The Topology of Preferential Attachment

Higher-Order Connectivity of Random Interactions

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Probabilists

Statisticians

Network Scientists

Topologist

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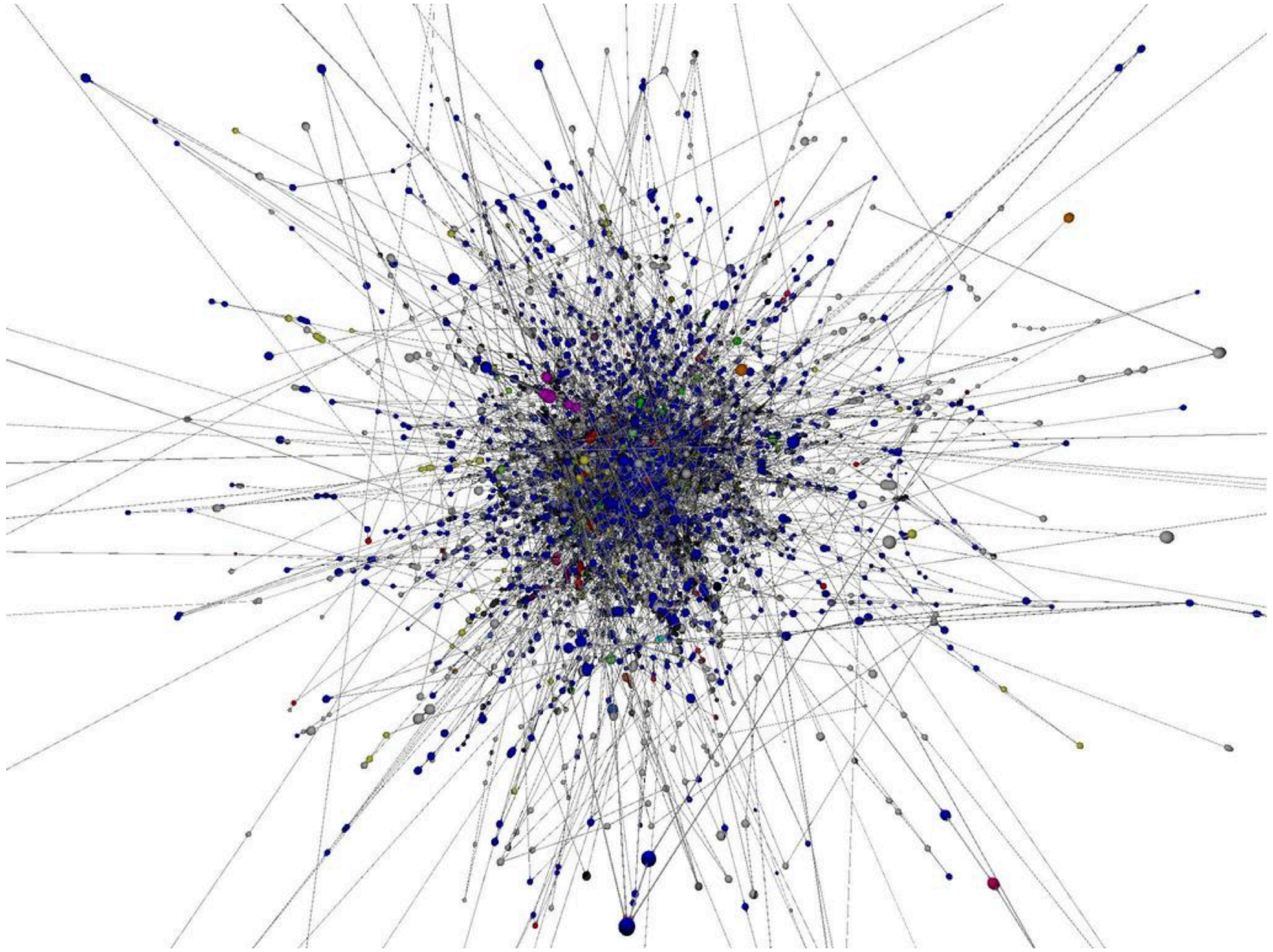
So, preferential attachment…

- Highly connected hubs



(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Highly connected hubs
- Dense core of hubs?

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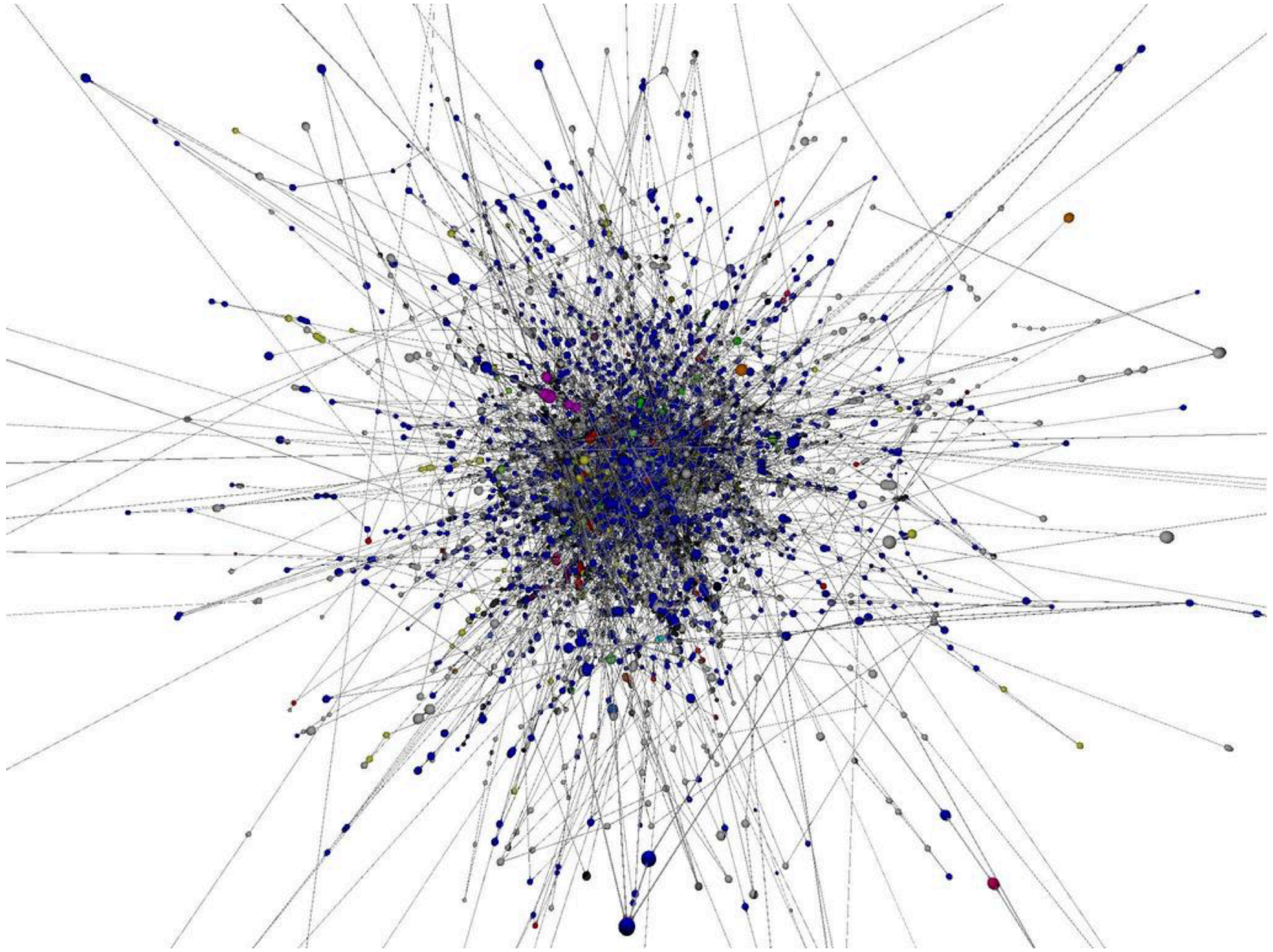
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Highly connected hubs
- Dense core of hubs?
- Beyond pairwise connections?

- —> topological properties



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Agenda

preferential attachment

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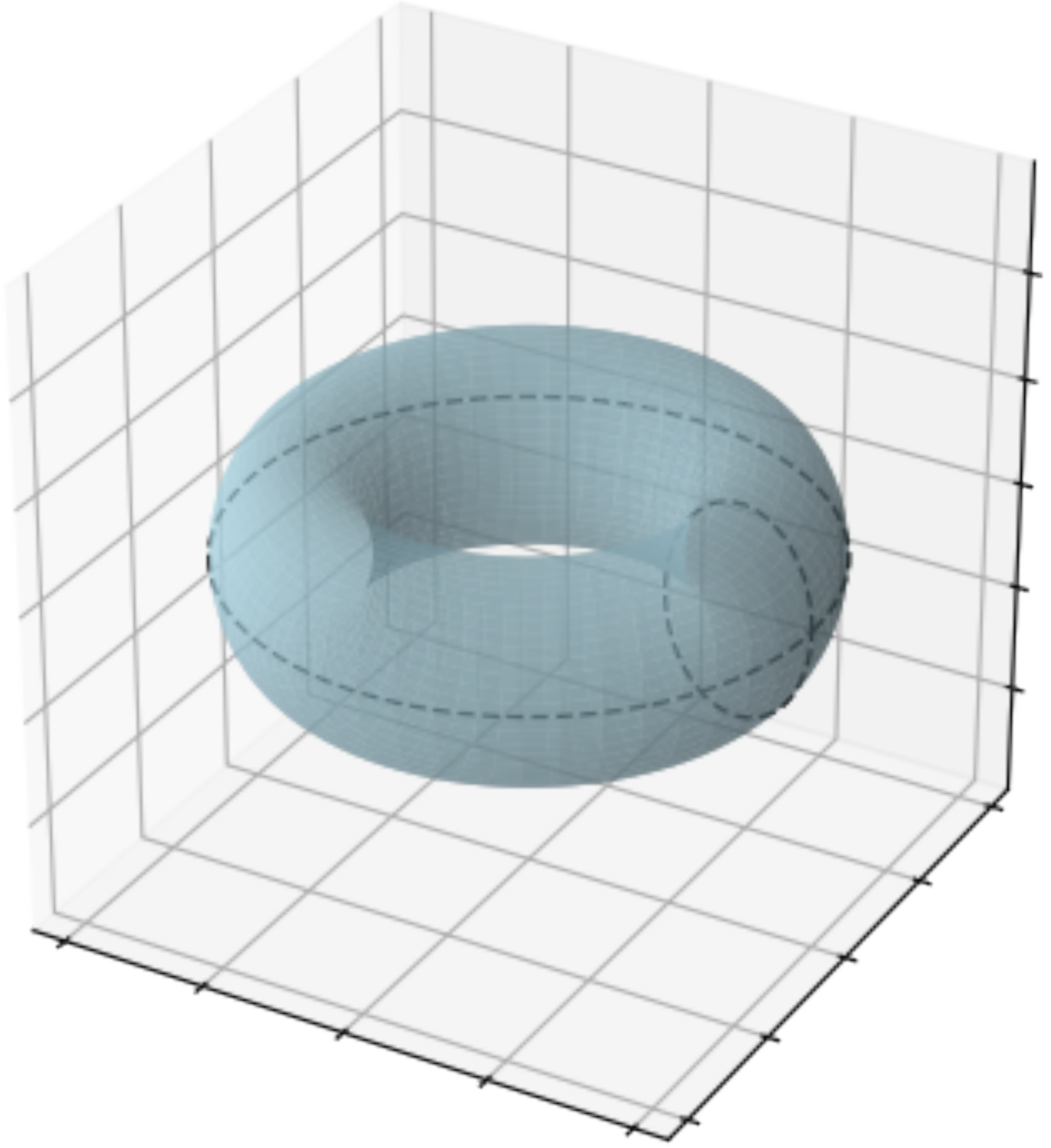
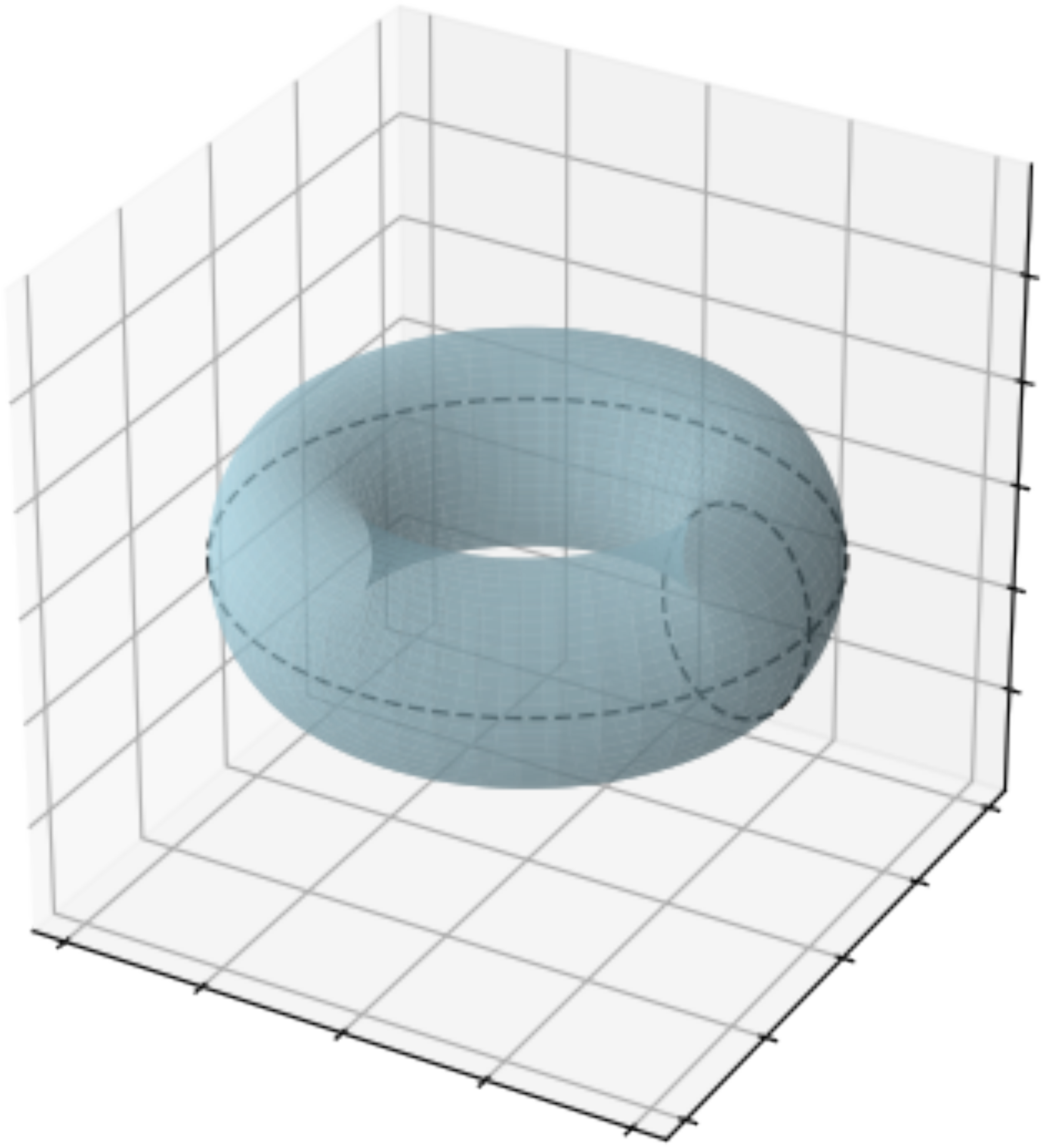
preferential attachment



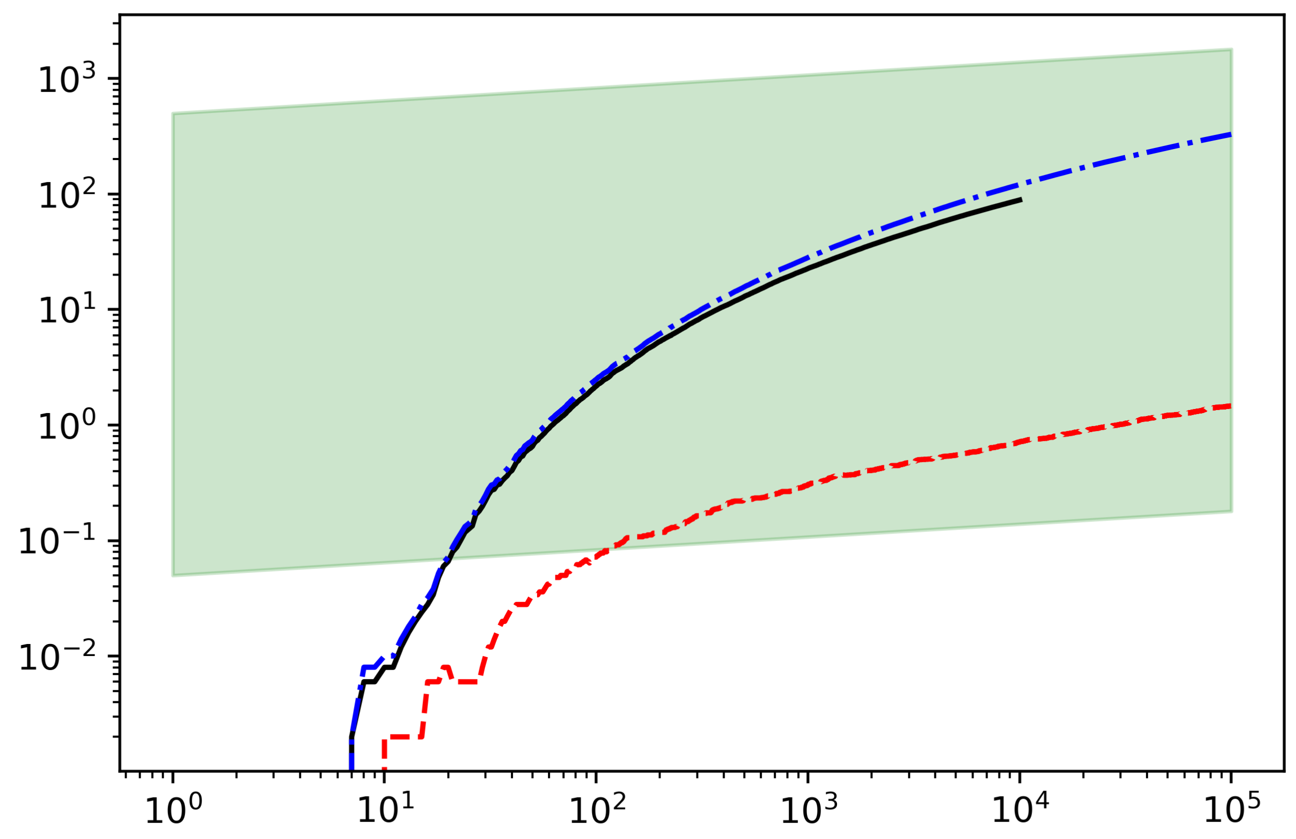
topology

Agenda

preferential attachment



topology

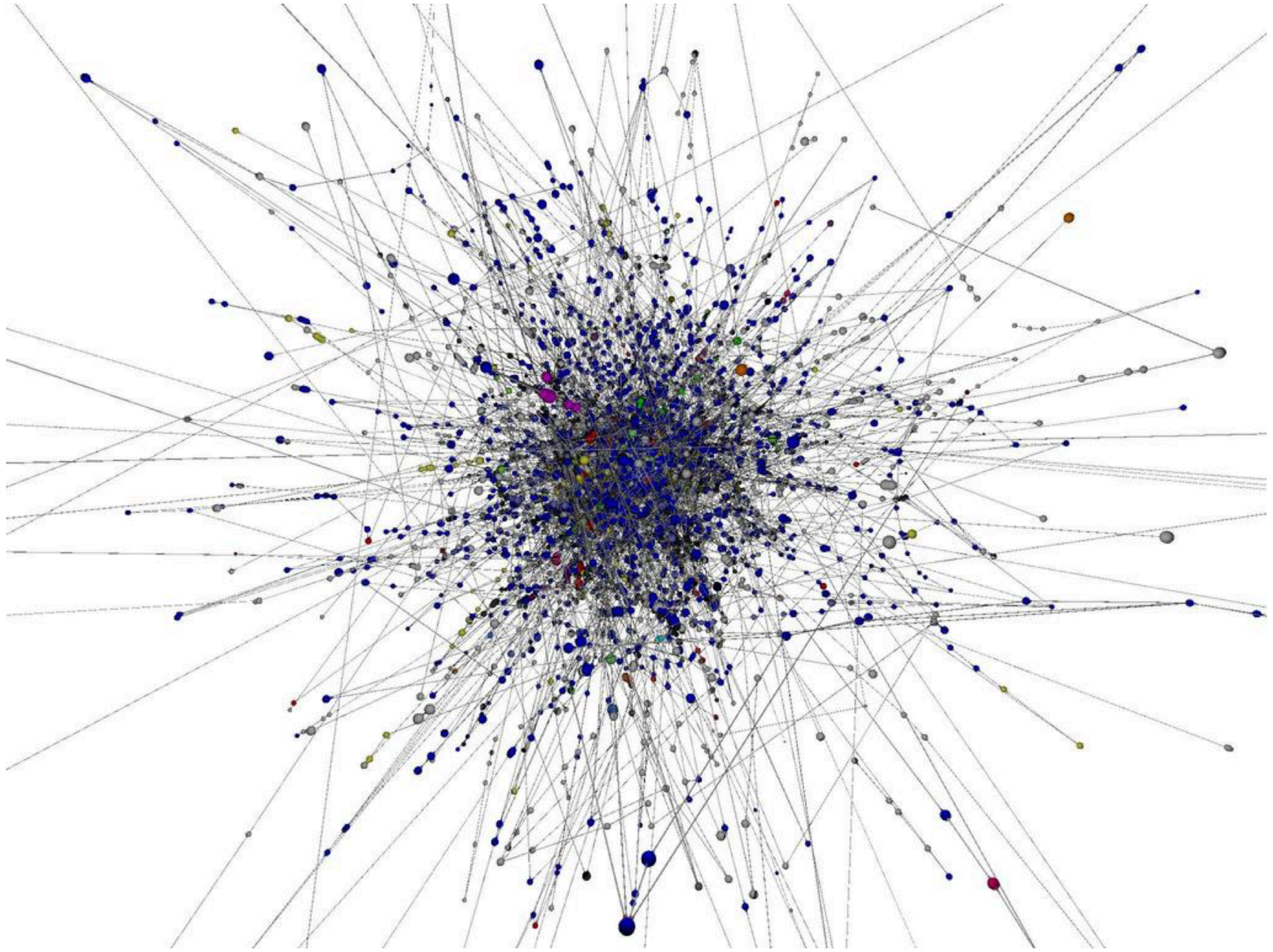


our result

I. Preferential Attachment

Preferential Attachment

[Albert and Barabasi 1999]



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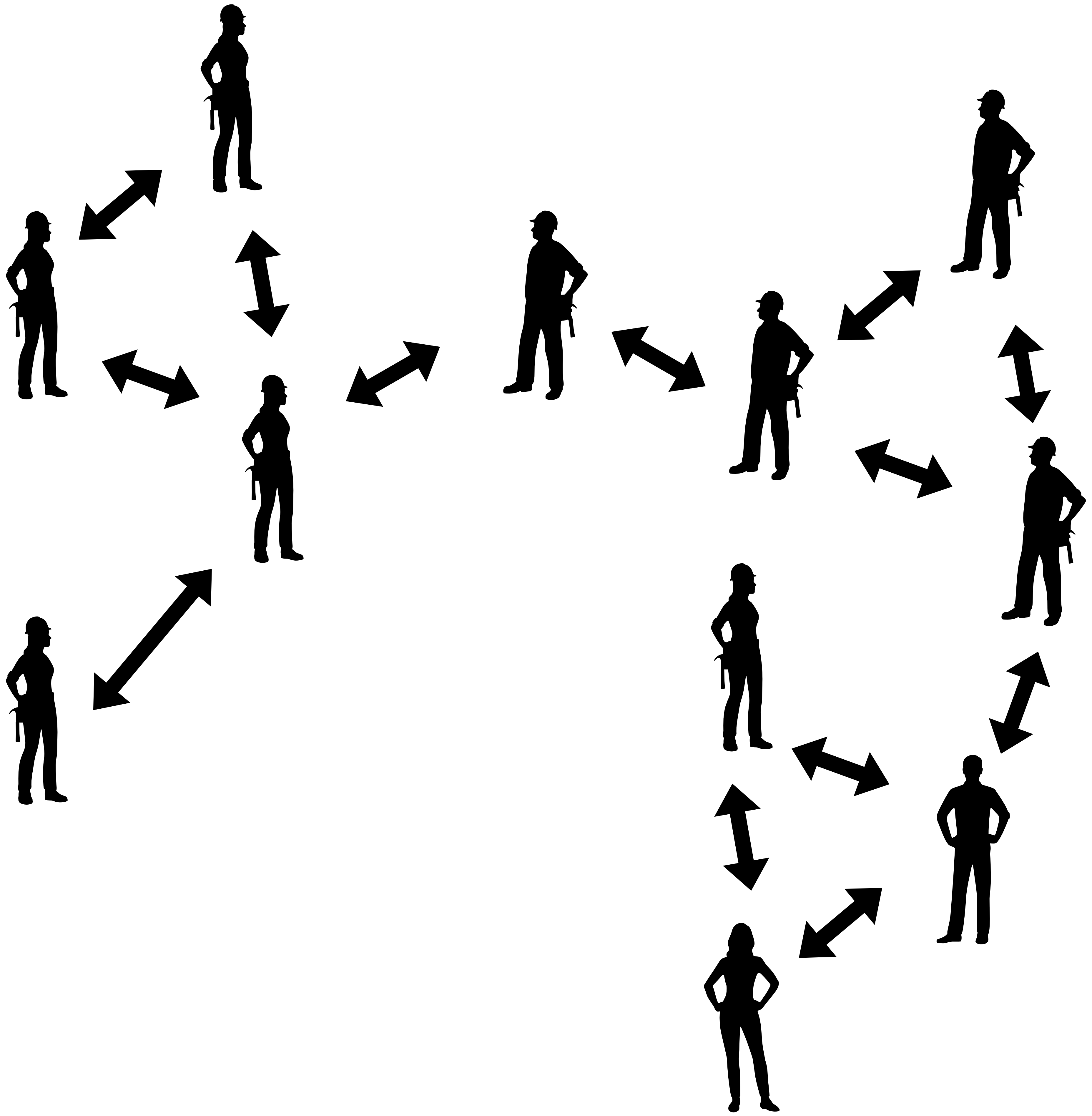

Preferential Attachment
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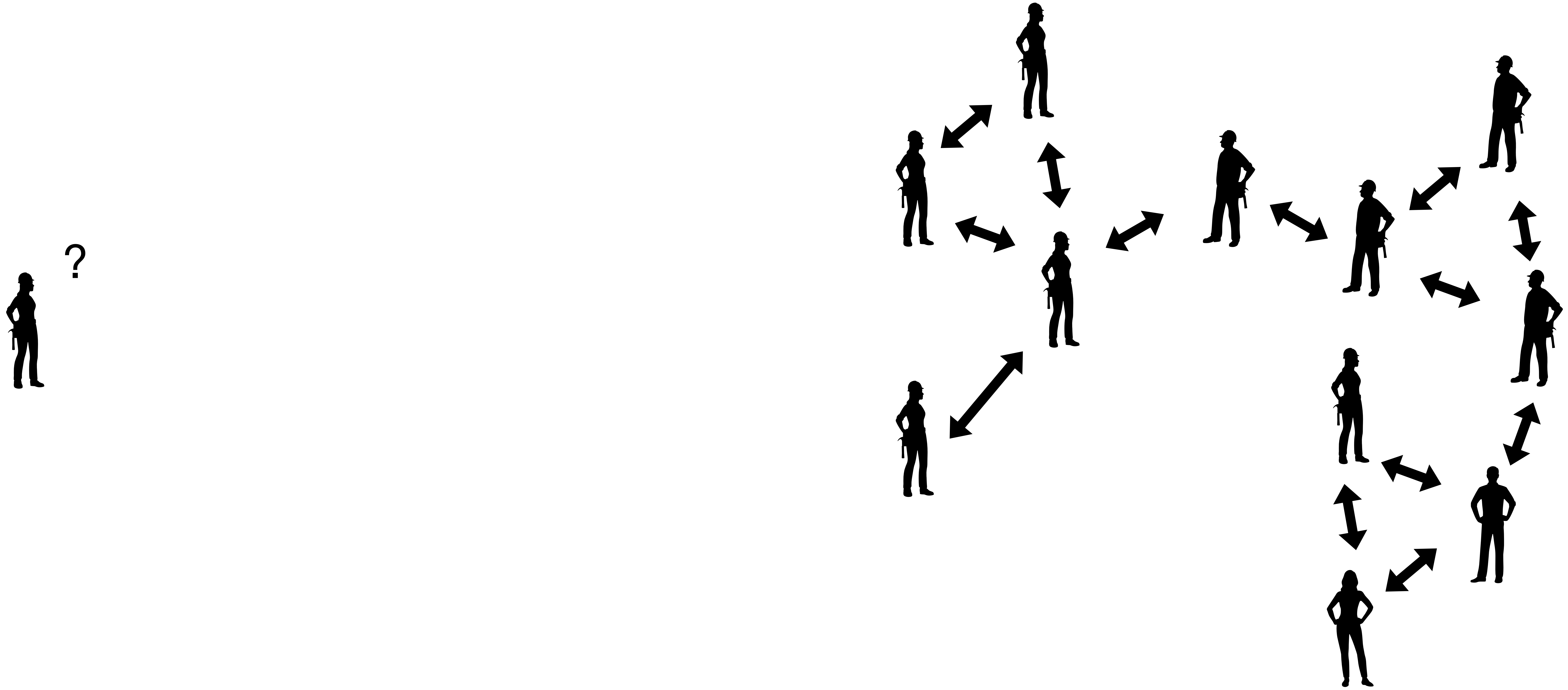

