Topology of Scale-Free Complexes — Homology & Homotopy

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joint work with Gennady Samorodnitsky, Christina Yu, Caroline He and Avian Misra

Preferential Attachment Clique Complexes

Inductively built **random graph** with ***T* nodes**

Each node *v* connected to ***m* previous nodes**

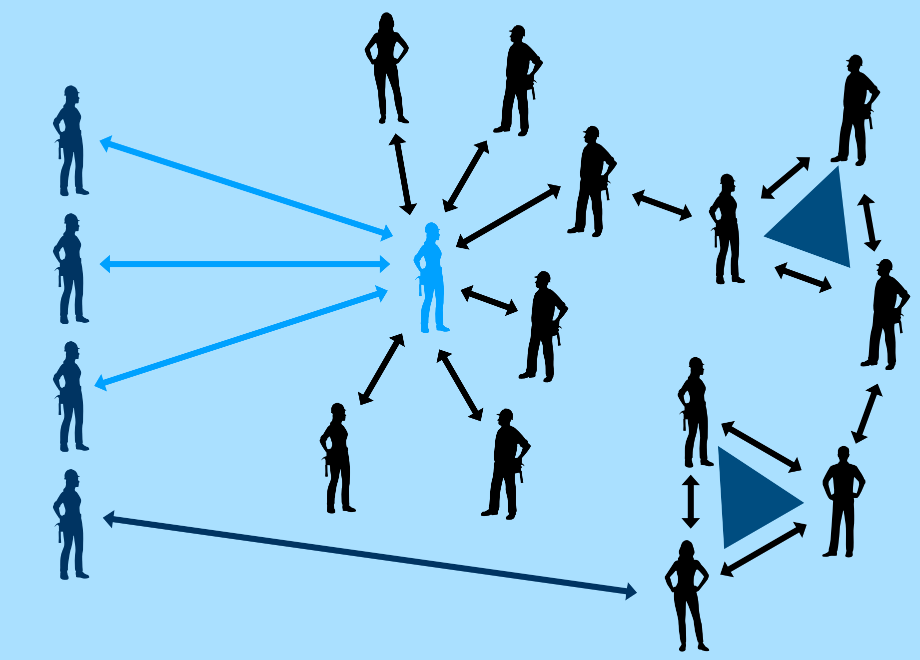
$P(v \to j) \propto$ deg $j + \delta$, with **tuning parameter** $\delta \in (-m, 0)$

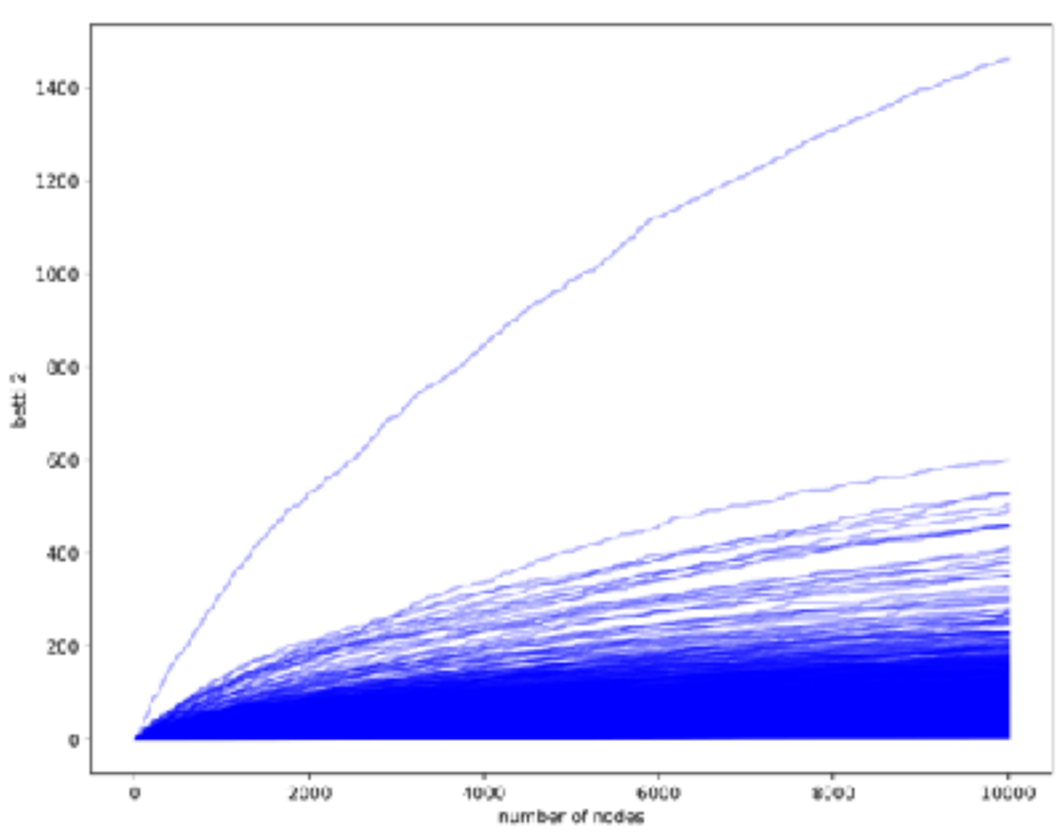
Collapse repeated edges and build **clique complex**

