The Topology of Preferential Attachment

Higher-Order Connectivity of Random Interactions

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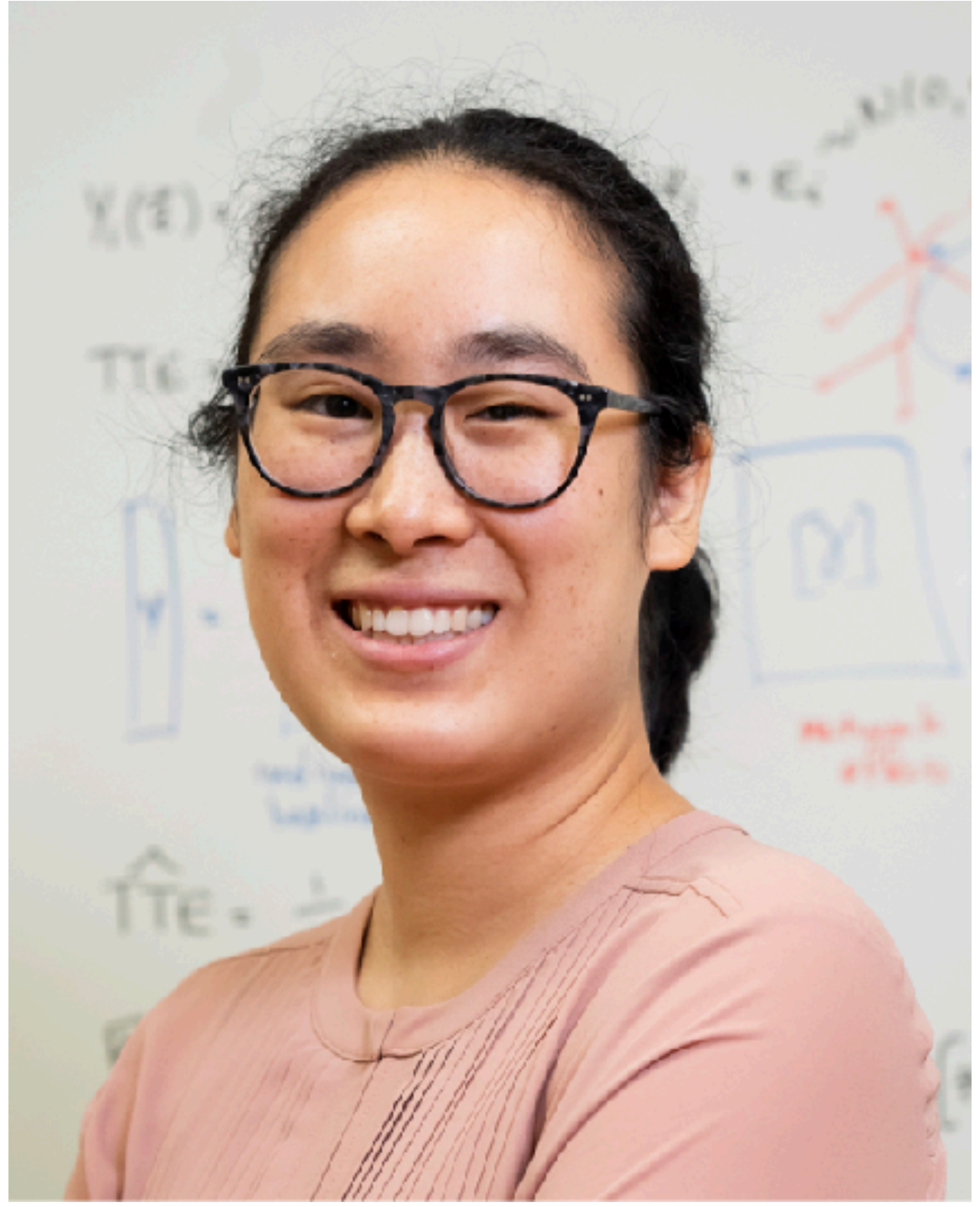
My Lovely Collaborators



Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

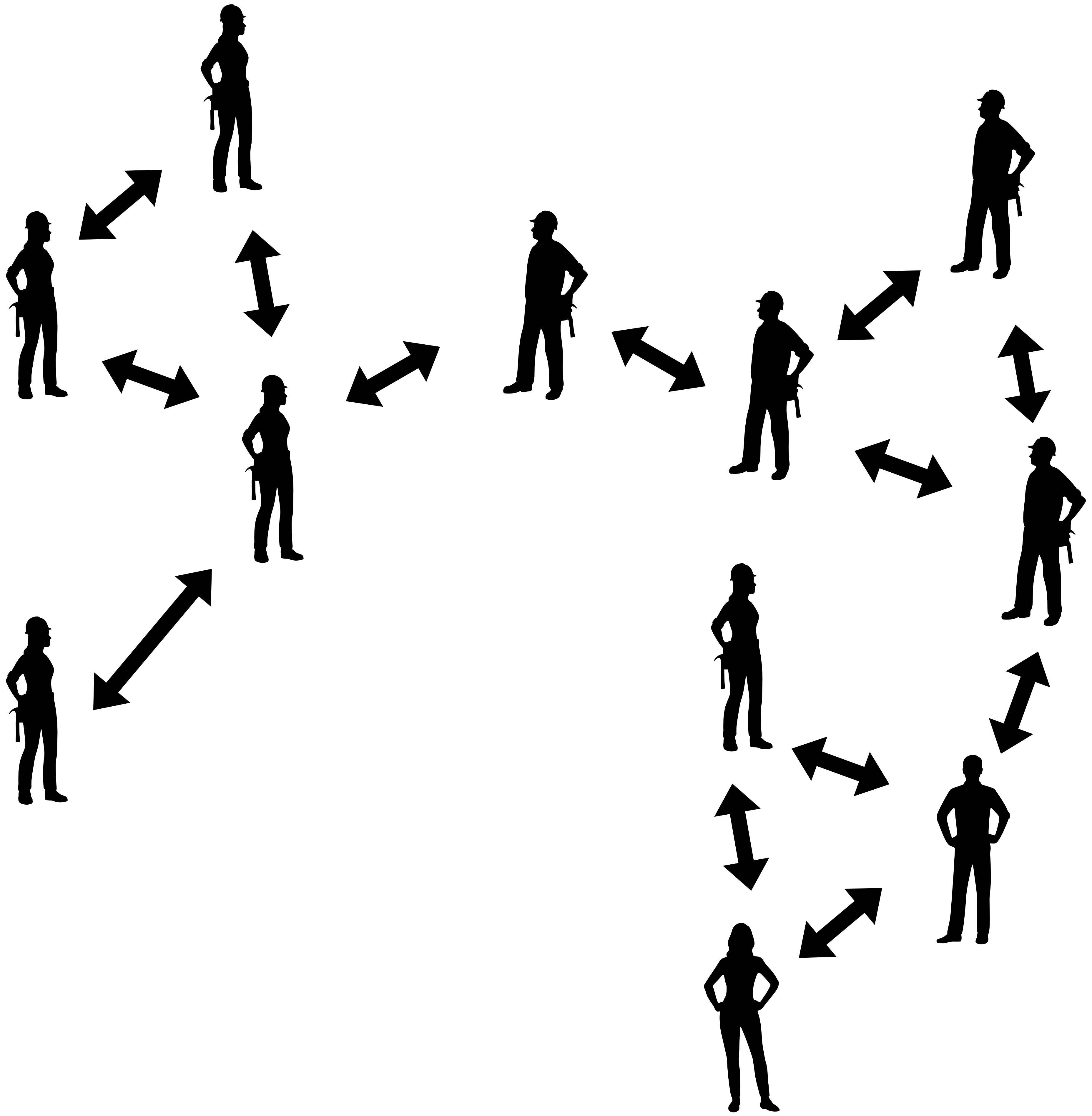
I. Preferential Attachment

Preferential Attachment

[Albert and Barabasi 1999]

