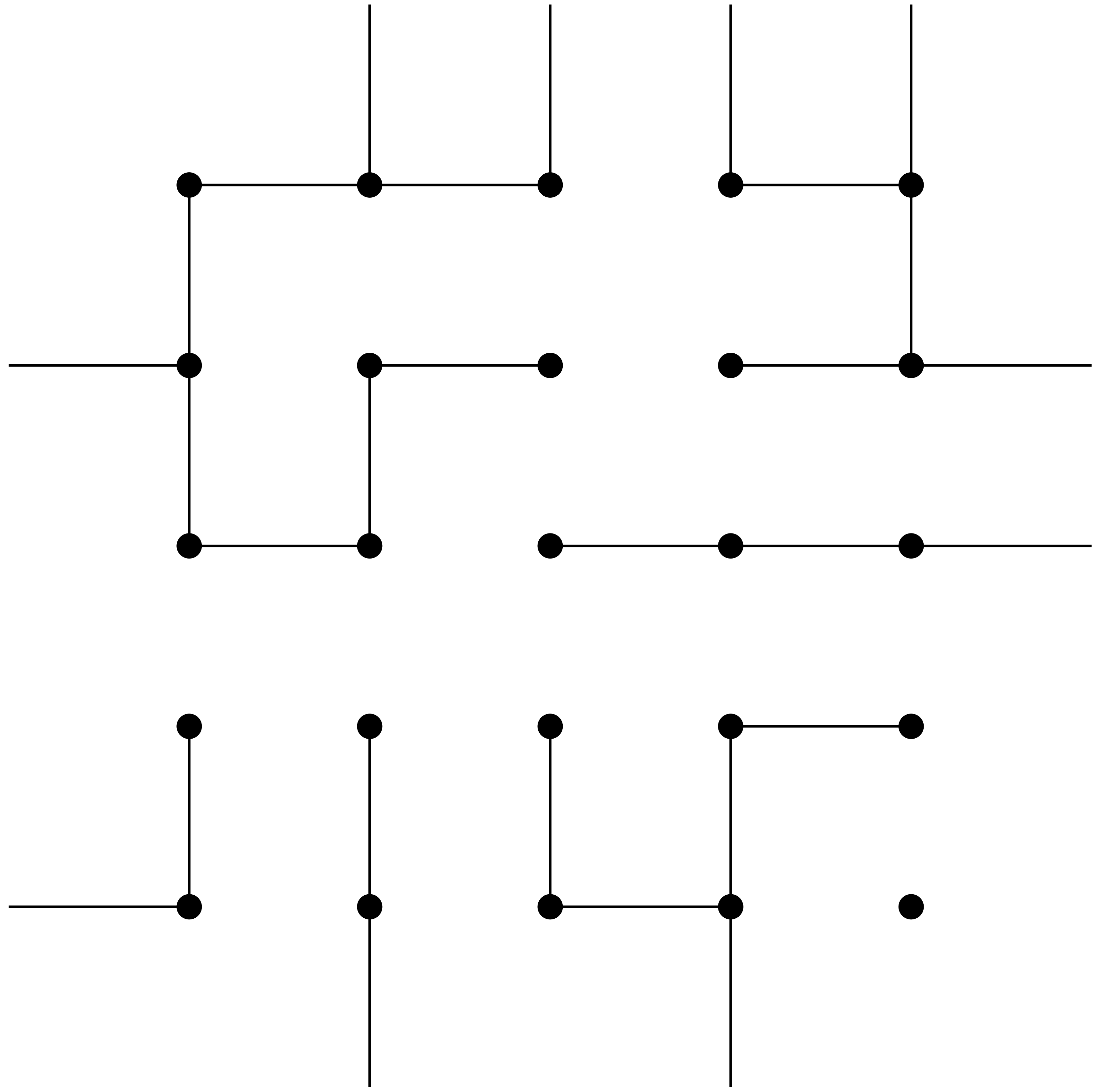
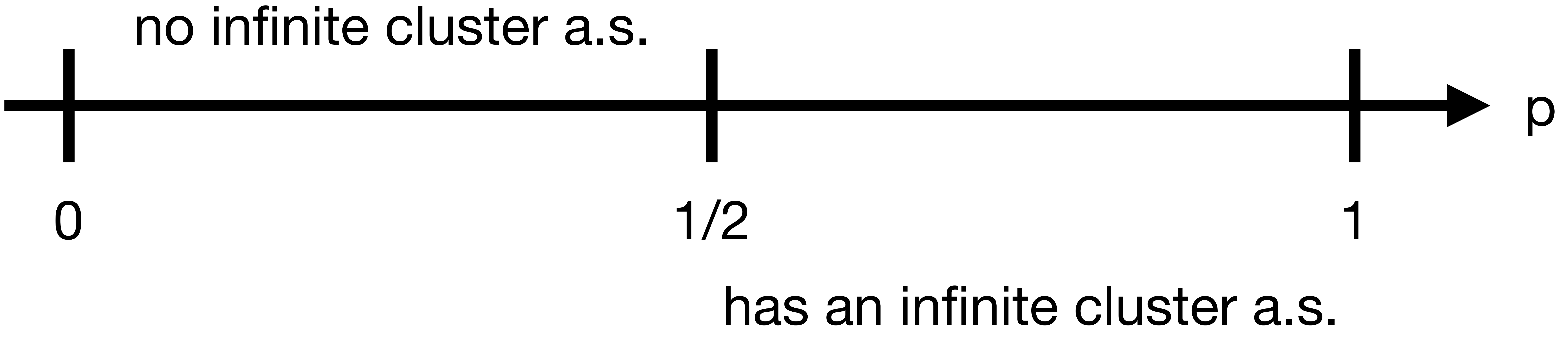


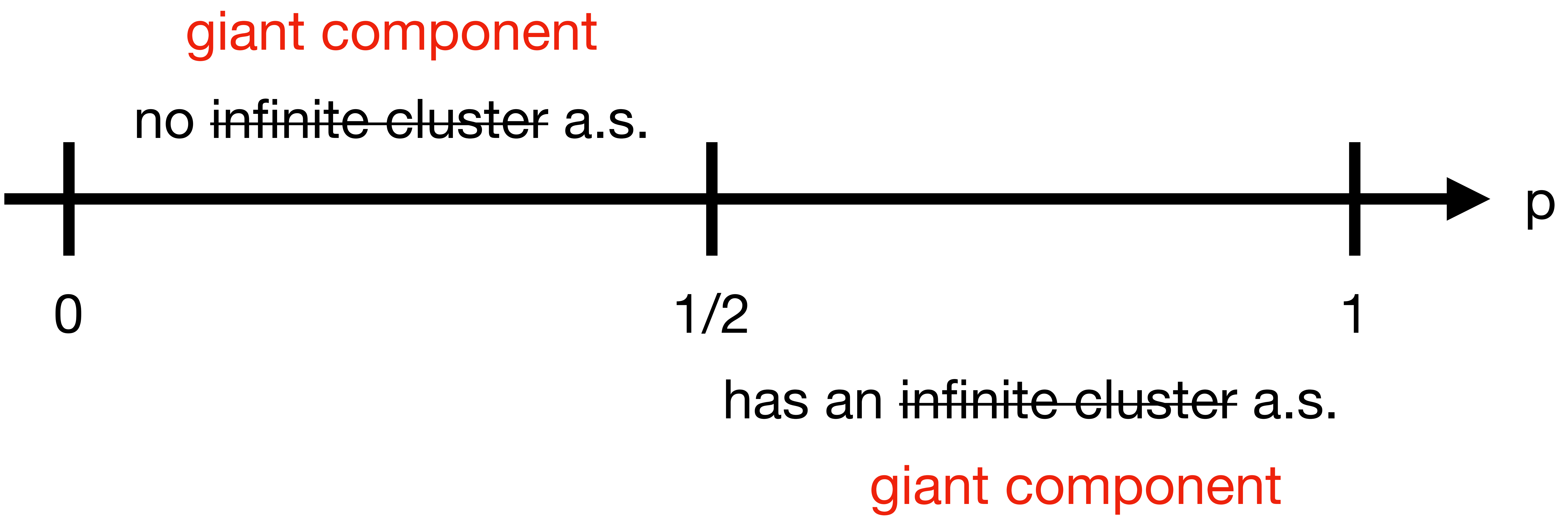
Phase Transition

[Harris 1960, Kesten 1980]



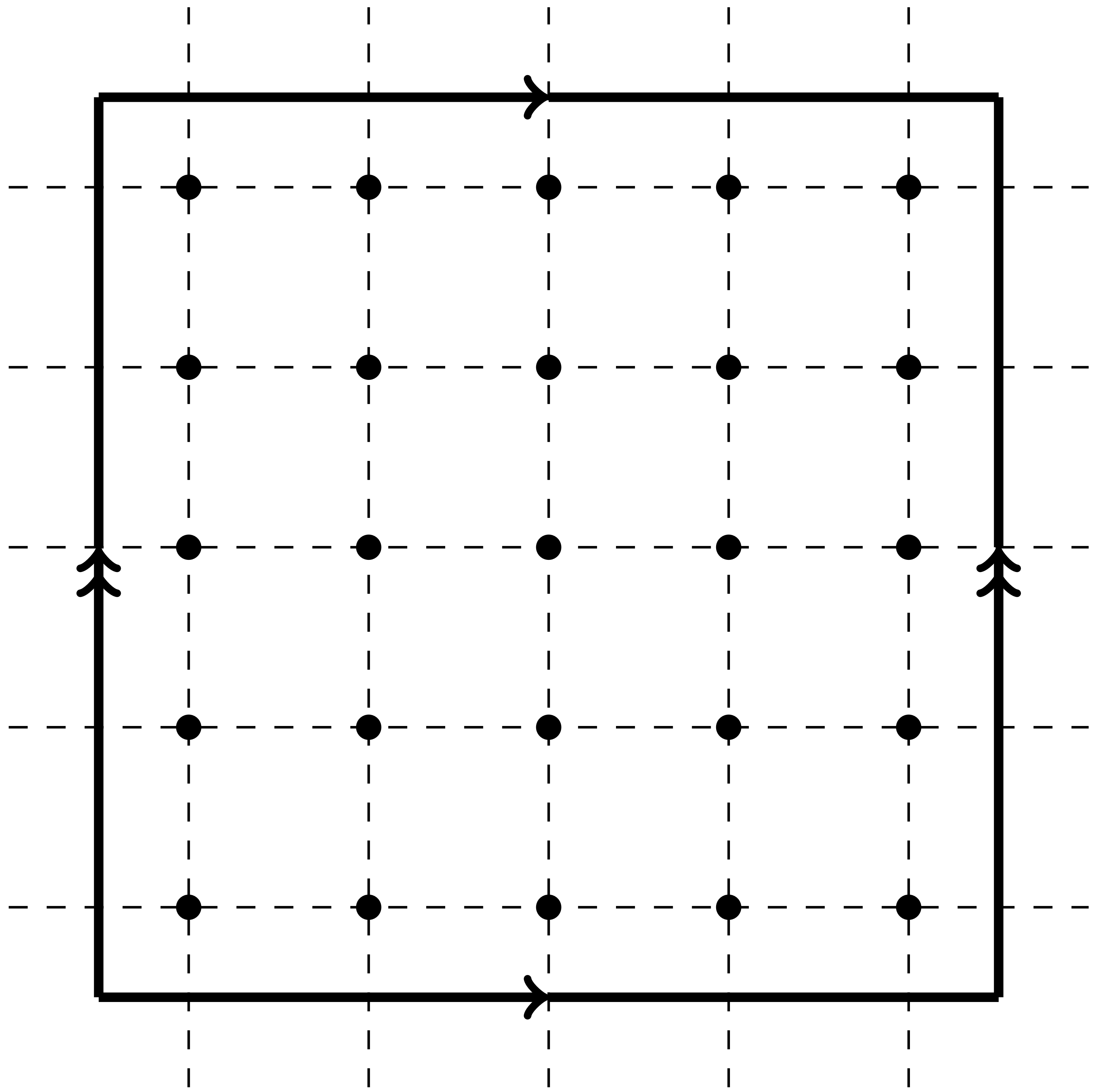
Phase Transition

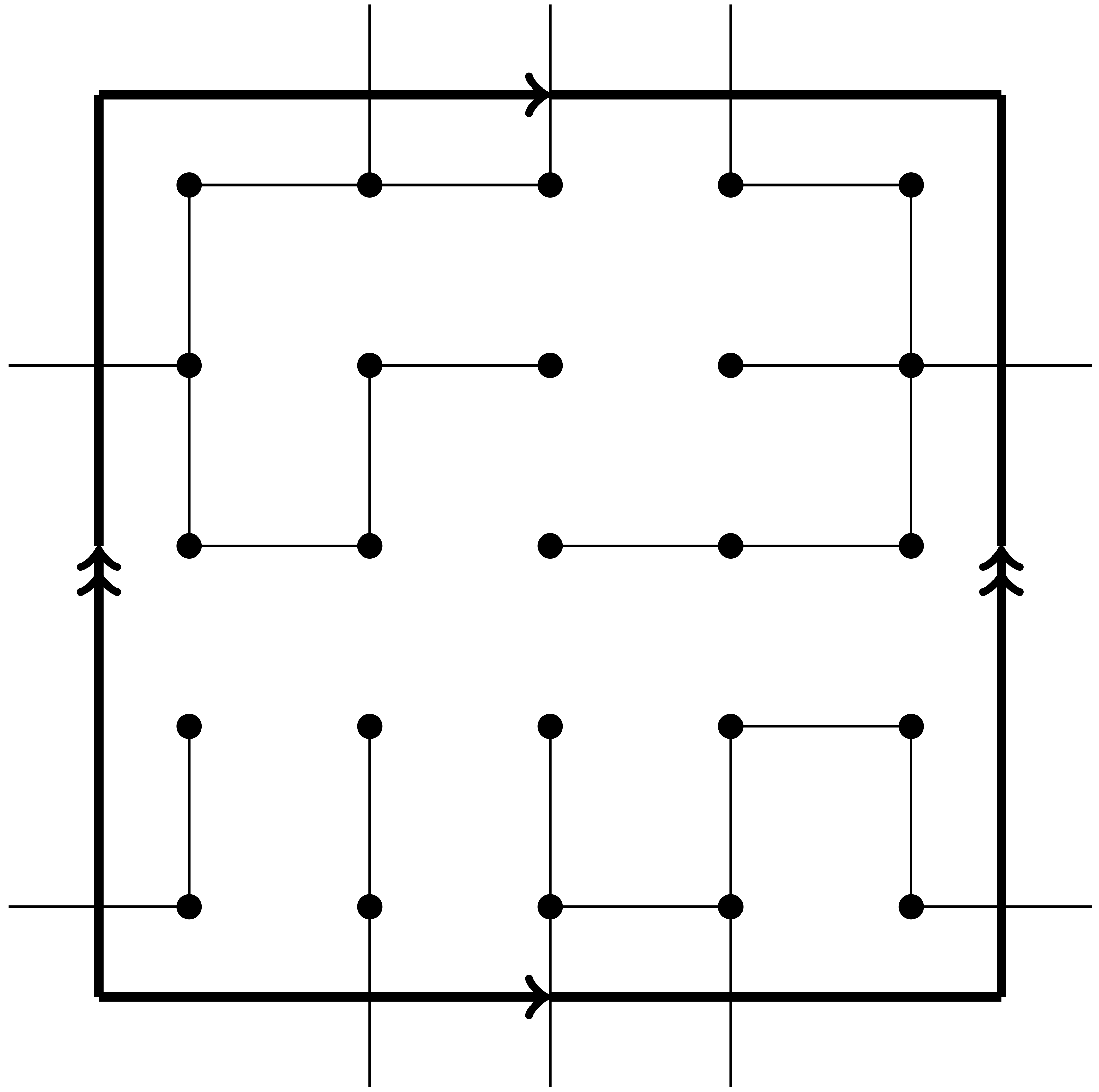
[Harris 1960, Kesten 1980]



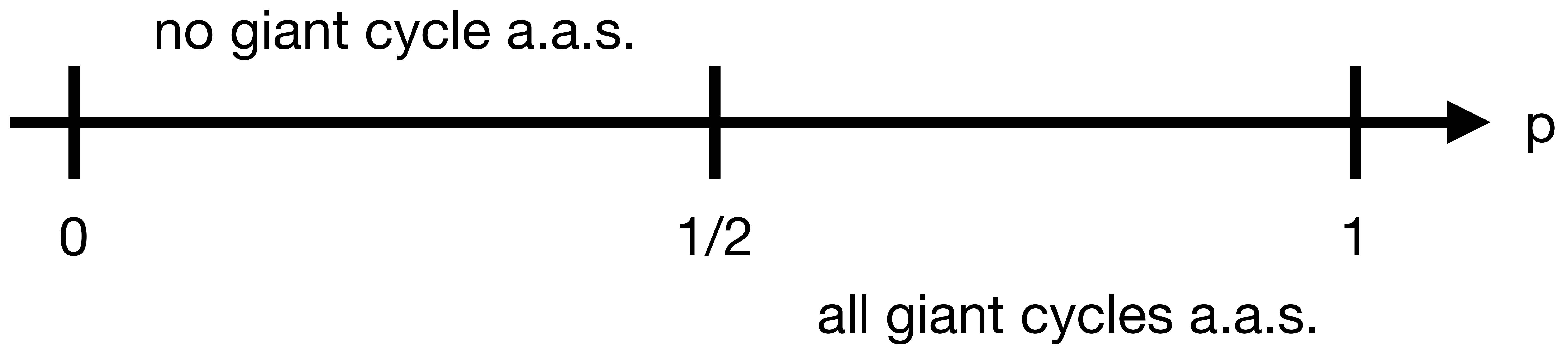
Giant Cycles?