Let $x = 1 - (2 + \delta/m)^{-1} \in (0, 1/2)$.

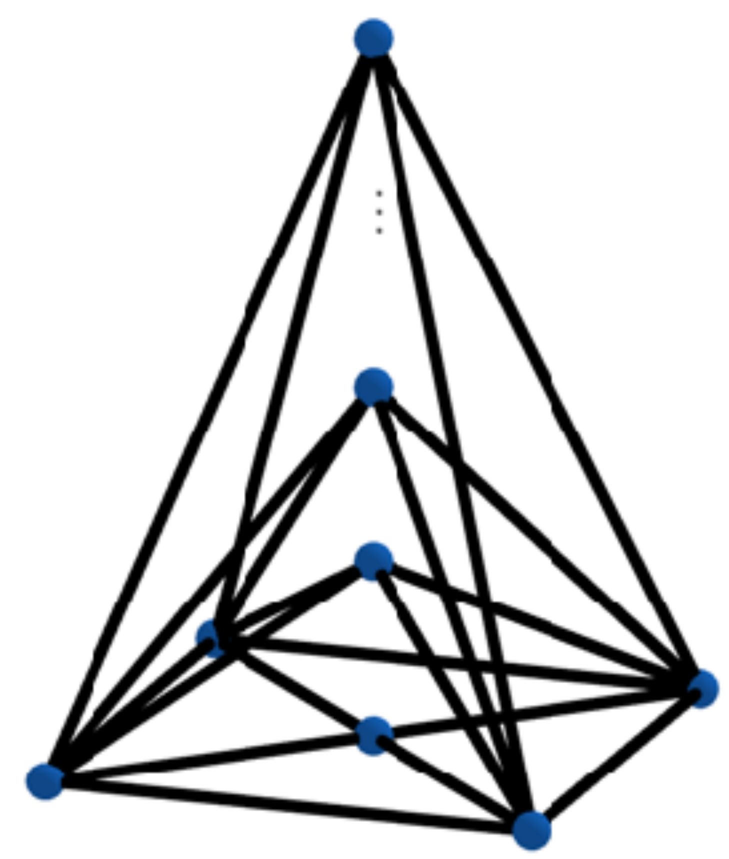
Then *x* decreases with the **preferential attachment strength**, as

$P(T \to$ early node$) \approx T^{-x}$.





evolution of β_2 as the number of nodes increases



repeatedly coned squares

Betti Numbers of Finite Complexes

Dimension 1 $\beta_1(X^{(T)}) = (m-1)T + o(T)$ with high prob.

Higher Dimensions Let $x = 1 - (2 + \delta/m)^{-1}$. For $q \geq 2$,

$$c_\varepsilon T^{1-2qx} \leq \beta_q(X^{(T)}) \leq C_\varepsilon T^{1-2qx}$$

with probability at least $1 - \varepsilon$ for some constants $c_\varepsilon, C_\varepsilon > 0$ if $1 - 2qx > 0$.

Intuition Coned squares dominate; boundaries are rare.