Preferential Attachment

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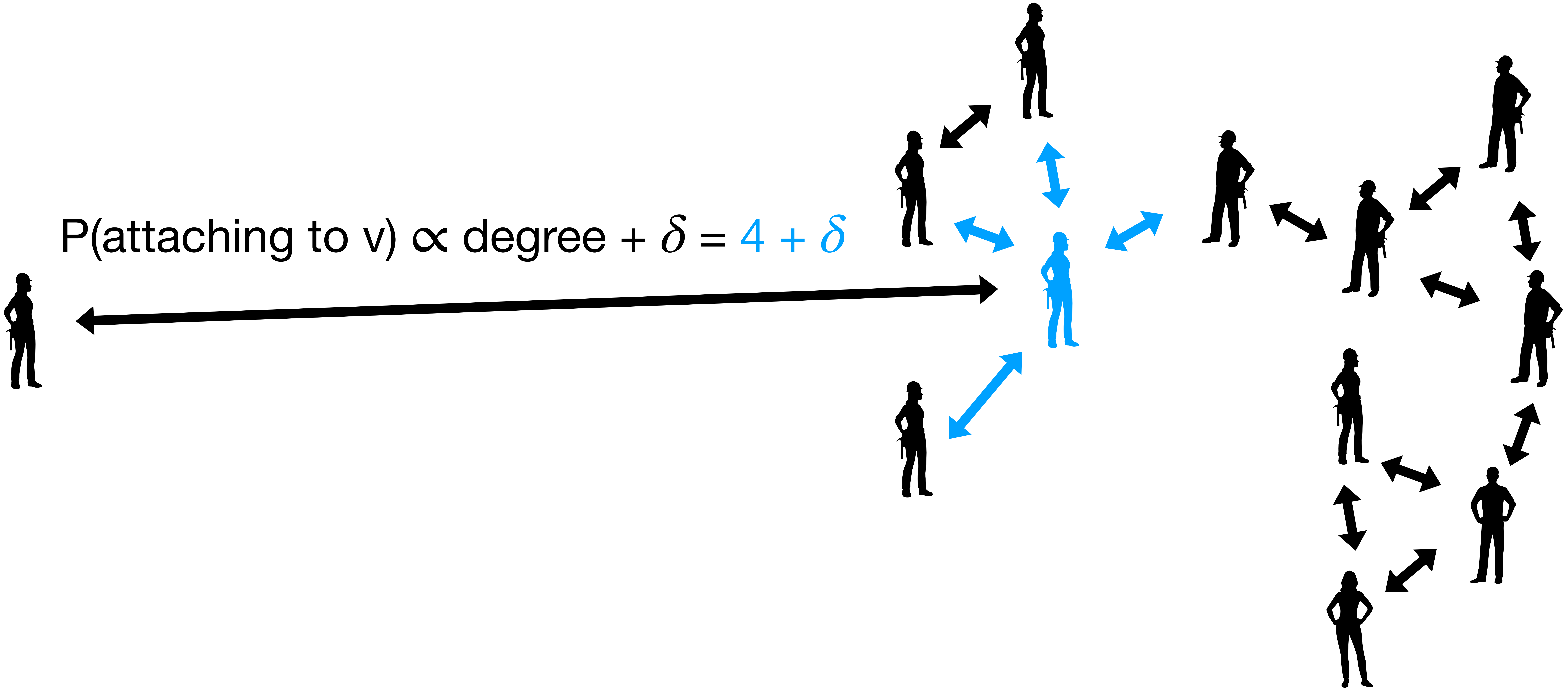
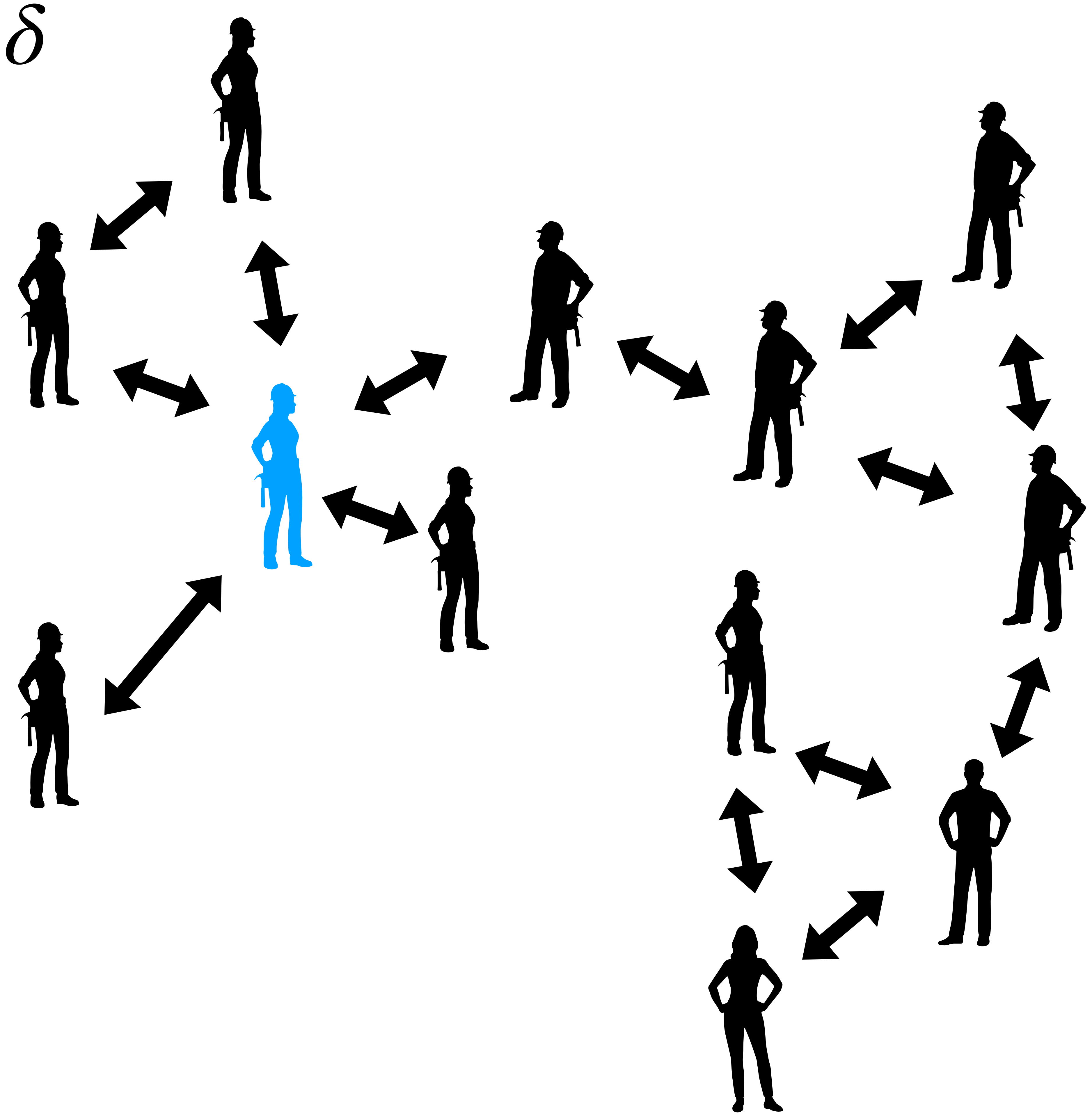


Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) ∝ degree + a tuning parameter δ



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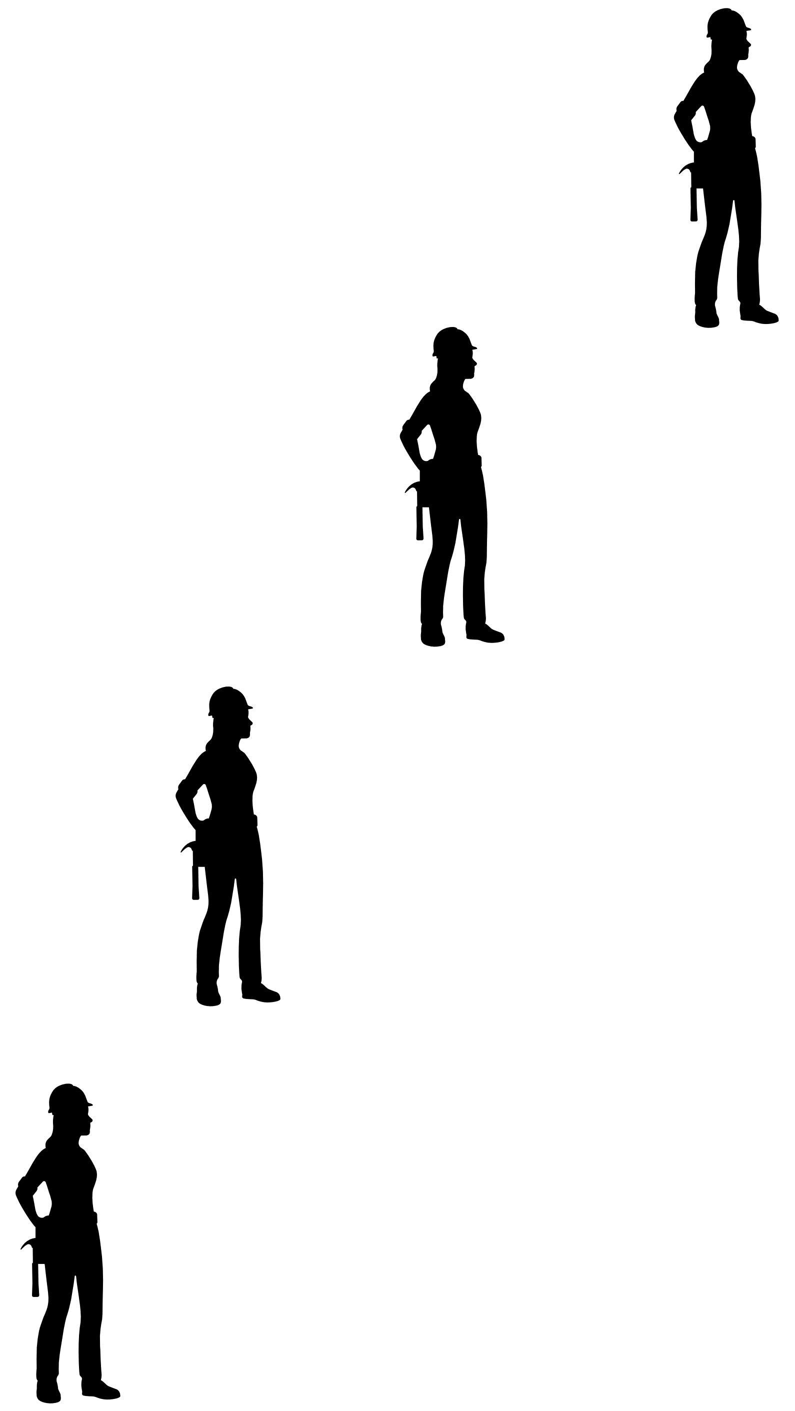

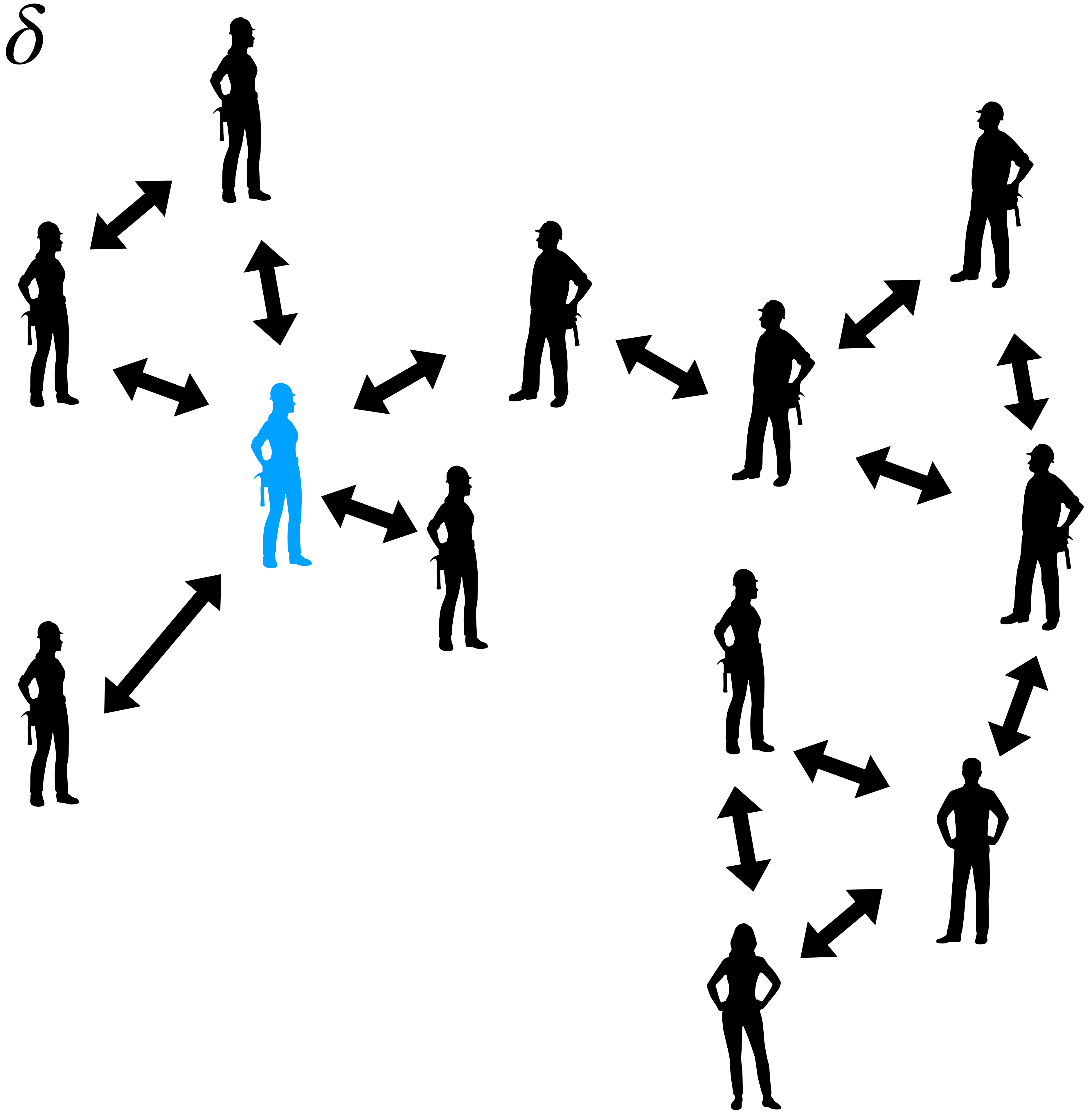

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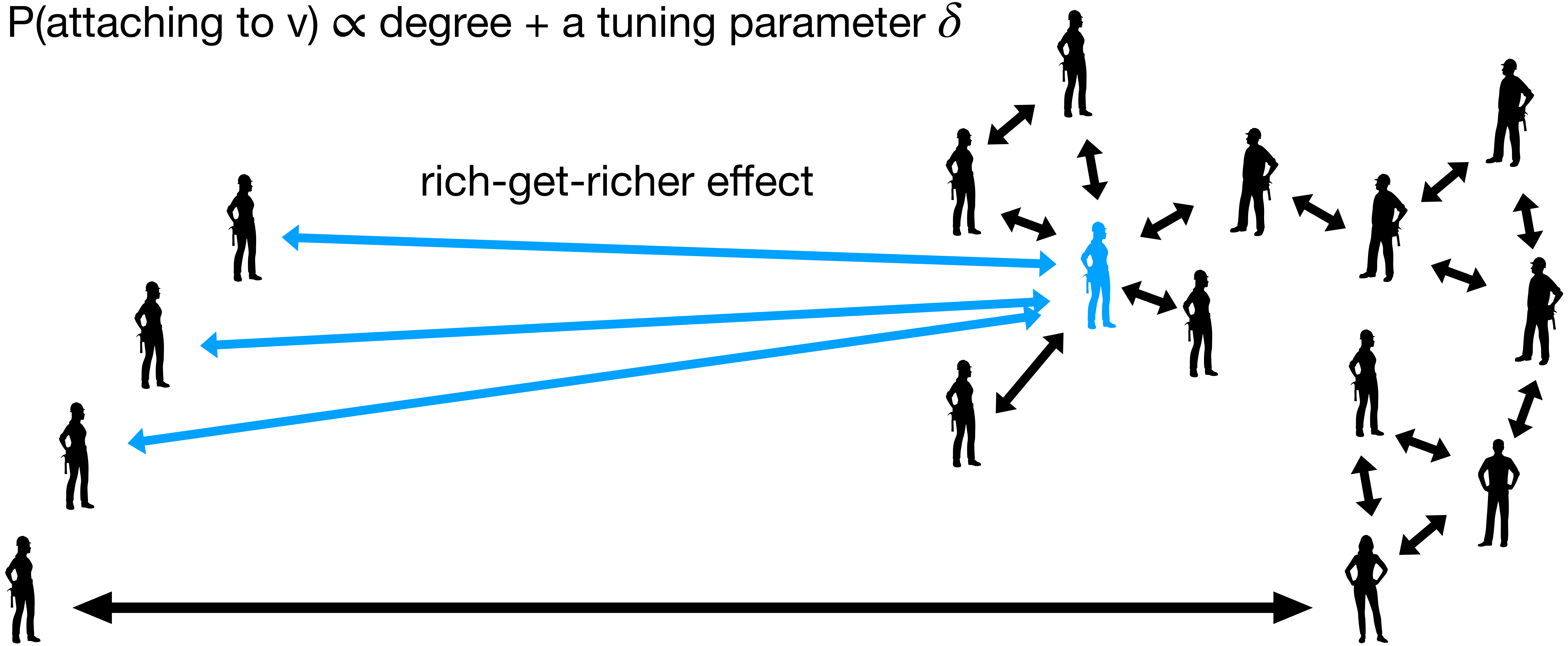


What do we know?

What do we know?

- Scale-freeness and Degree distribution
  [Barabasi and Albert 1999; Dorogovtsev, Mendes and Samukhin 2000; Krapivsky, Redner and Leyvraz 2000]

pmf of degree distribution

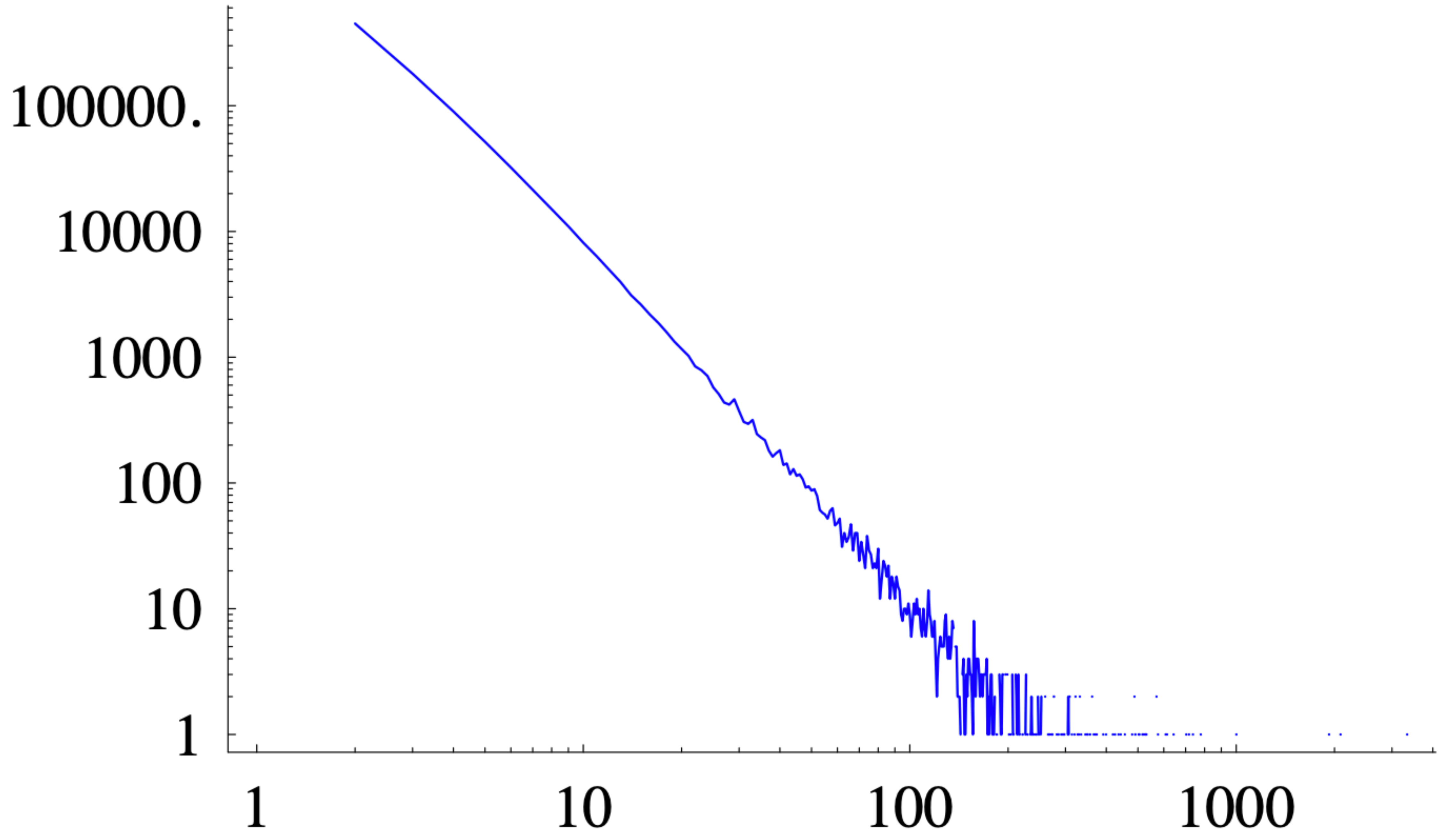


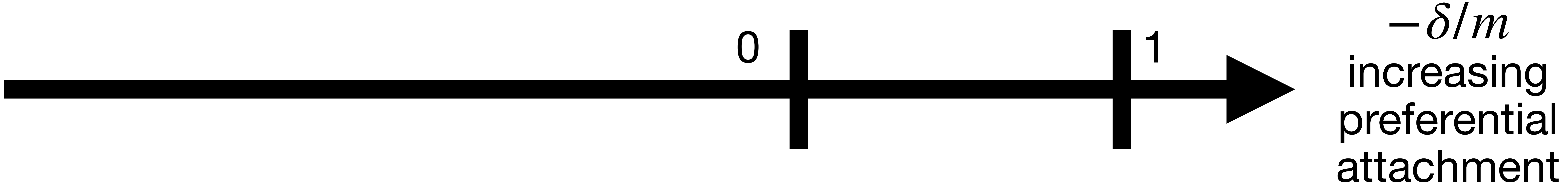
Fig 8.3 of R. Hofstad (2013).
Random Graphs and Complex Networks.
https://doi.org/10.1017/9781316779422

Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



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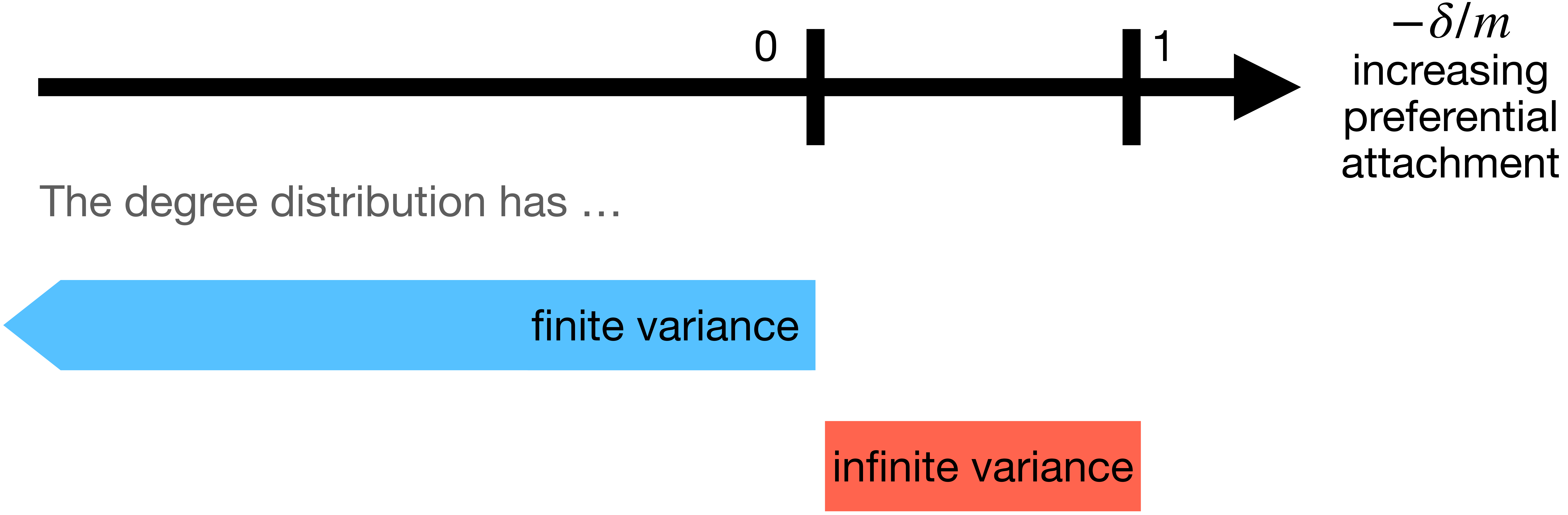
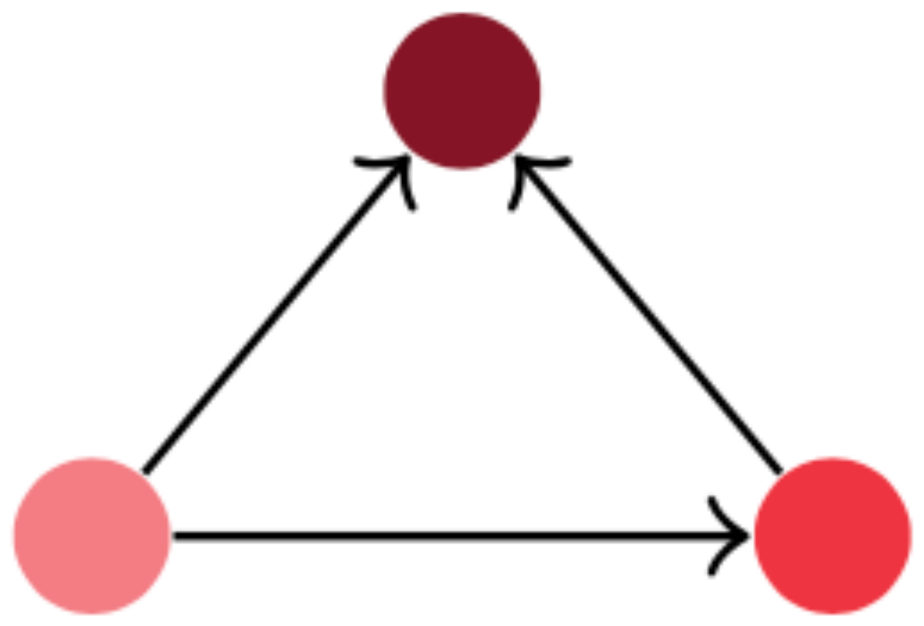


What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013, Garavaglia and Stegehuis 2019]

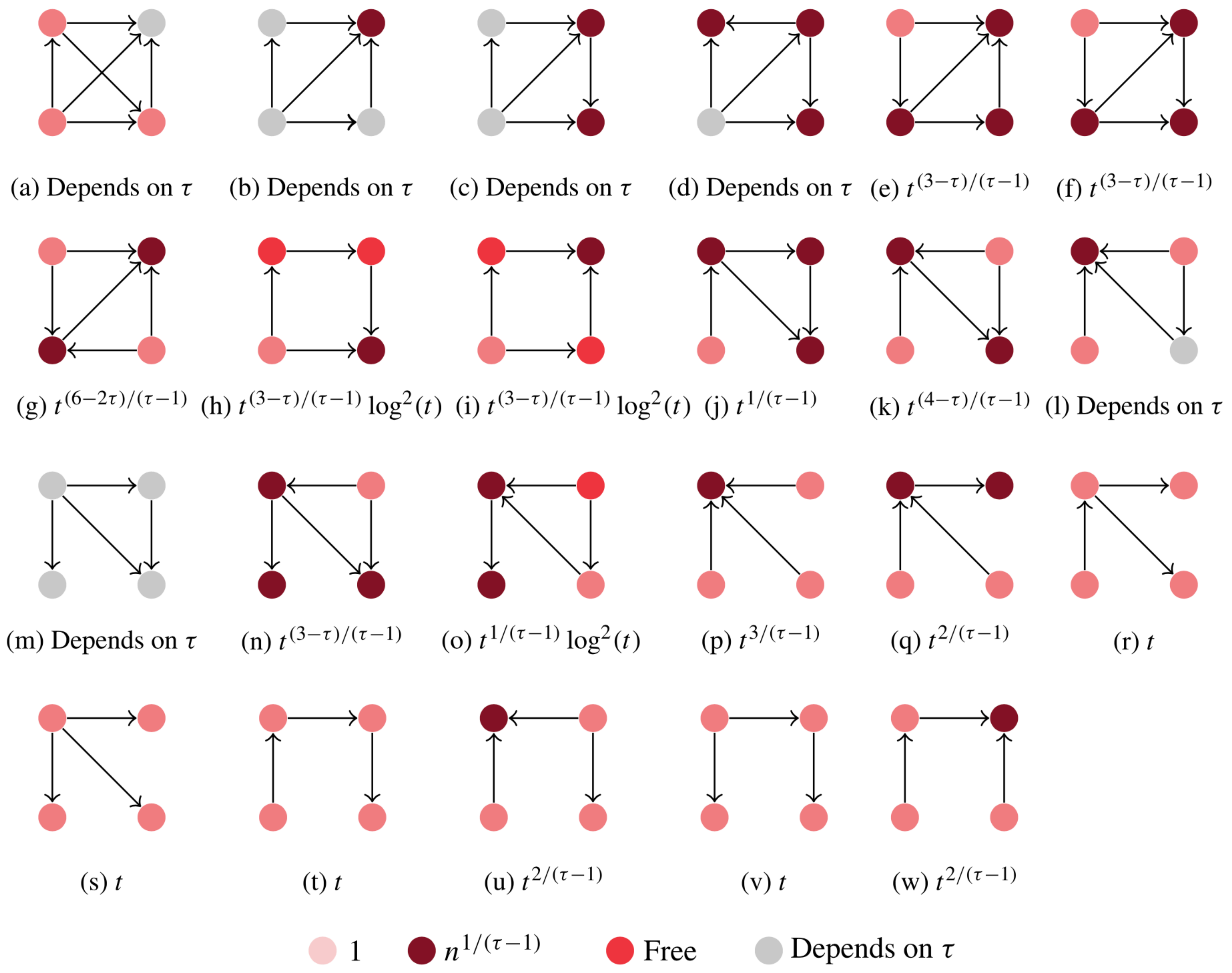


(a) $t^{(3-\tau)/(\tau-1)} \log(t)$

Fig 2 of A. Garavaglia and C. Stegehuis (2019). Subgraphs in Preferential Attachment Models. https://doi.org/10.1017/apr.2019.36

What do we know?