Bernoulli Bond Percolation

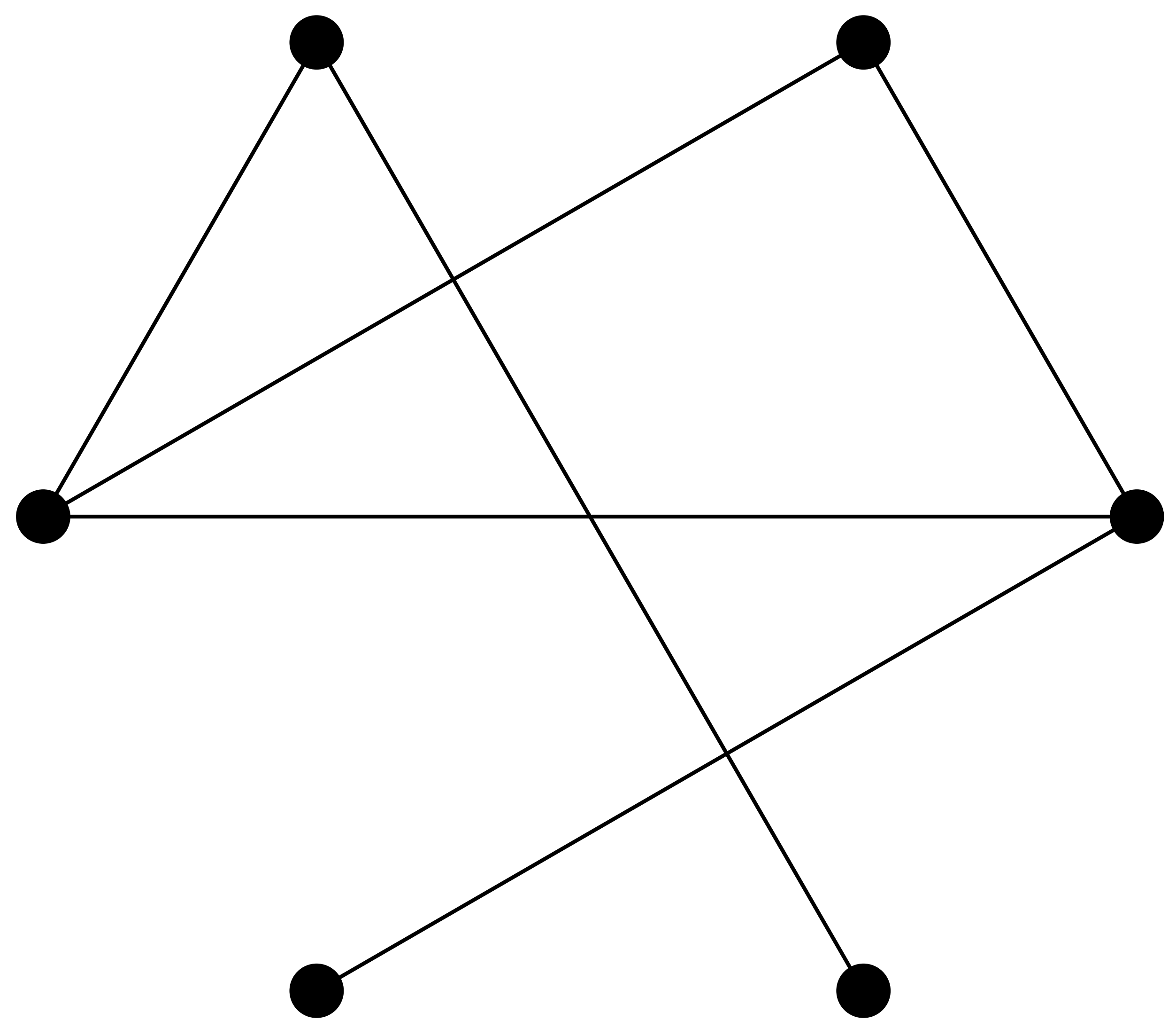




Phase Transition
[Duncan-Kahle-Schweinhart, 2021]



Tapas de Random Topology



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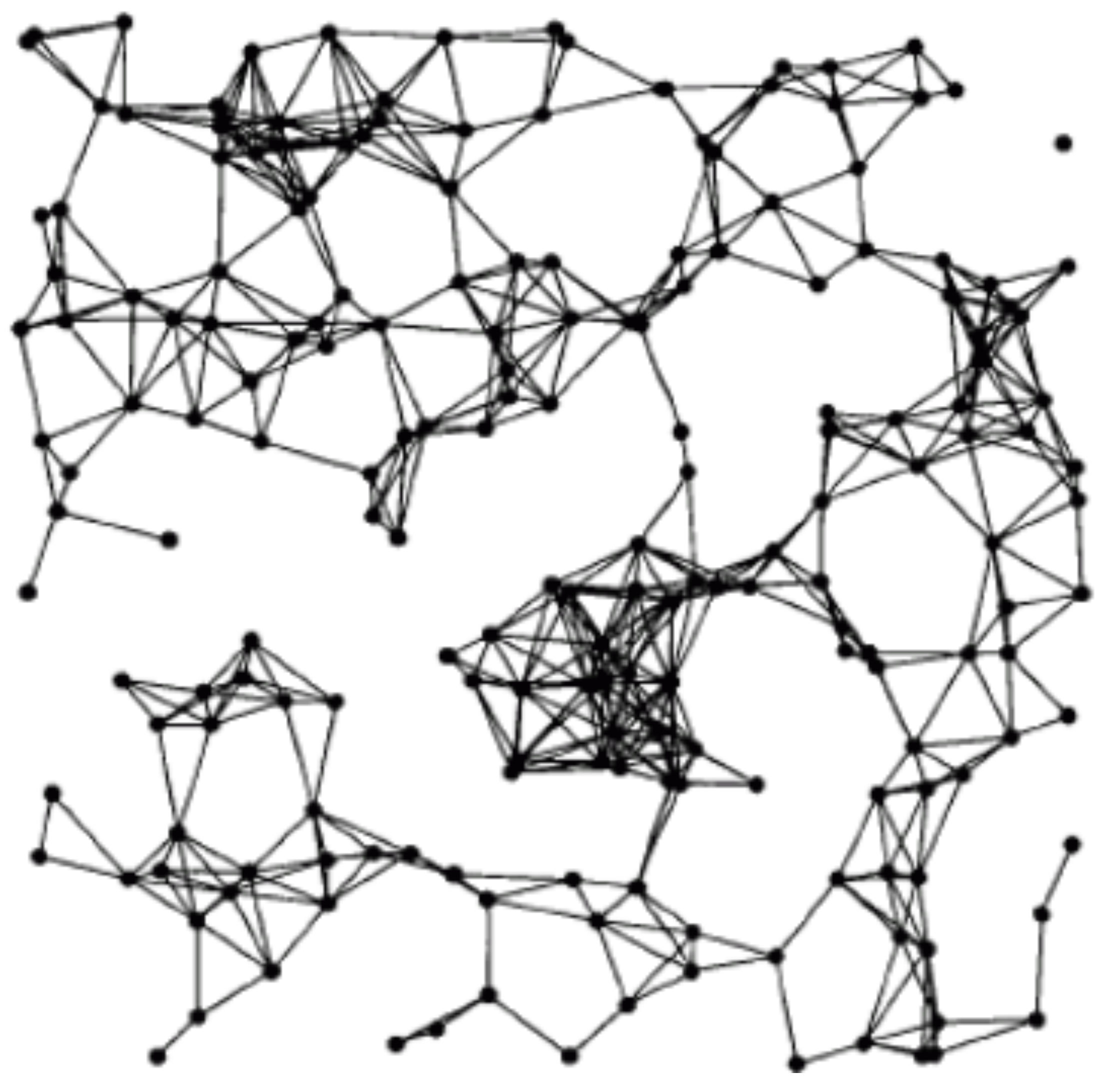
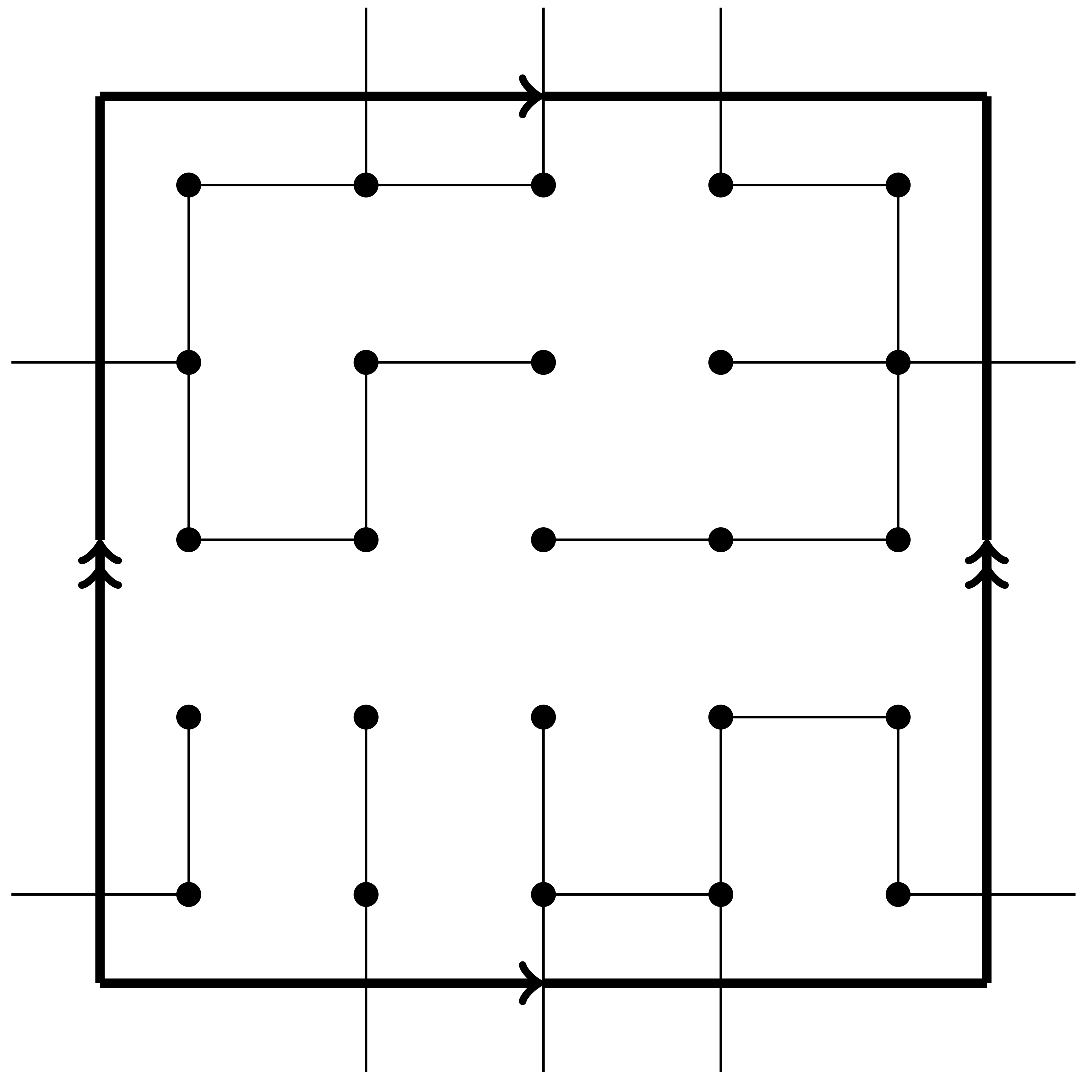


Geometric Complexes



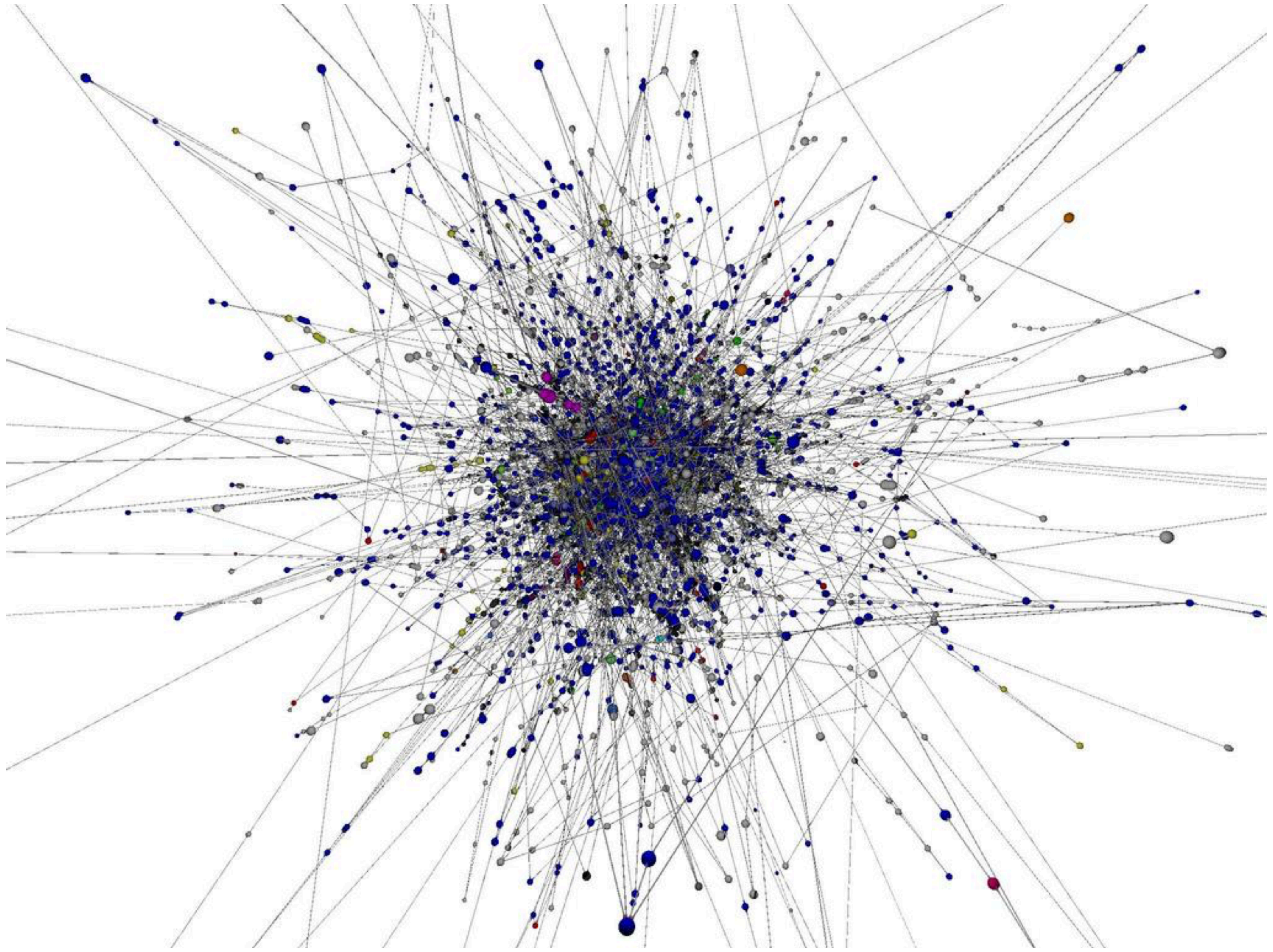
Topological Percolation

II. Preferential Attachment

Beyond independence and homogeneity

Independent and identically distributed?

Independent and identically distributed?



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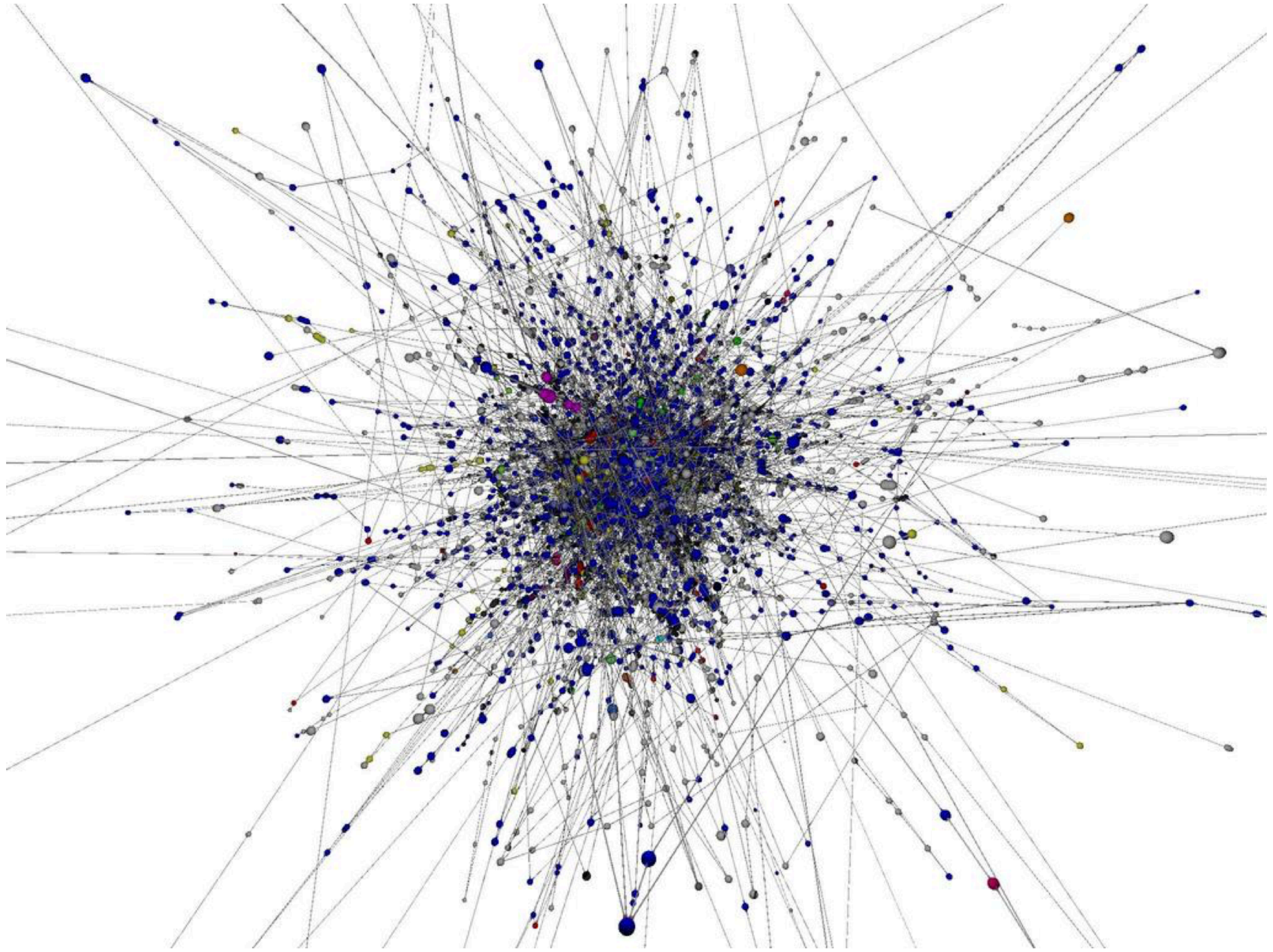
Preferential Attachment

[Albert and Barabasi 1999]



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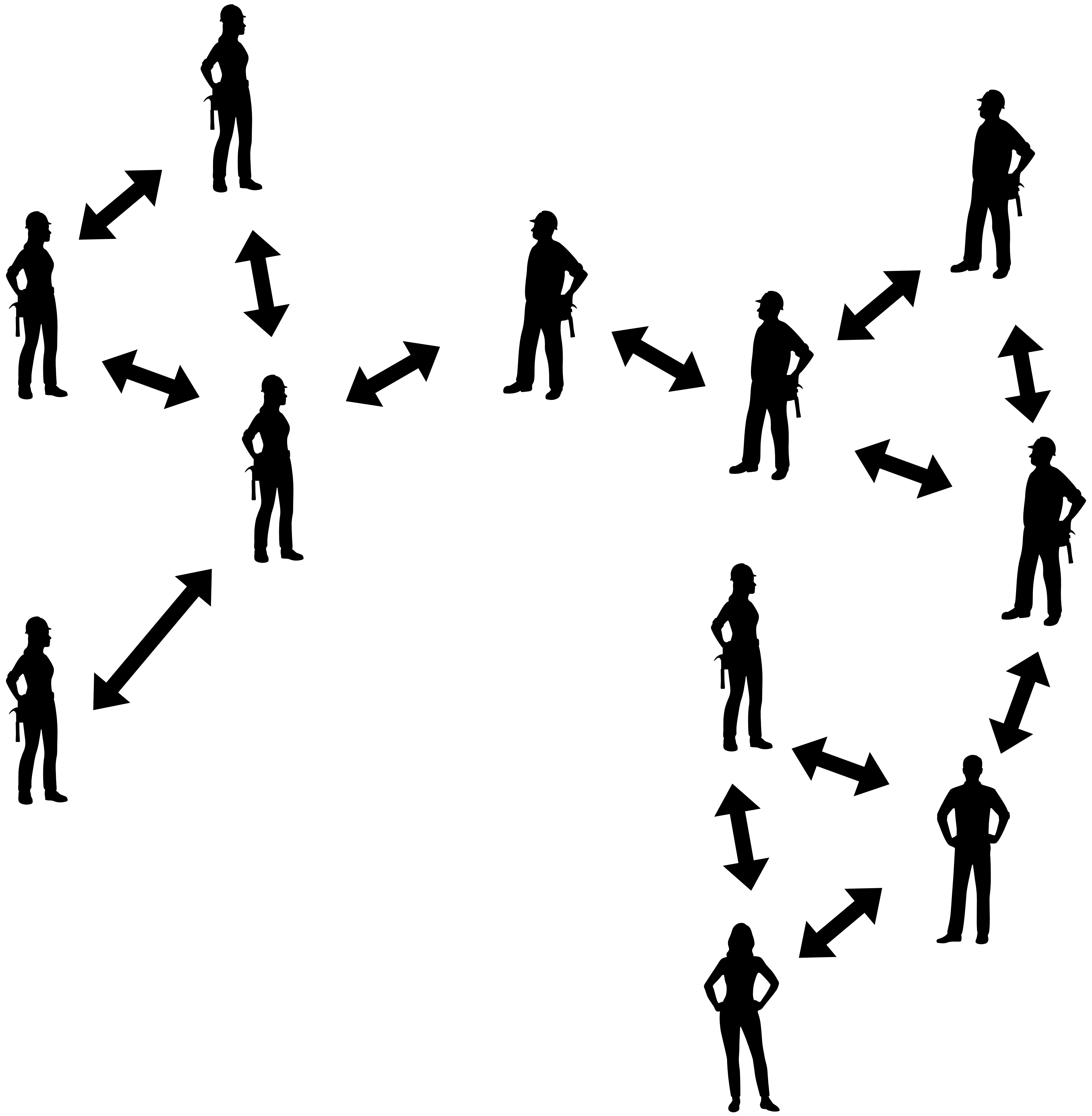