Homotopy-Connectedness of Infinite Complexes

Strong Preferential Attachment

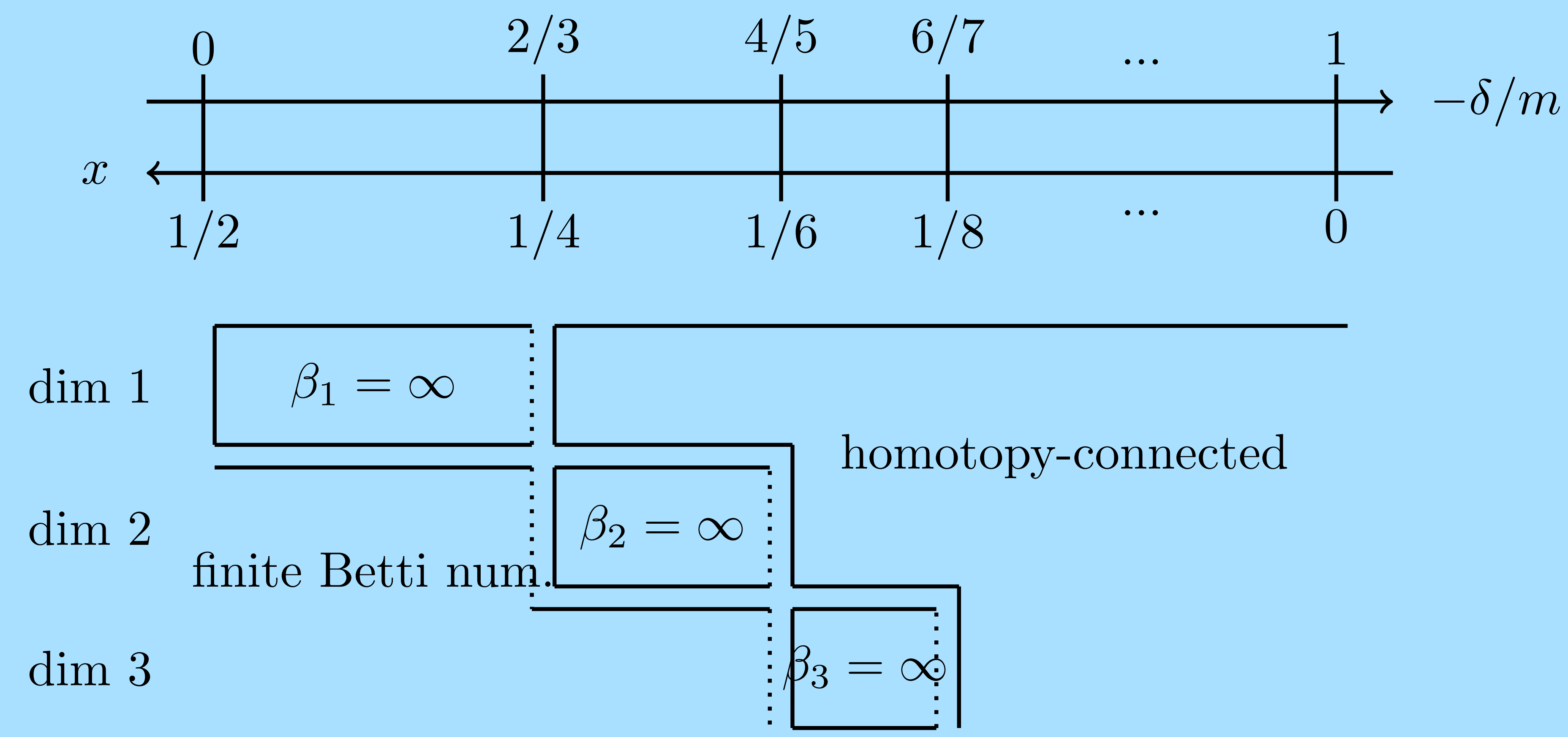
$X^{(\infty)}$ is *q*-homotopy-connected almost surely if $x \leq (2q+2)^{-1}$.

Weak Preferential Attachment

$\beta_q(X^{(\infty)};$ field$) = \infty$ almost surely if $(2q+2)^{-1} < x \leq (2q)^{-1}$.

Intuition

Each small subset of nodes eventually has a common neighbor.



Proof Techniques

- Observe the clique complex is an iterated **mapping cone**.
- Apply **minimal-cycle** [Kahle 2009] and **graph-counting** [Garavaglia and Steghuis 2019] arguments.
- Use **Barmark's criterion** for homotopy-connectedness [Farber, 2023].

Preprint on Homology of Finite Complexes

- Siu C., Samorodnitsky G., Yu C.L. and He R.: The asymptotic of the expected Betti numbers of preferential attachment clique complexes (2023)

Cited Works on this Poster

- Farber M.: Large simplicial complexes: universality, randomness, and ampleness. *Journal of Applied and Computational Topology* (2023).
- Garavaglia A. and Steghuis C.: Subgraphs in preferential attachment models. *Advances in Applied Probability*, 51(3), 898 — 926 (2019).
- Kahle M.: Topology of random clique complexes. *Discrete Mathematics*, 309(6): 1658 — 1671 (2009).