- subgraph counts [Garavaglia and Stegehuis 2019]



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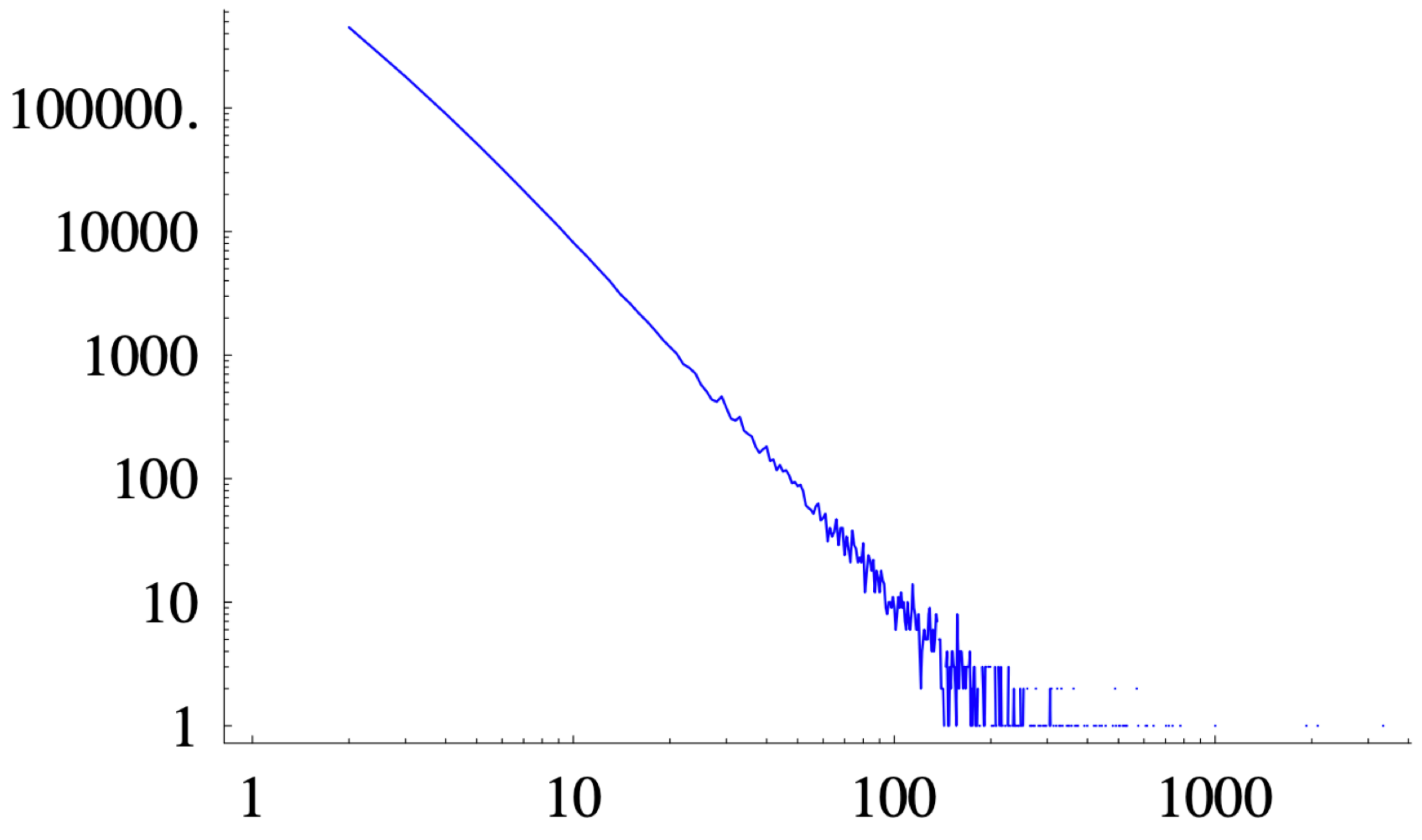
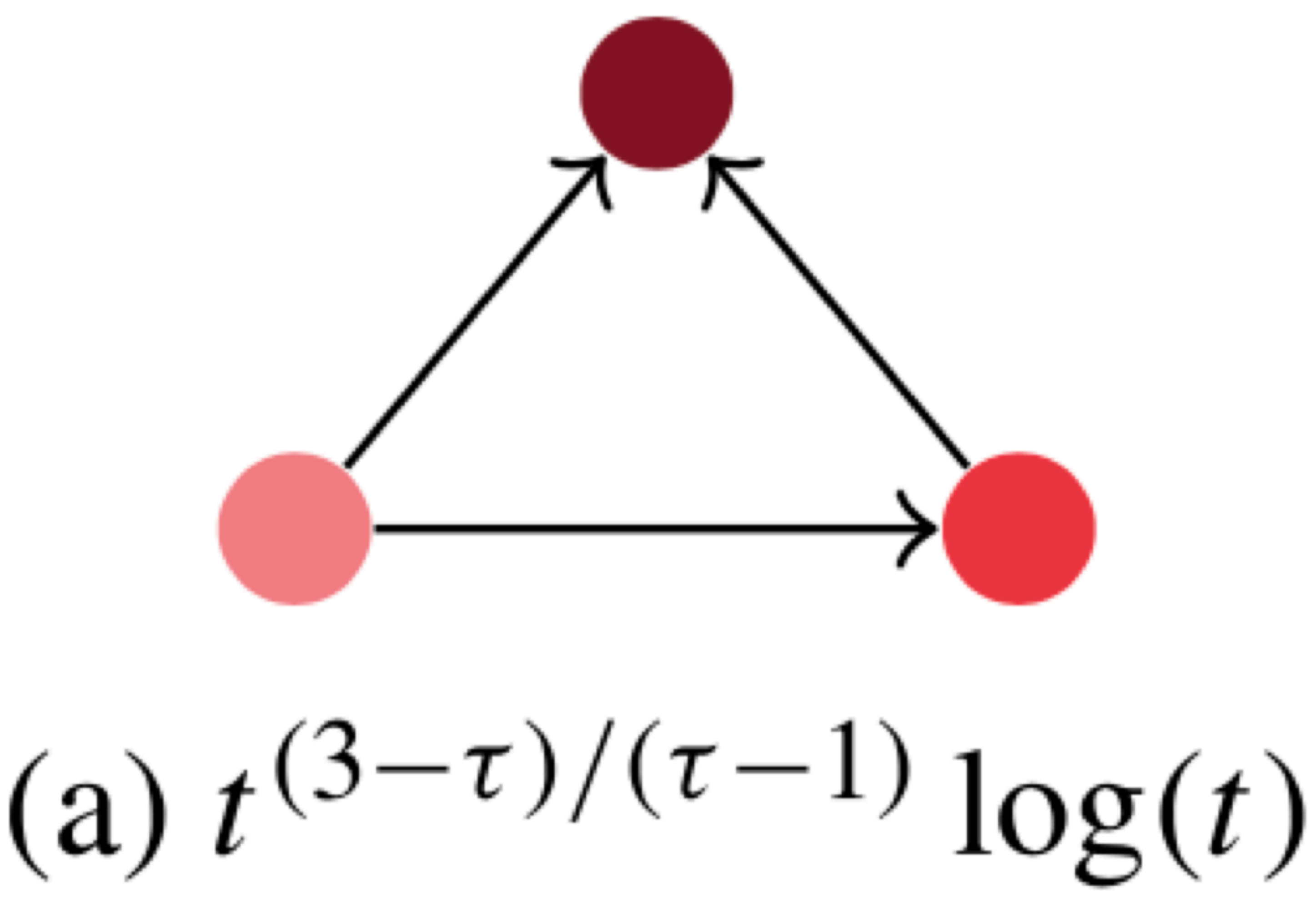
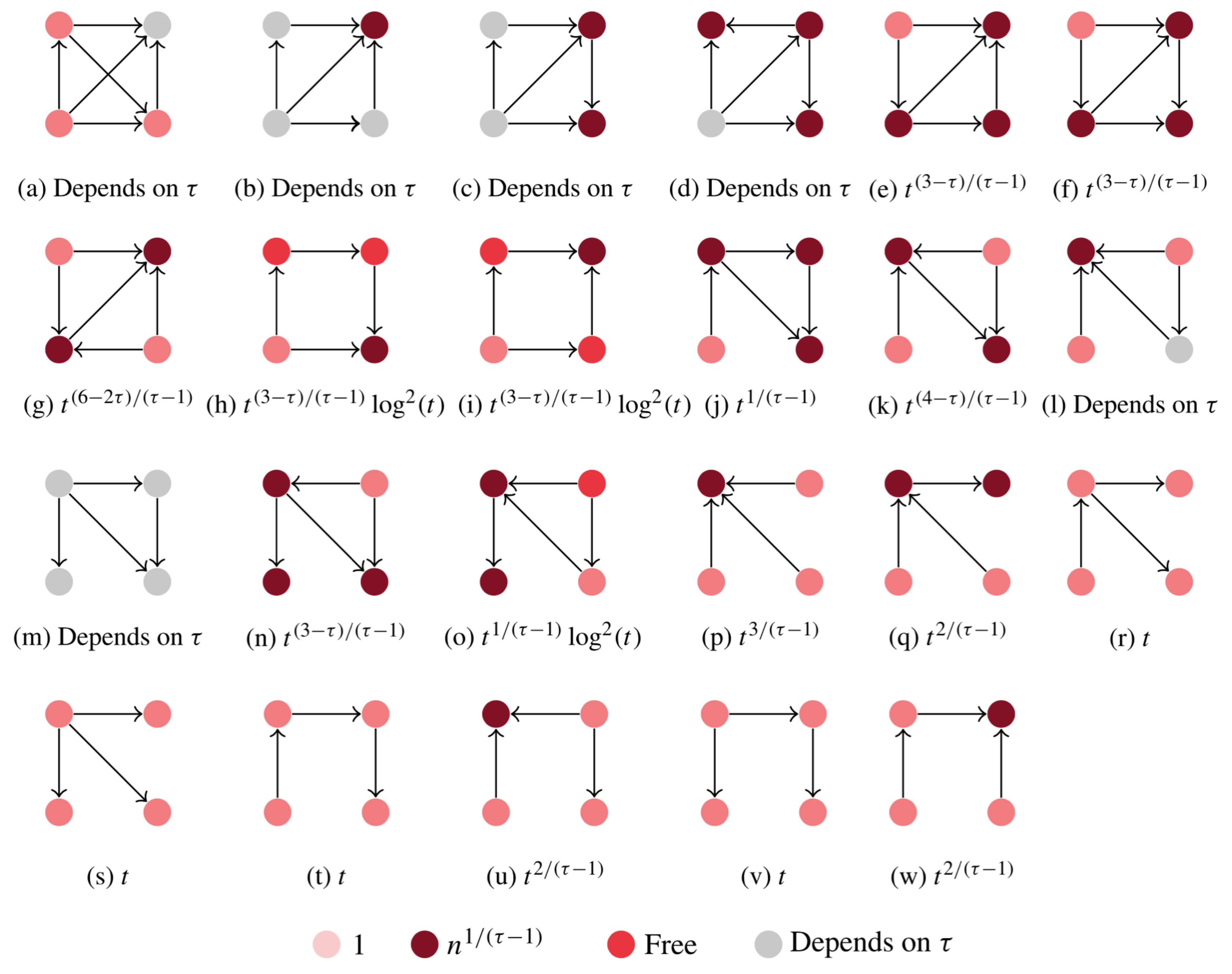


degree distribution



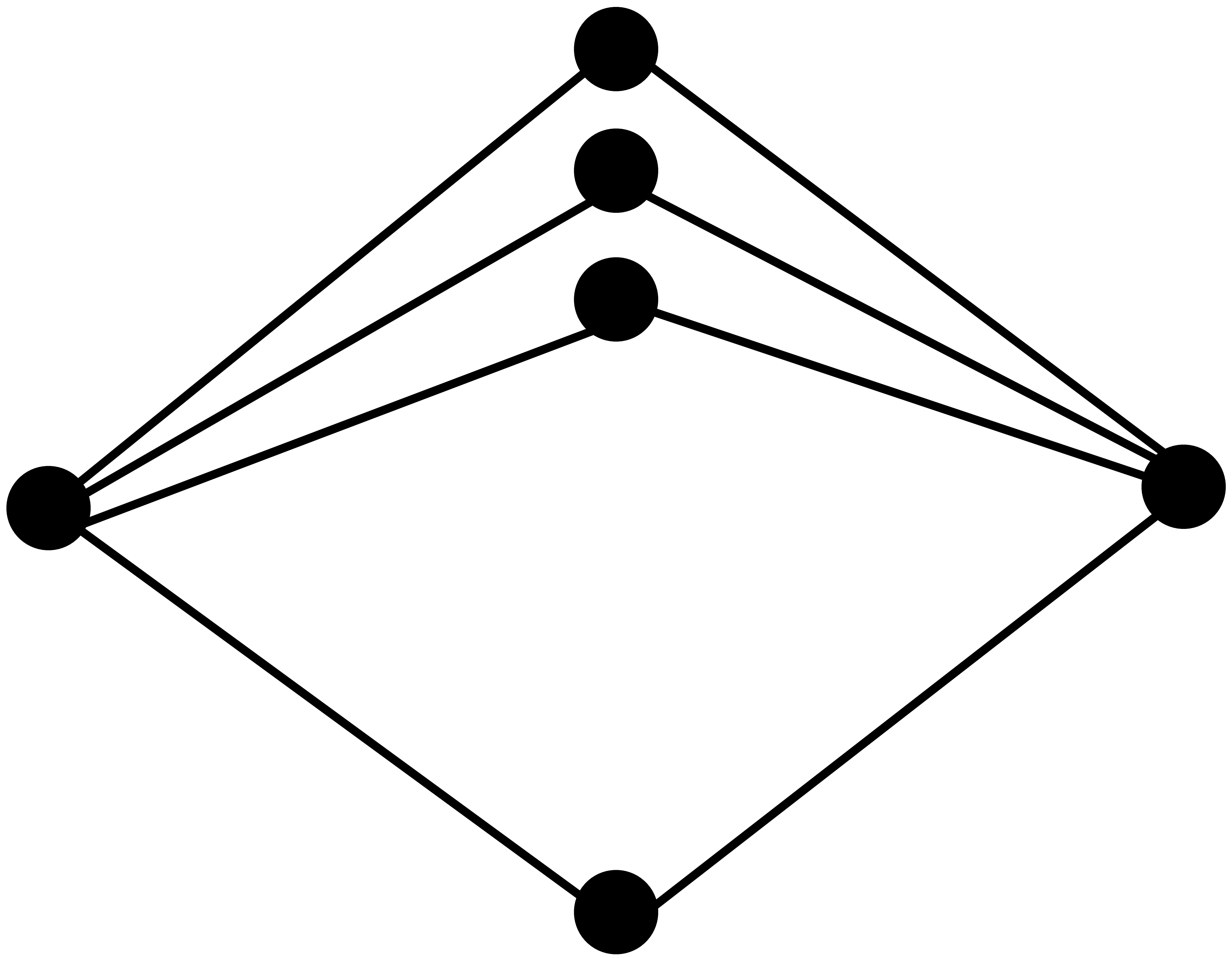
triangle counts



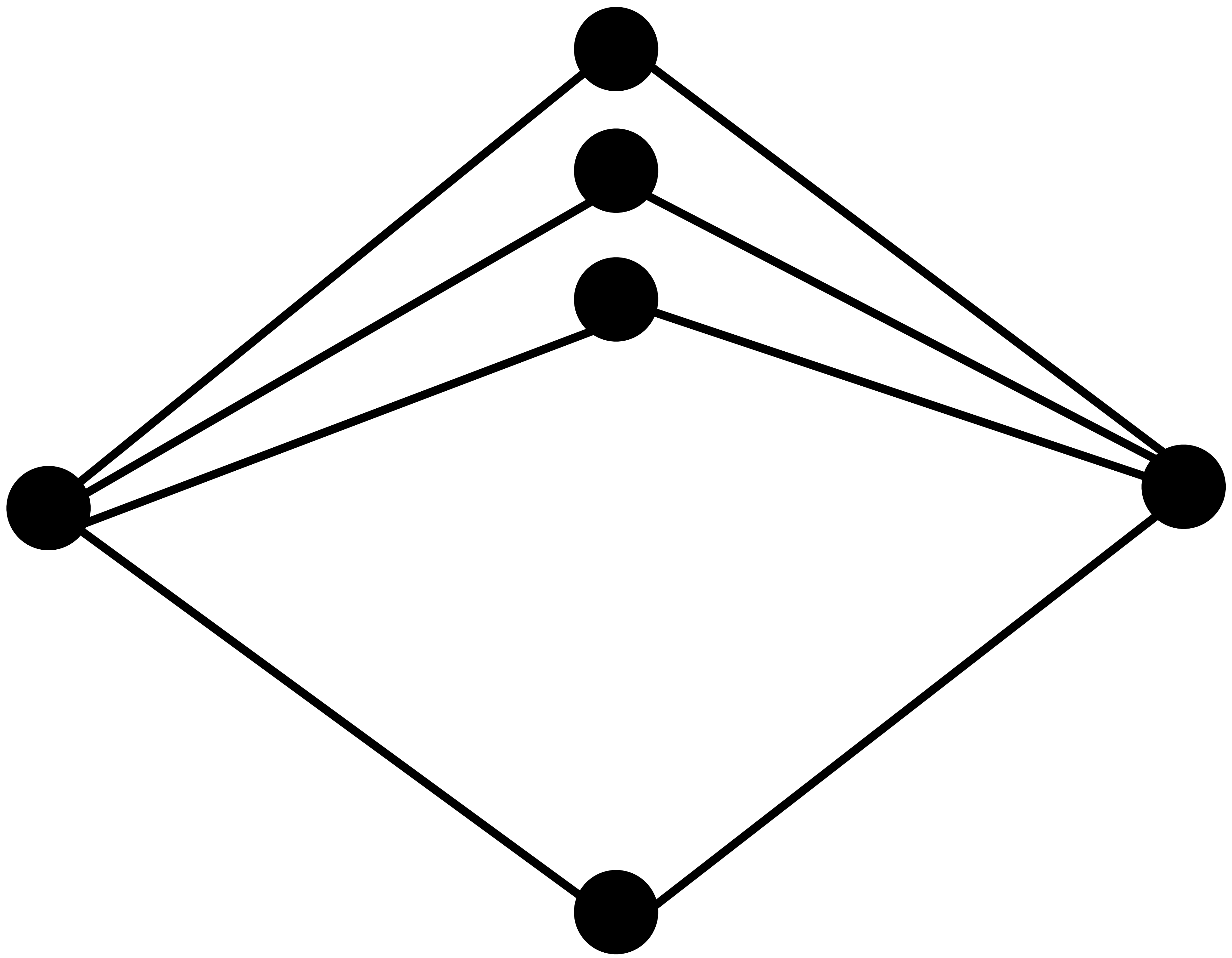
subgraph counts

What should we count?
And how?

Paths from left to right?

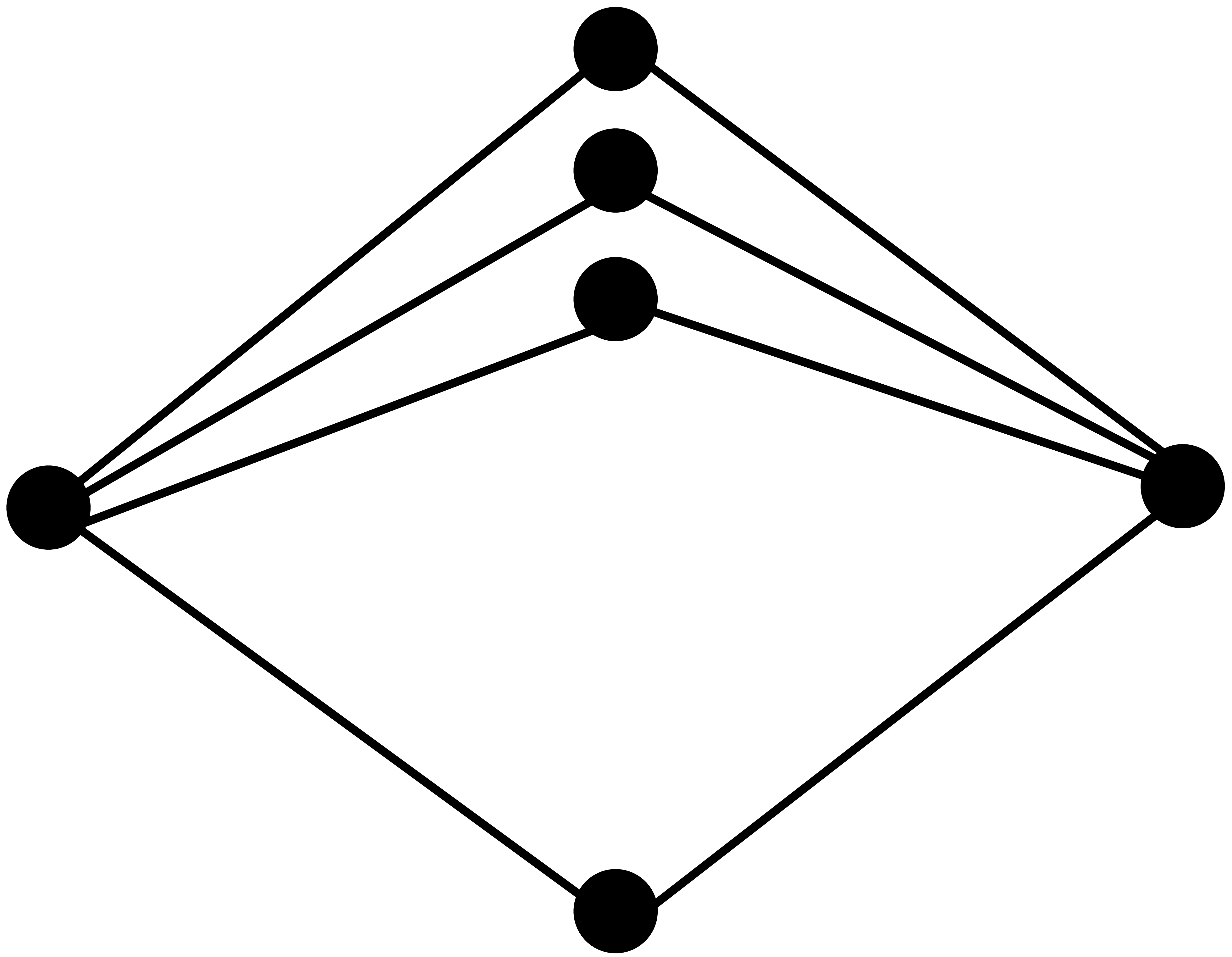


Paths from left to right?



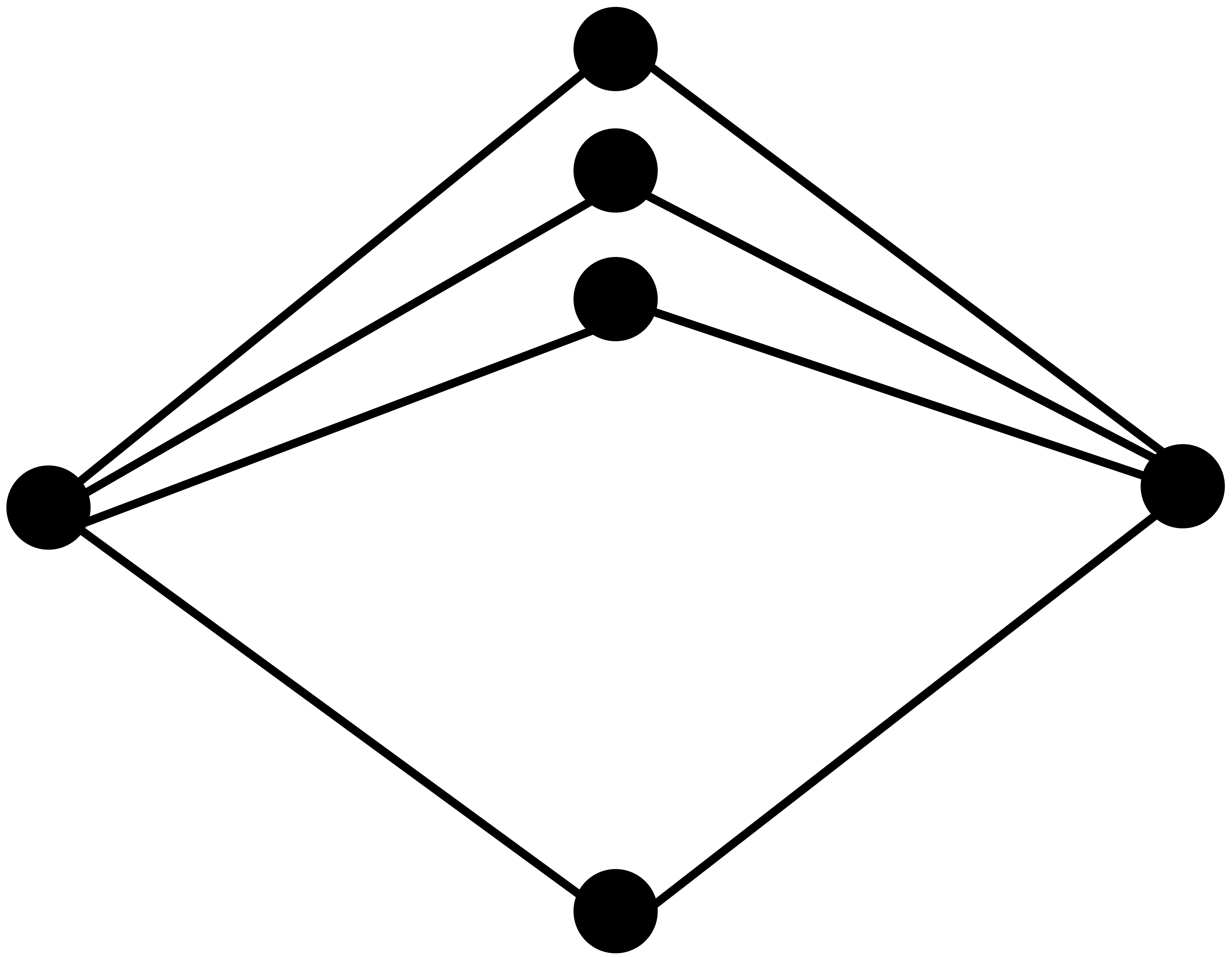
- backtracking?

Paths from left to right?



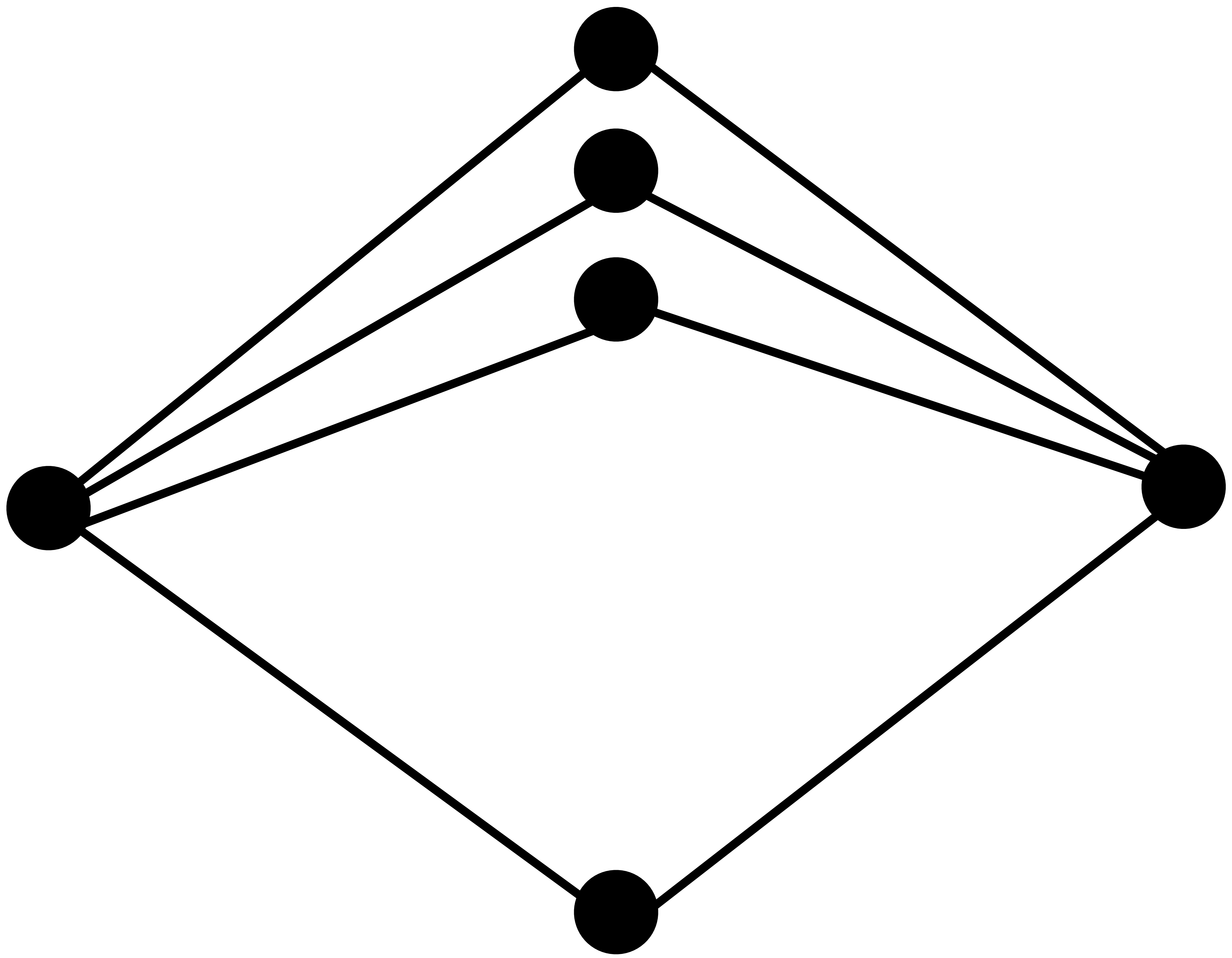
- backtracking?
- concatenating with loops?

Paths from left to right?



- backtracking?
- concatenating with loops?
- how to count loops?

Paths from left to right?



- backtracking?
- concatenating with loops?
- how to count loops?
  - backtracking?
  - concatenating with other loops?

II. Into Topology

Counting everything in every dimension all at once

Betti numbers count repeated connections “in all dimensions”.

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GOOD ✓

“correct” way to count things

Betti numbers count repeated connections "in all dimensions".

GOOD ✓

"correct" way to count things

homological algebra

Betti numbers count repeated connections “in all dimensions”.