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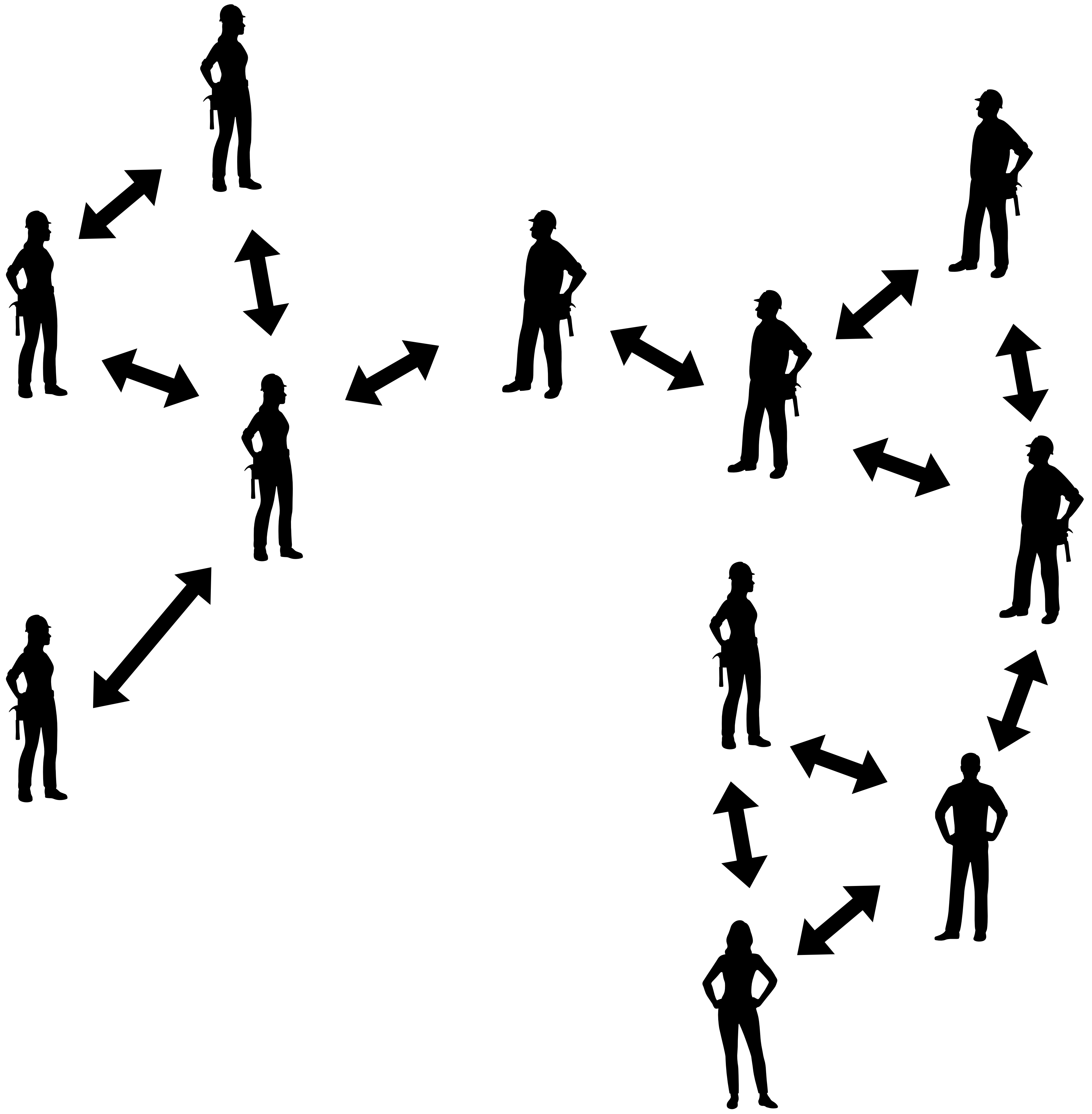

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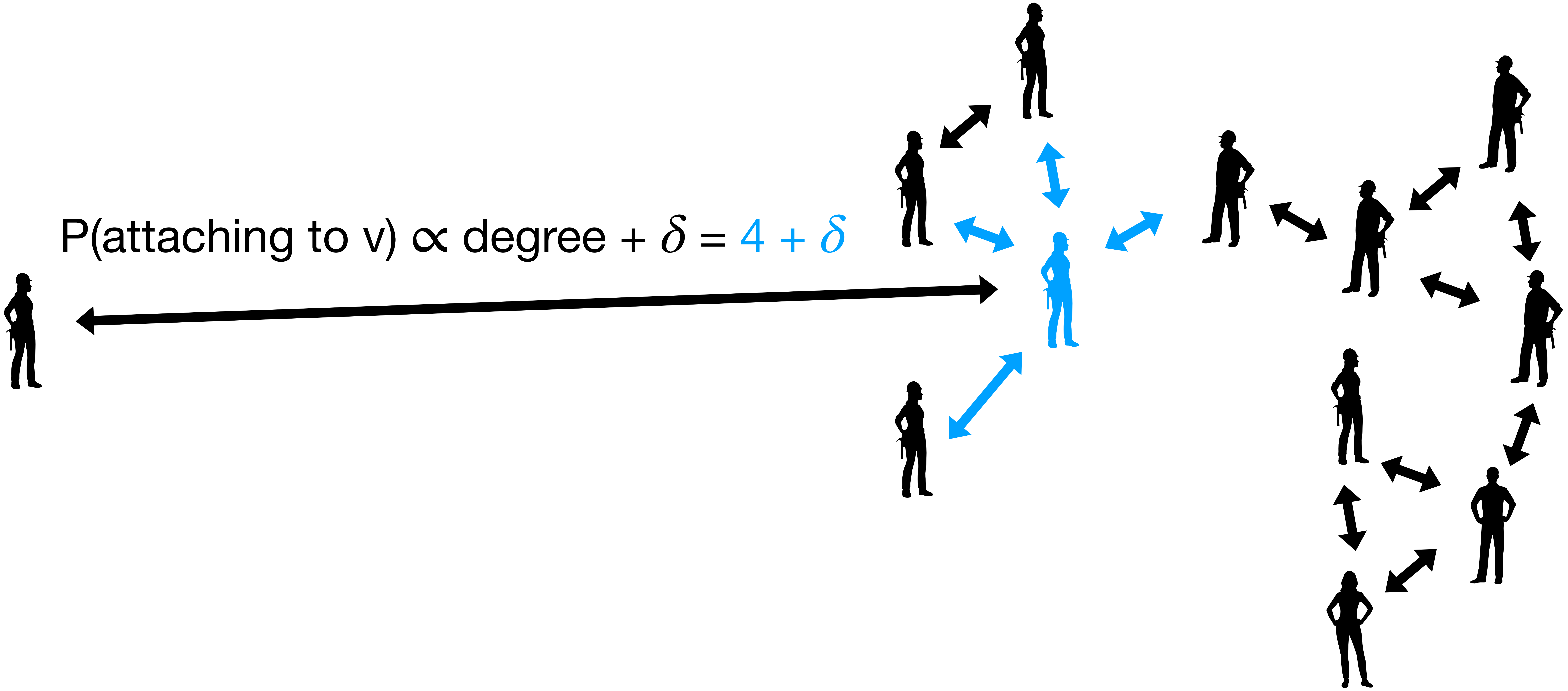
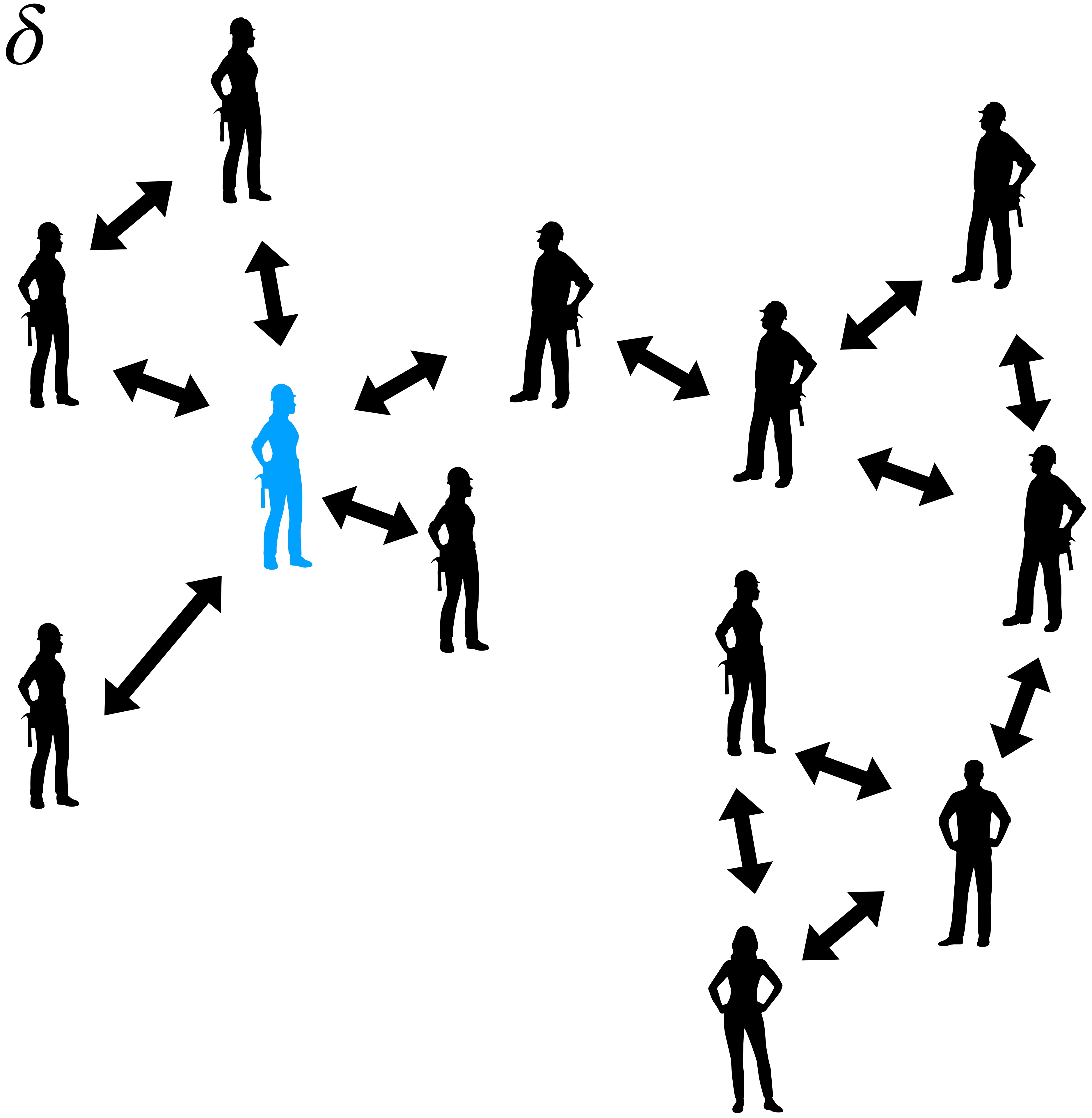


Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) ∝ degree + a tuning parameter δ



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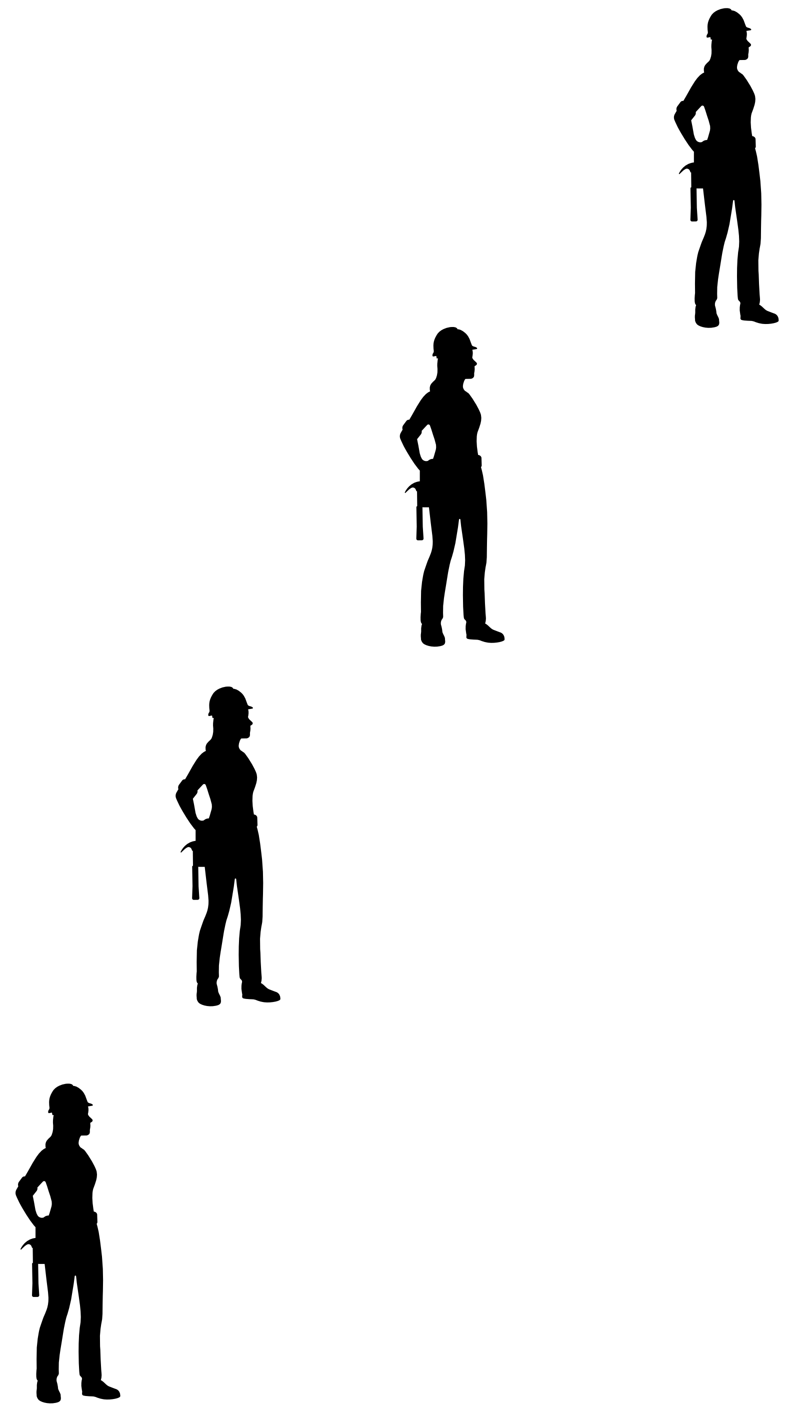

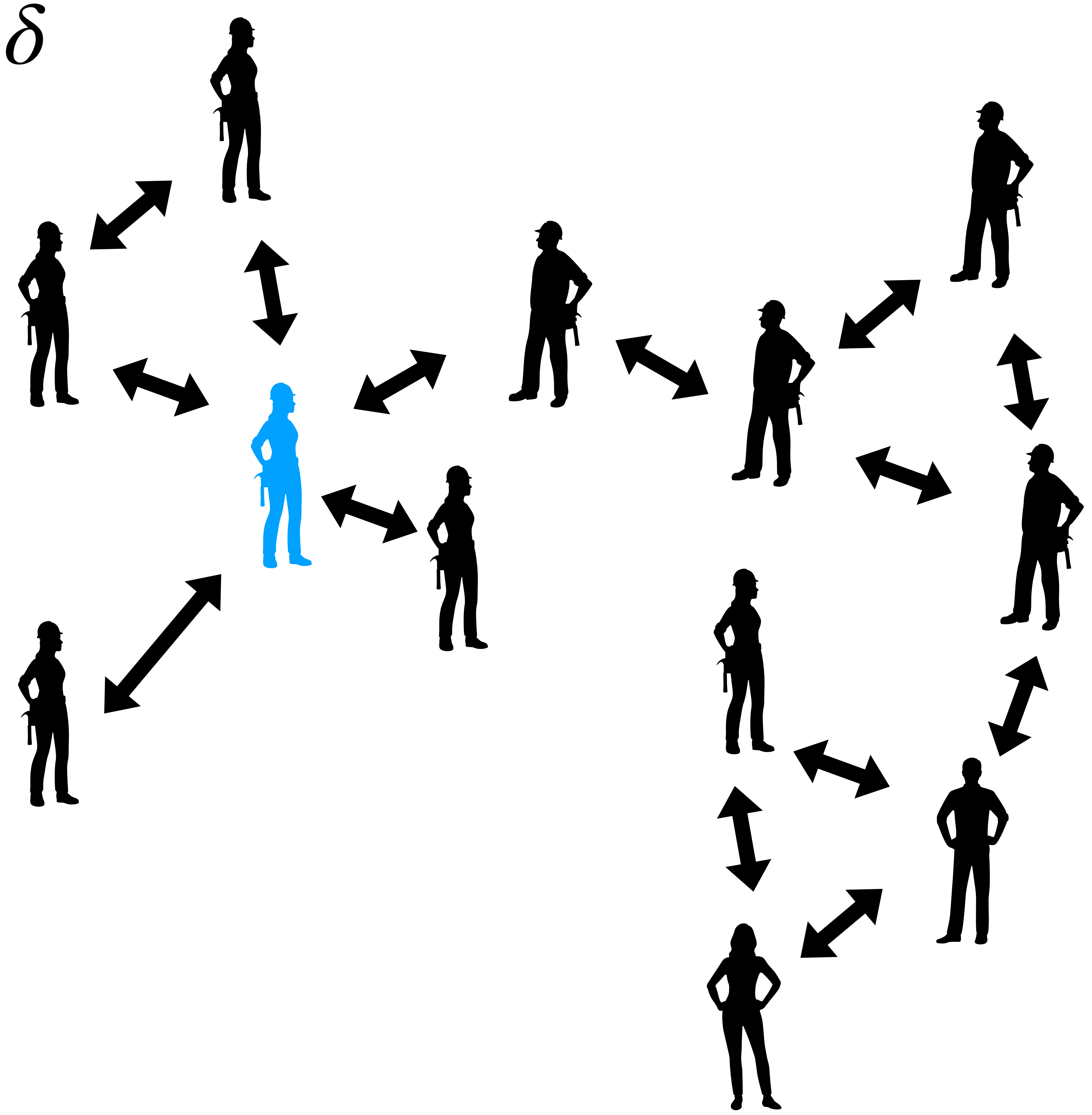

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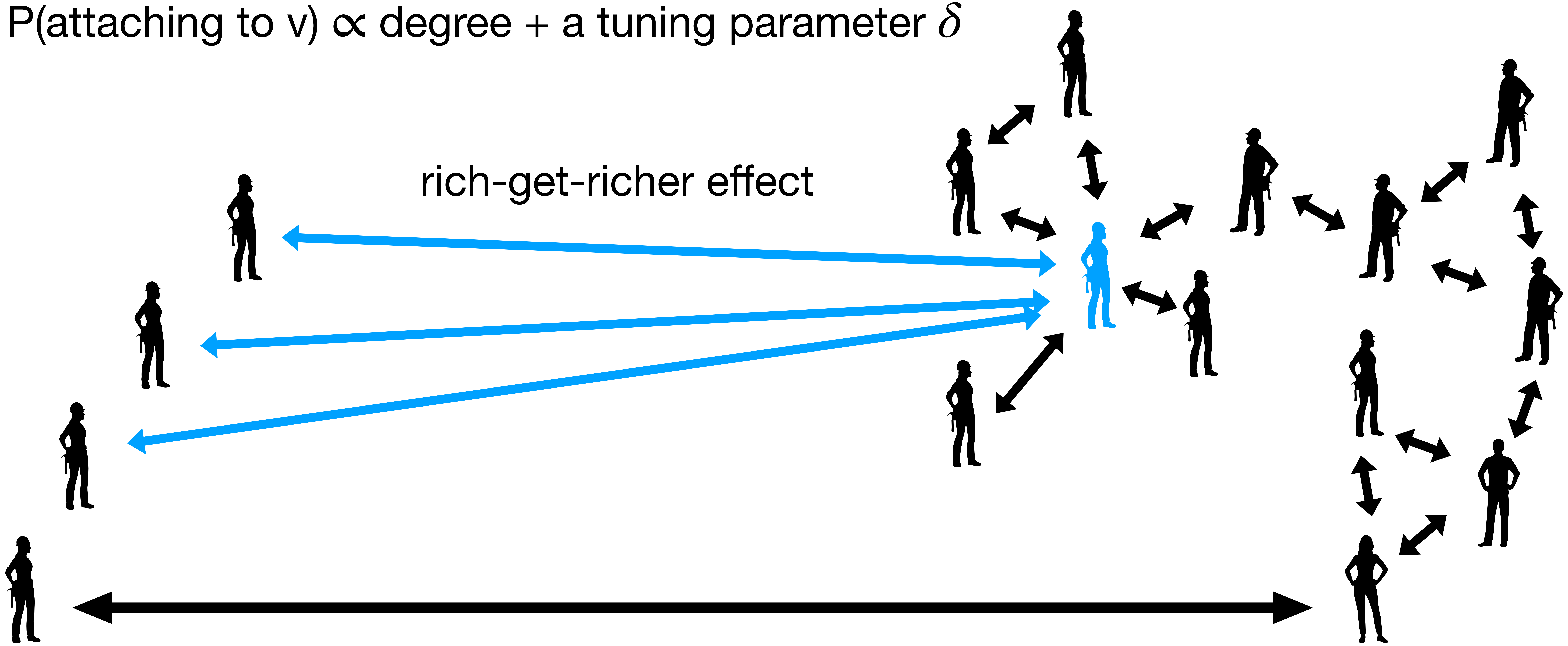

What do we know?