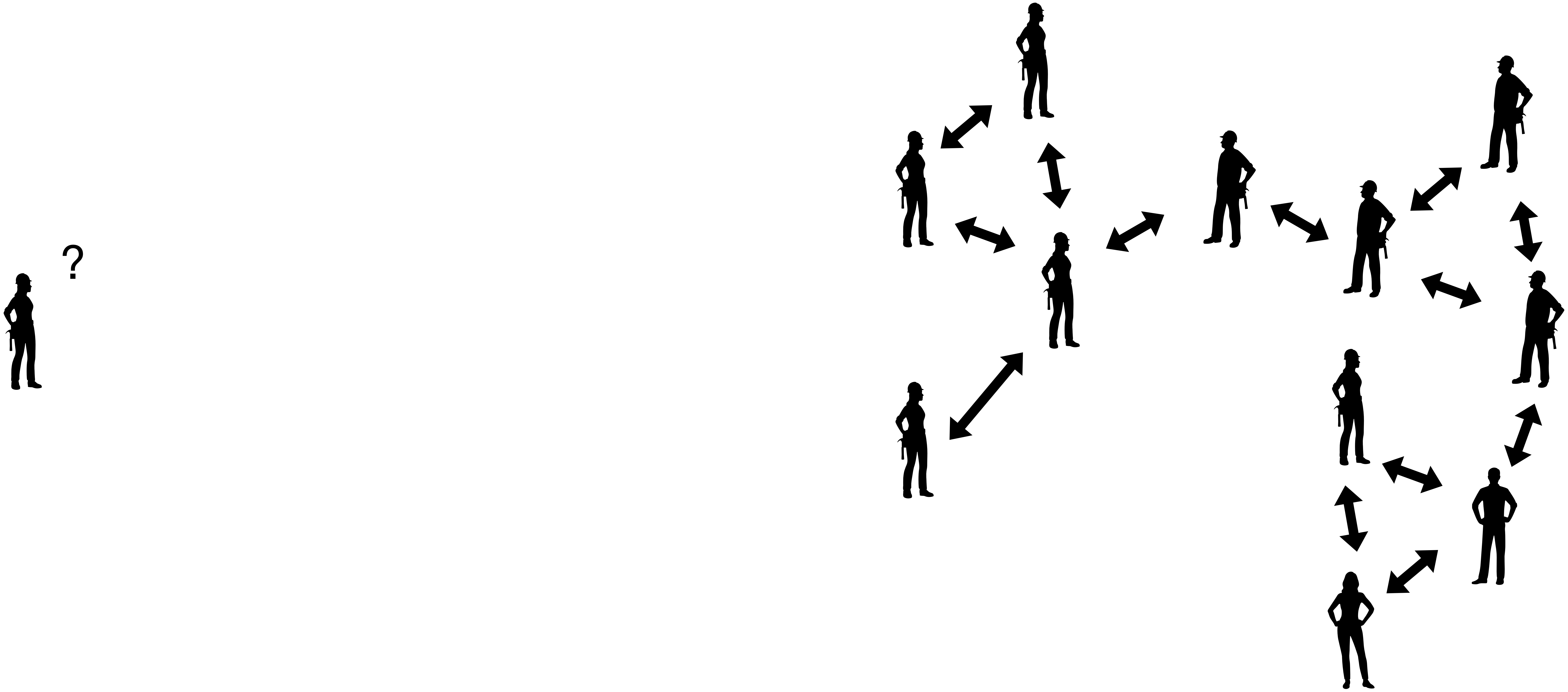
(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Preferential Attachment
[Albert and Barabasi 1999]