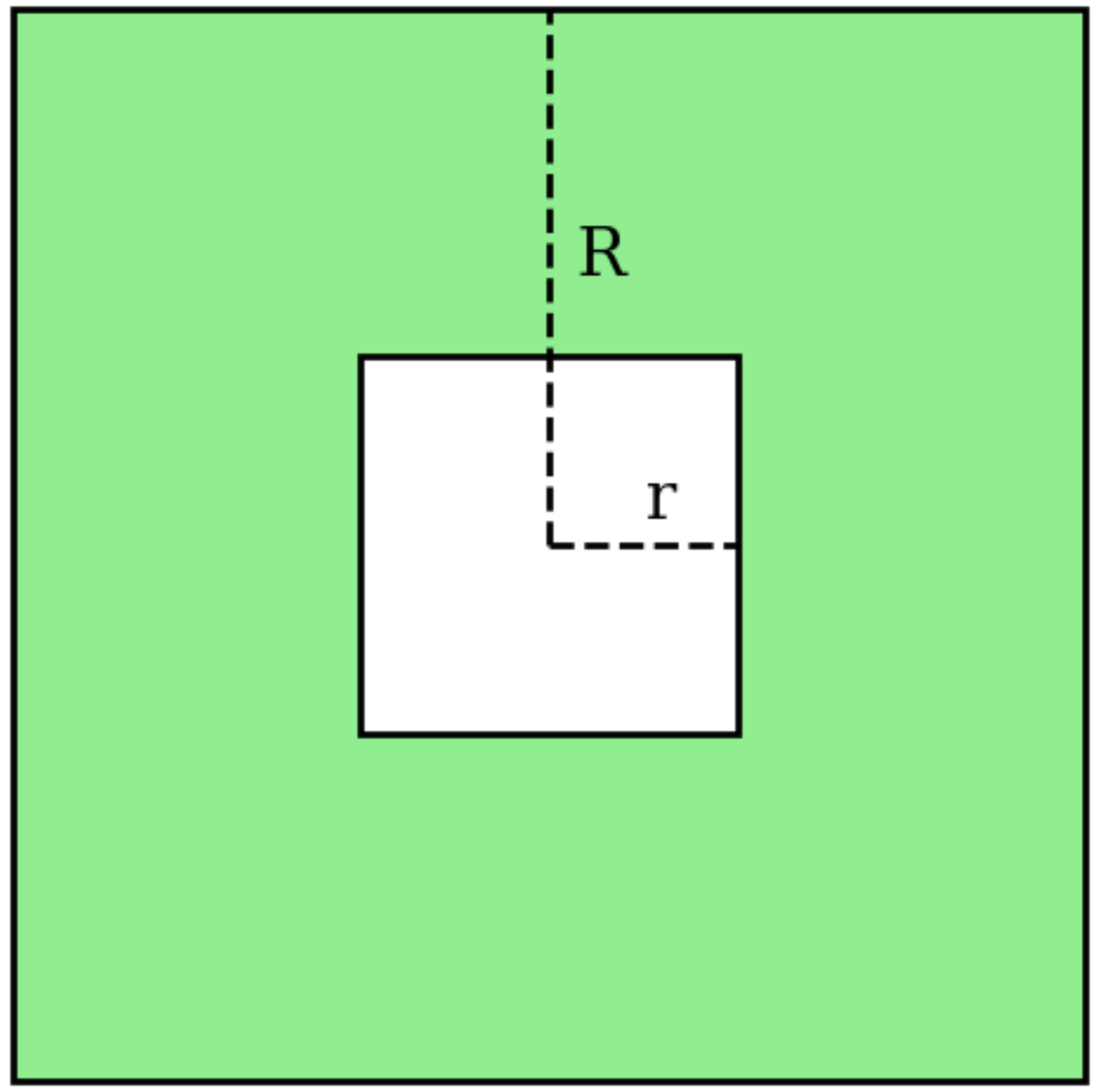
GOOD ✓	BAD ✗
“correct” way to count things	hard to write down
homological algebra	hard to do

Betti numbers β_k

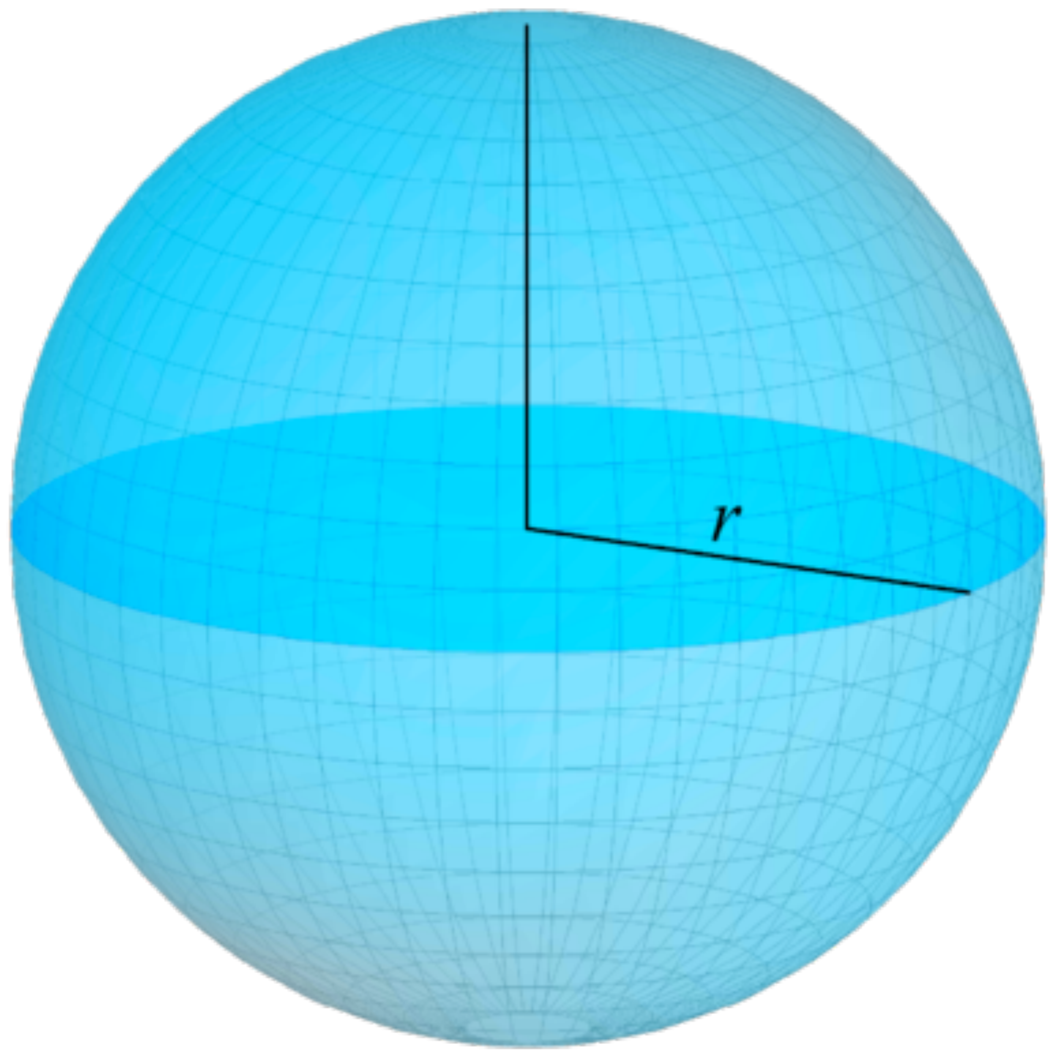
- Repeated connections?
- Holes?

Betti numbers β_k

Count of Holes



$\beta_1 = 1:$ 1 loop

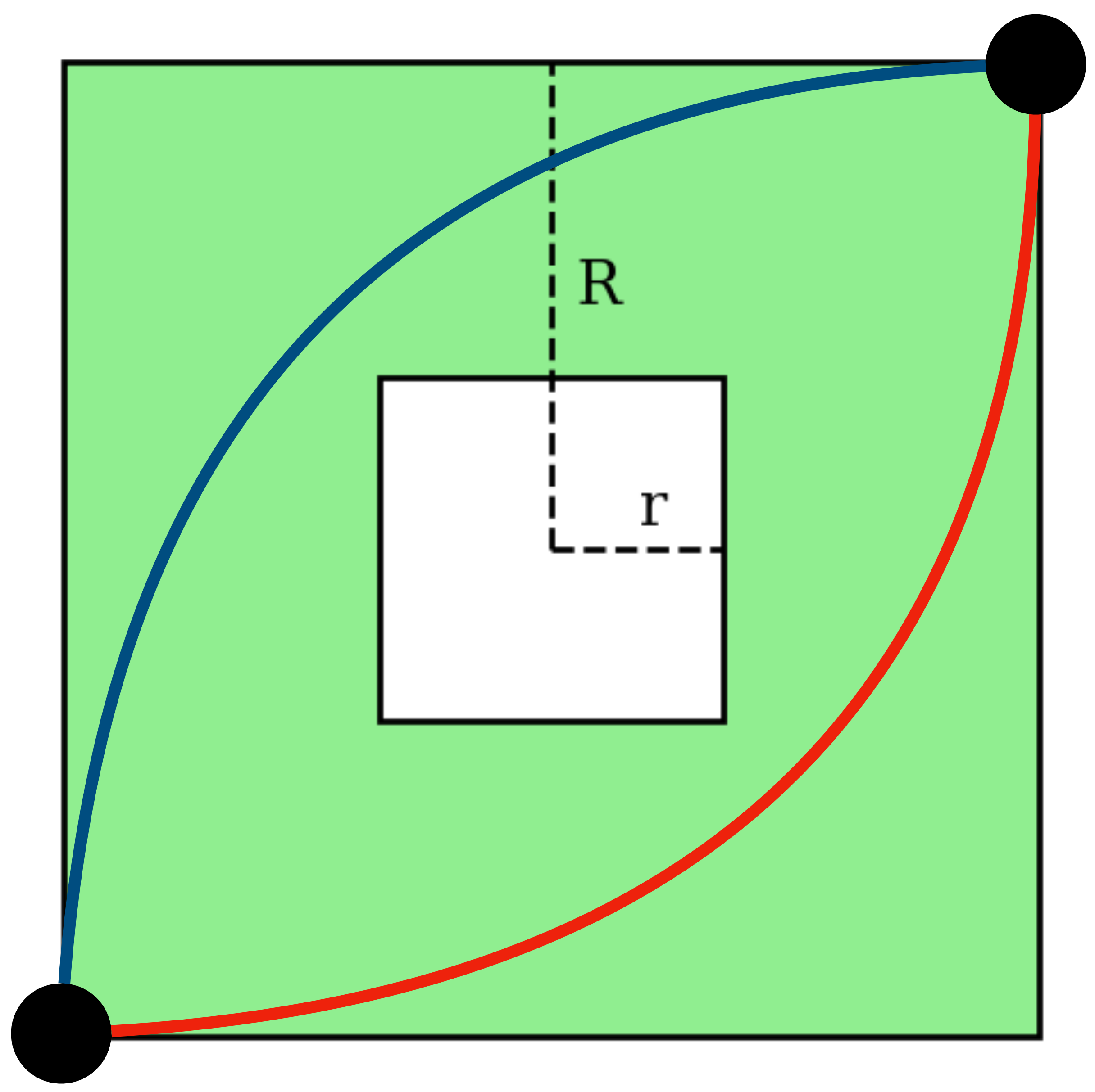


$\beta_1 = 0:$ 0 loop

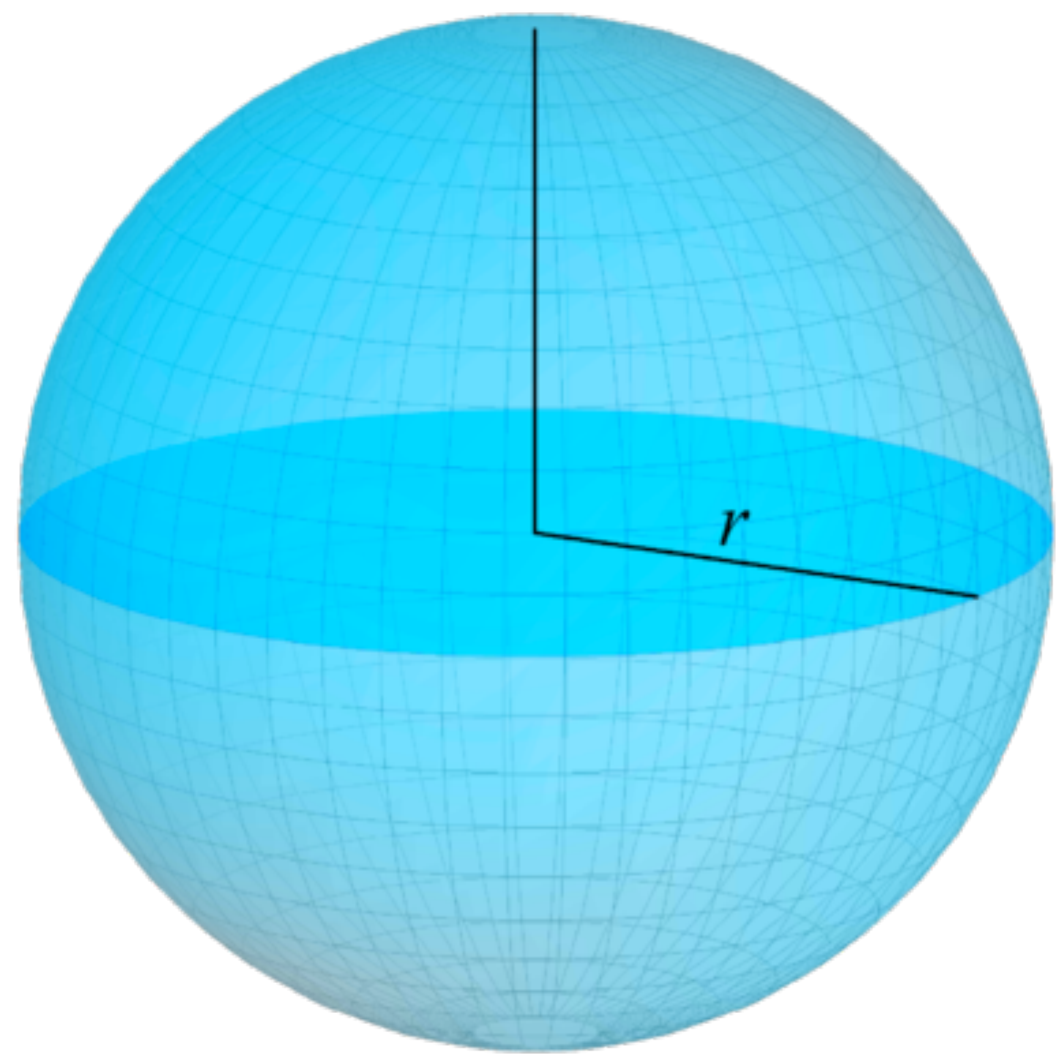
$\beta_2 = 1:$ 1 cavity

Betti numbers

Count of Repeated Connections



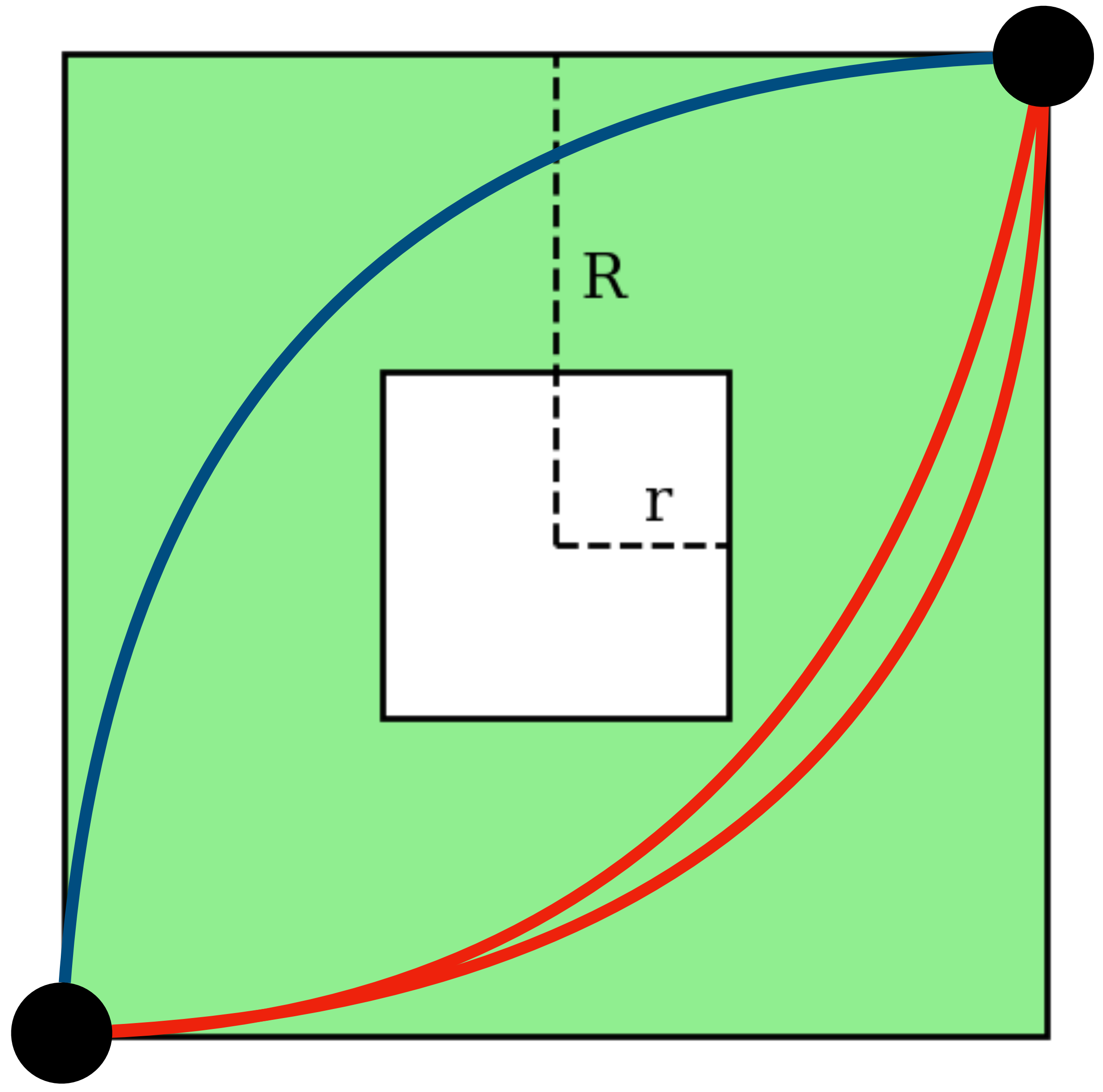
1 alternative path



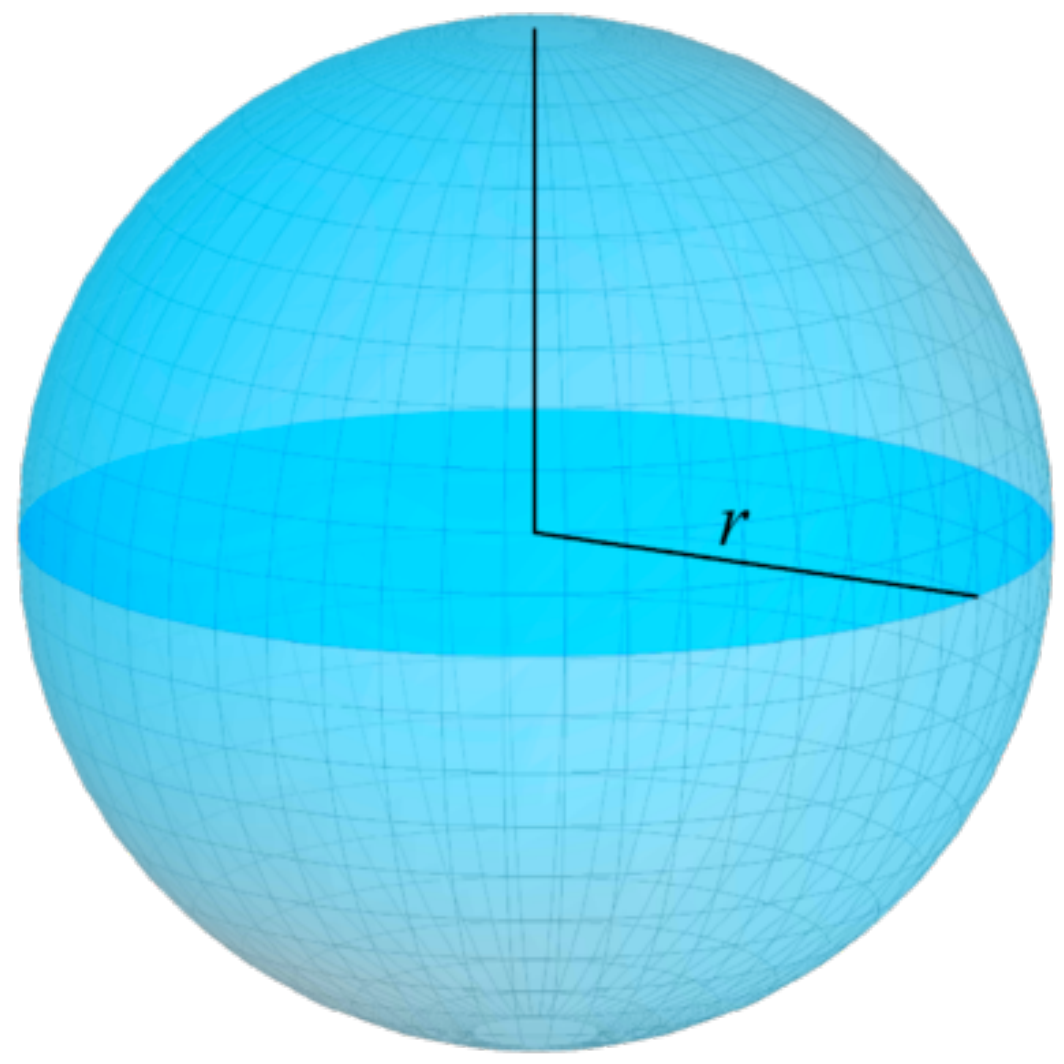
0 loop
1 cavity

Betti numbers

Count of (Independent) Repeated Connections



1 alternative path

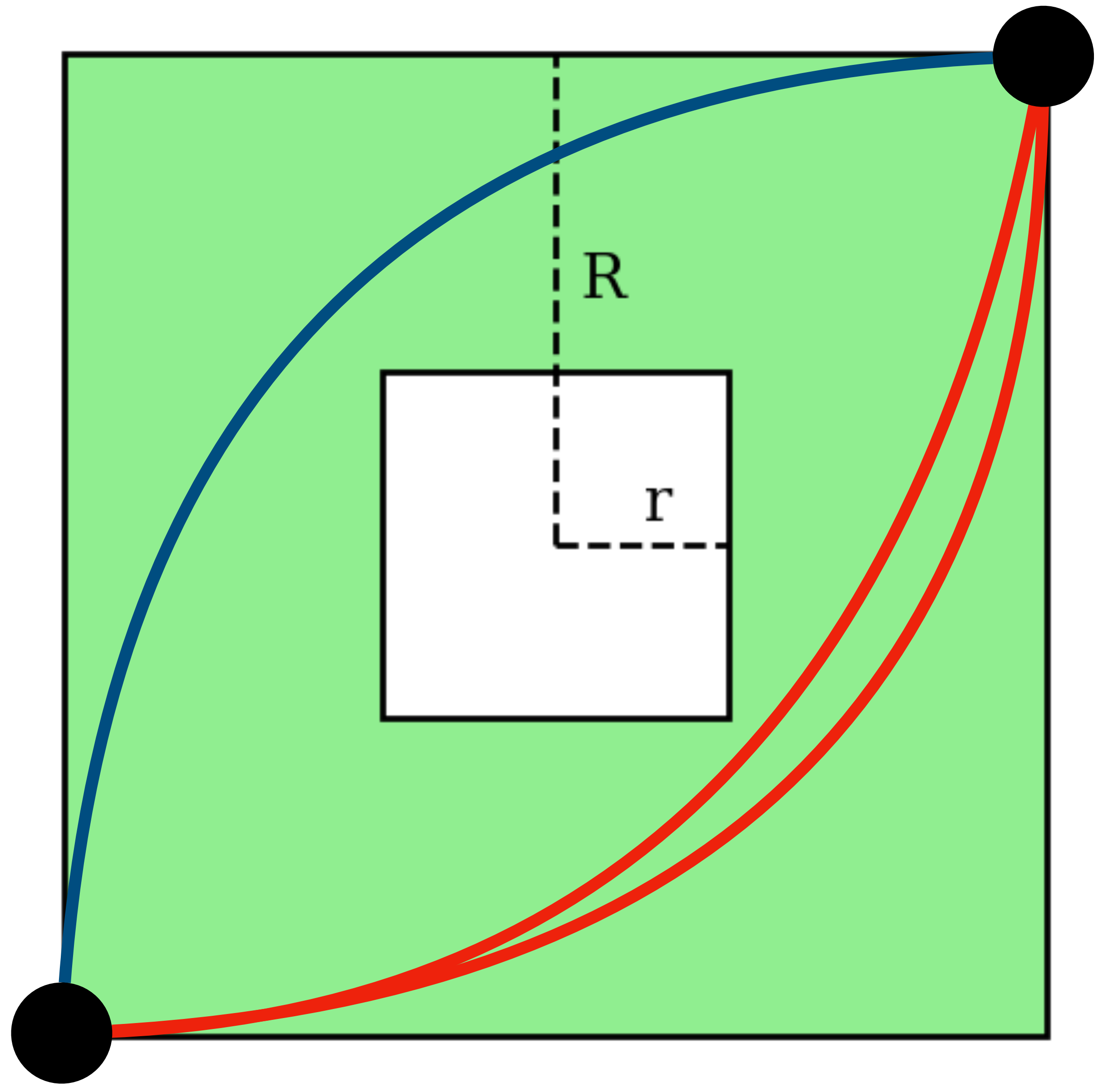


0 loop

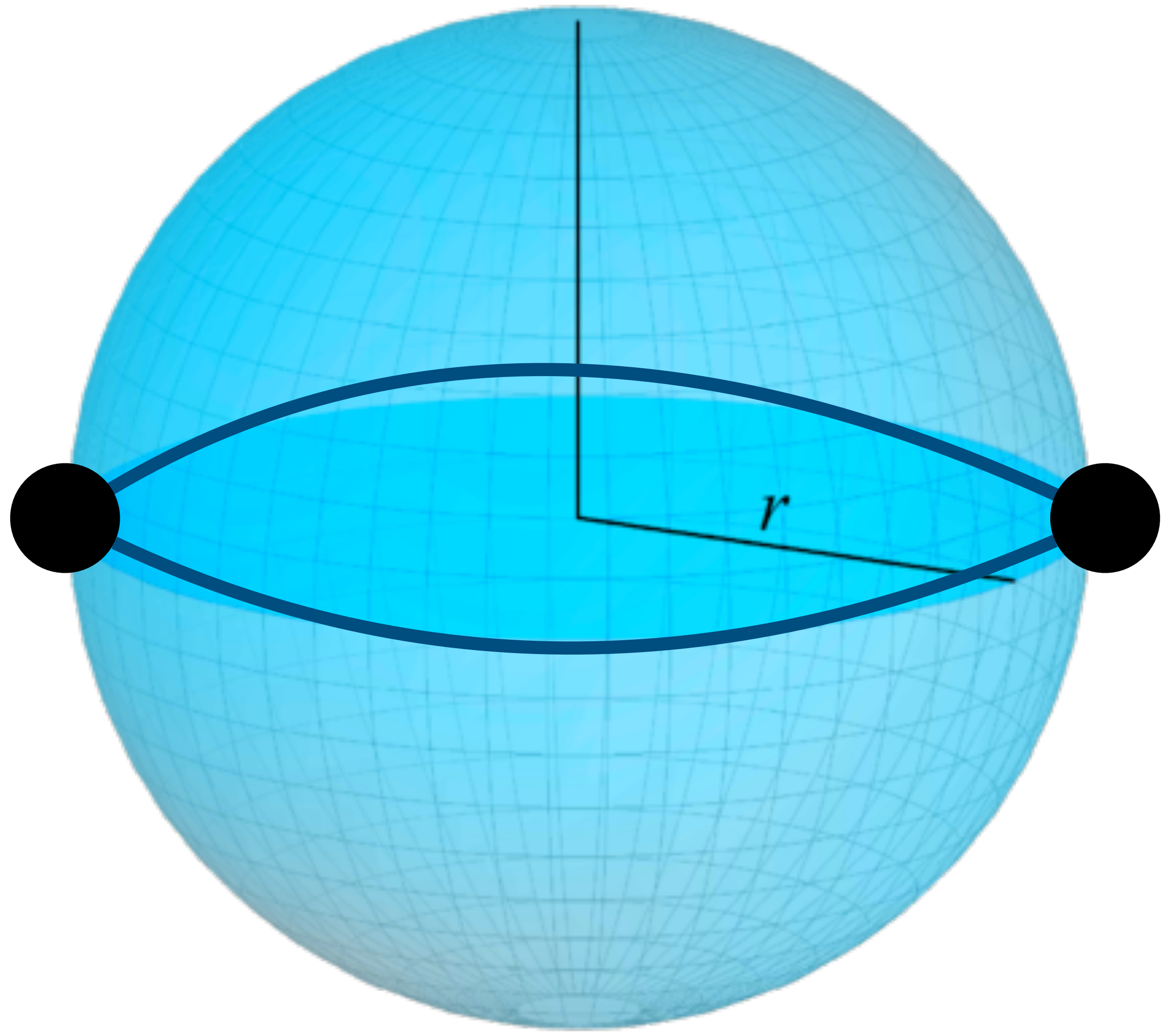
1 cavity

Betti numbers

Count of (Independent) Repeated Connections



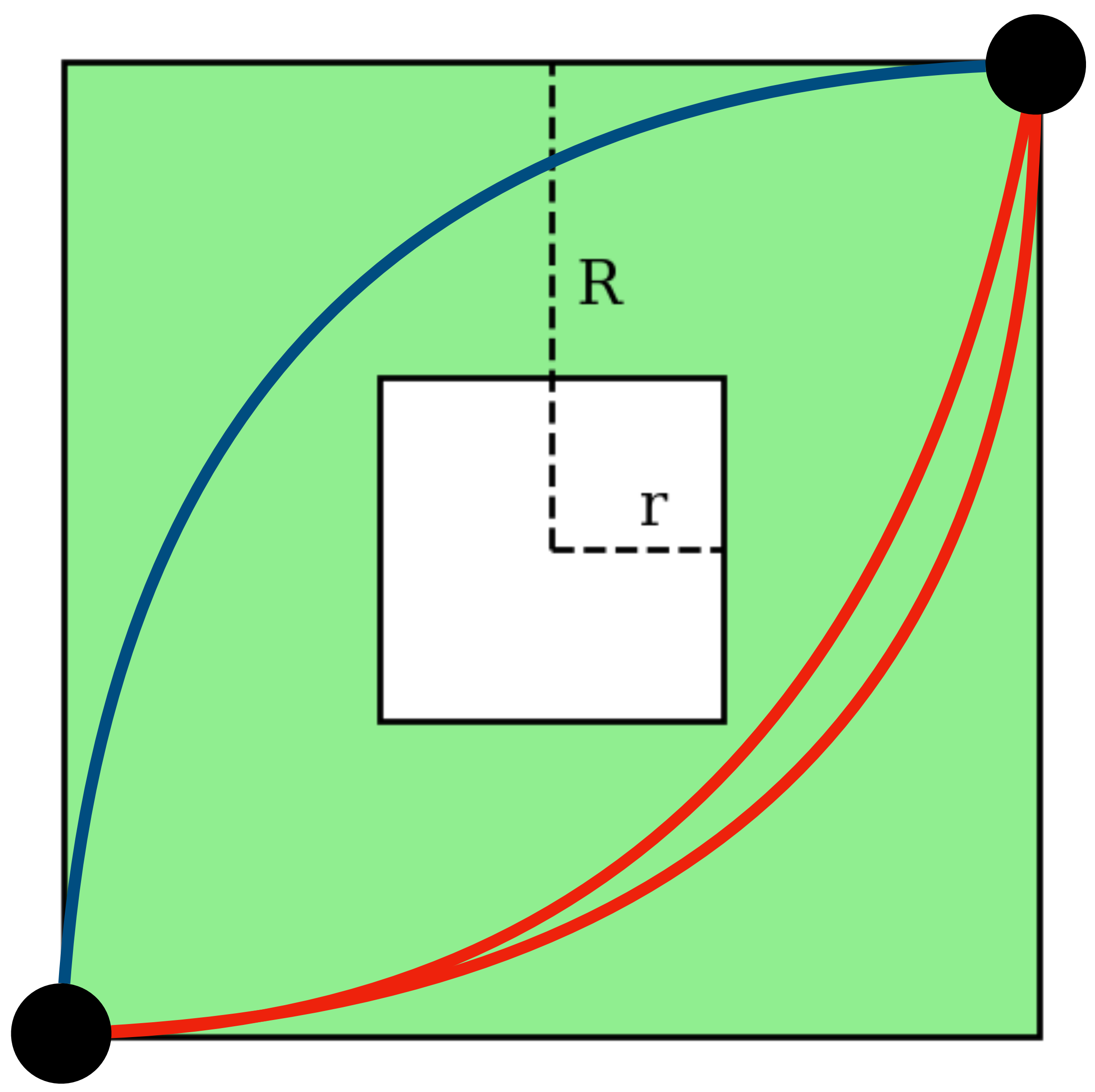
1 alternative path



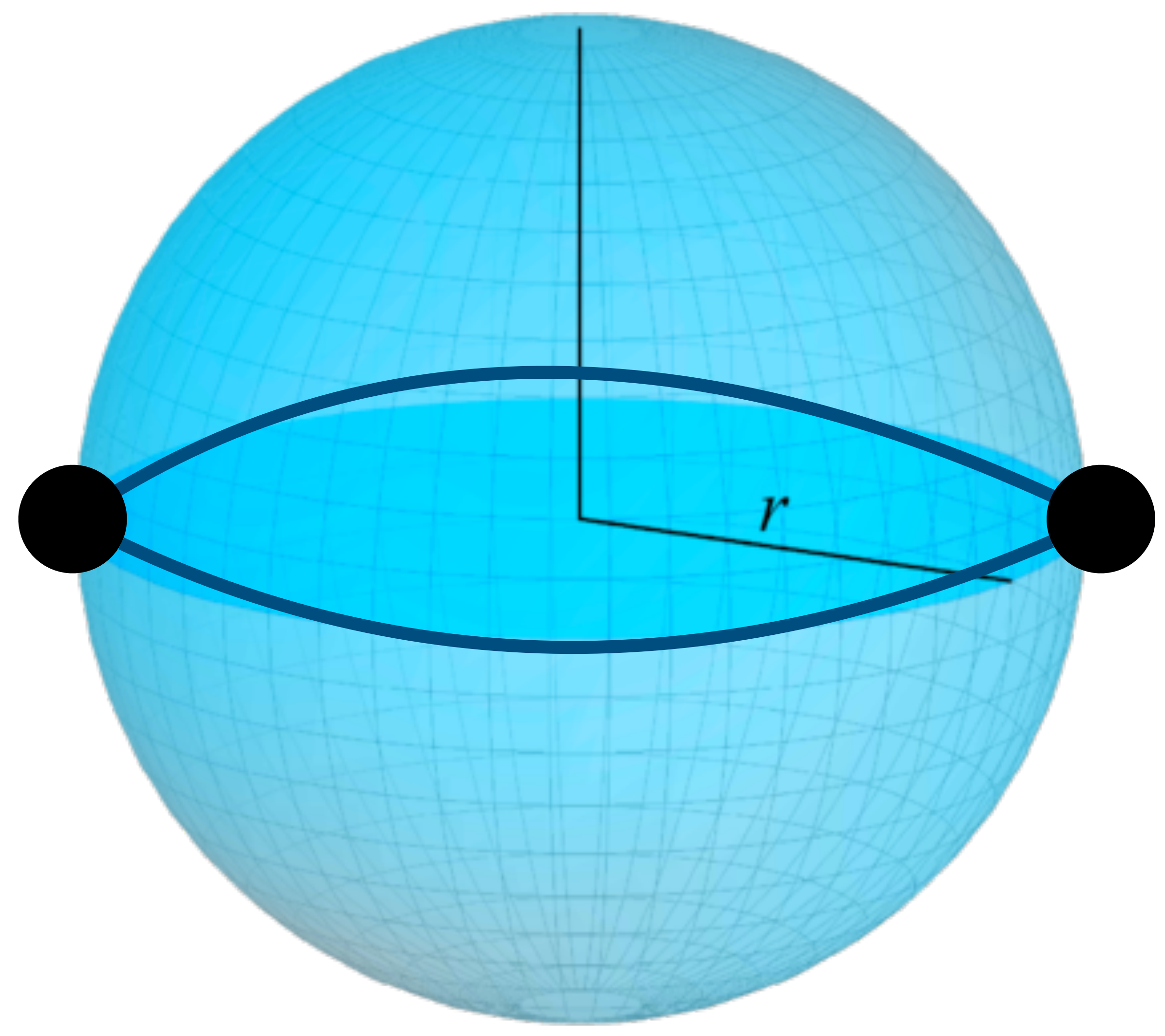
0 alternative path (slide through upper hemisphere)
1 cavity