What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]

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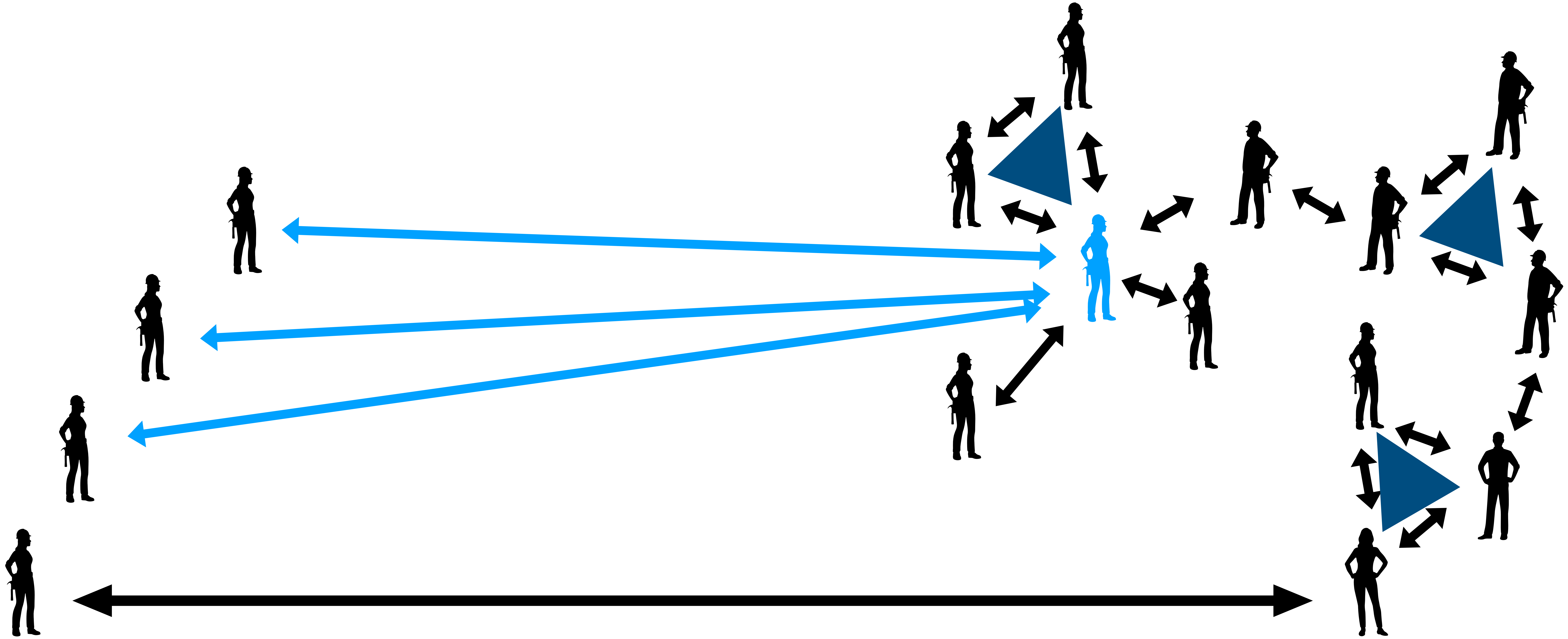
- subgraph counts [Garavaglia and Steghuis 2019]

What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]
- and more…

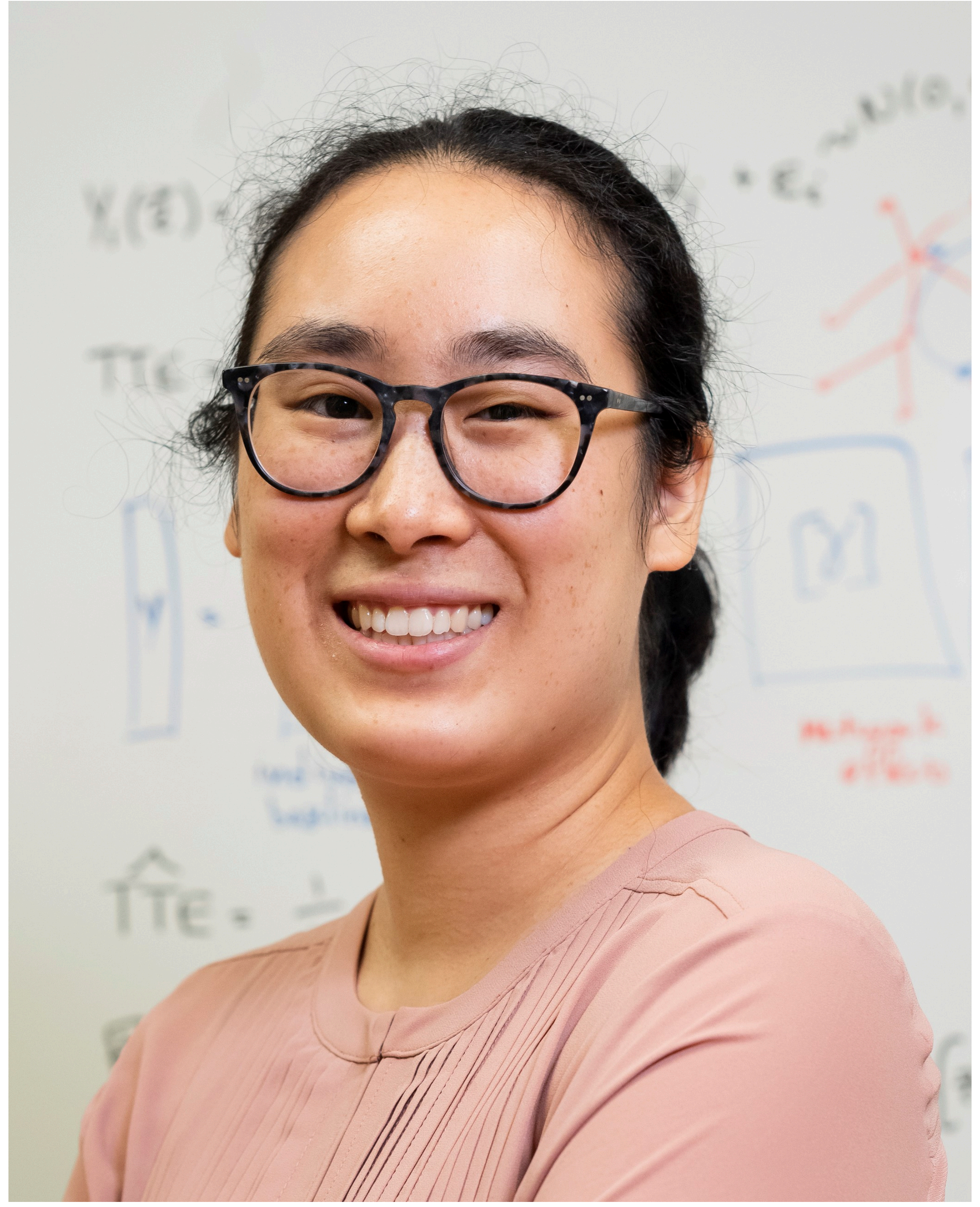
Clique Complex

aka Flag Complex



III Topology of Preferential Attachment

My Lovely Collaborators



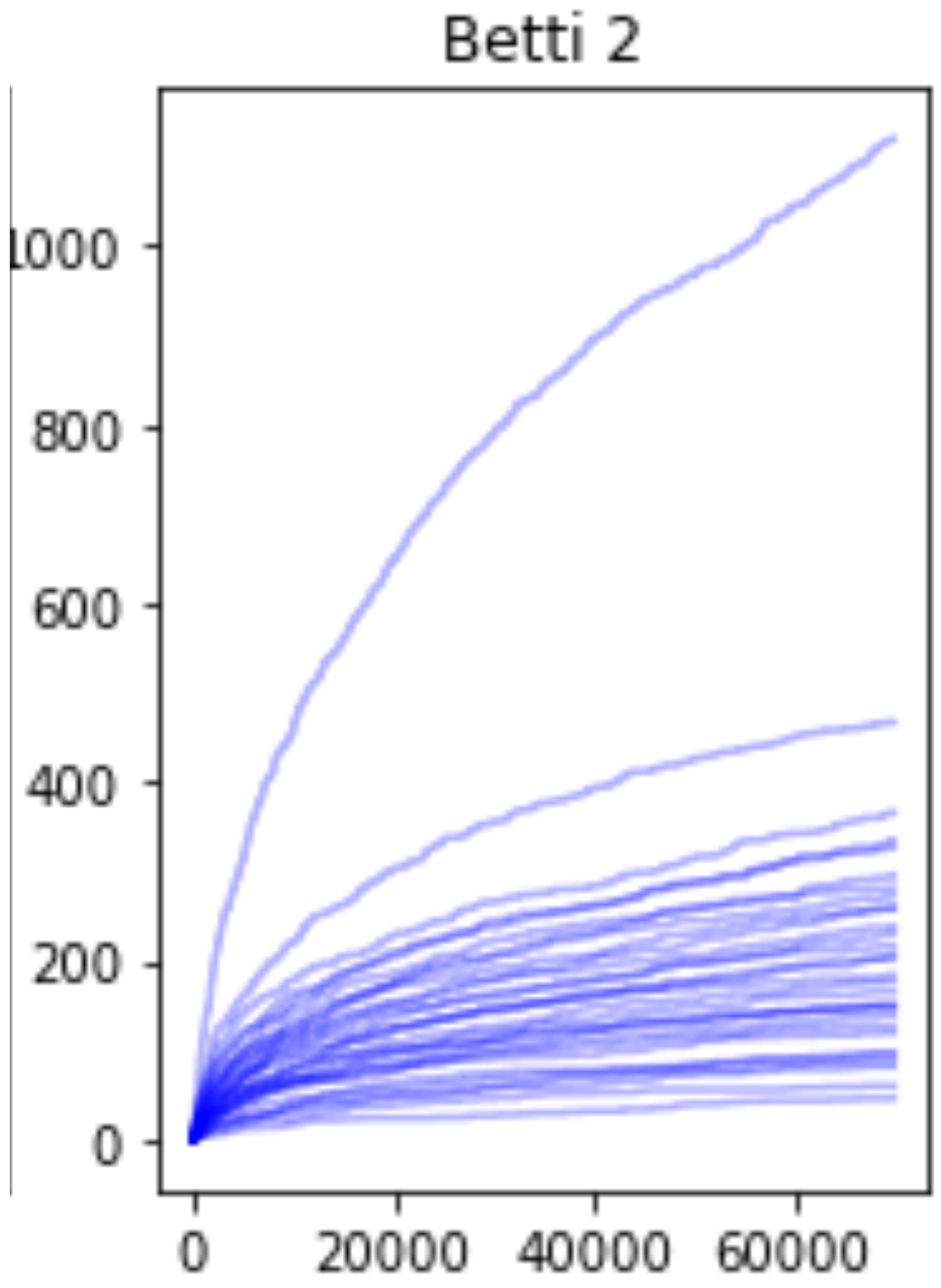
Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

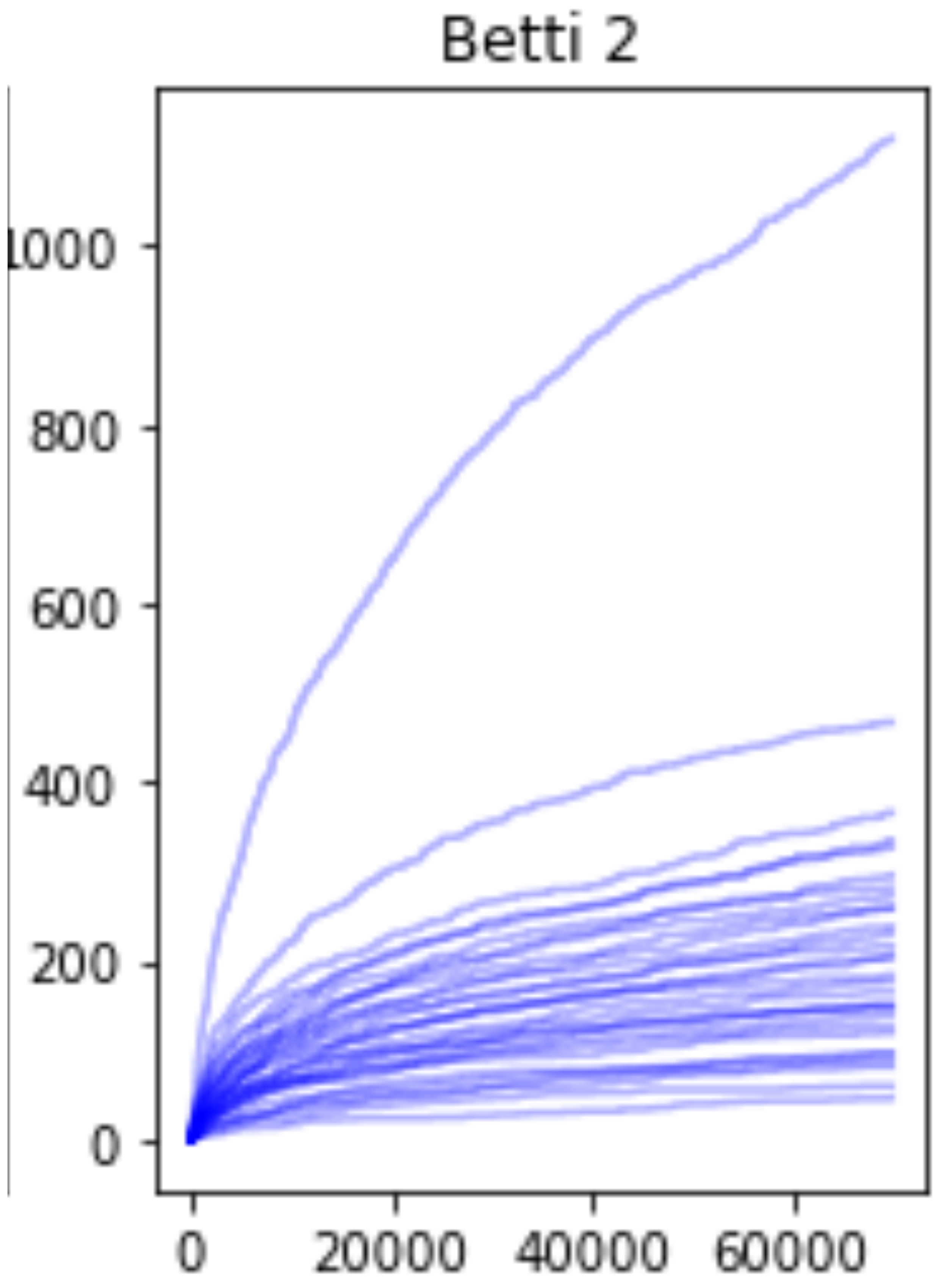
Expected Betti Number $E[\beta_q]$

Expected Betti Number $E[\beta_q]$



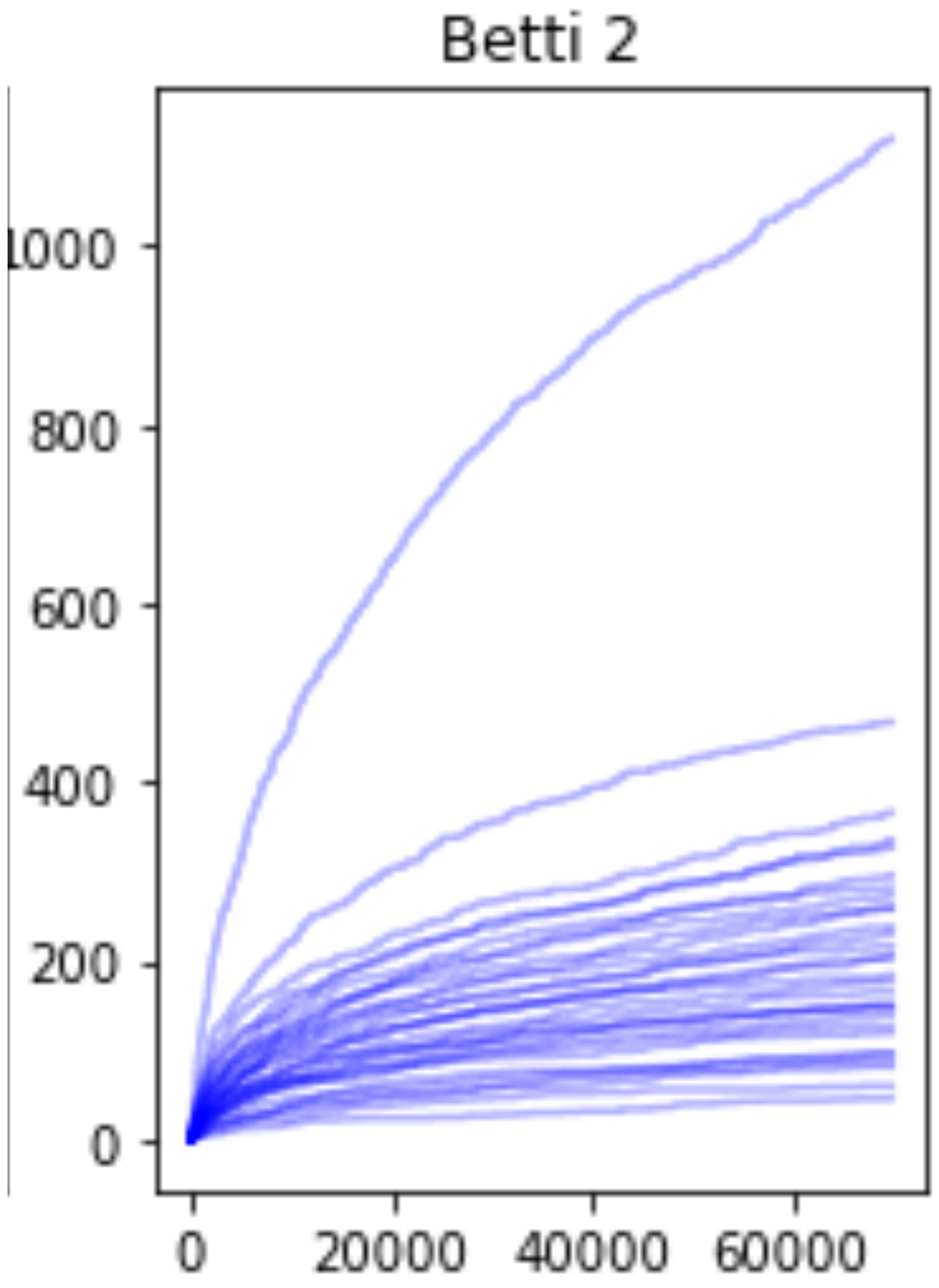
Expected Betti Number $E[\beta_q]$

- increasing trend



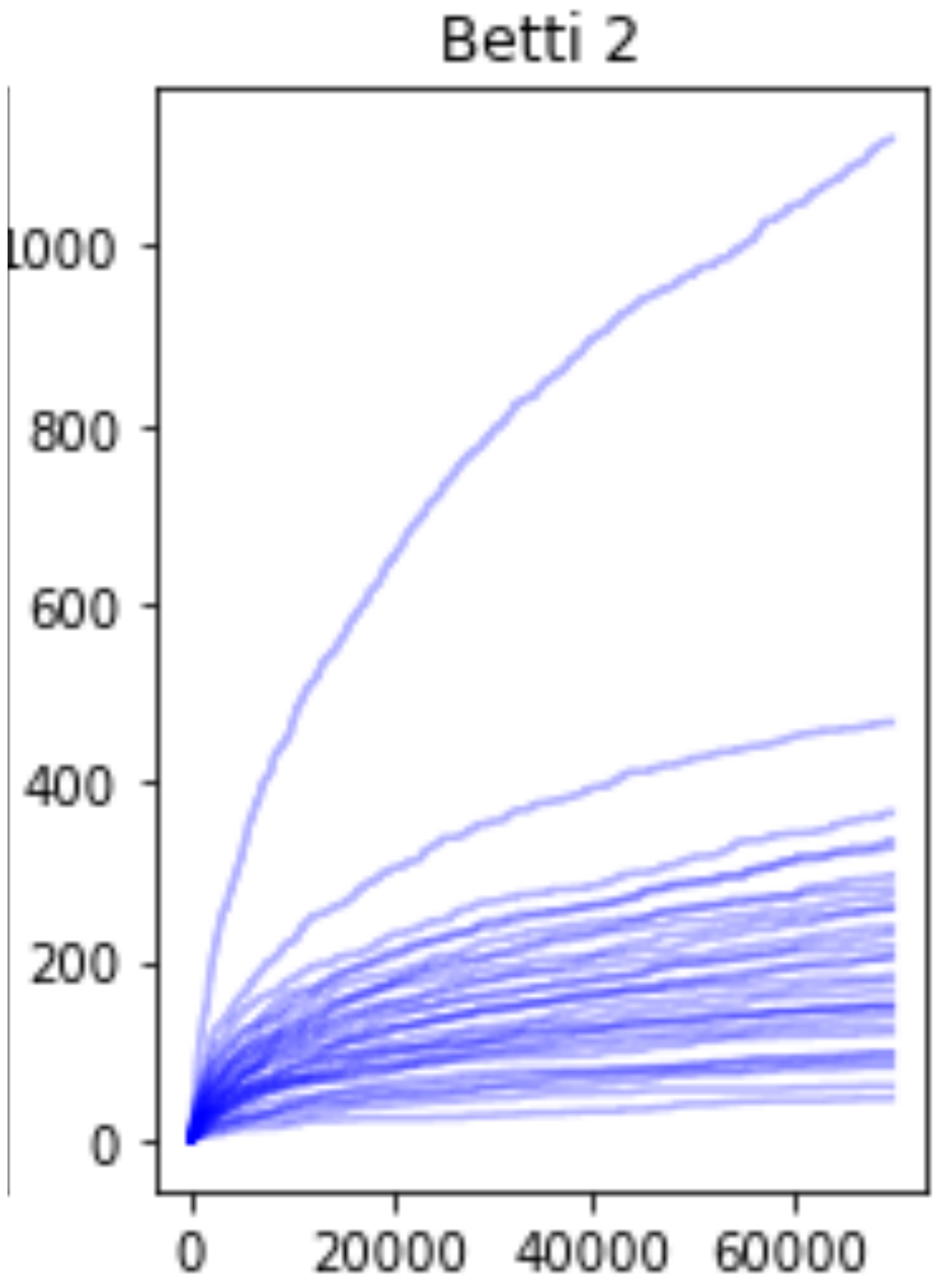
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth



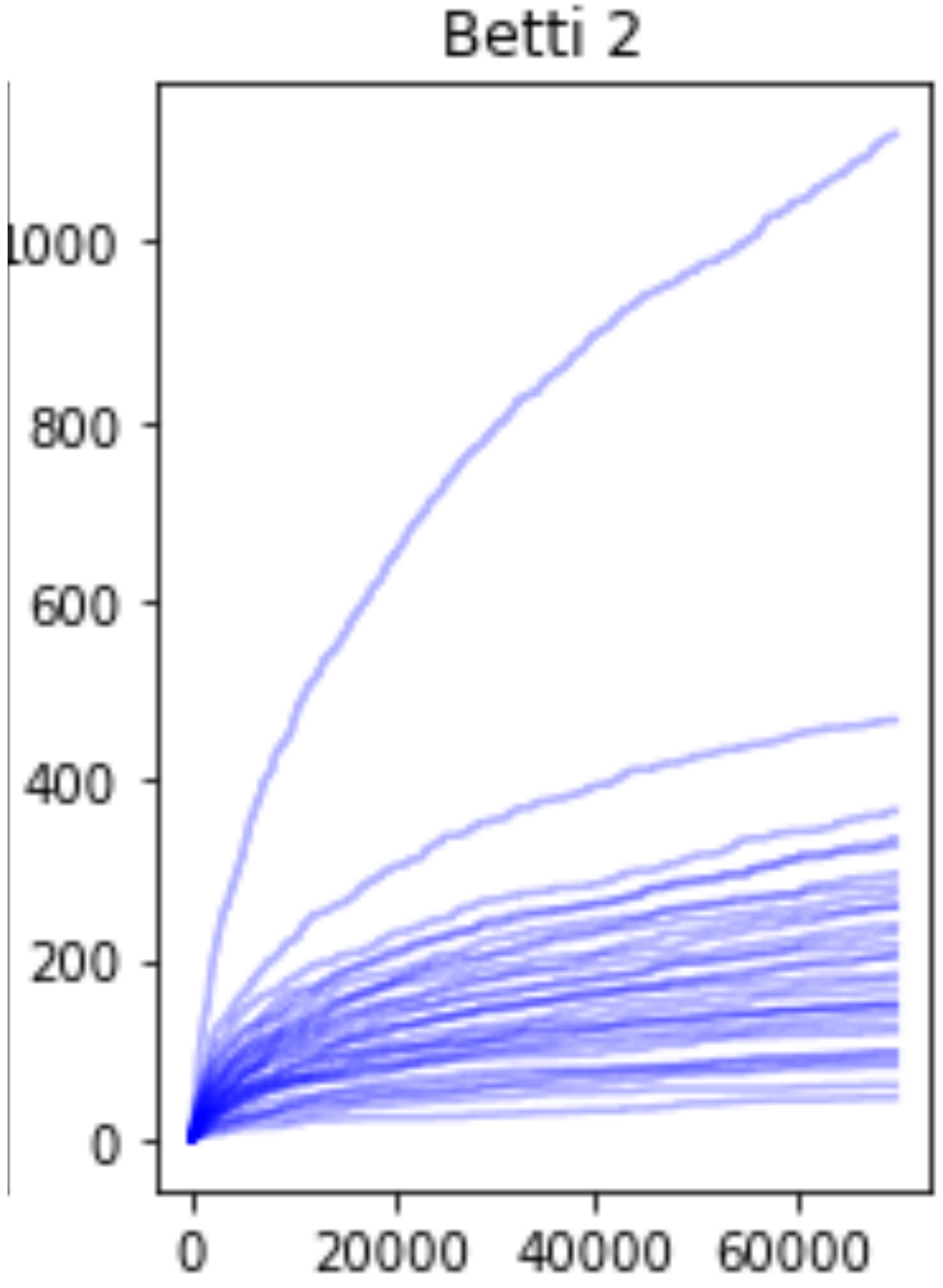
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth
- outlier



Expected Betti Number $E[\beta_q]$

- $c(\text{num of nodes}^{1-4x}) \leq E[\beta_2] \leq C(\text{num of nodes}^{1-4x})$
  under mild assumptions
  - $x \in (0,1/2)$ depends on model parameters



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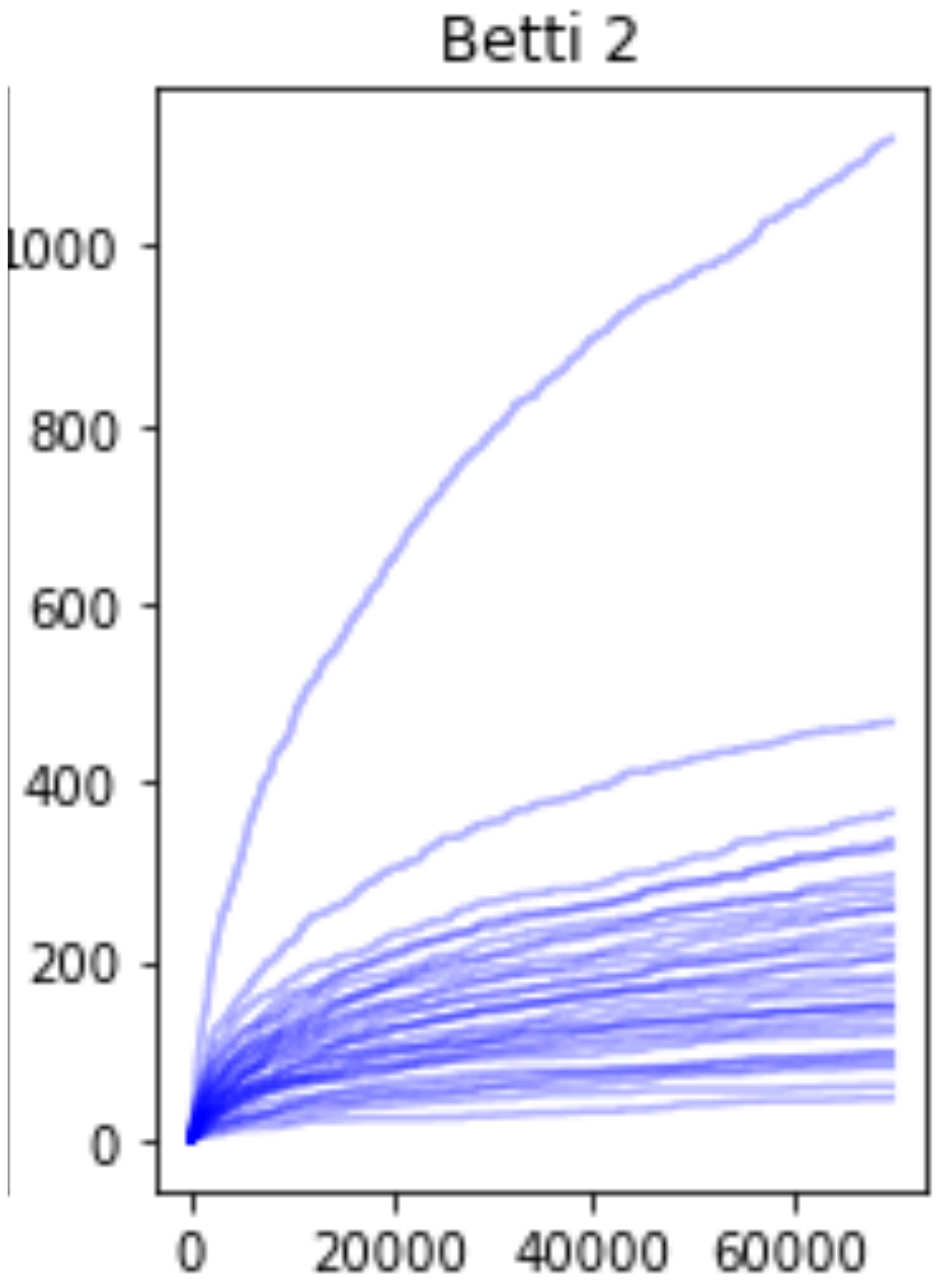

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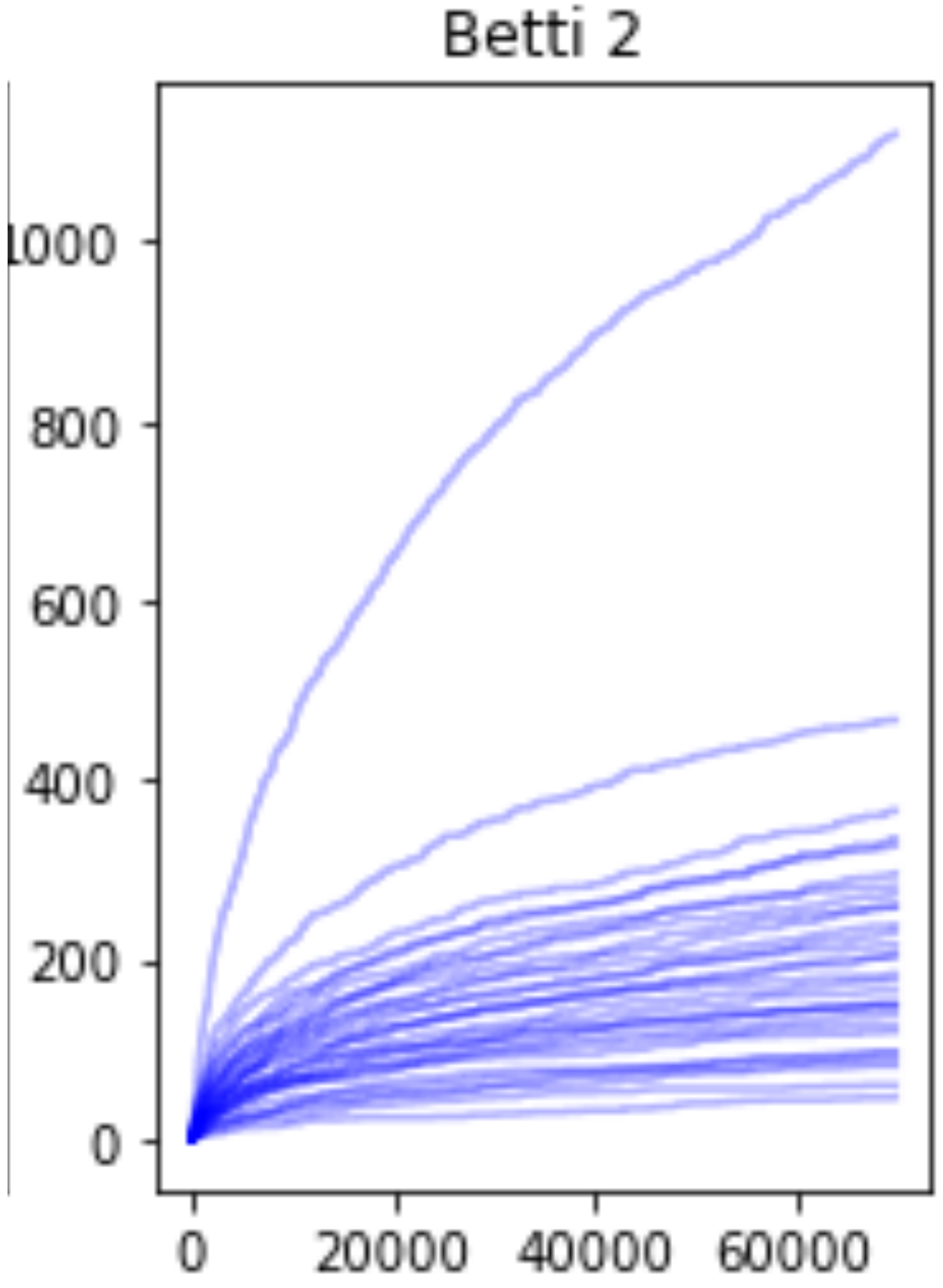
- c(num of nodes^{1-2qx}) $\leq E[\beta_q] \leq C$(num of nodes^{1-2qx}) for $q \geq 2$ if $1 - 2qx > 0$

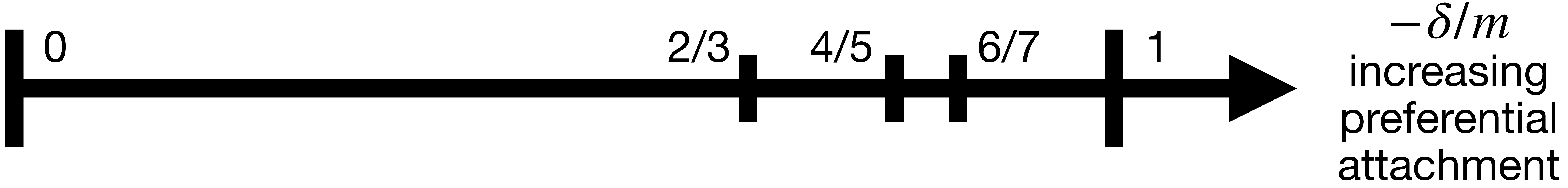


Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



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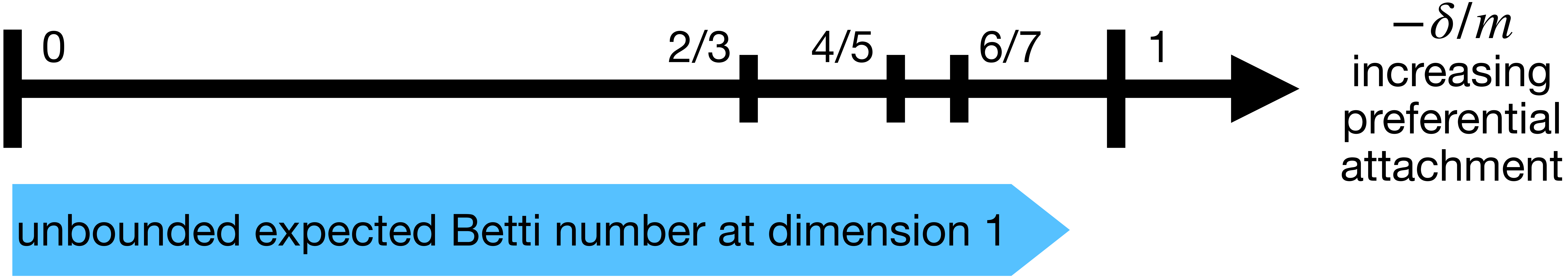

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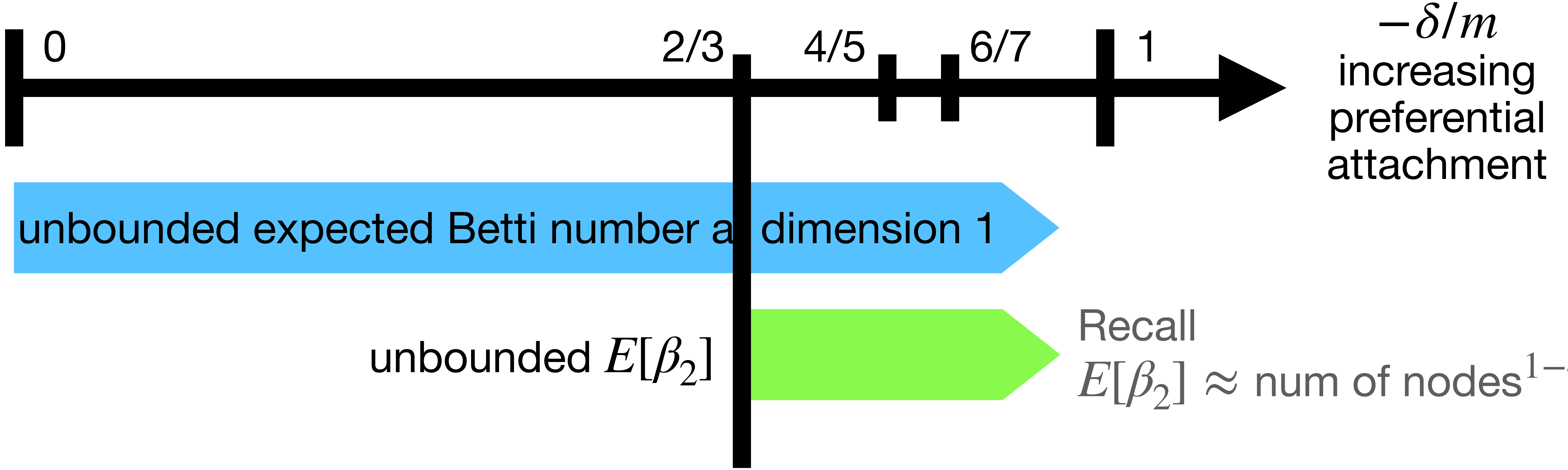