Betti numbers

Count of (Independent) Repeated Connections

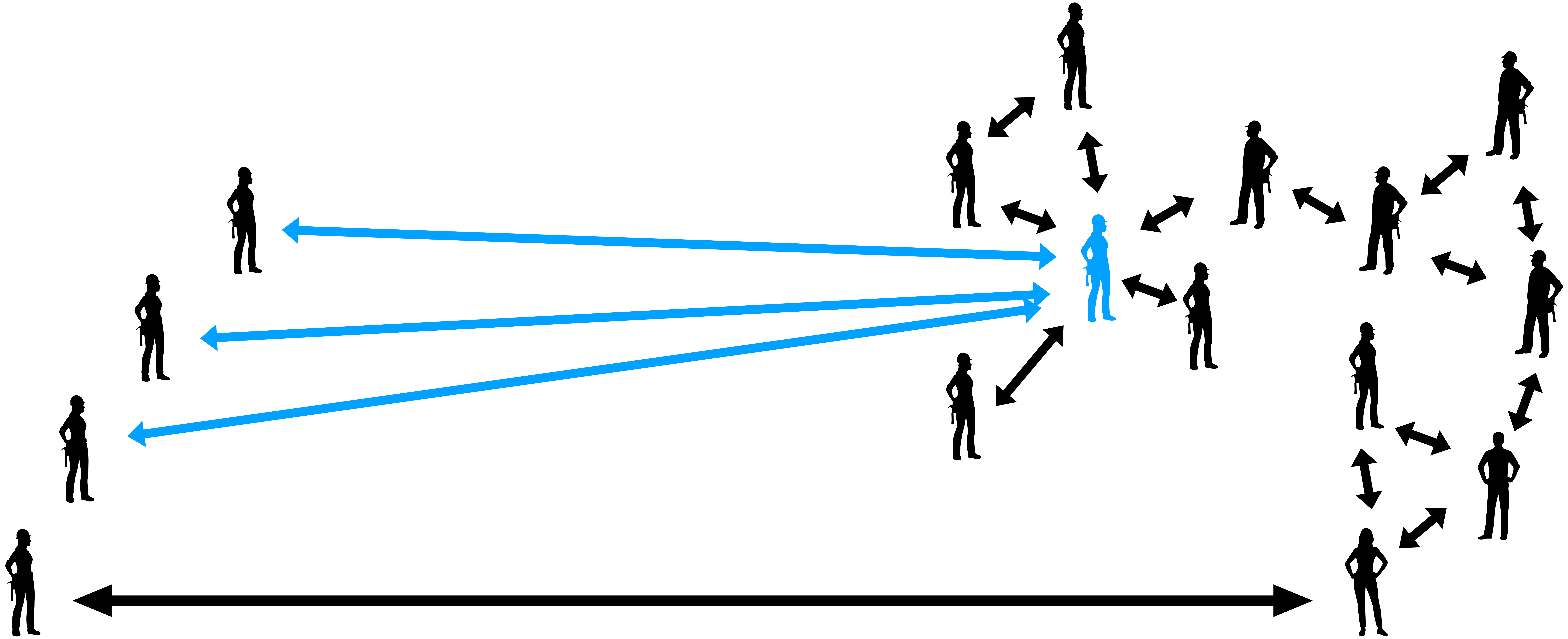


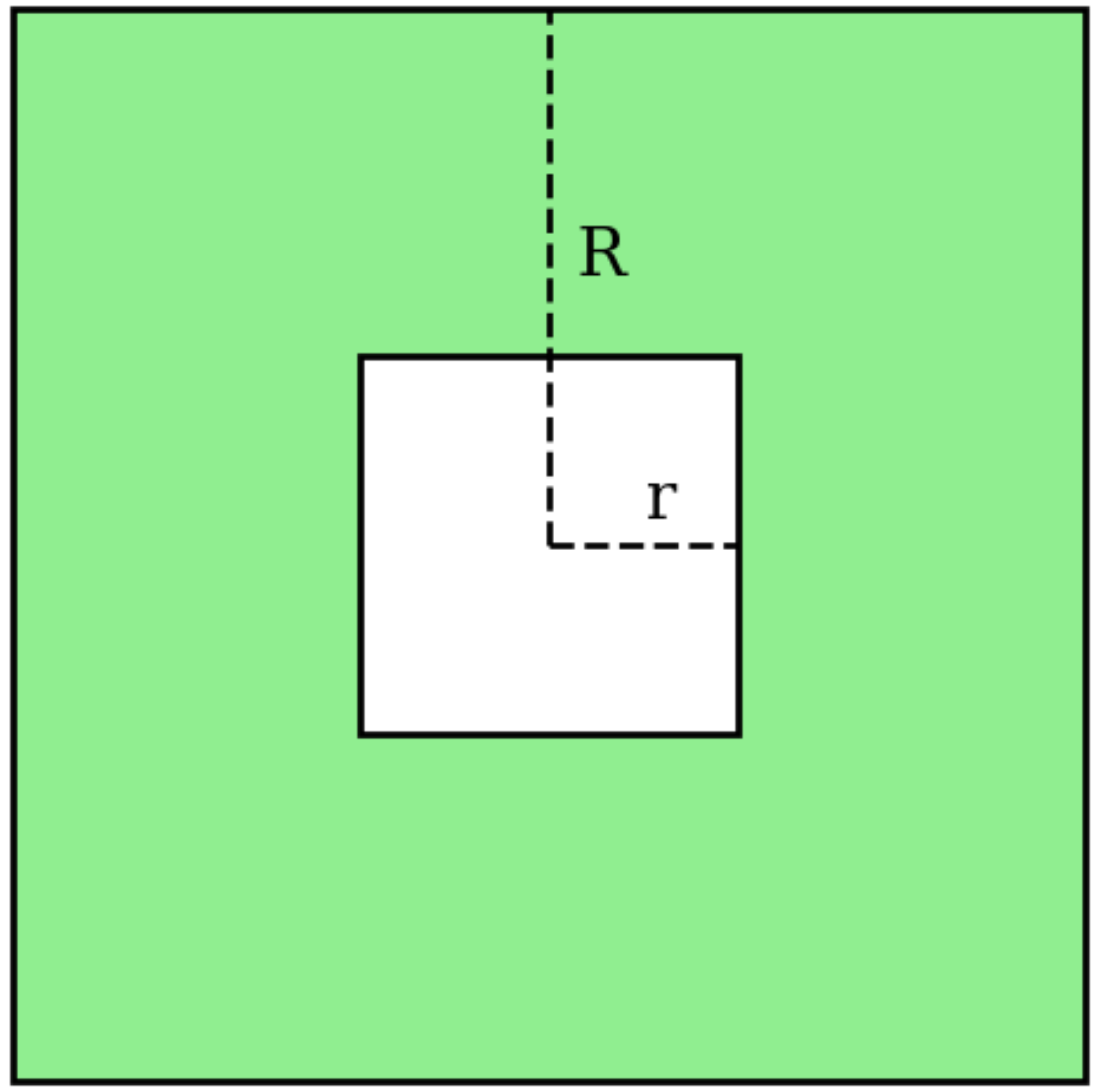
1 alternative path

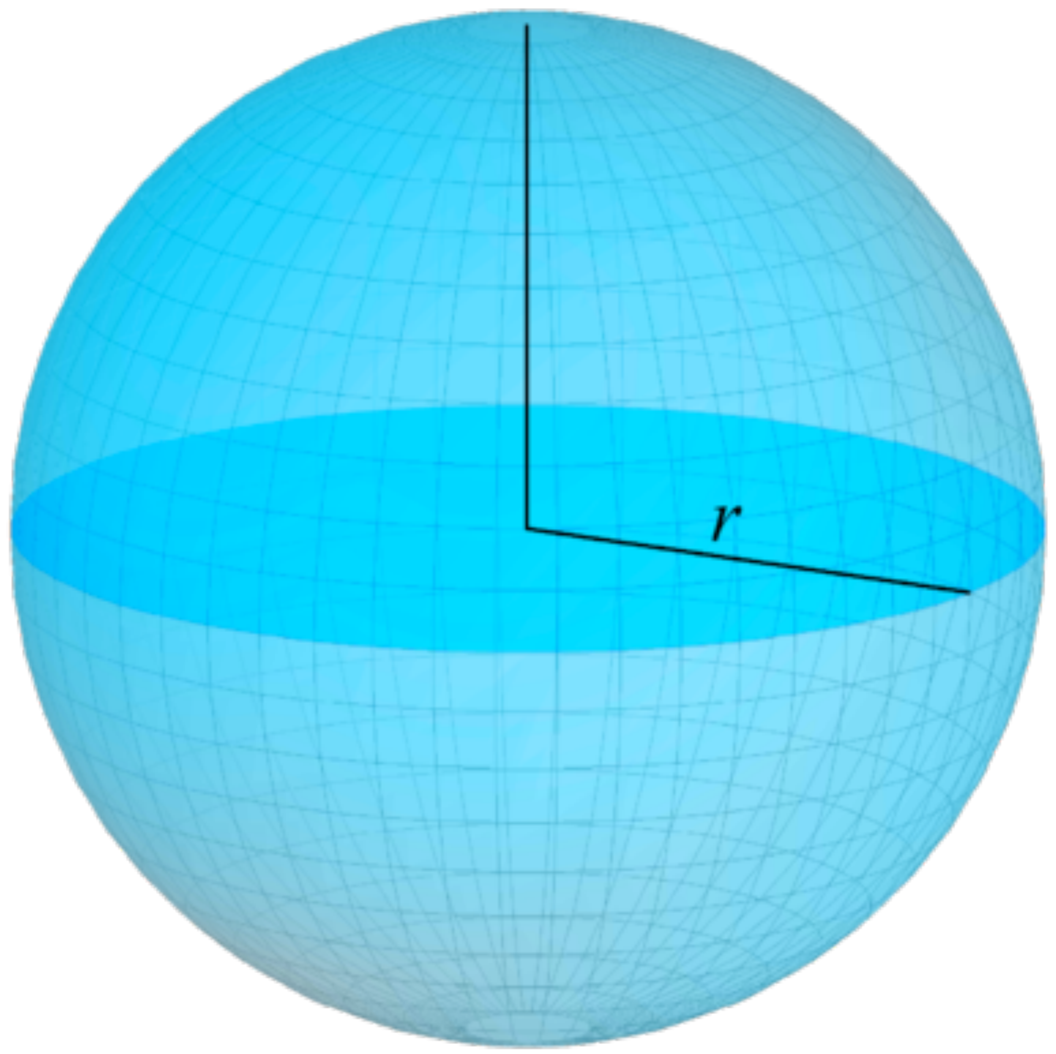


0 alternative path (slide through upper hemisphere)
1 alternative way to slide a path

Betti numbers count repeated connections “in all dimensions”.




R
r


r

Combinatorial Objects?

Simplicial Complex

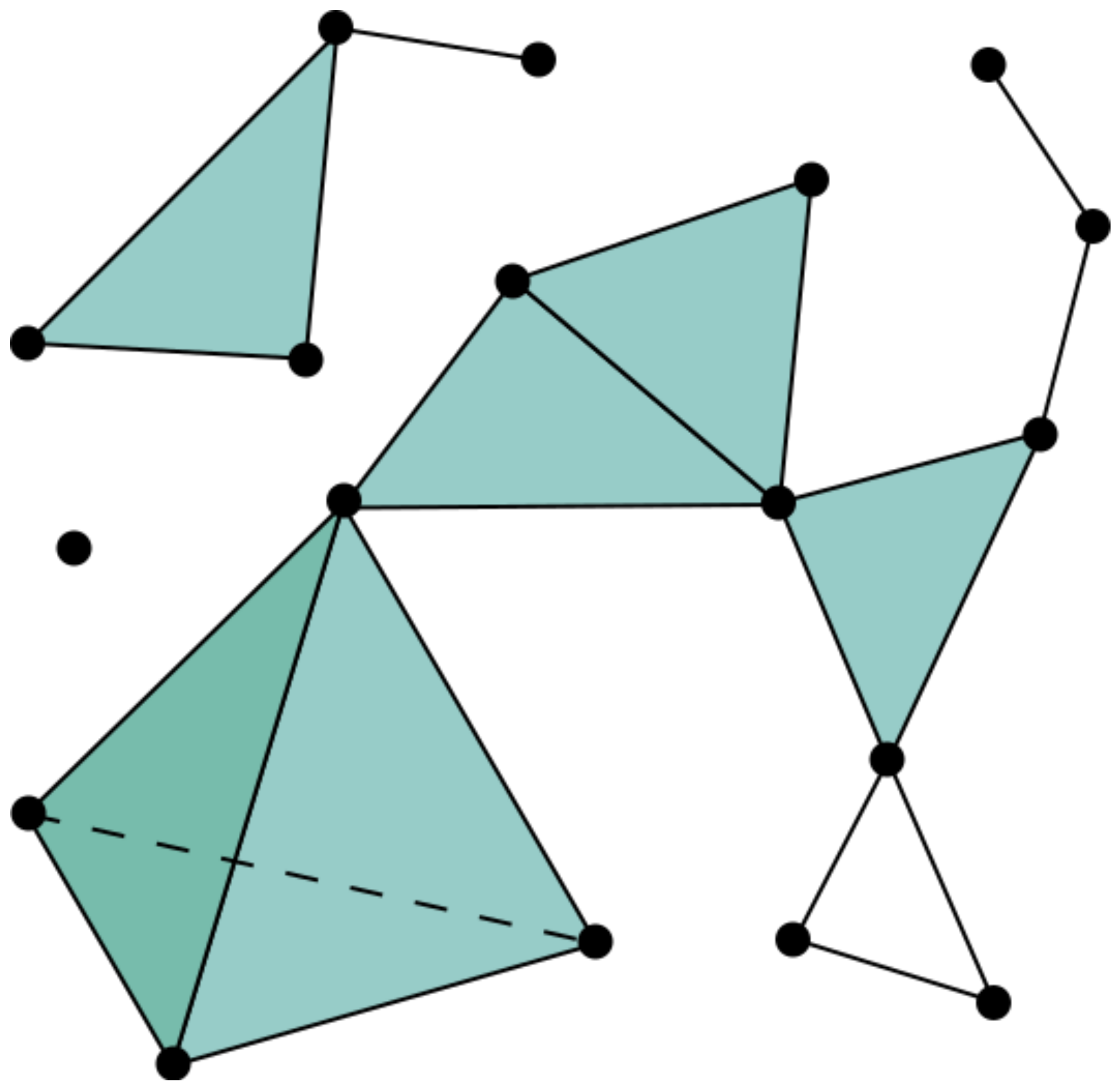
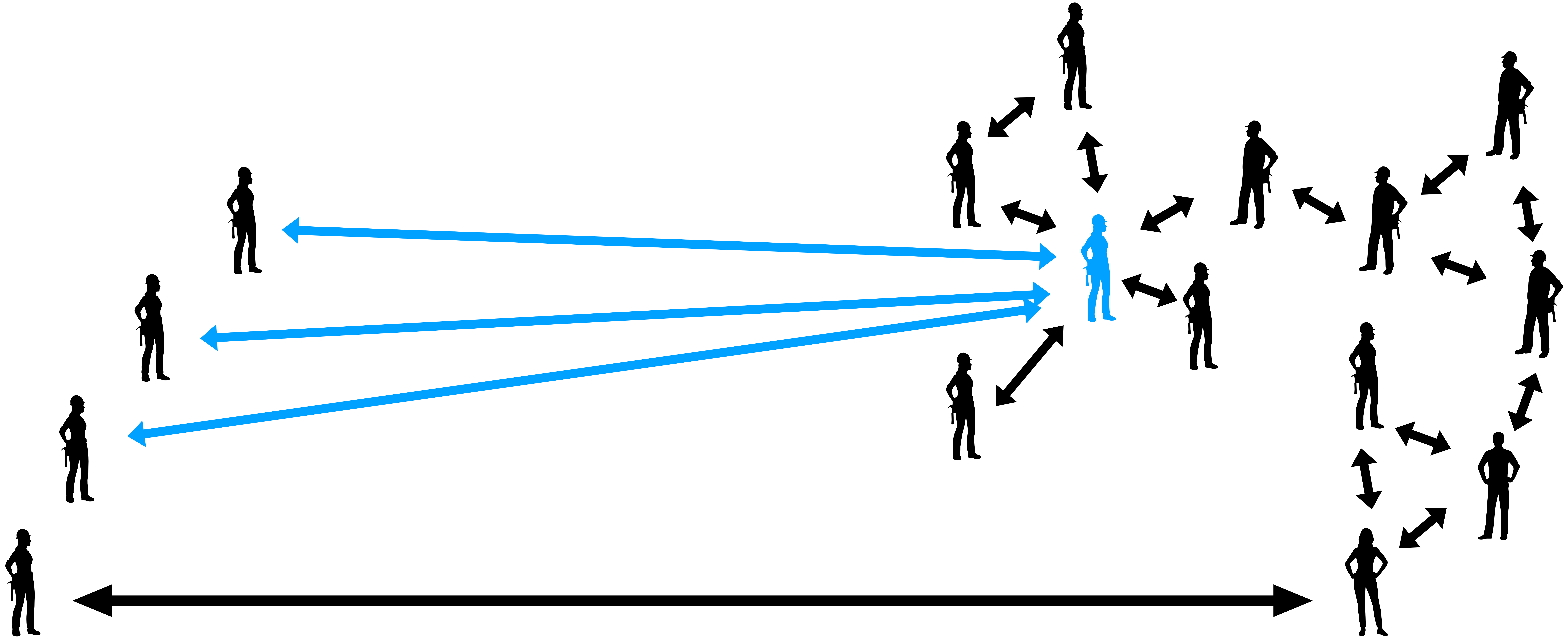


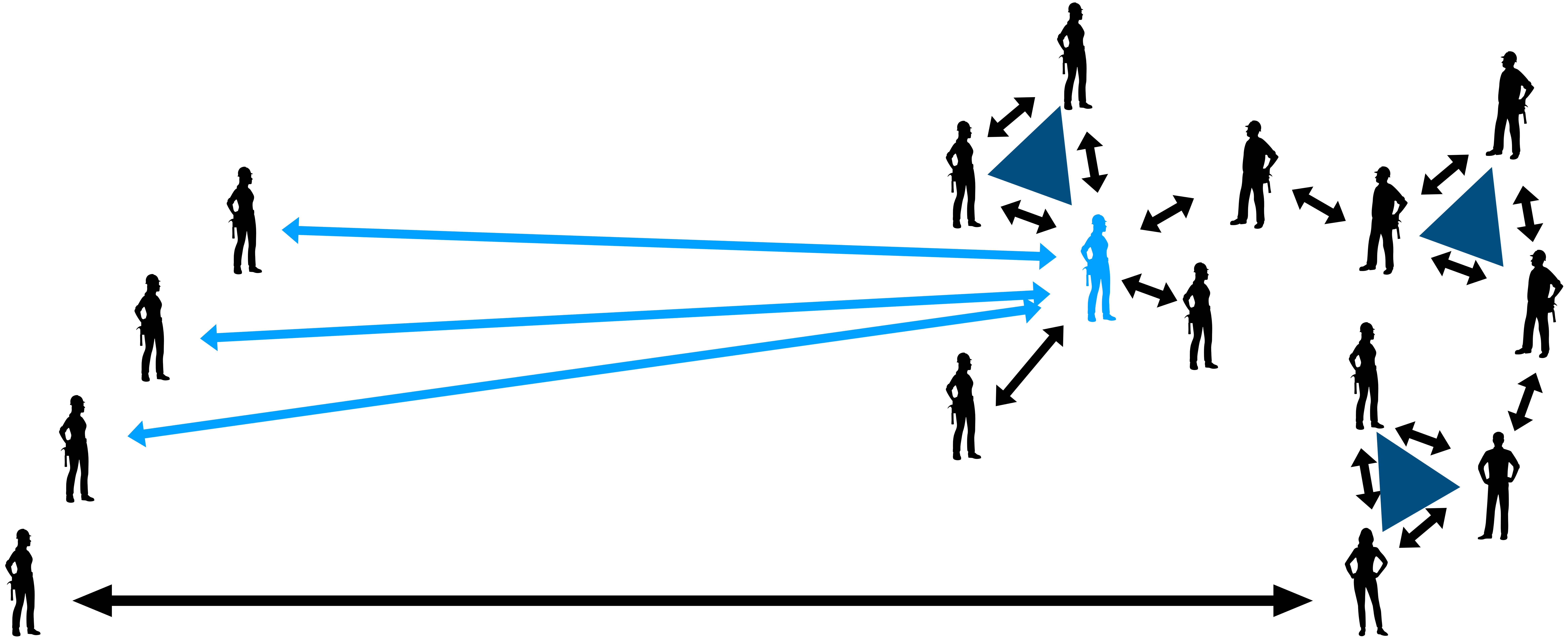
image credit: calm

Graphs?