Recall

$E[\beta_2] \approx$ num of nodes^{1-4x}

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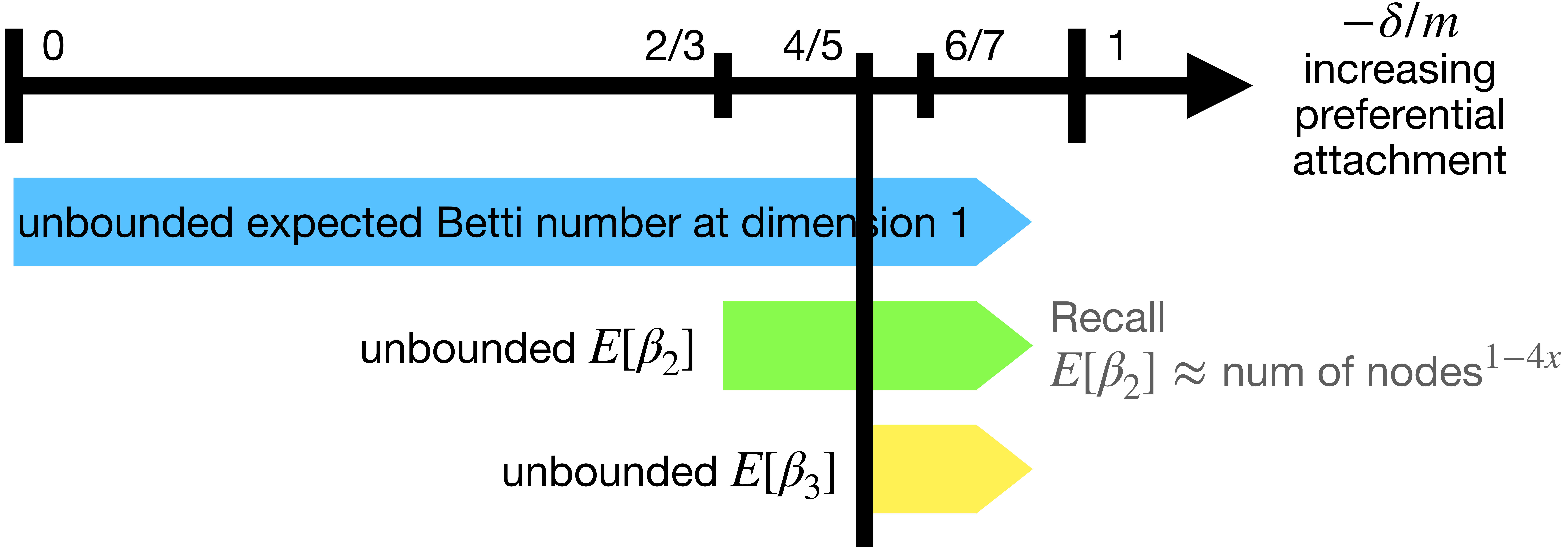

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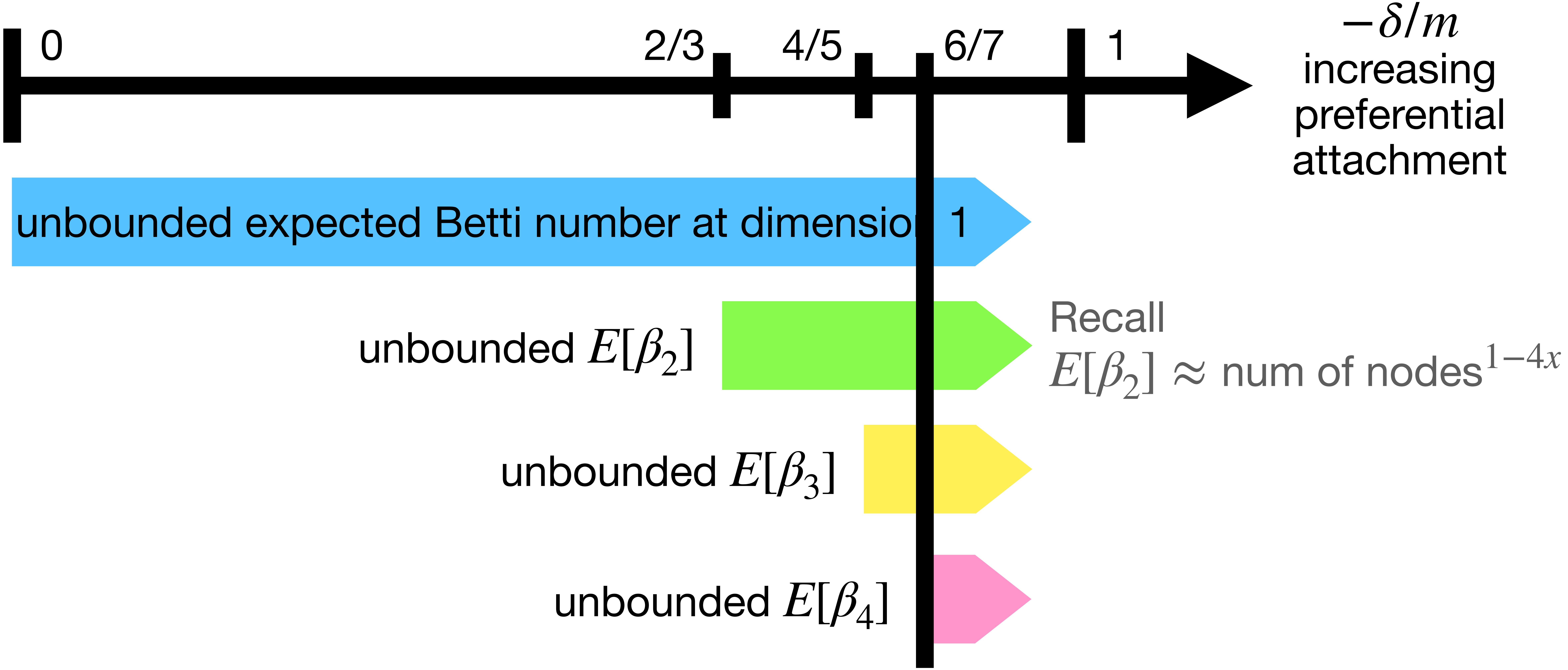

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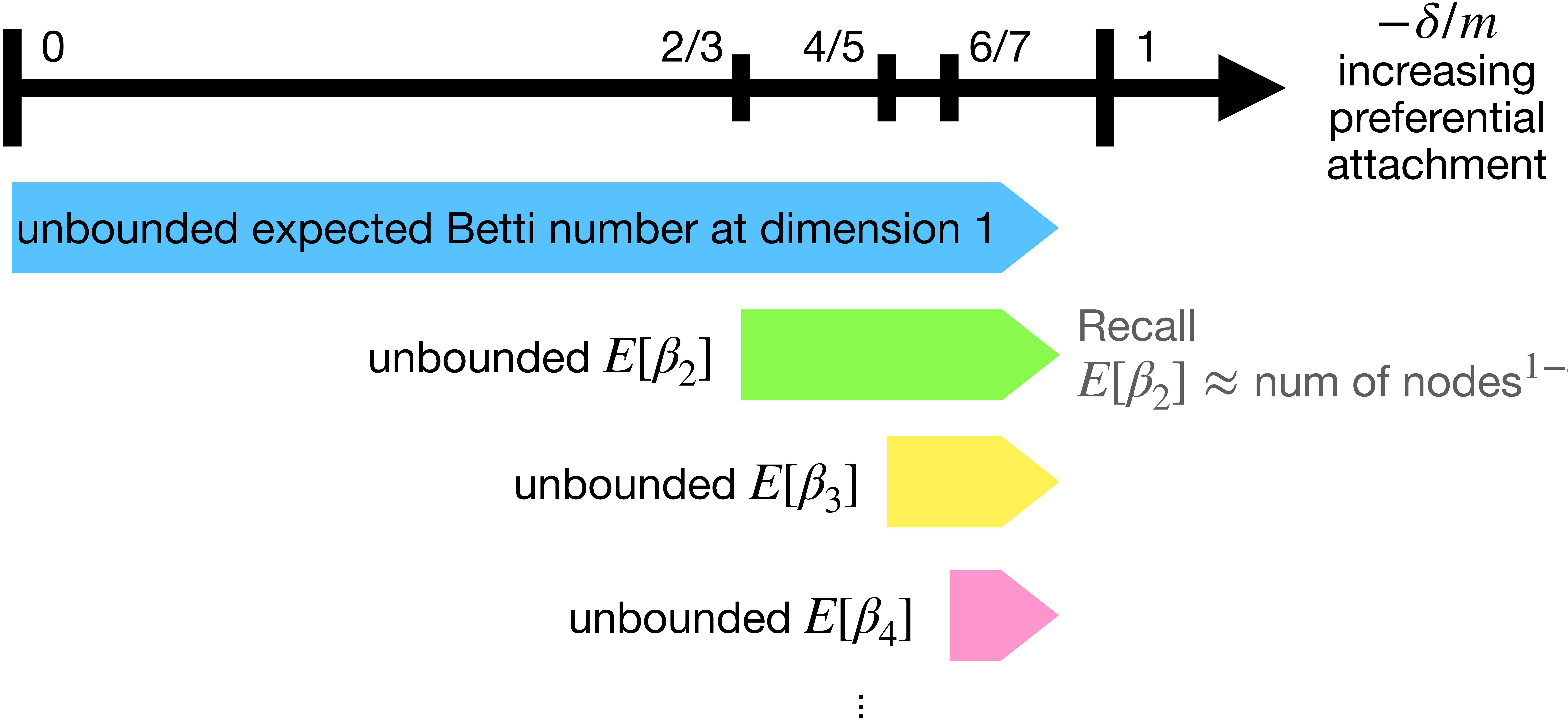
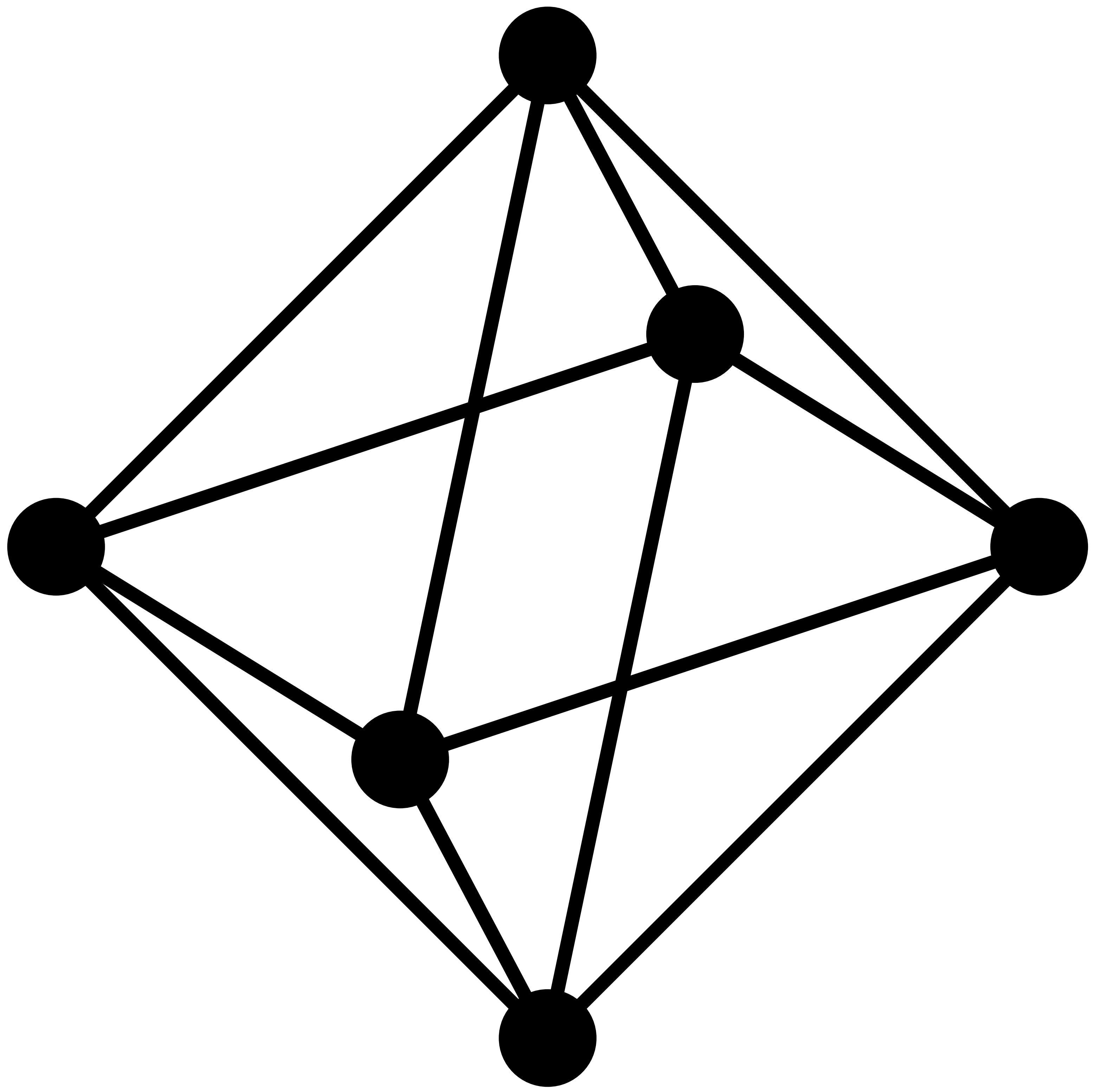


Recall
$E[\beta_2] \approx$ num of nodes^{1-4x}

Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

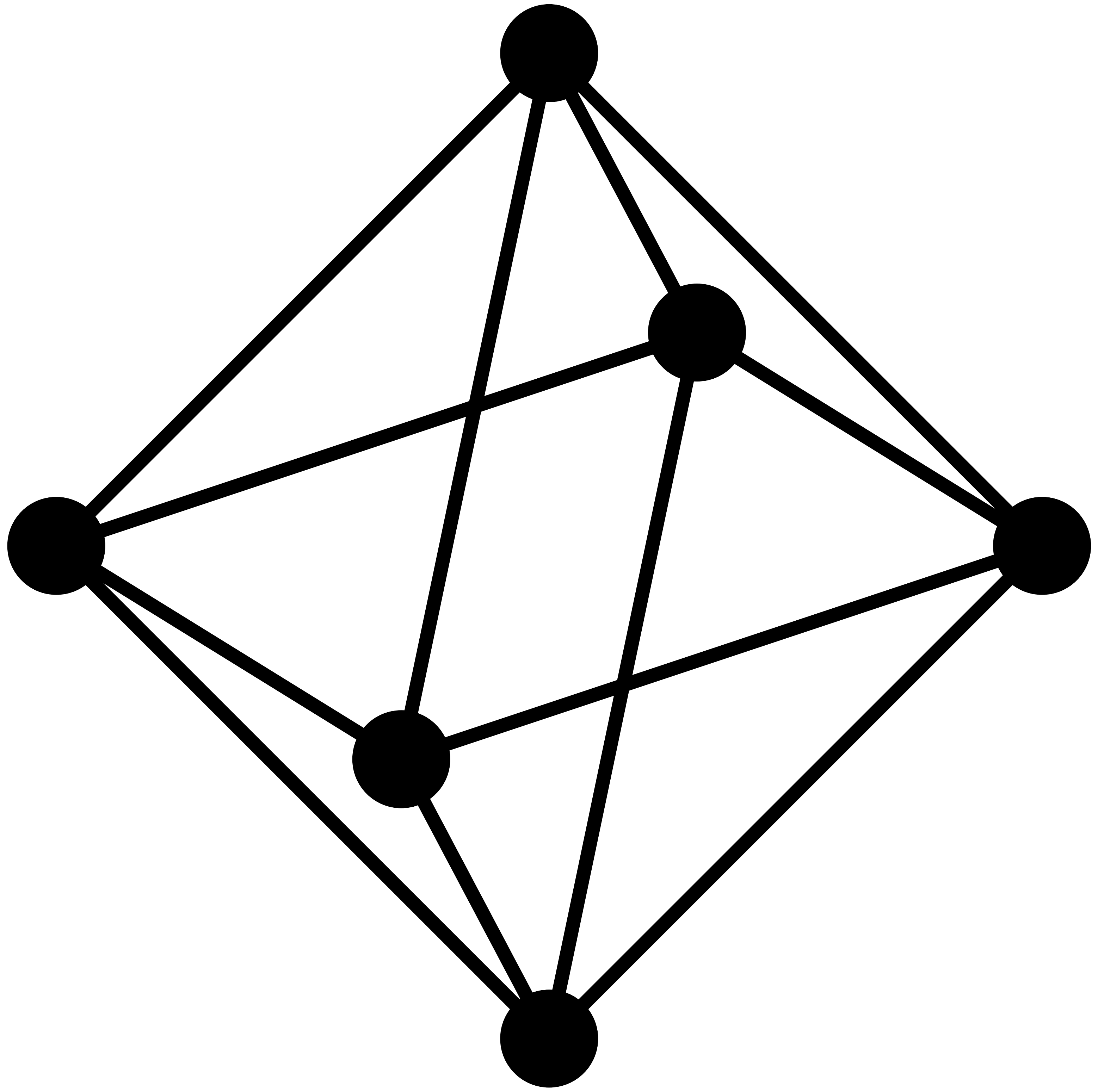
Proof?

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



$\beta_2 = 1$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



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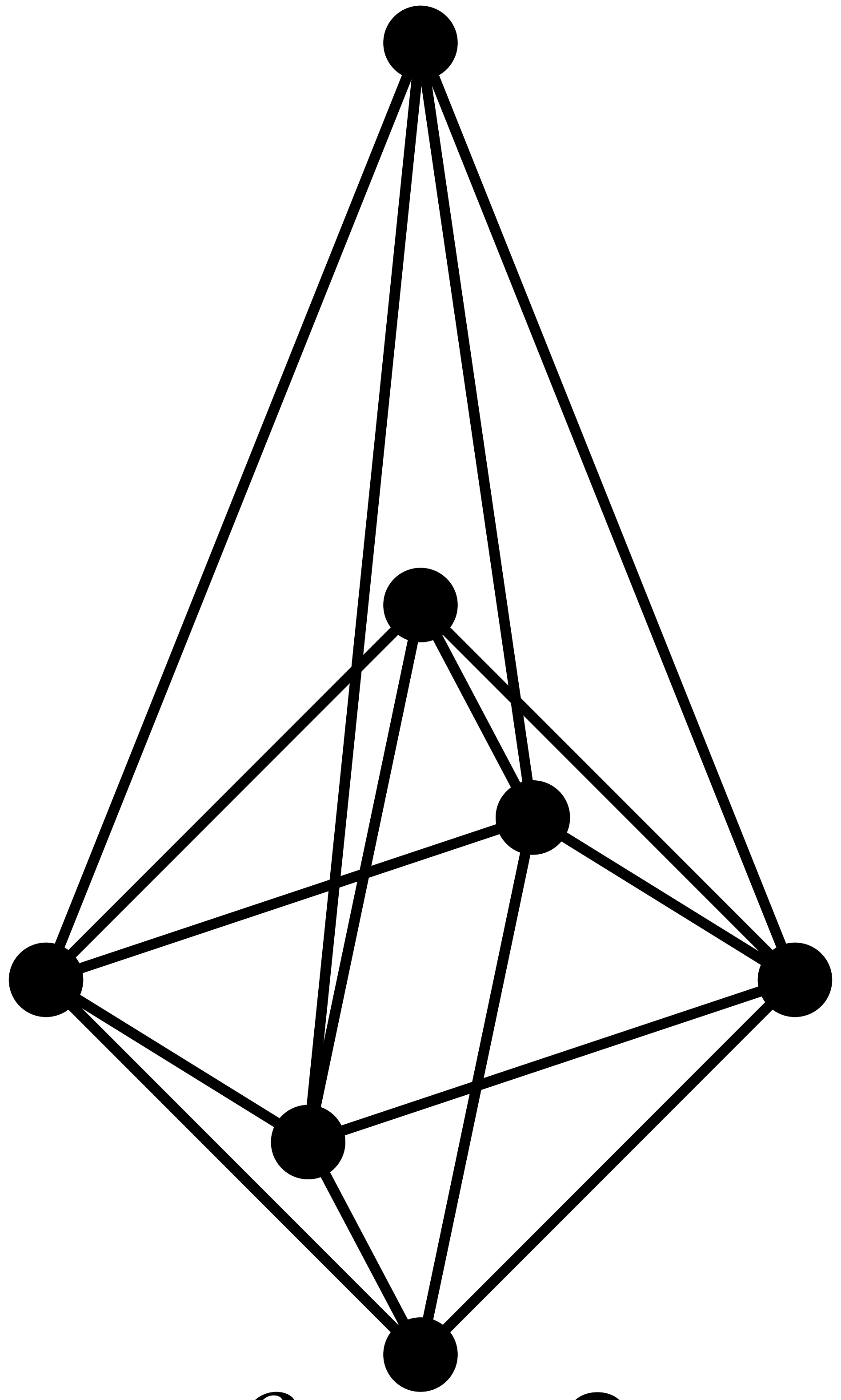
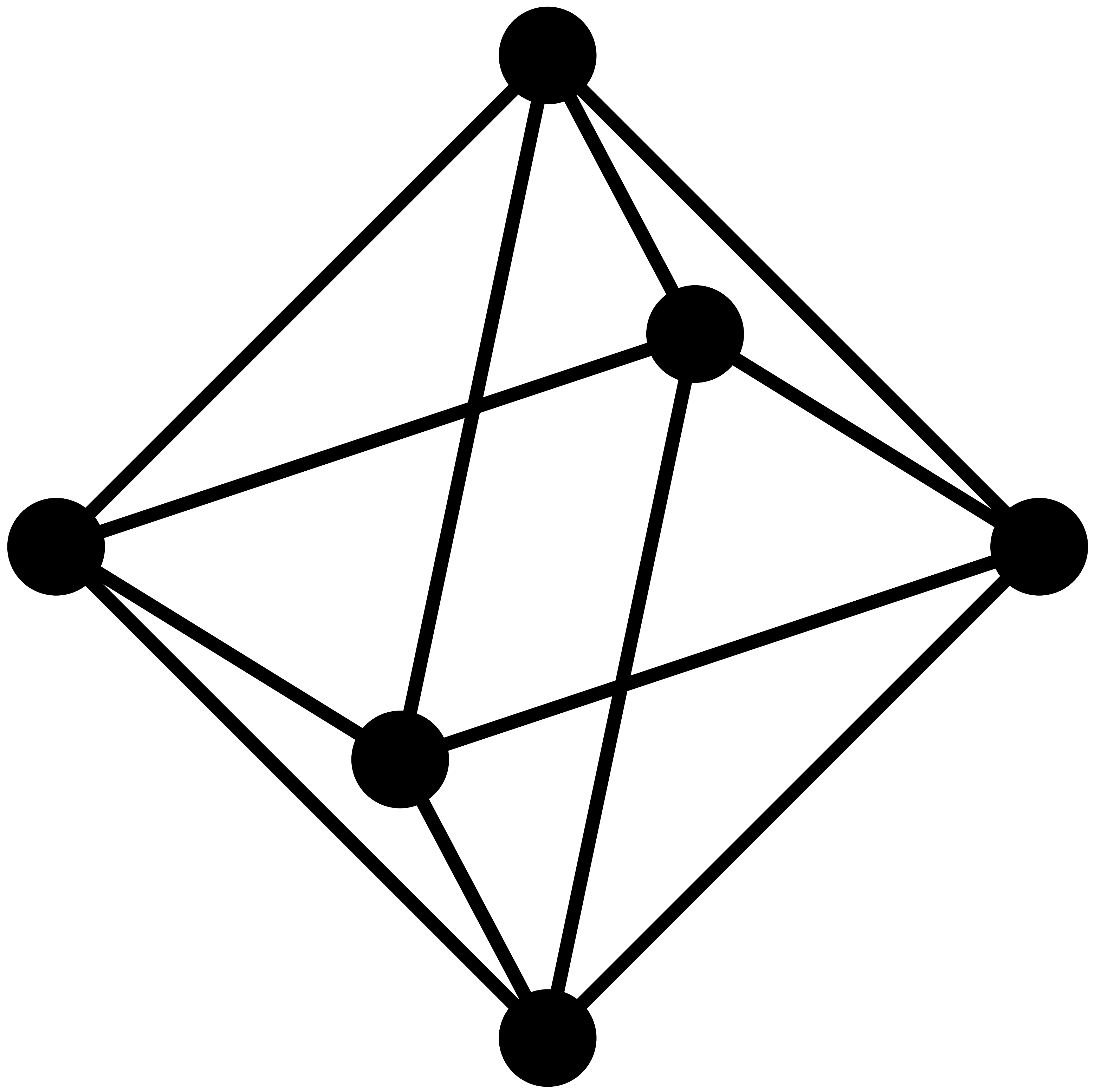


$\beta_2 = 2$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



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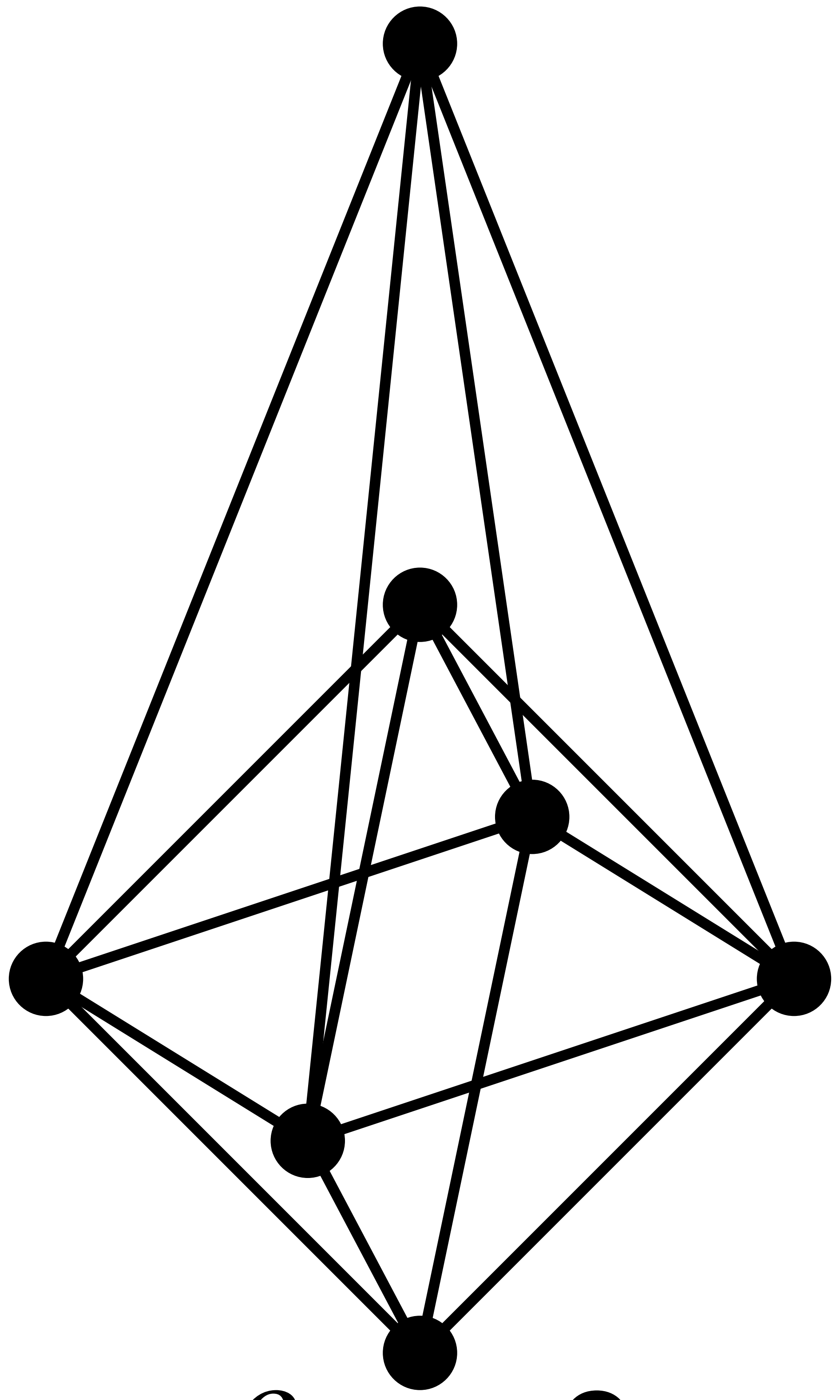

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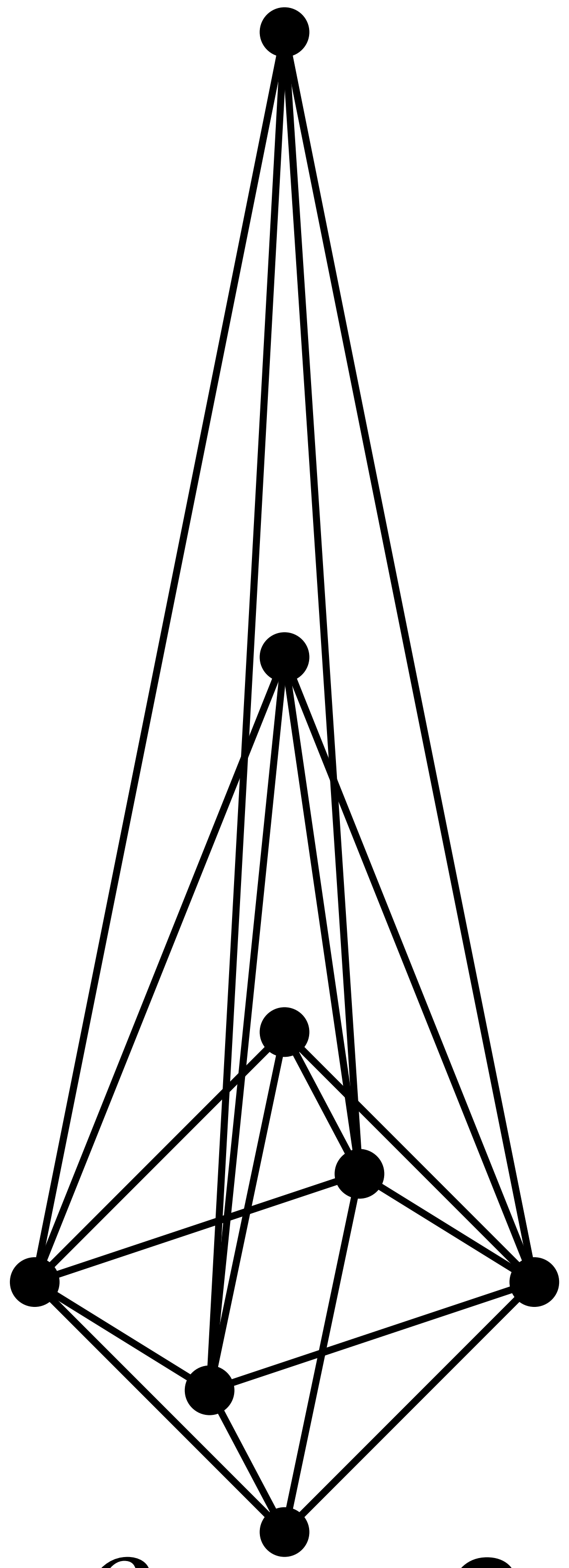

$\beta_2 = 3$

Subtleties

- Need homological algebra to relate Betti numbers with counts

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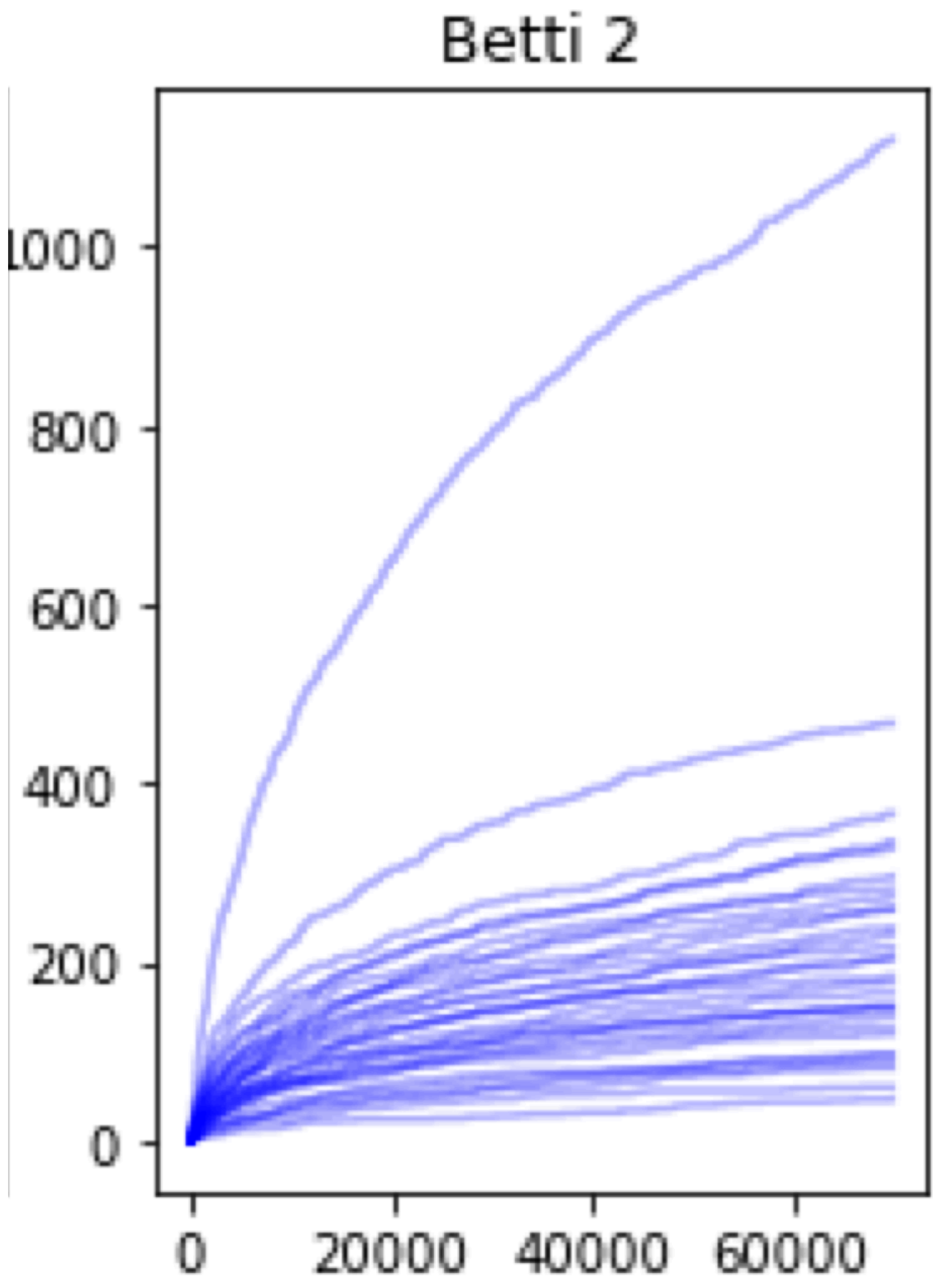
- Need homological algebra to relate Betti numbers with counts
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results in the language of homological algebra
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs

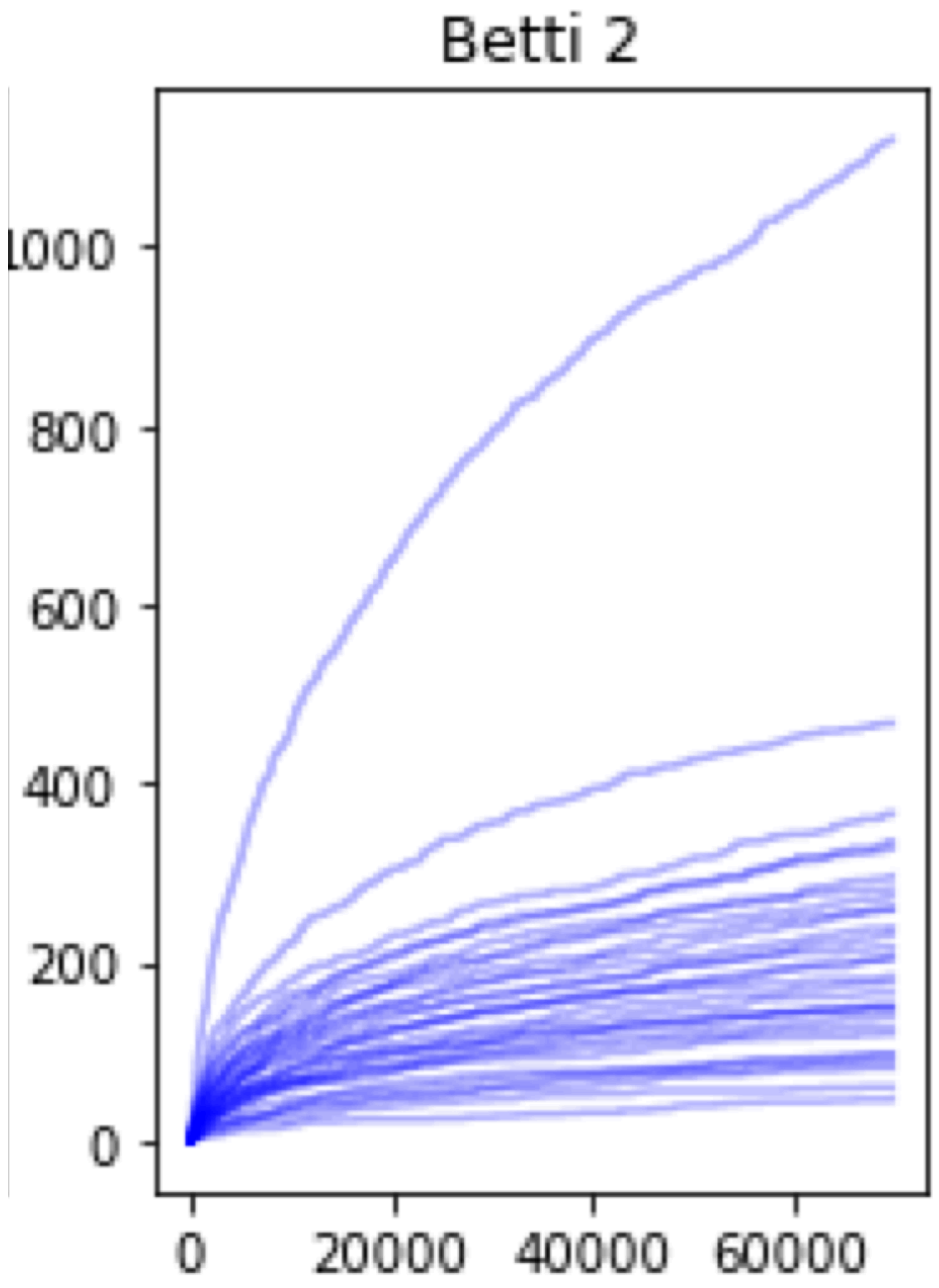
Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

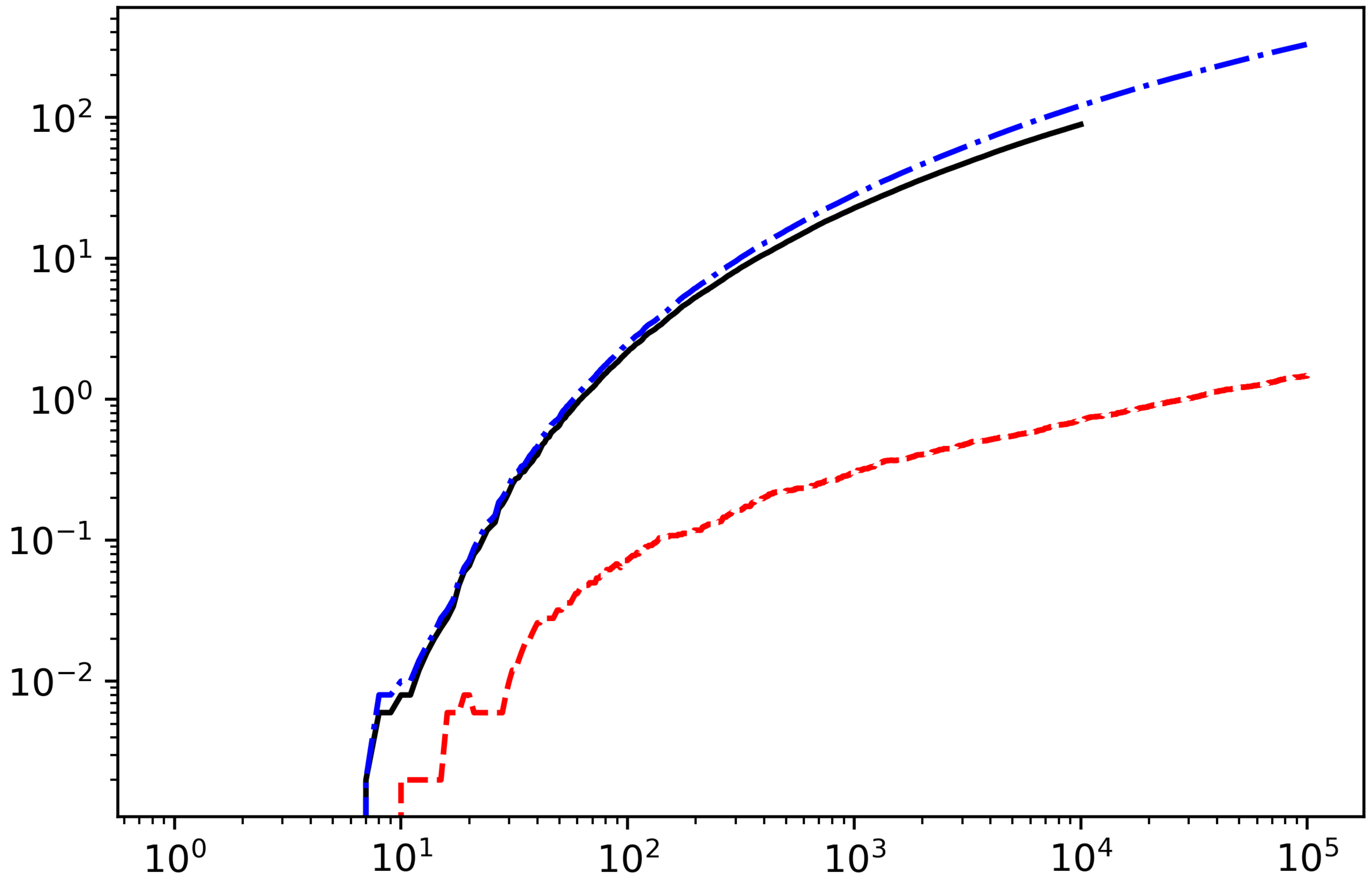
In practice???