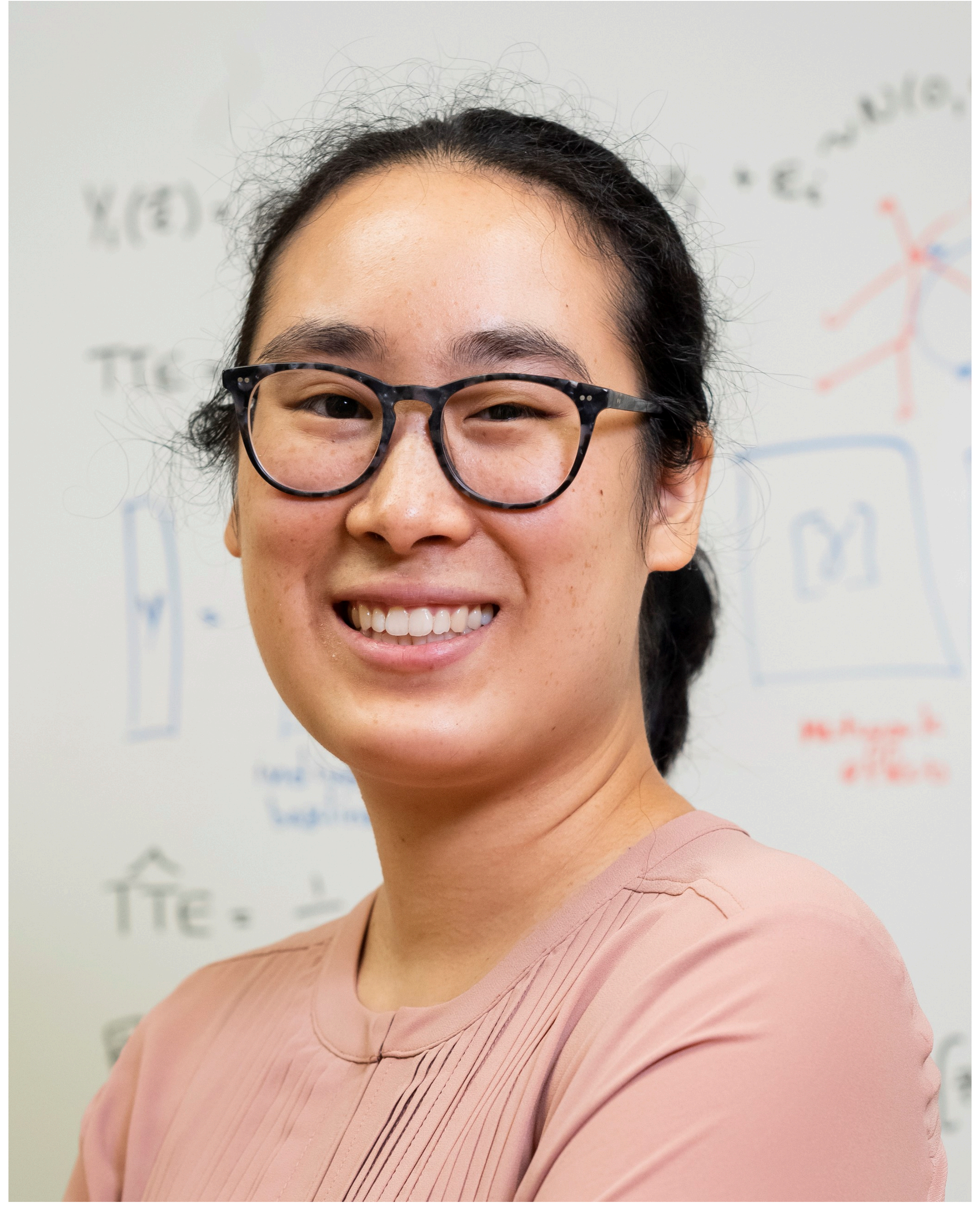
Clique Complex

aka Flag Complex



III Topology of Preferential Attachment

My Lovely Collaborators



Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

Expected Betti Number $E[\beta_q]$

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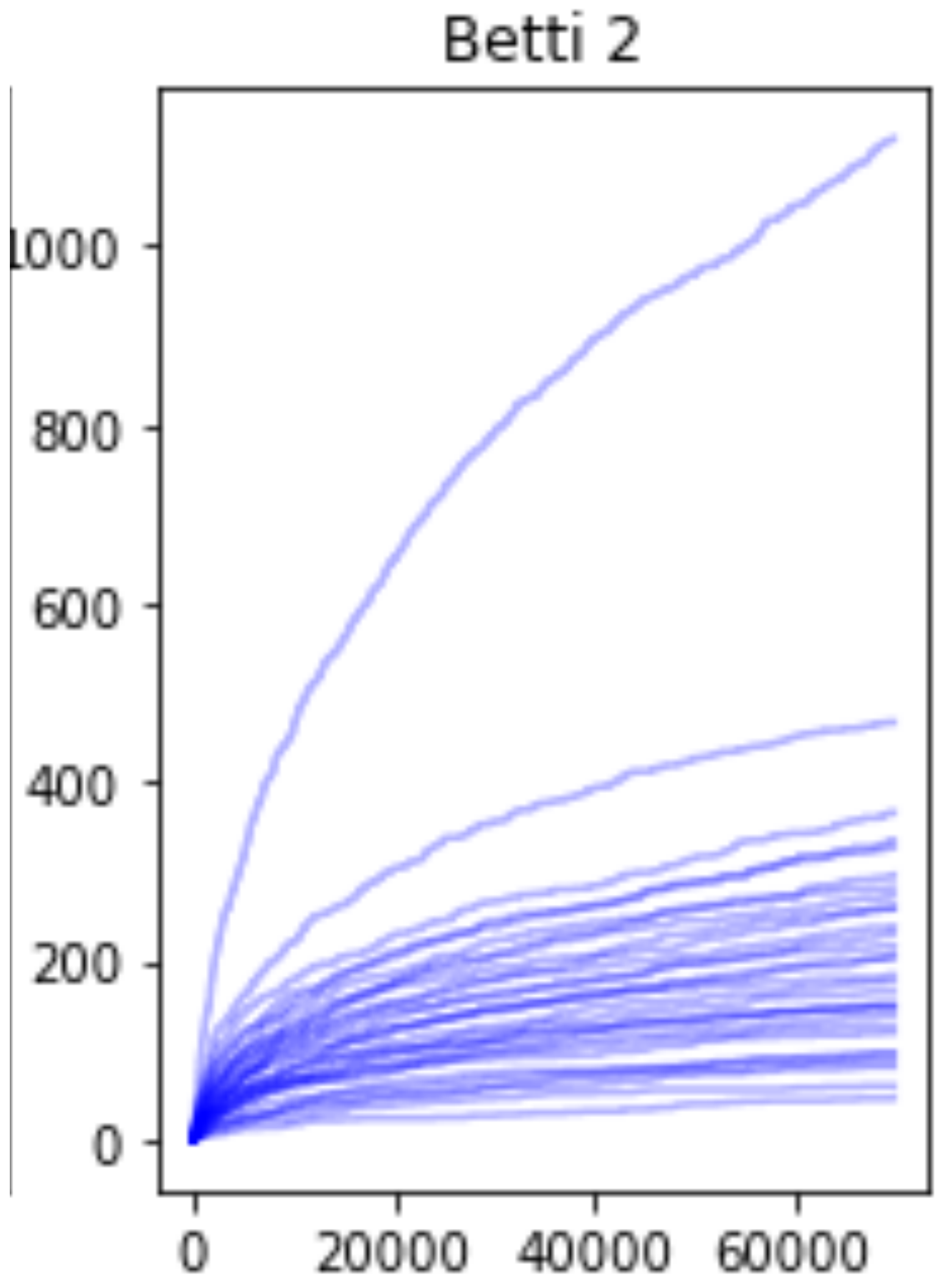
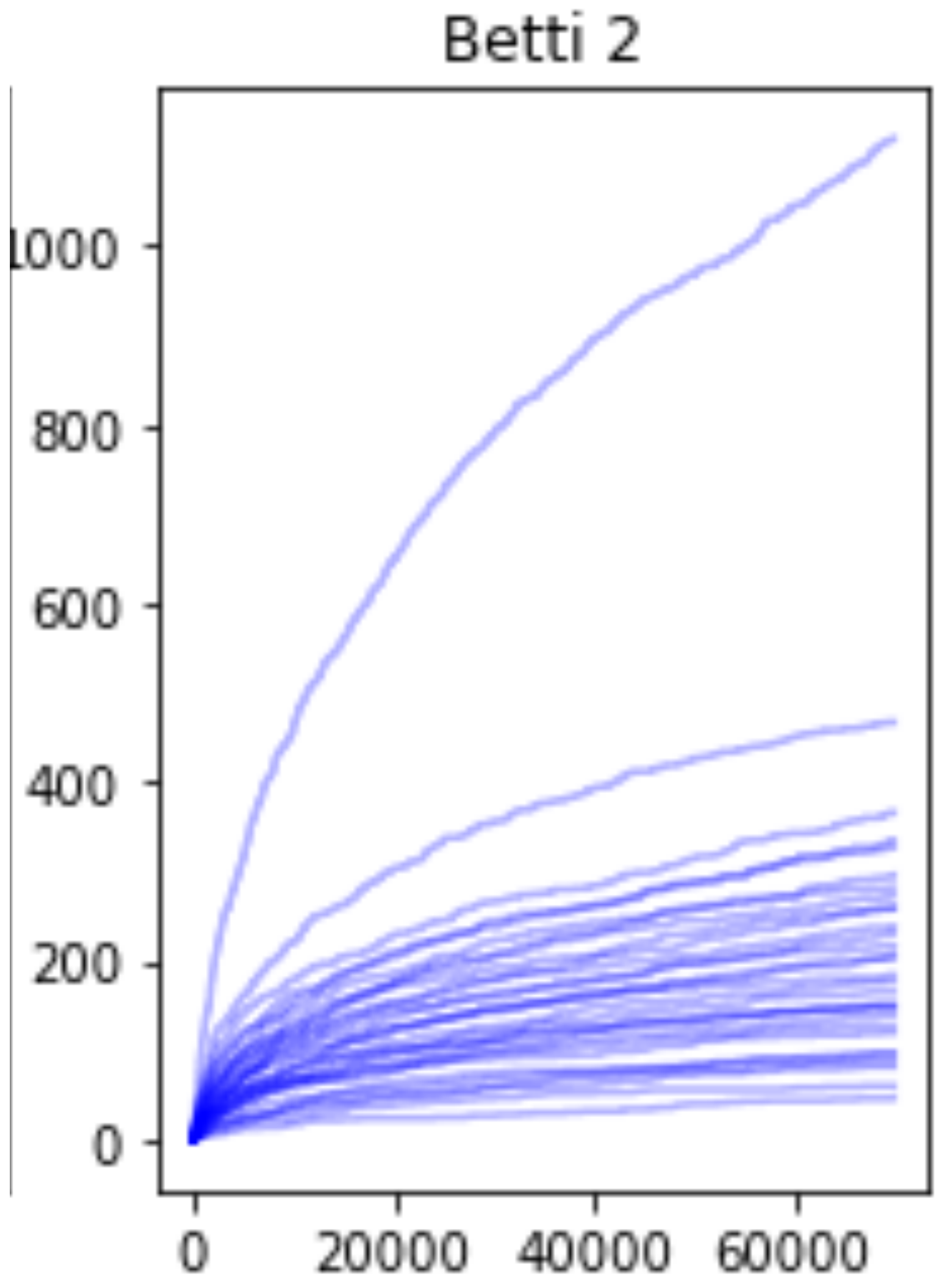


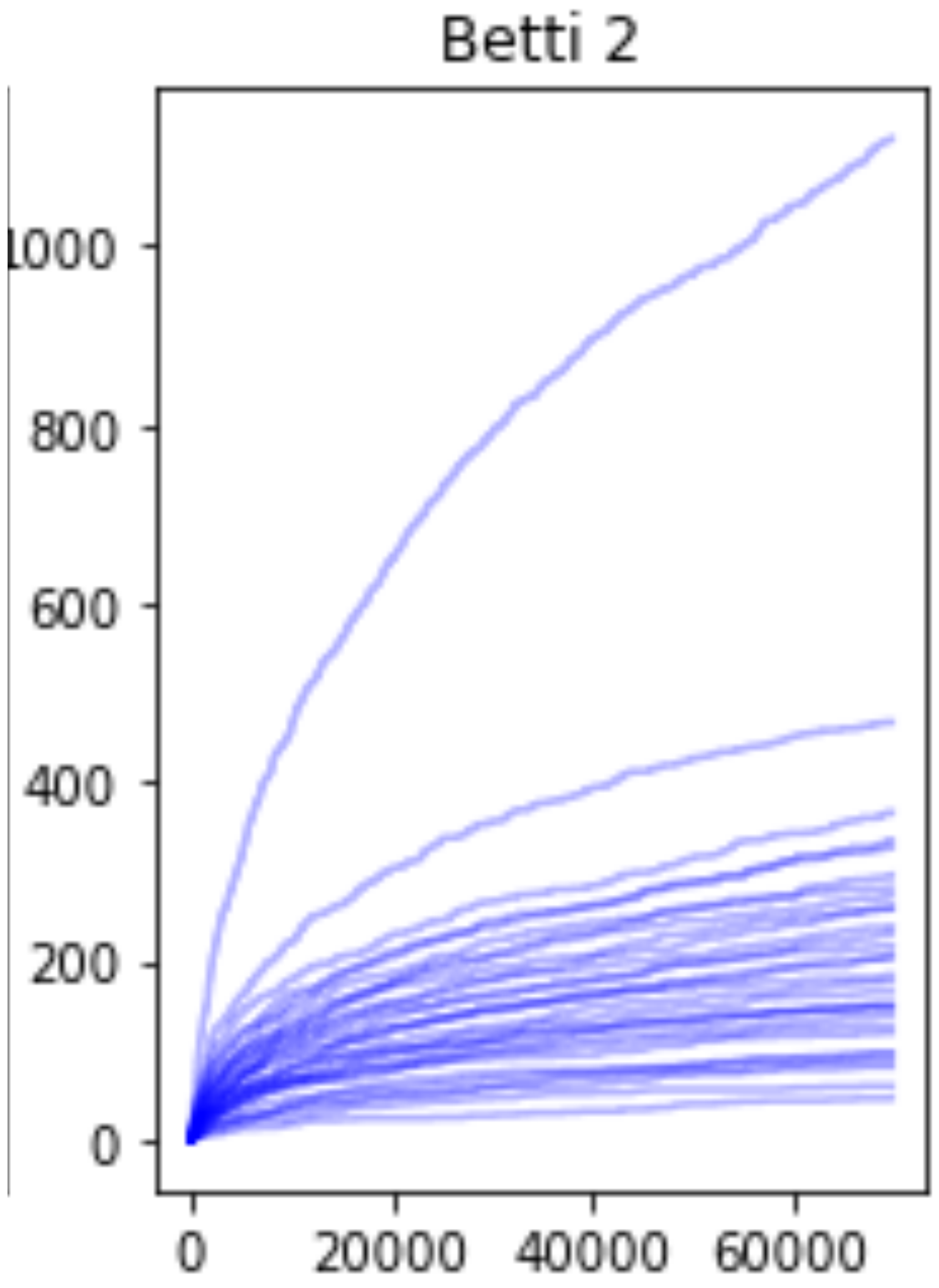
Expected Betti Number $E[\beta_q]$

- increasing trend



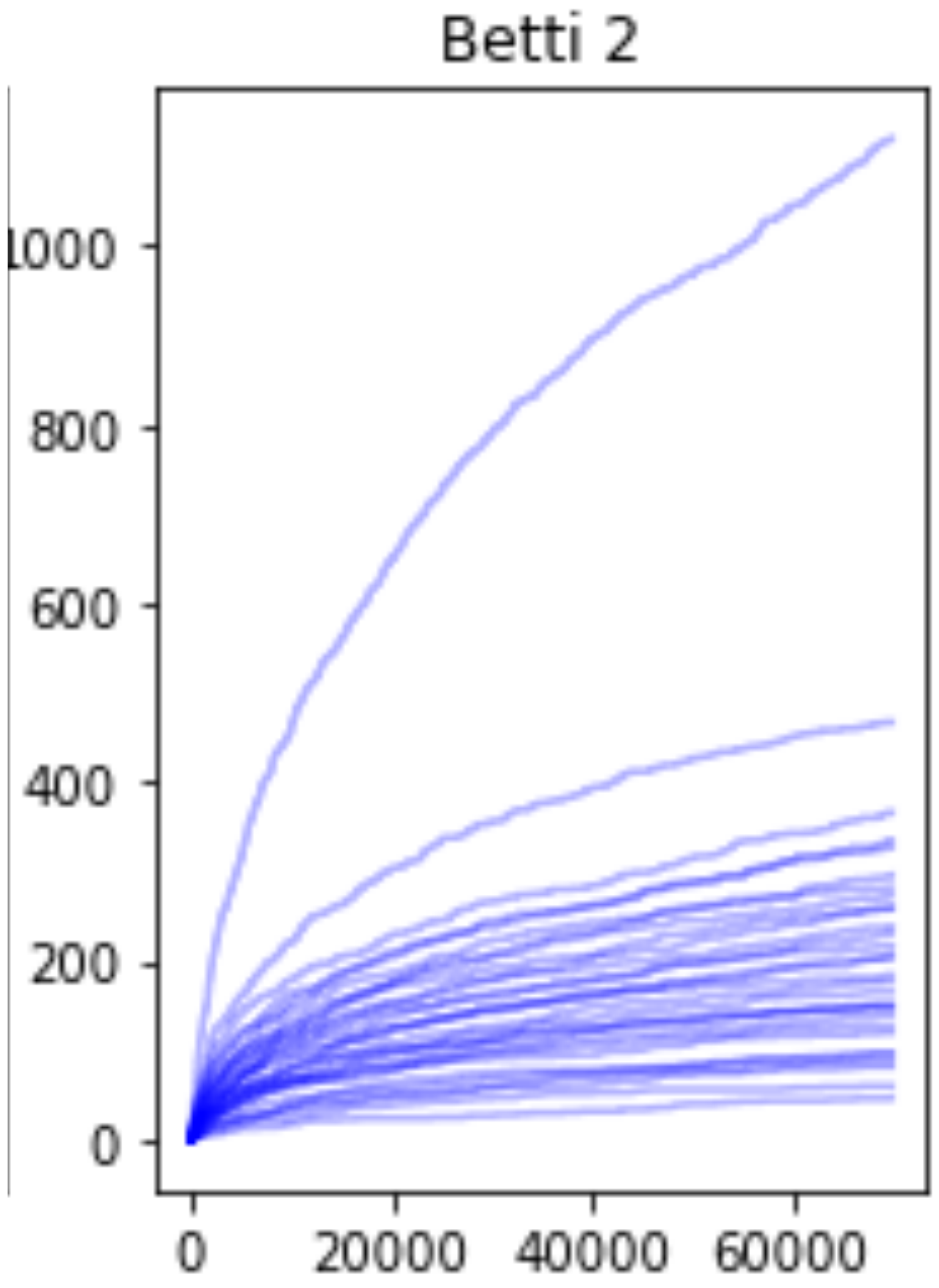
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth



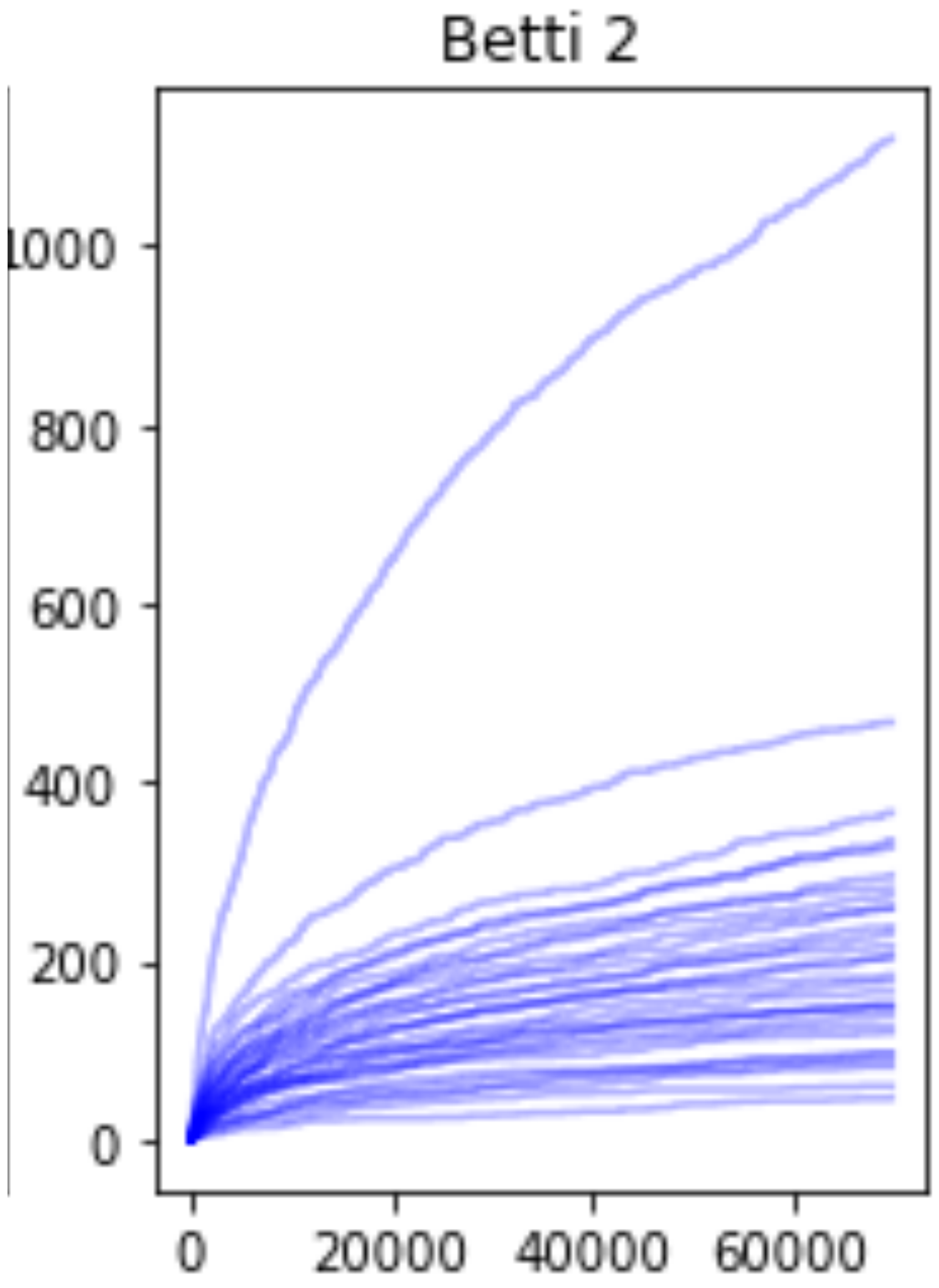
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth
- outlier



Expected Betti Number $E[\beta_q]$

- $c(\text{num of nodes}^{1-4x}) \leq E[\beta_2] \leq C(\text{num of nodes}^{1-4x})$
  under mild assumptions
  - $x \in (0,1/2)$ depends on model parameters



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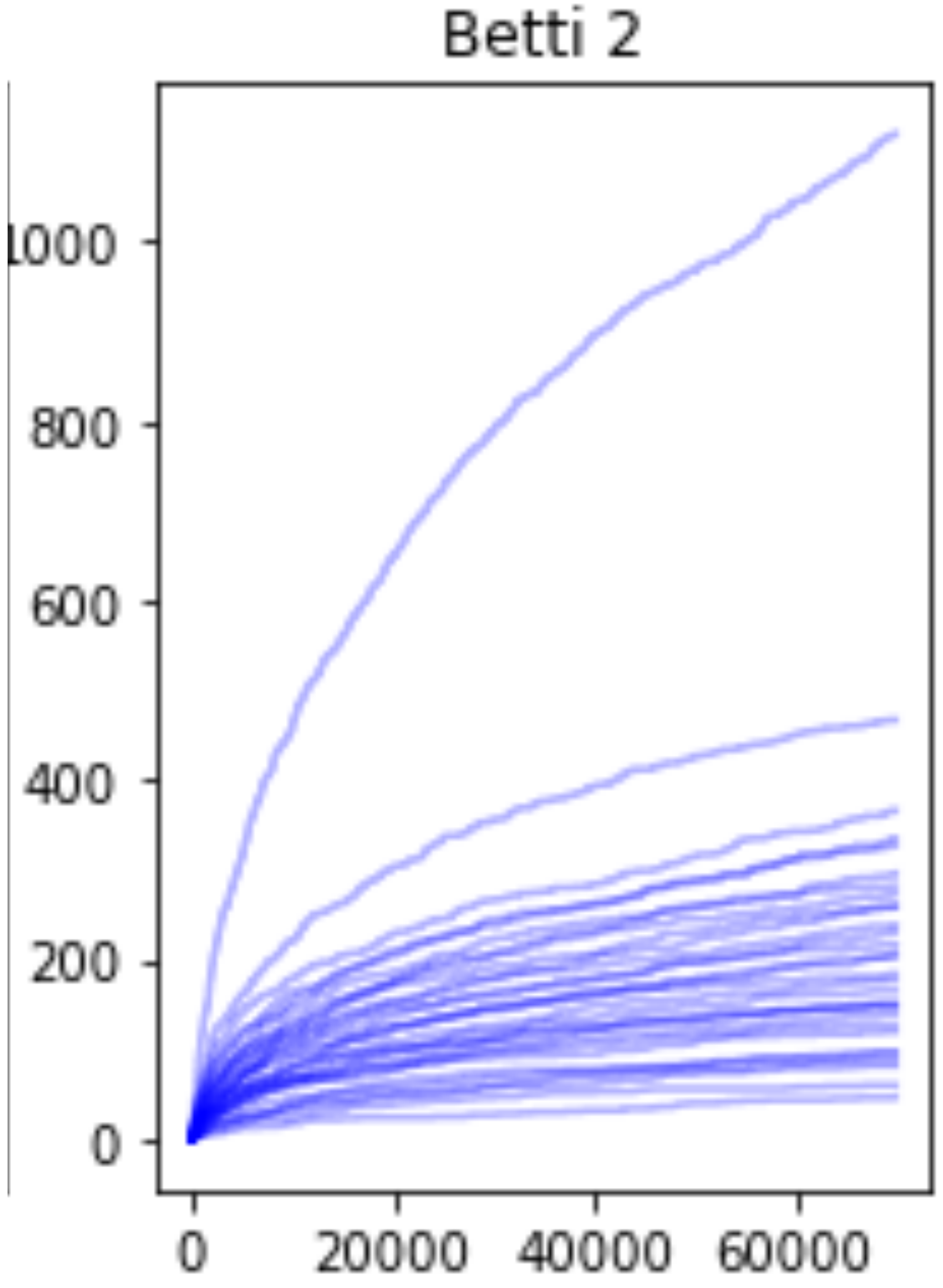
  - If $1 - 4x < 0$, then $E[\beta_2] \leq C$.



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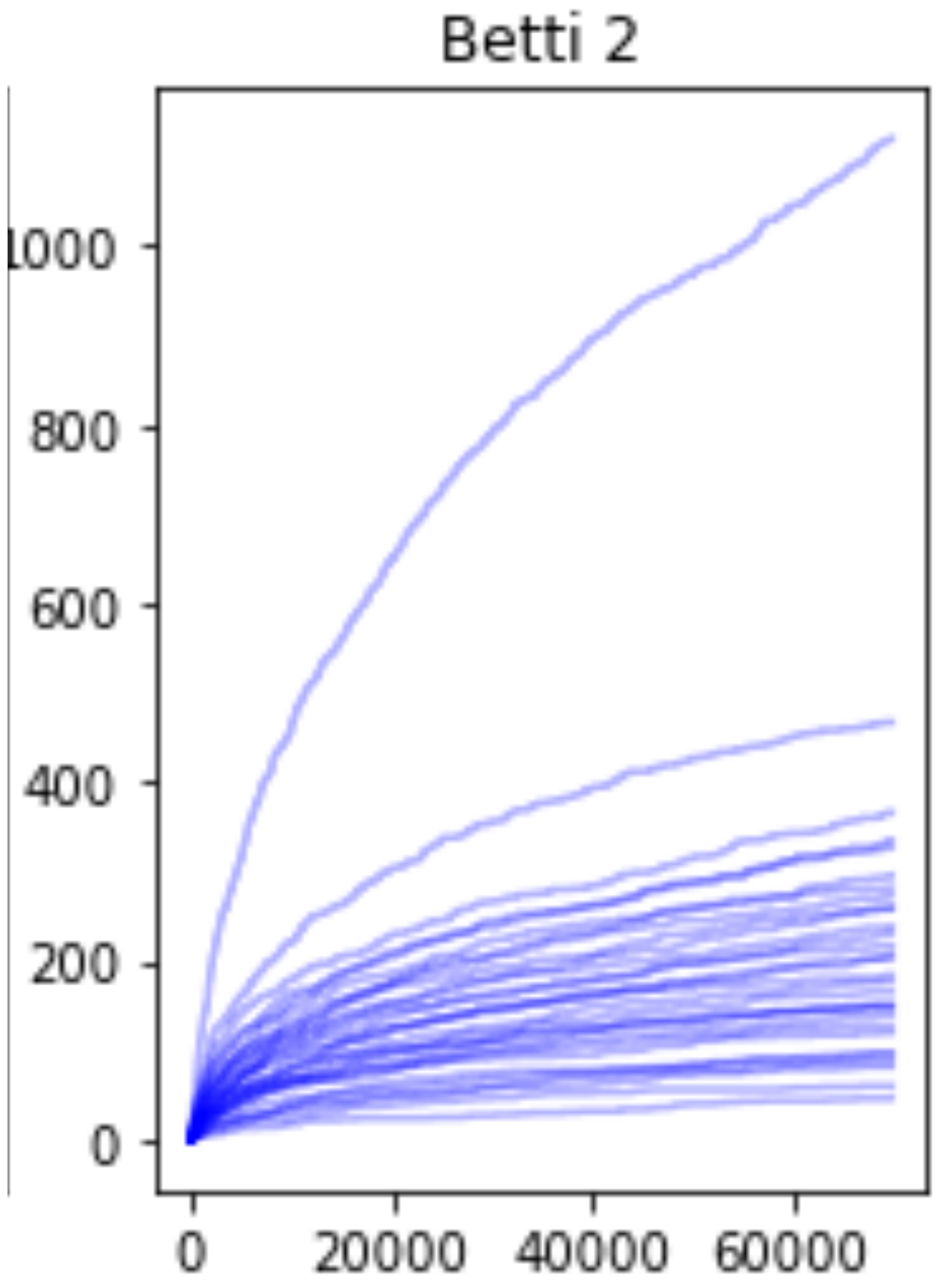
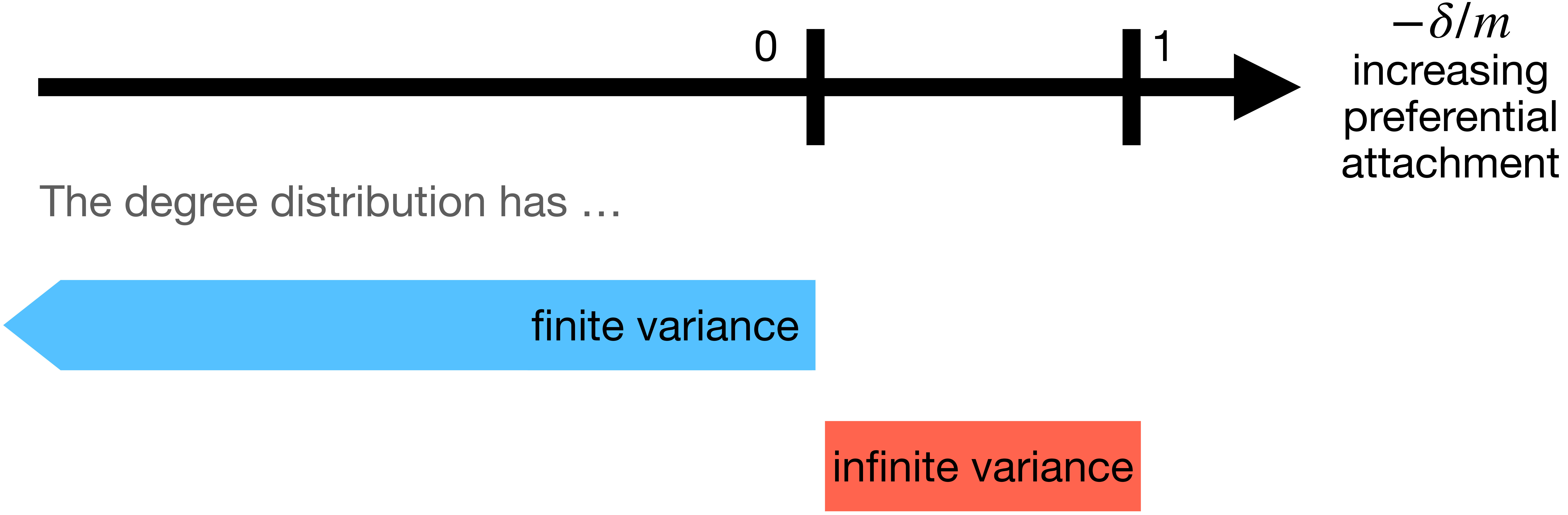
  under mild assumptions
  - $x \in (0,1/2)$ depends on model parameters
  - If $1 - 4x < 0$, then $E[\beta_2] \leq C$.
- $c(\text{num of nodes}^{1-2qx}) \leq E[\beta_q] \leq C(\text{num of nodes}^{1-2qx})$ for $q \geq 2$ if $1 - 2qx > 0$



Phase transition

Recall
P(attaching to v) ∝ degree + δ
m = number of edges per new node



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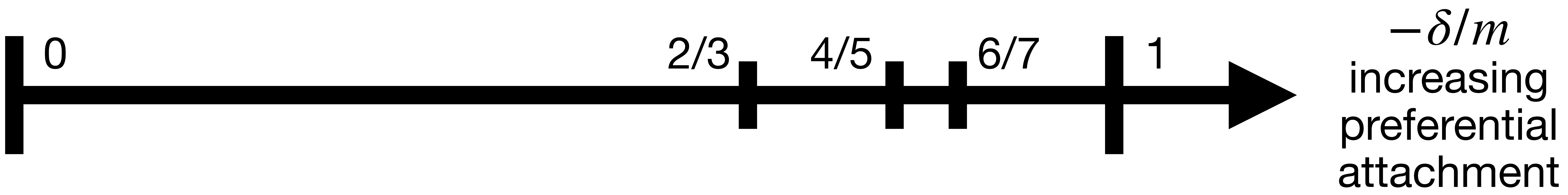

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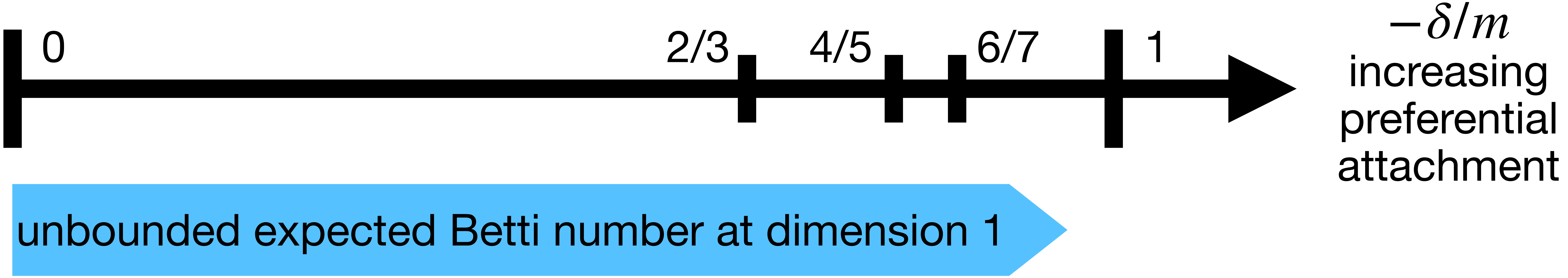

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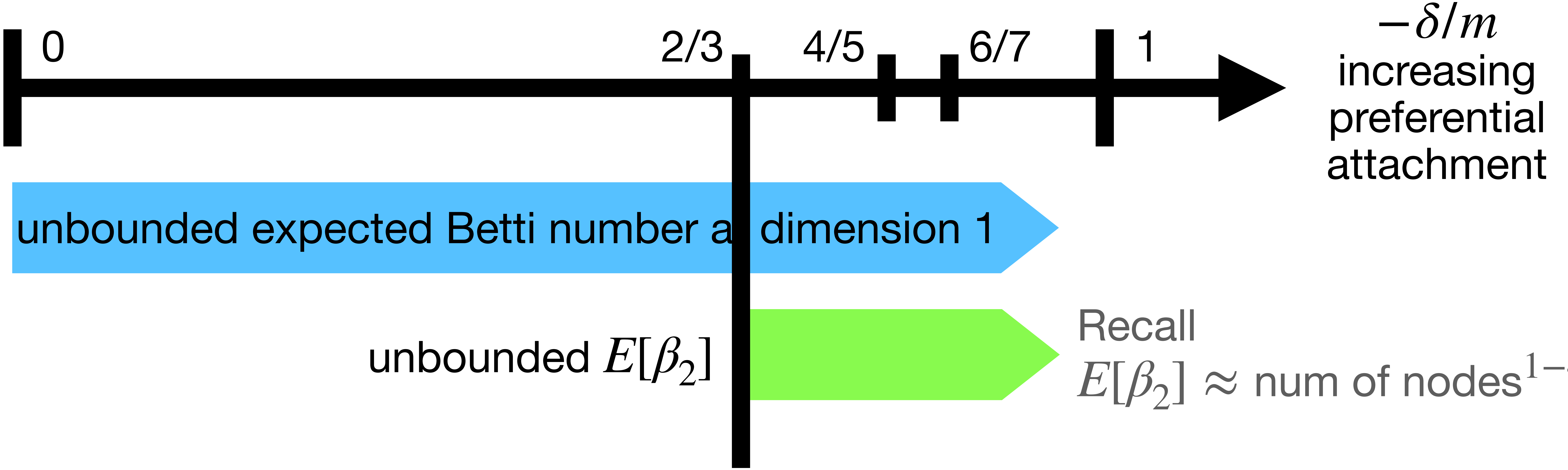

Recall
$E[\beta_2] \approx$ num of nodes^{1-4x}

Phase transition

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P(attaching to v) ∝ degree + δ

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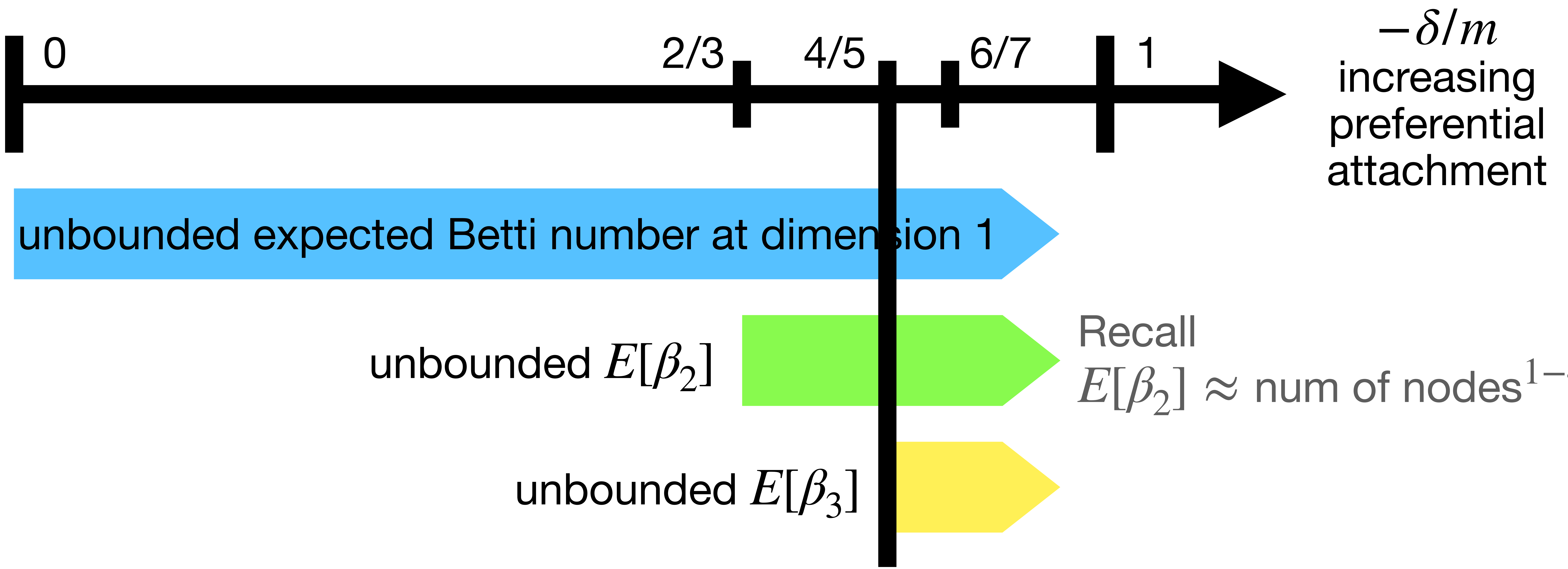

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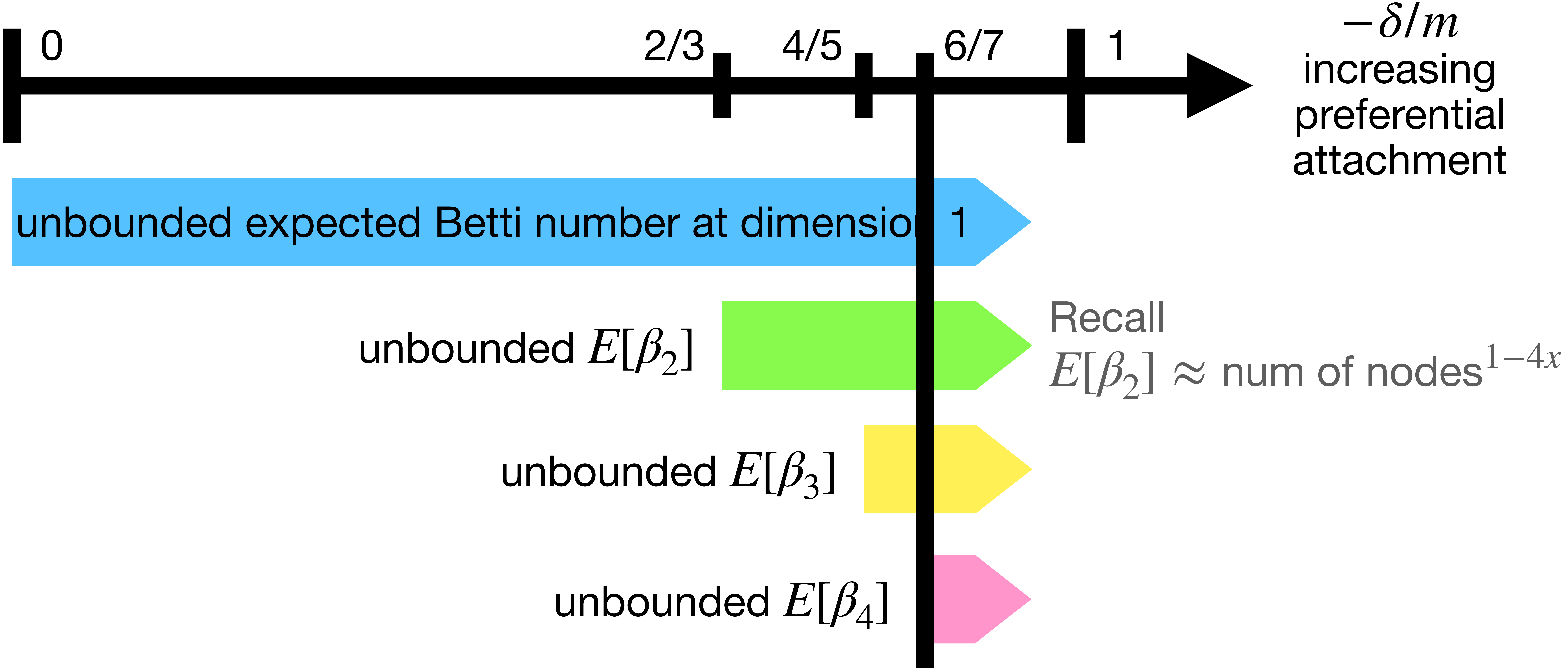

Phase transition

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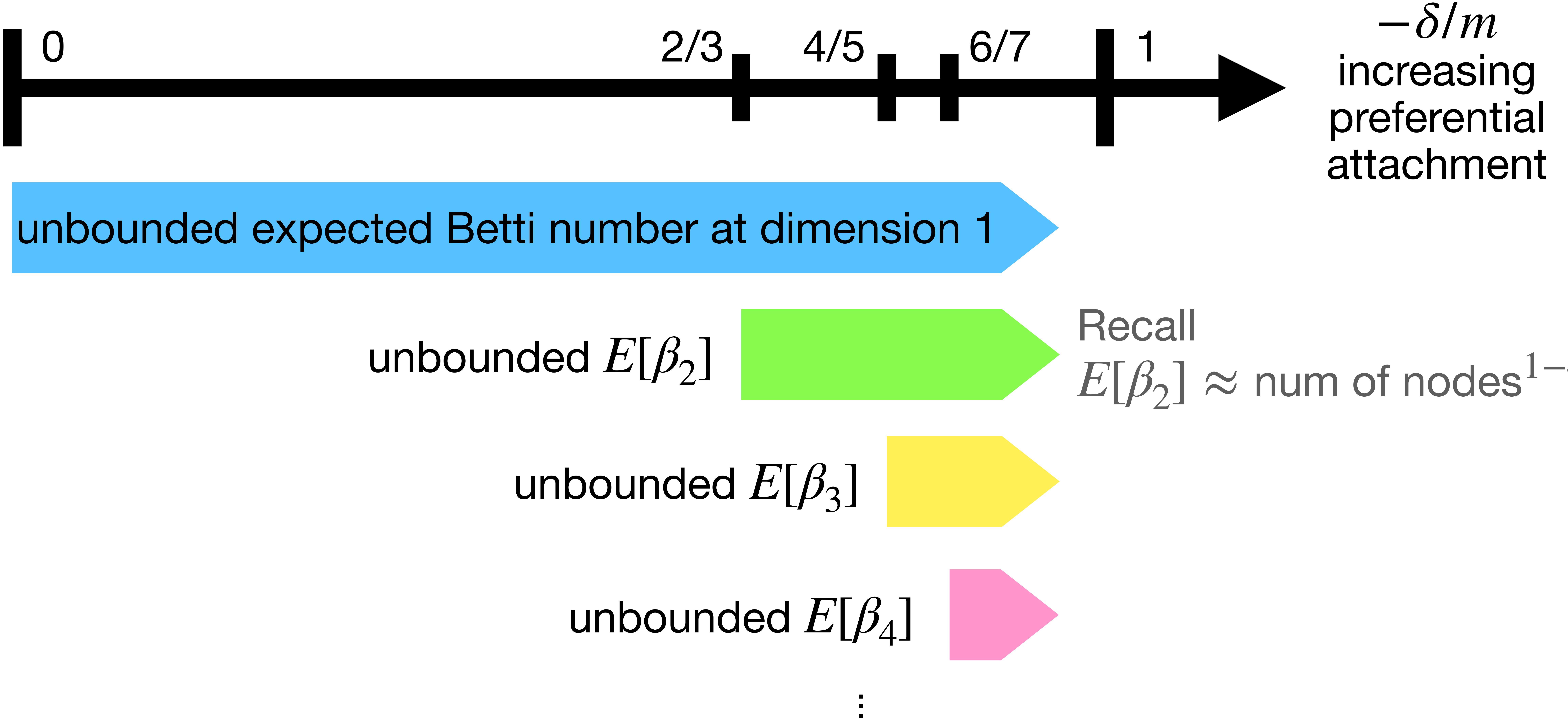
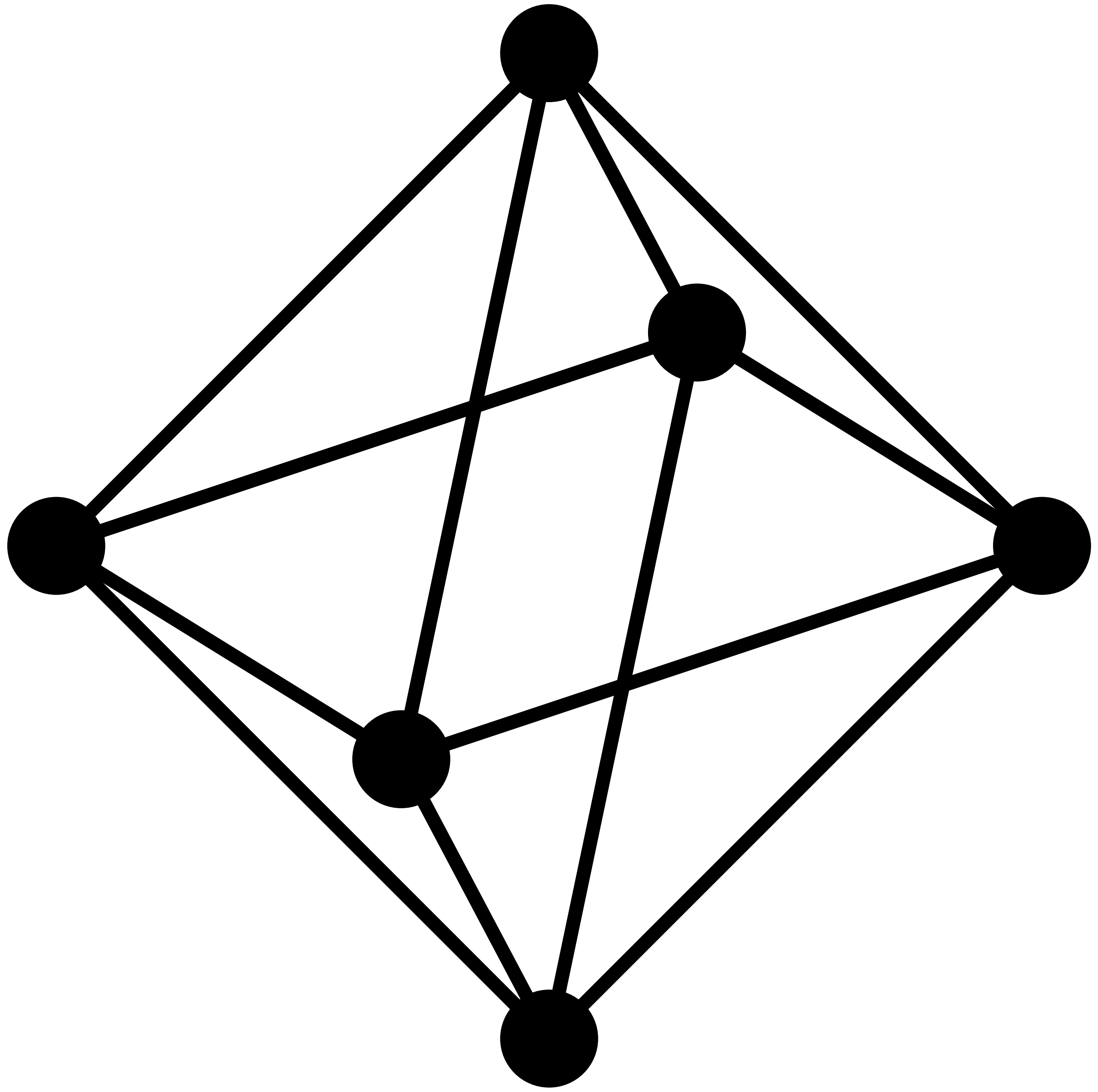


Recall
$E[\beta_2] \approx$ num of nodes^{1-4x}

Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

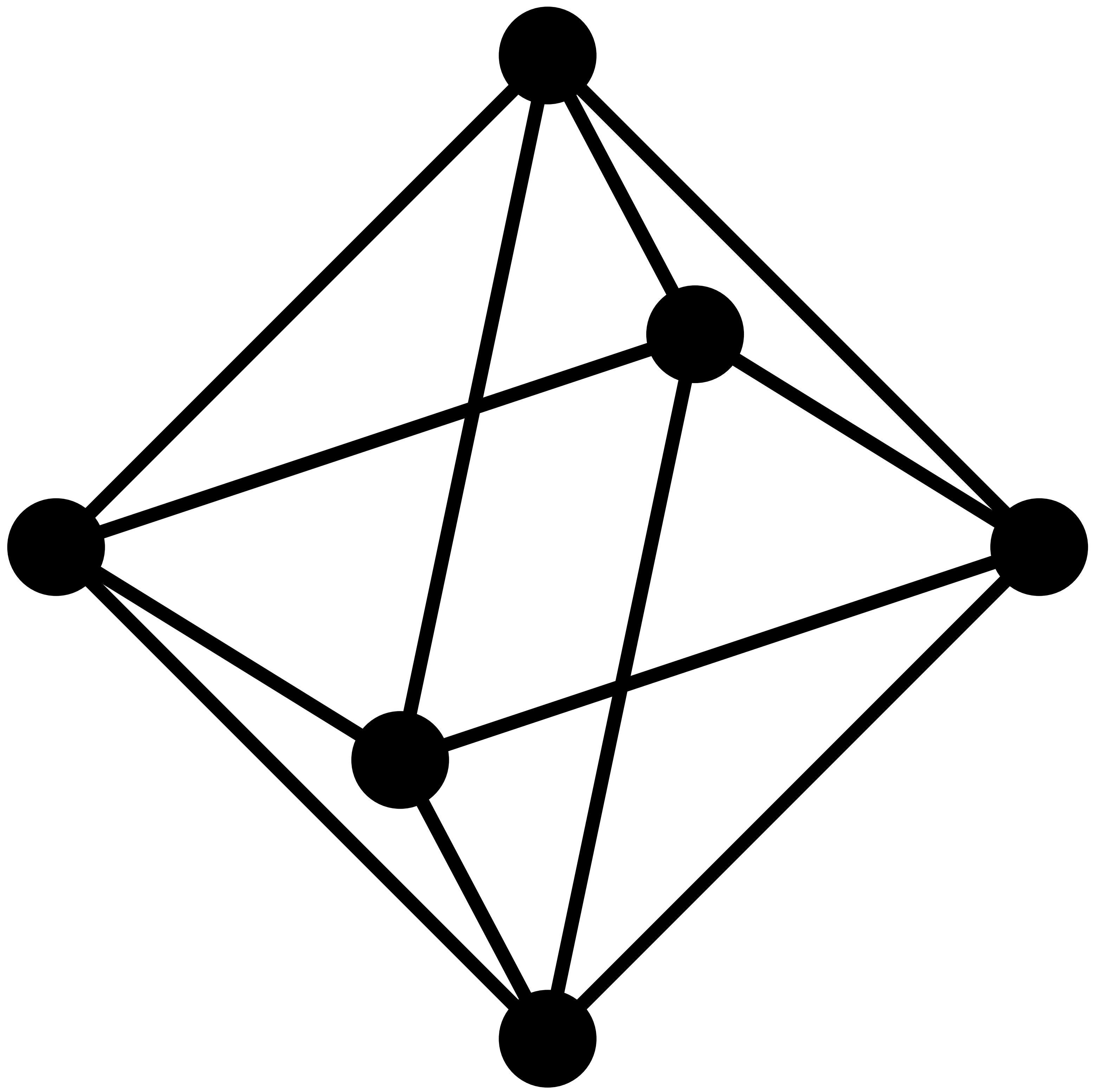
Proof?

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



$\beta_2 = 1$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



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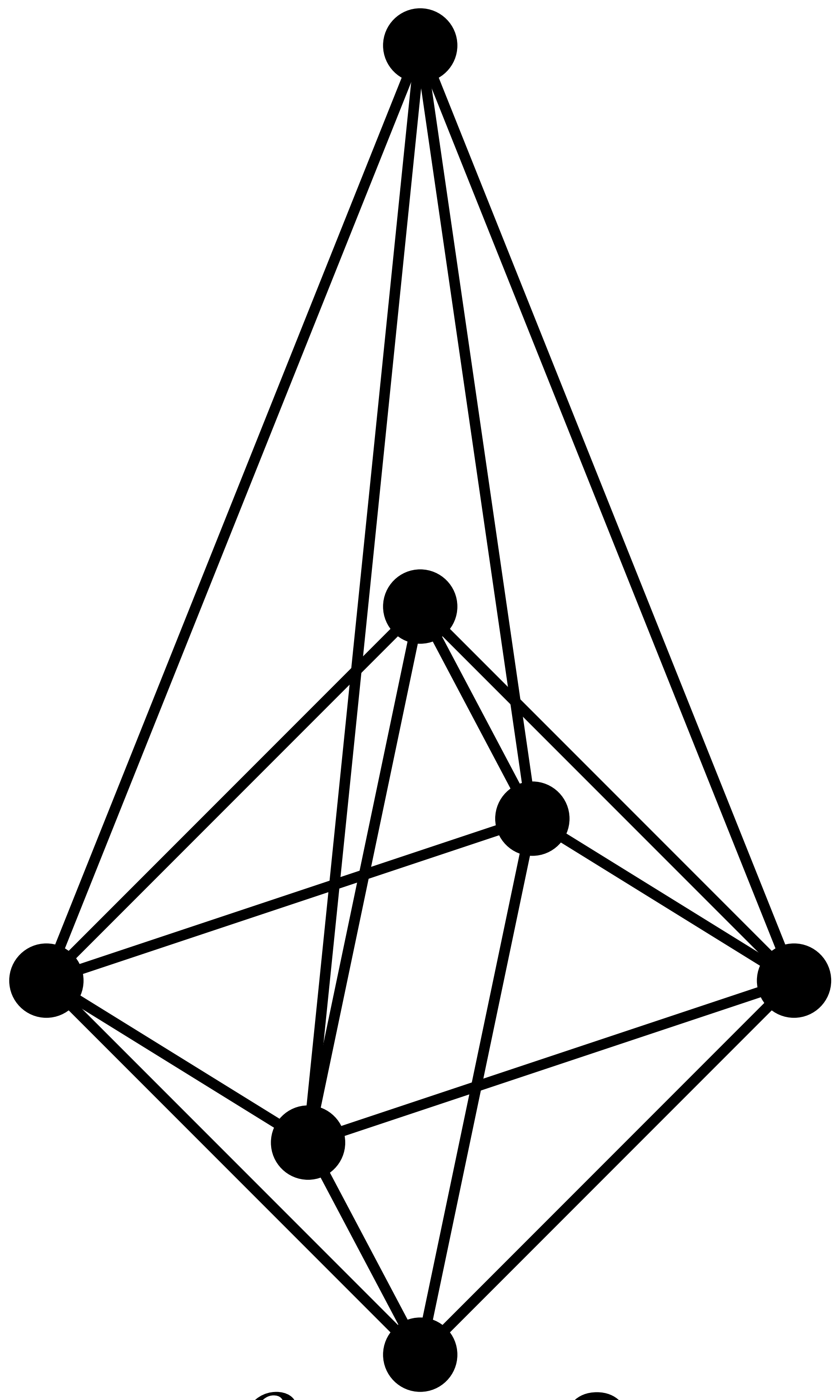
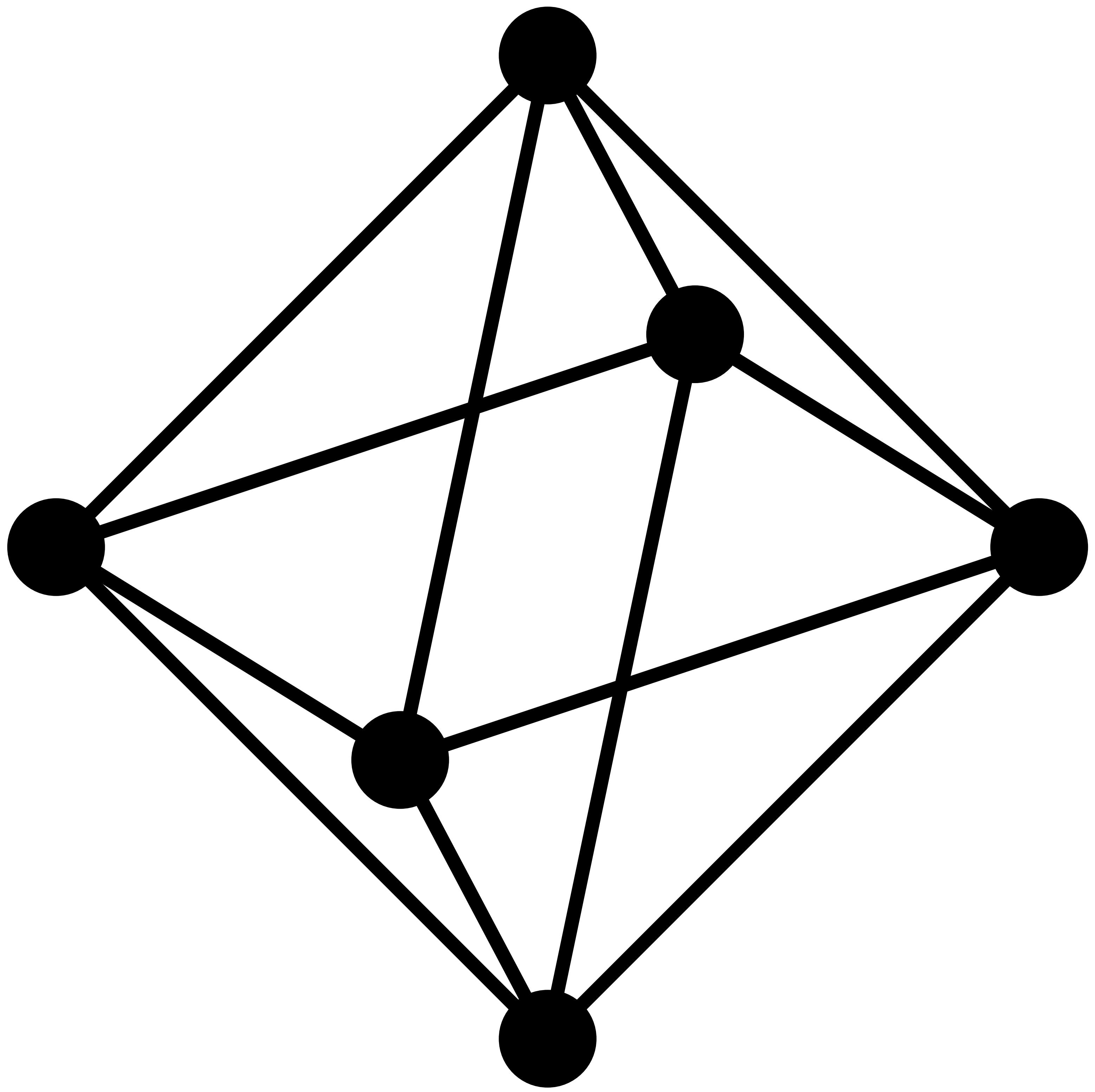
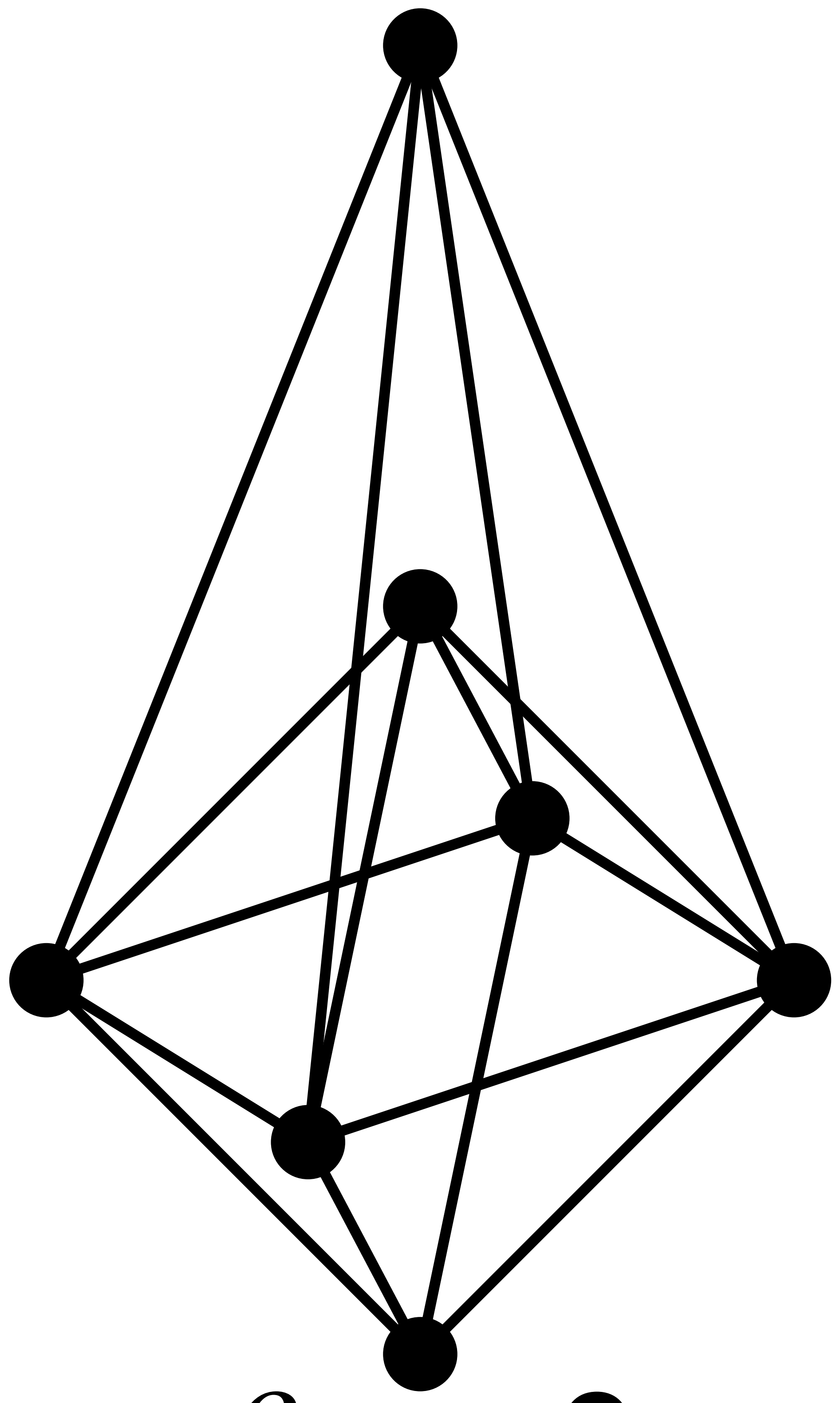


$\beta_2 = 2$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



$\beta_2 = 1$



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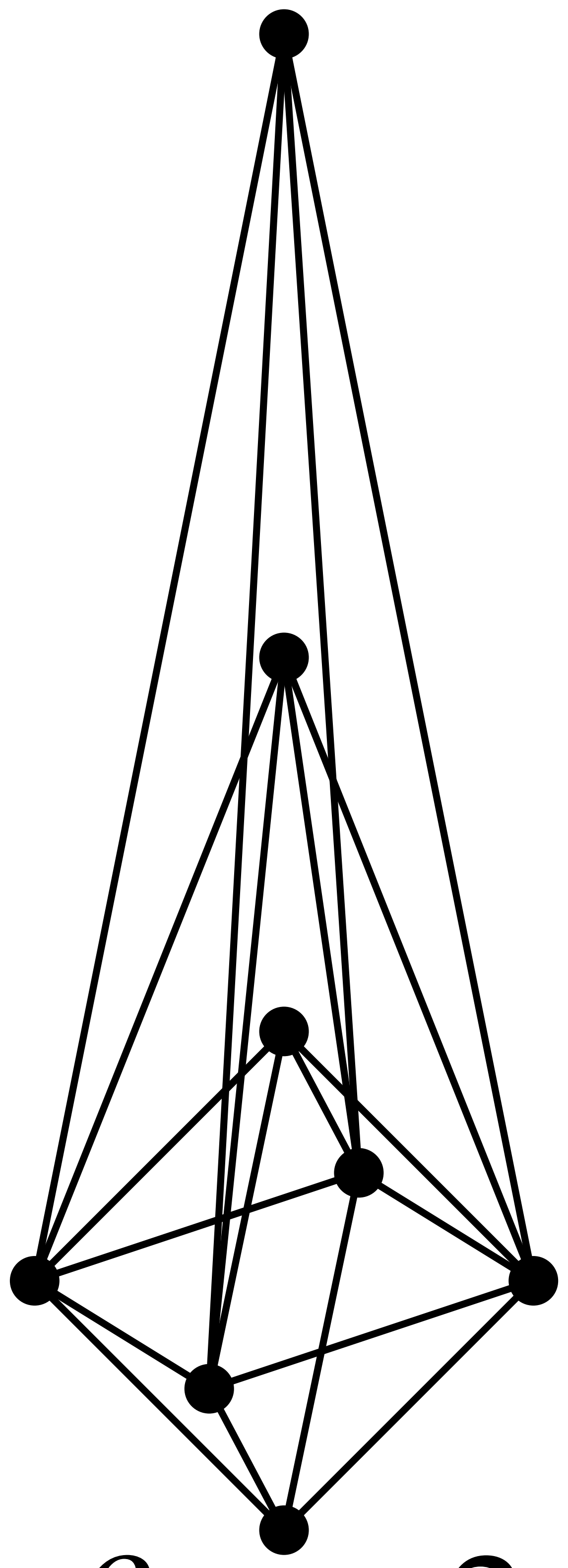