$E[\beta_2] \approx$ **num of nodes**$^{1-4x}$

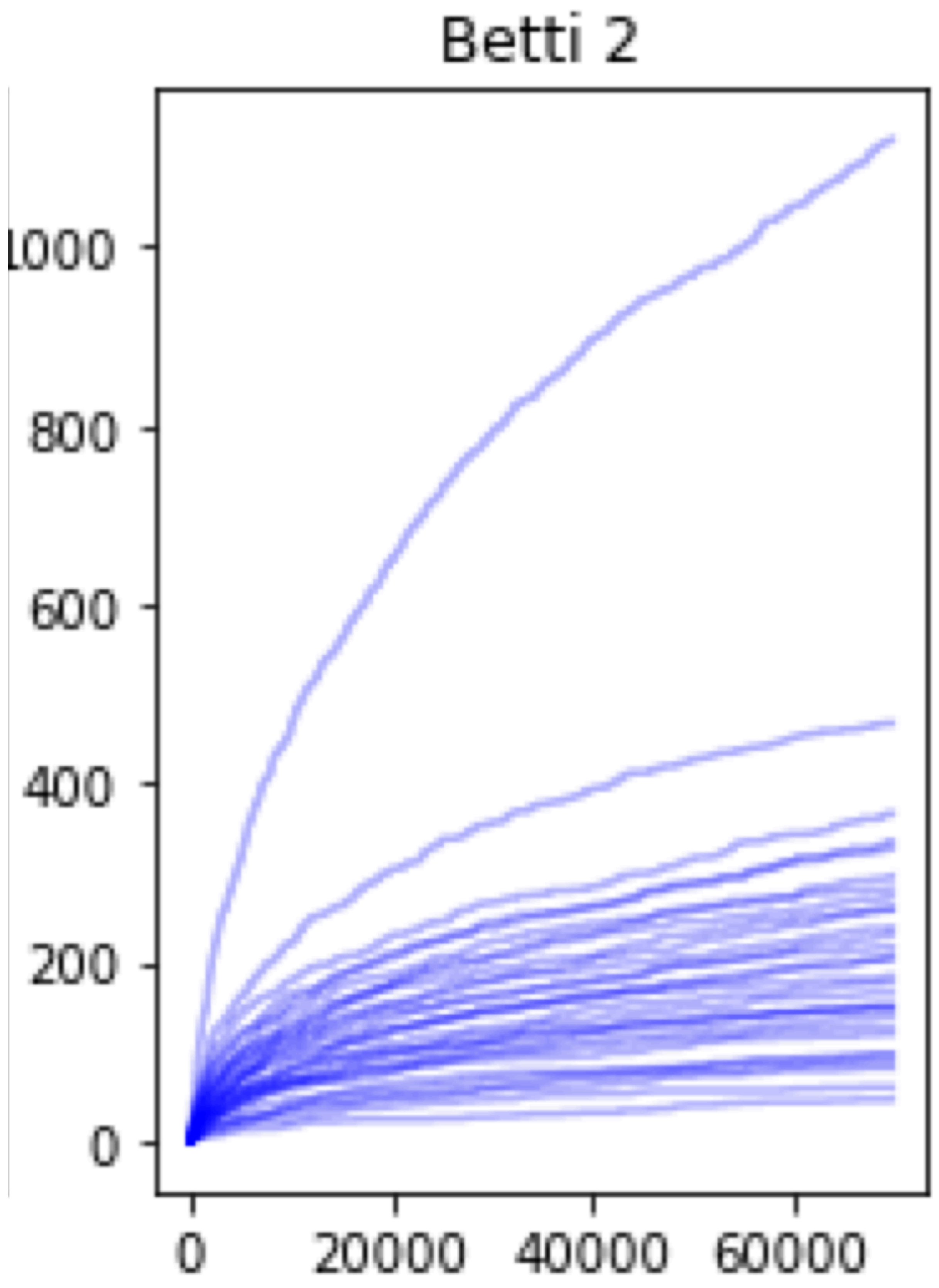


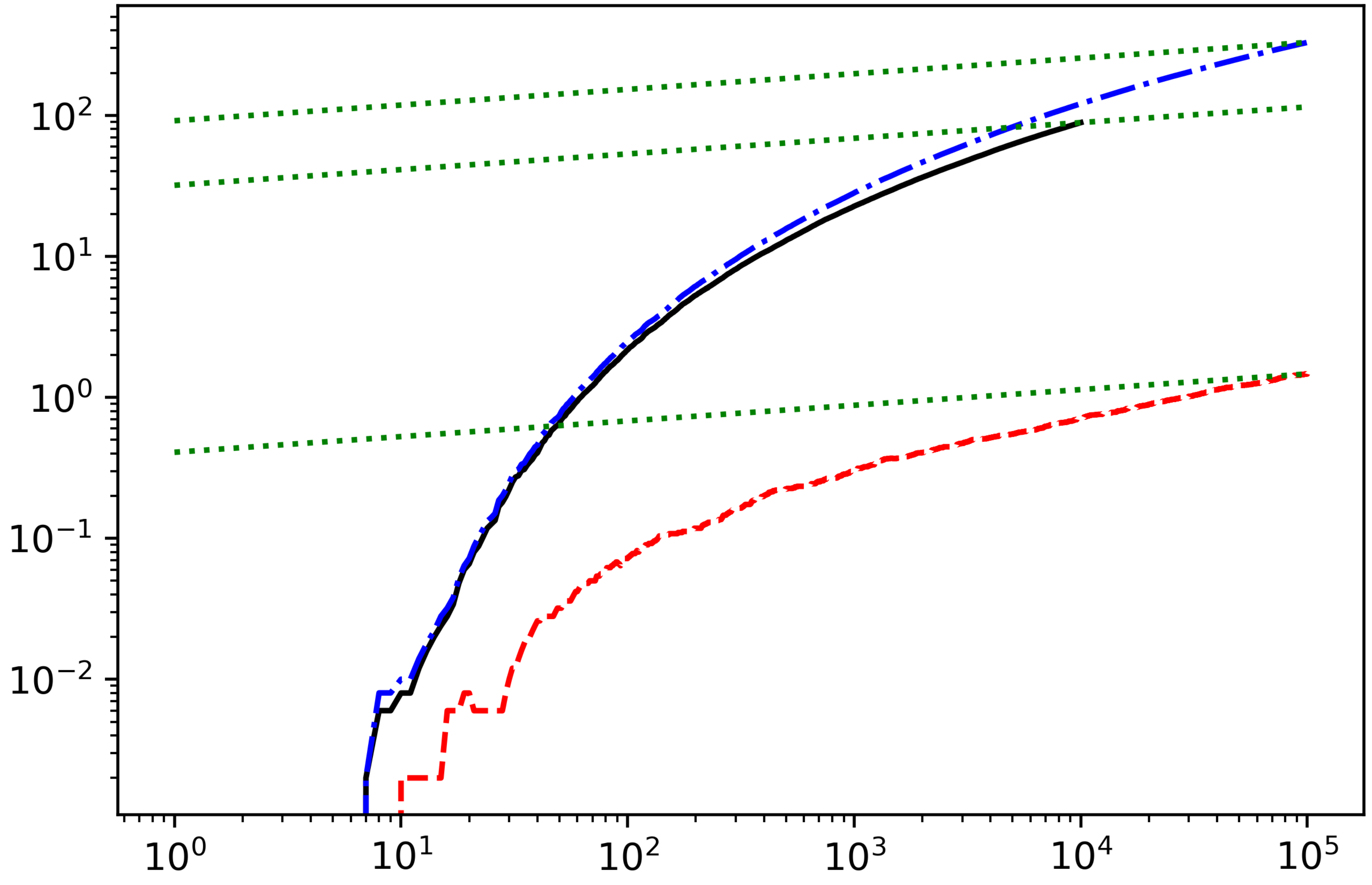
$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$





$E[\beta_2] \approx$ **num of nodes**$^{1-4x}$





IV. What lies ahead

order of magnitude of
expected Betti numbers

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

parameter estimation?

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order of magnitude of expected Betti numbers

simplicial preferential attachment?

parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

simplicial preferential attachment?

other non-homogeneous complexes?

What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

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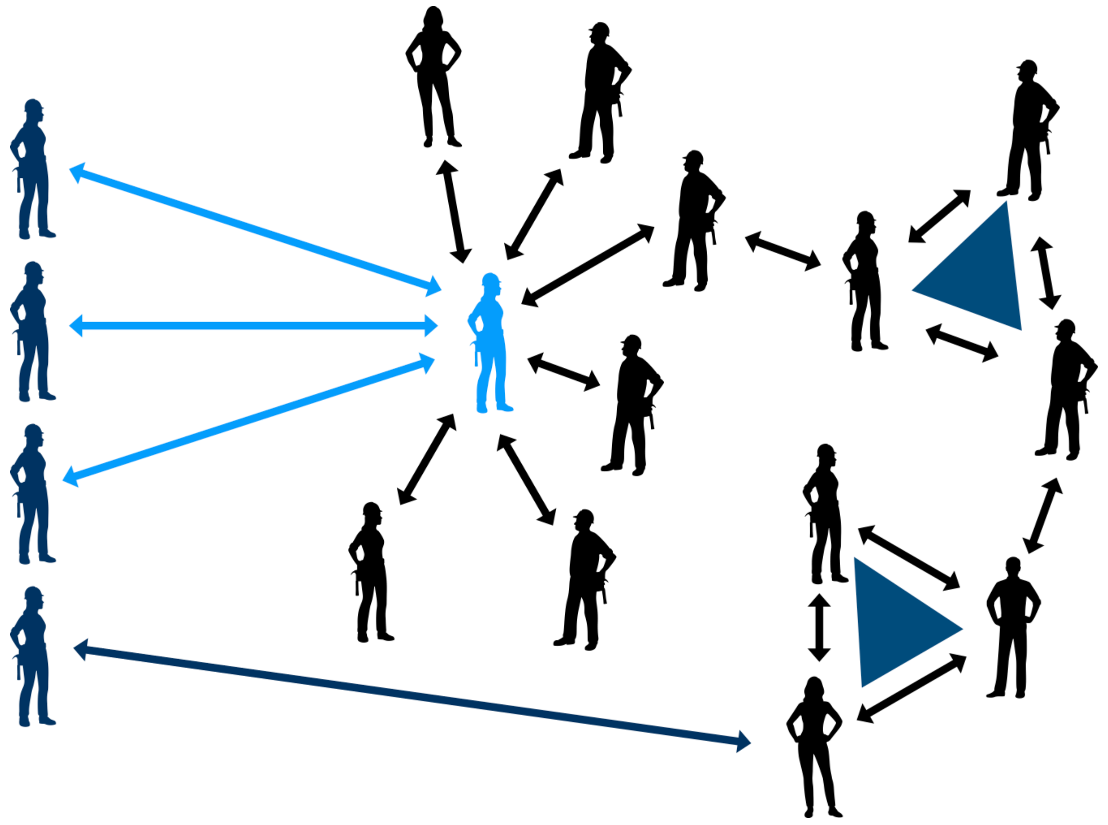
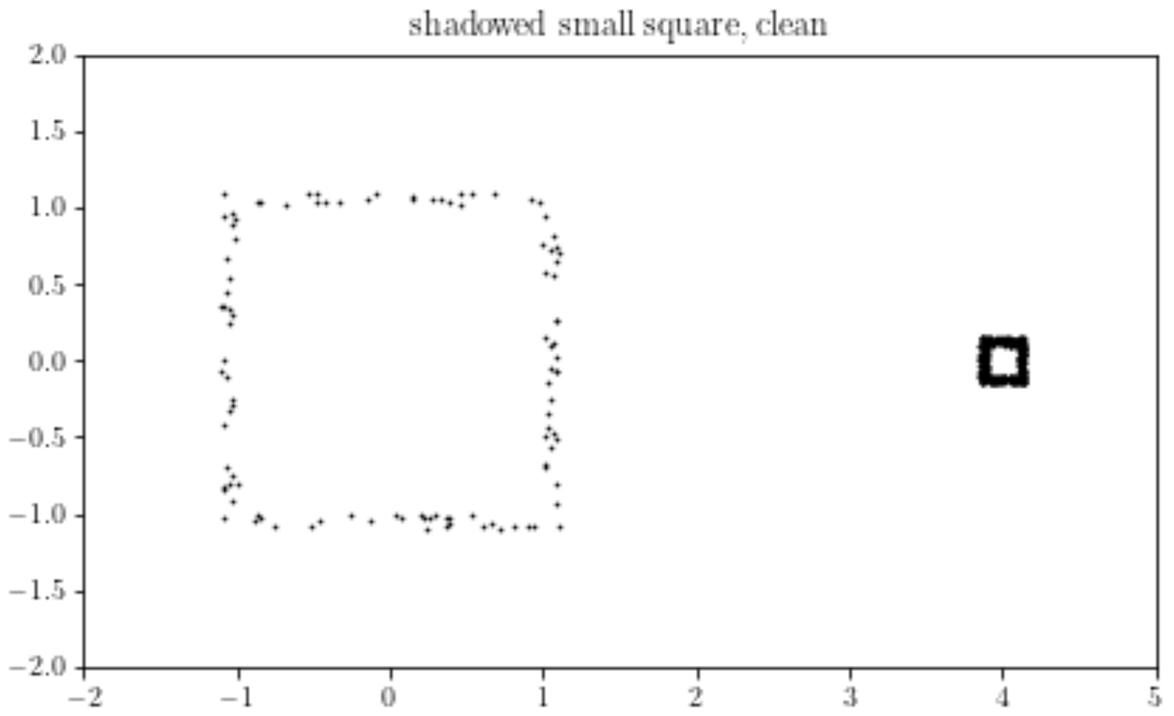
Cornell University



arxiv paper



my video about small holes

Gracias!

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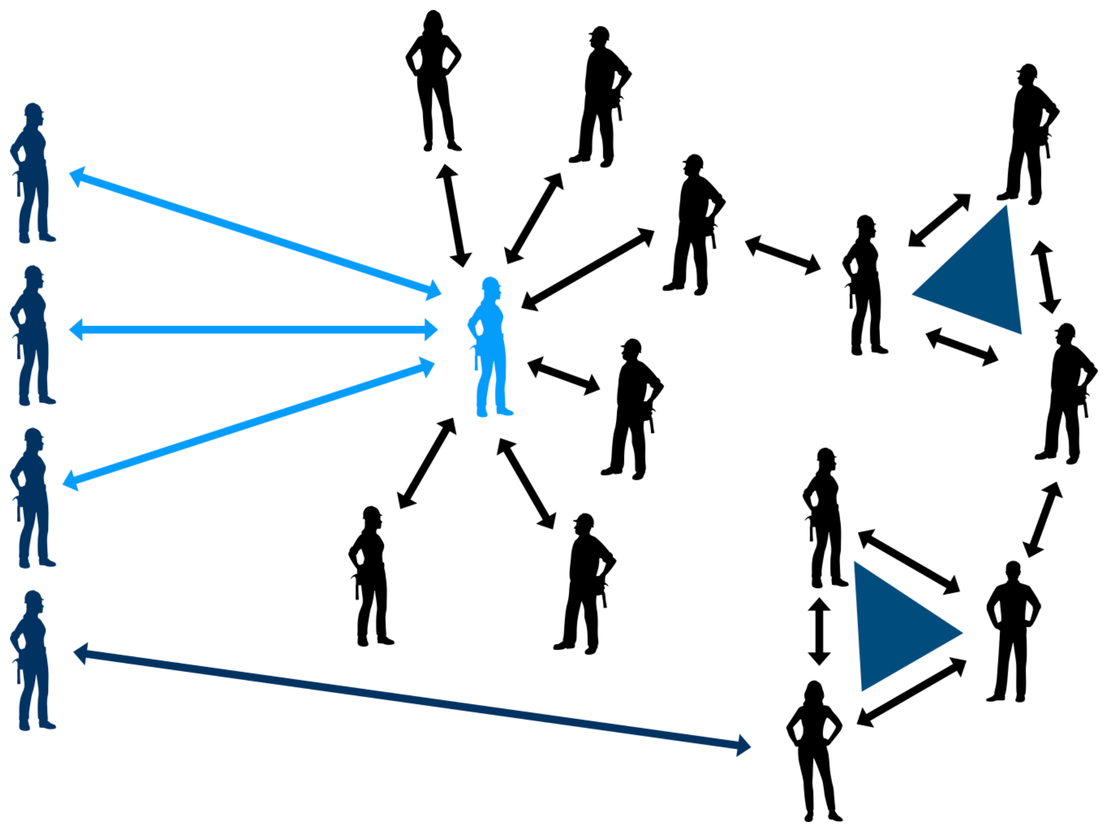

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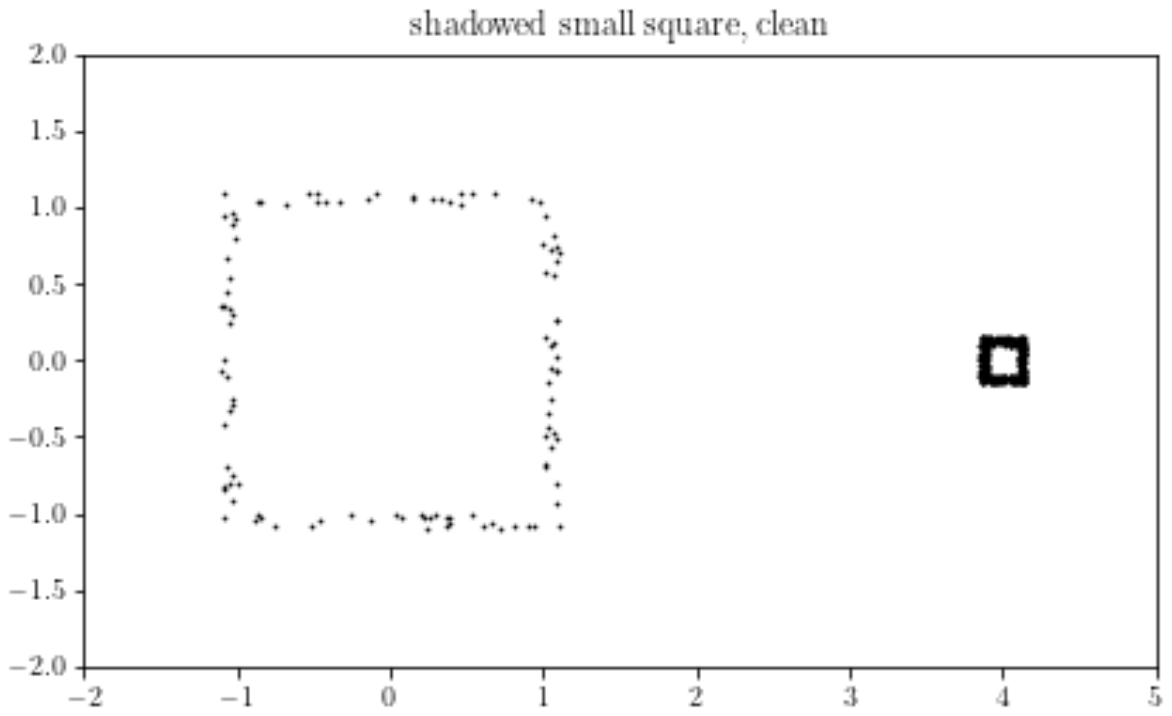

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