$\beta_2 = 3$

Subtleties

- Need homological algebra to relate Betti numbers with counts

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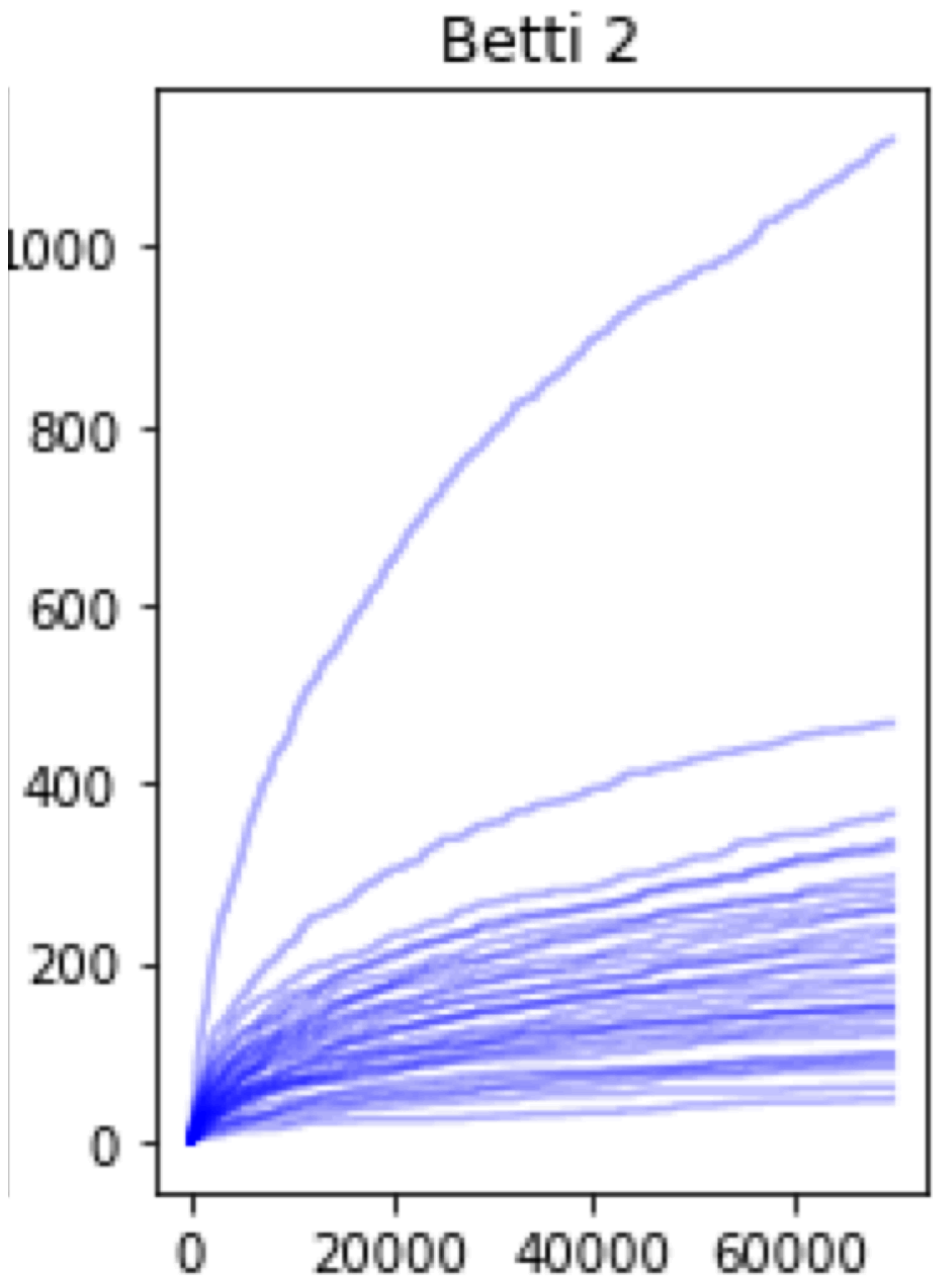
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results in the language of homological algebra
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs

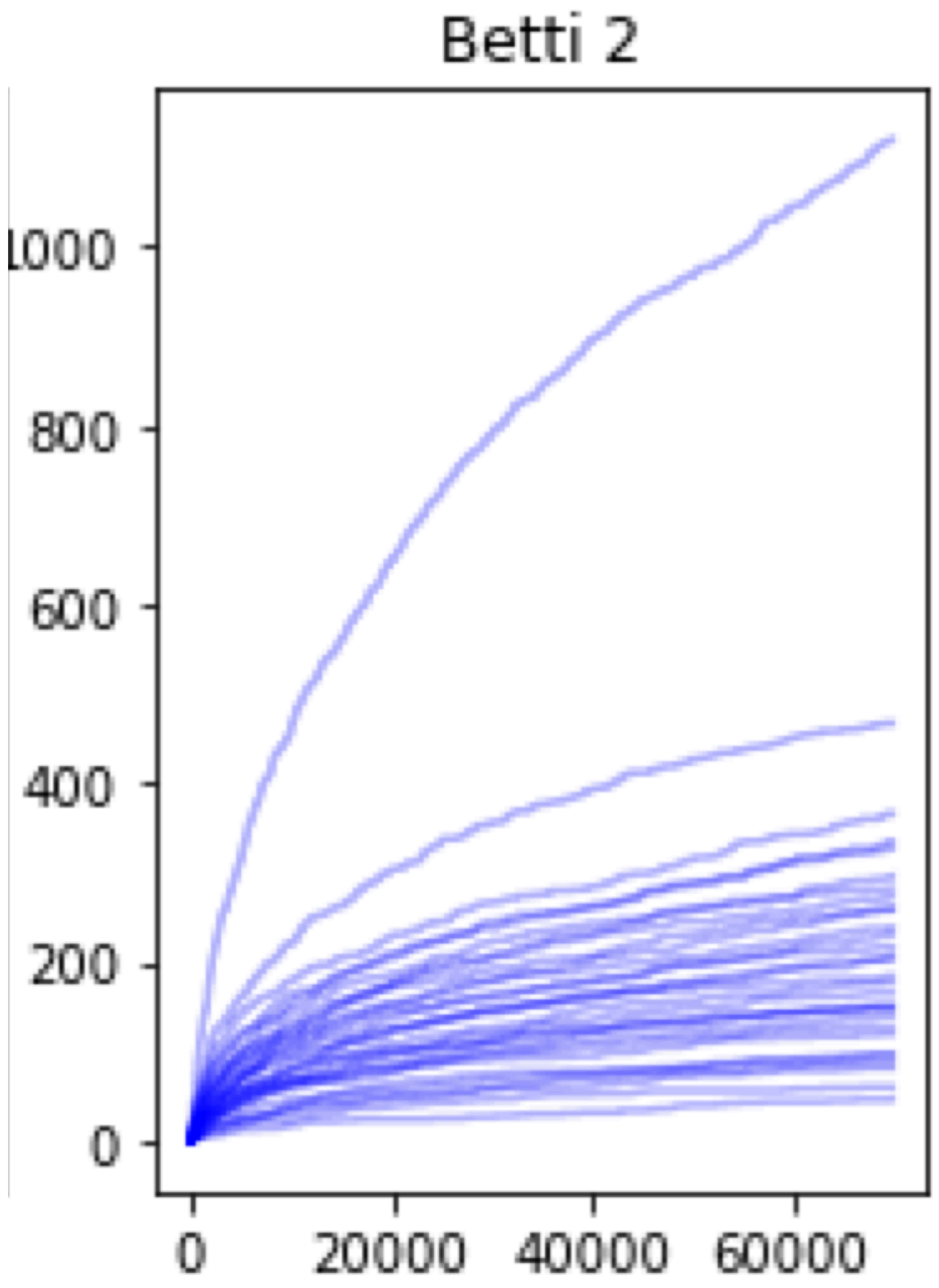
Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

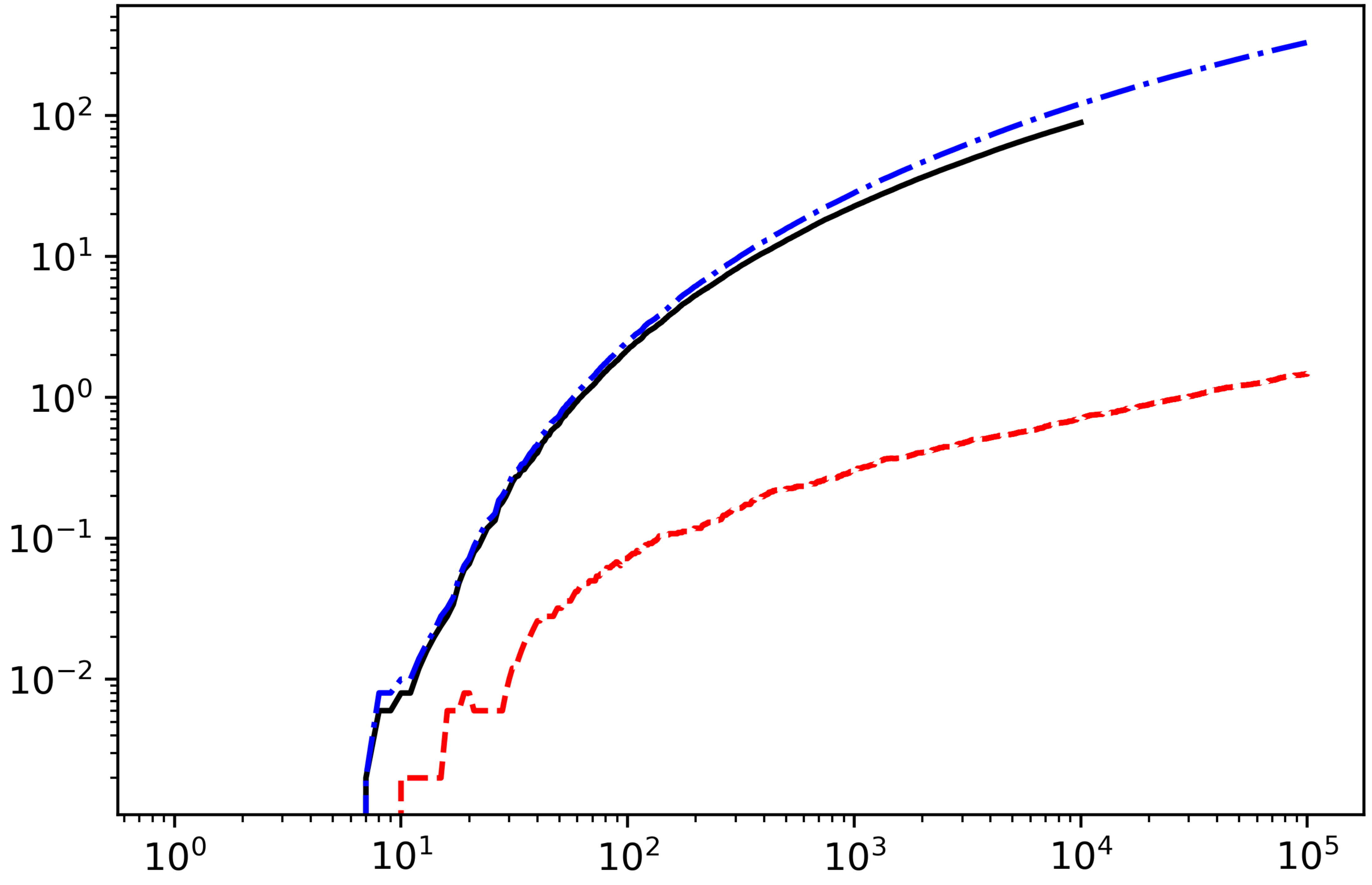
In practice???

$E[\beta_2] \approx$ **num of nodes**$^{1-4x}$

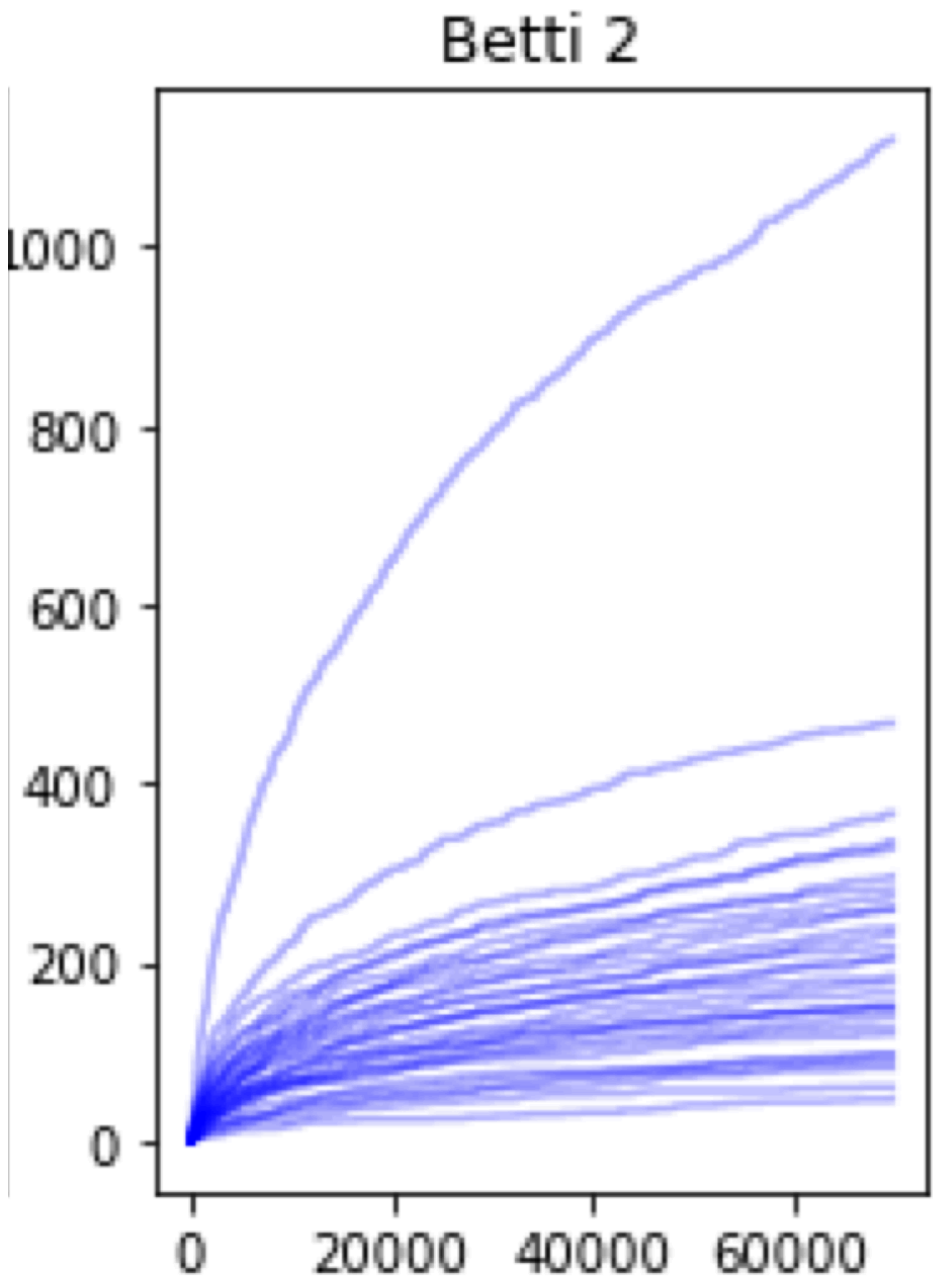


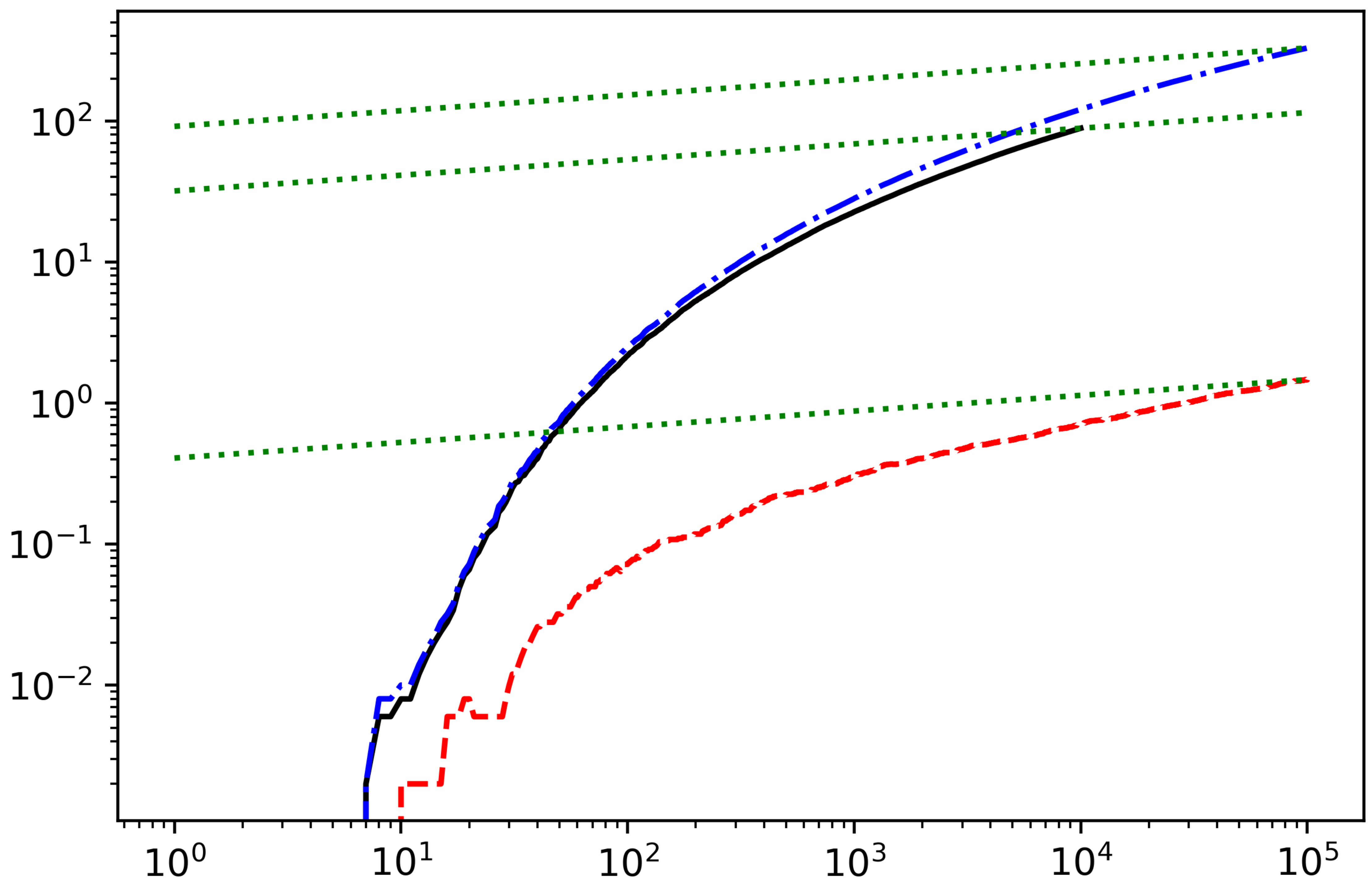
$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$





$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$





IV. What lies ahead

order of magnitude of
expected Betti numbers

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

parameter estimation?

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order of magnitude of expected Betti numbers

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order of magnitude of expected Betti numbers

simplicial preferential attachment?

parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

simplicial preferential attachment?

other non-homogeneous complexes?

What did we learn today?

- Random topology is cool.

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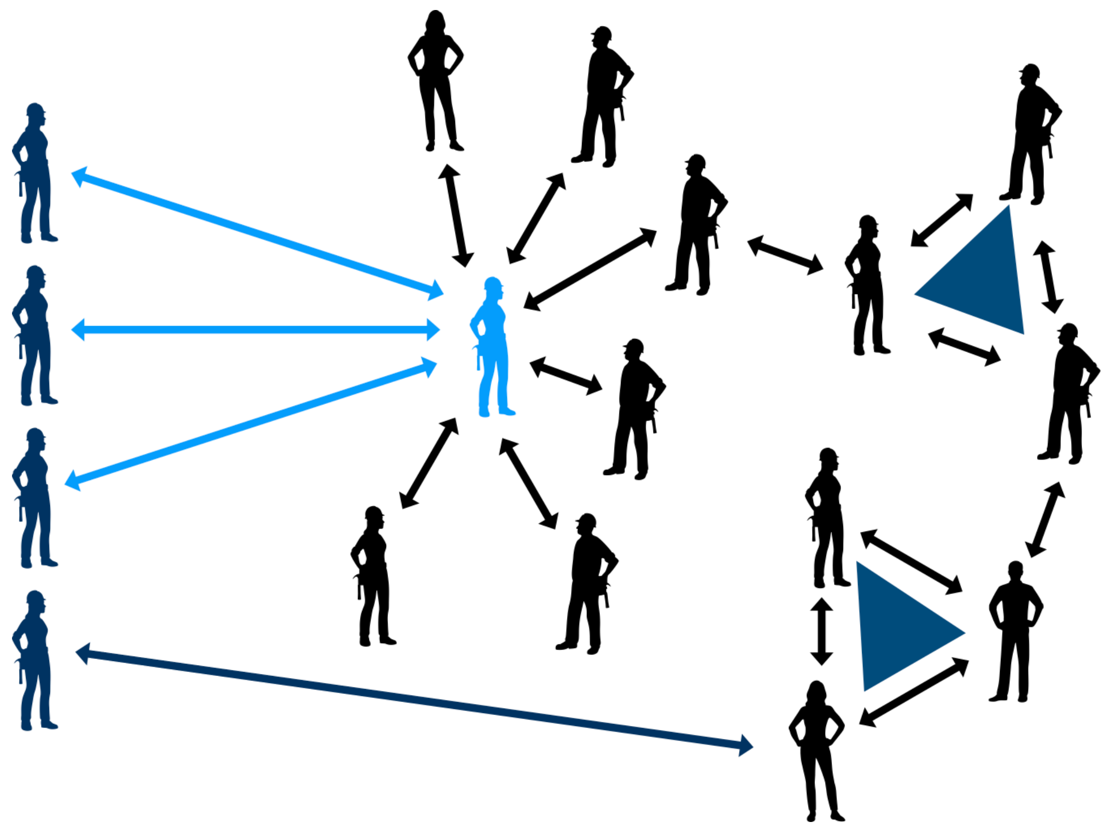
- Preferential attachment graph has interesting topology.

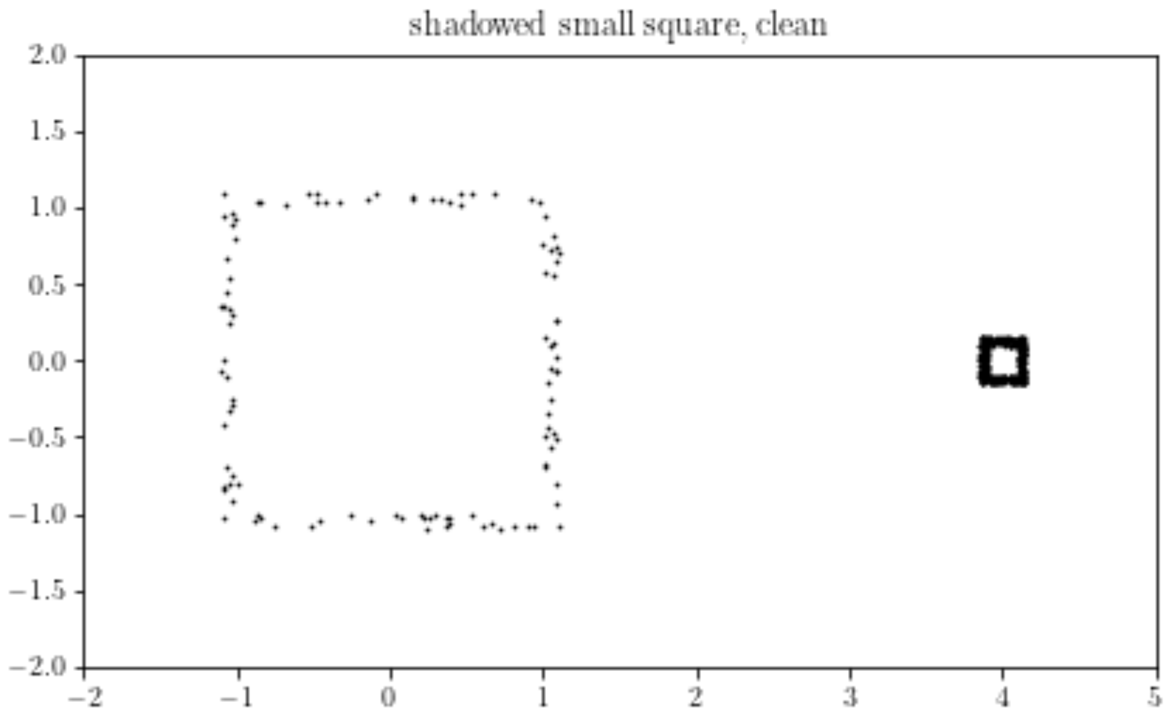
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

Chunyin Siu **cs2323@cornell.edu**
Cornell University



arxiv paper



my video about small holes

Thank you!

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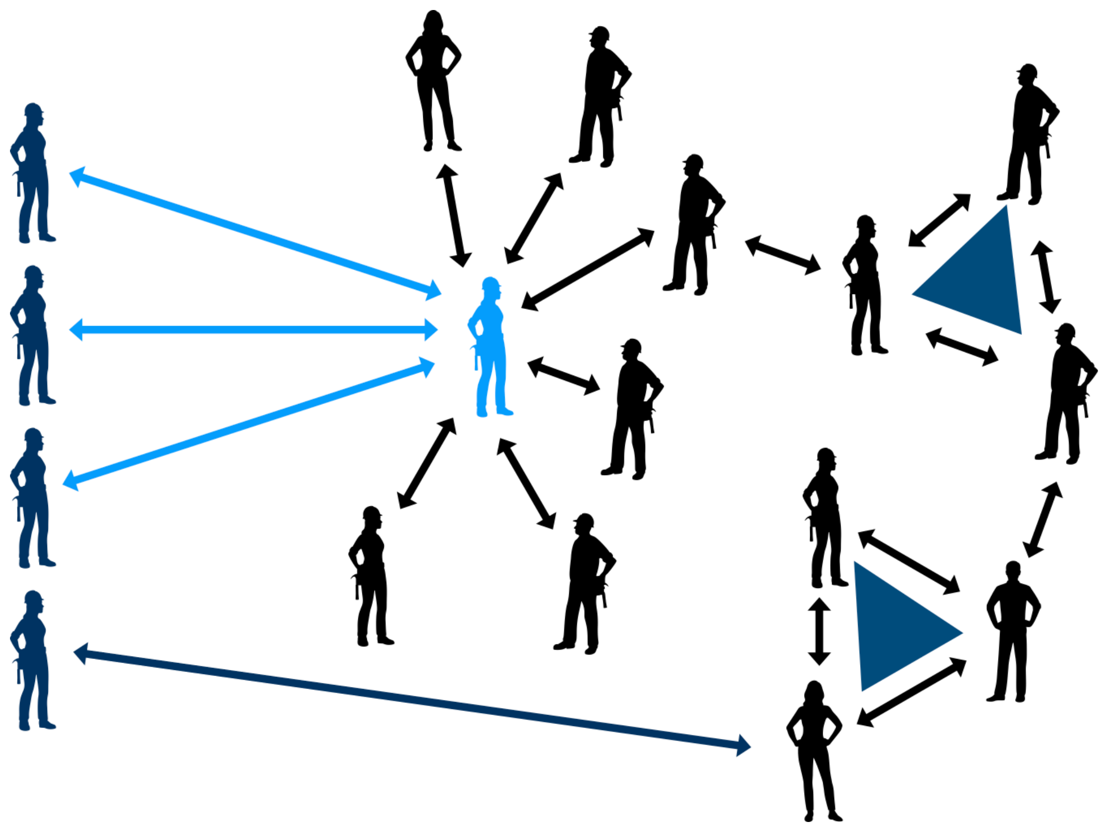

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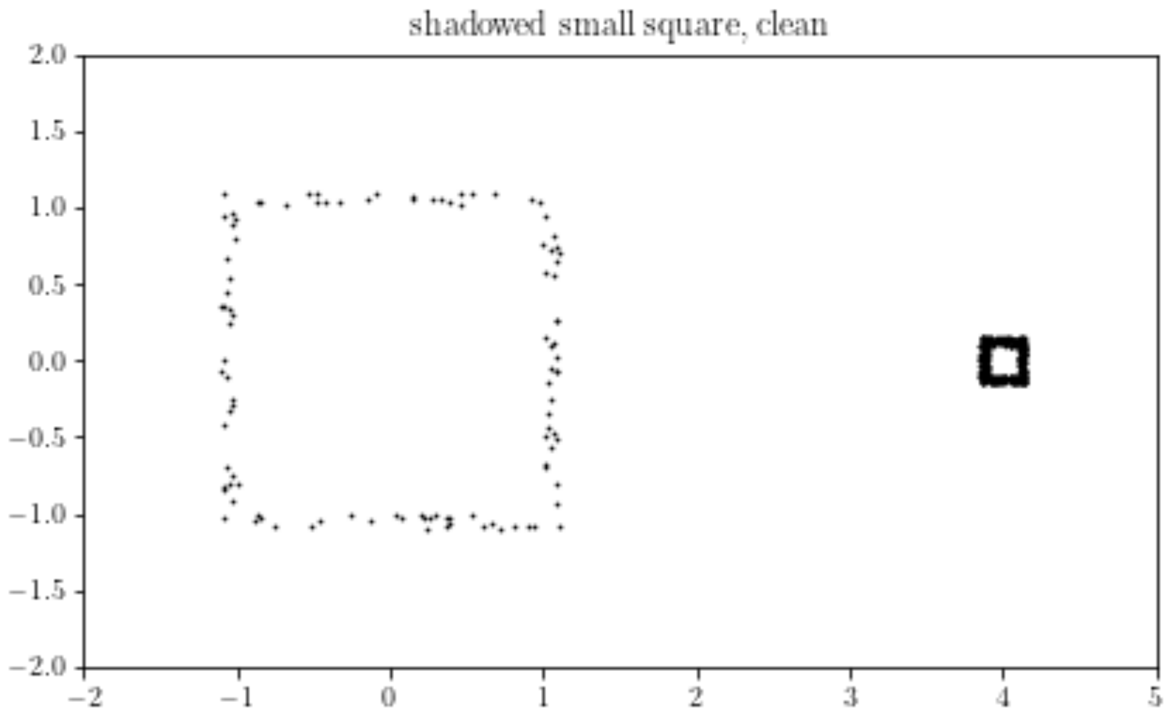

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