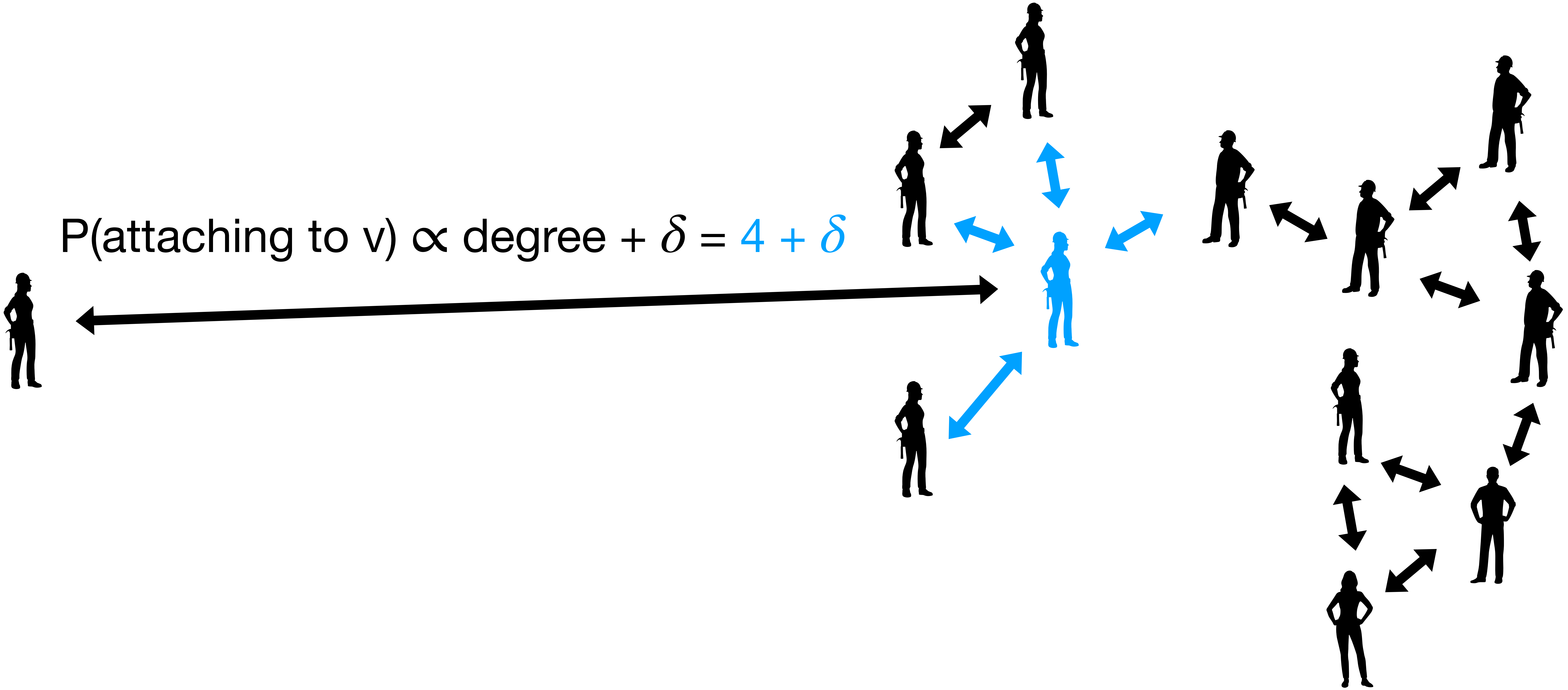
Preferential Attachment

[Albert and Barabasi 1999]



Preferential Attachment

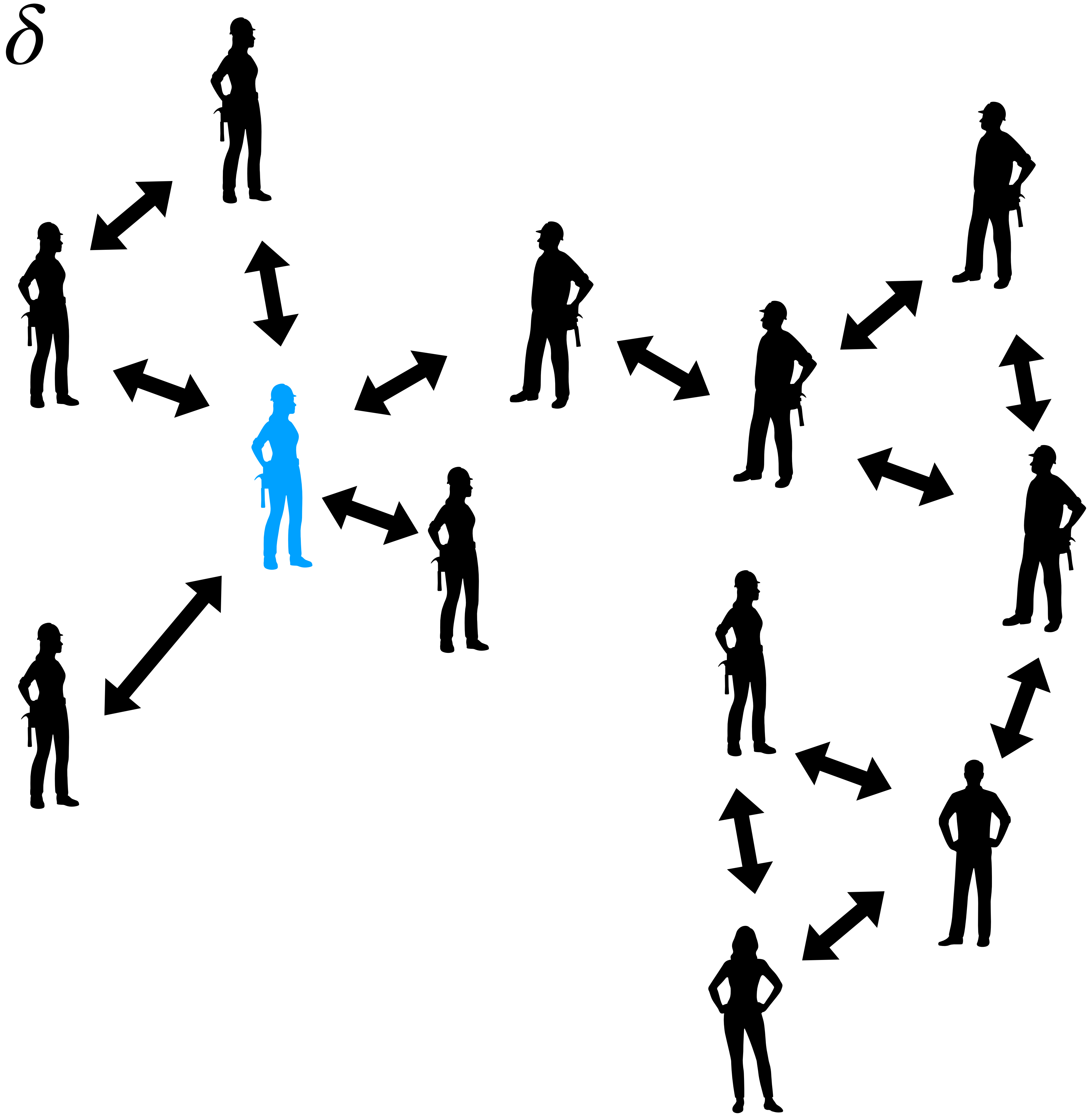
[Albert and Barabasi 1999]



Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) $\propto$ degree + a tuning parameter δ



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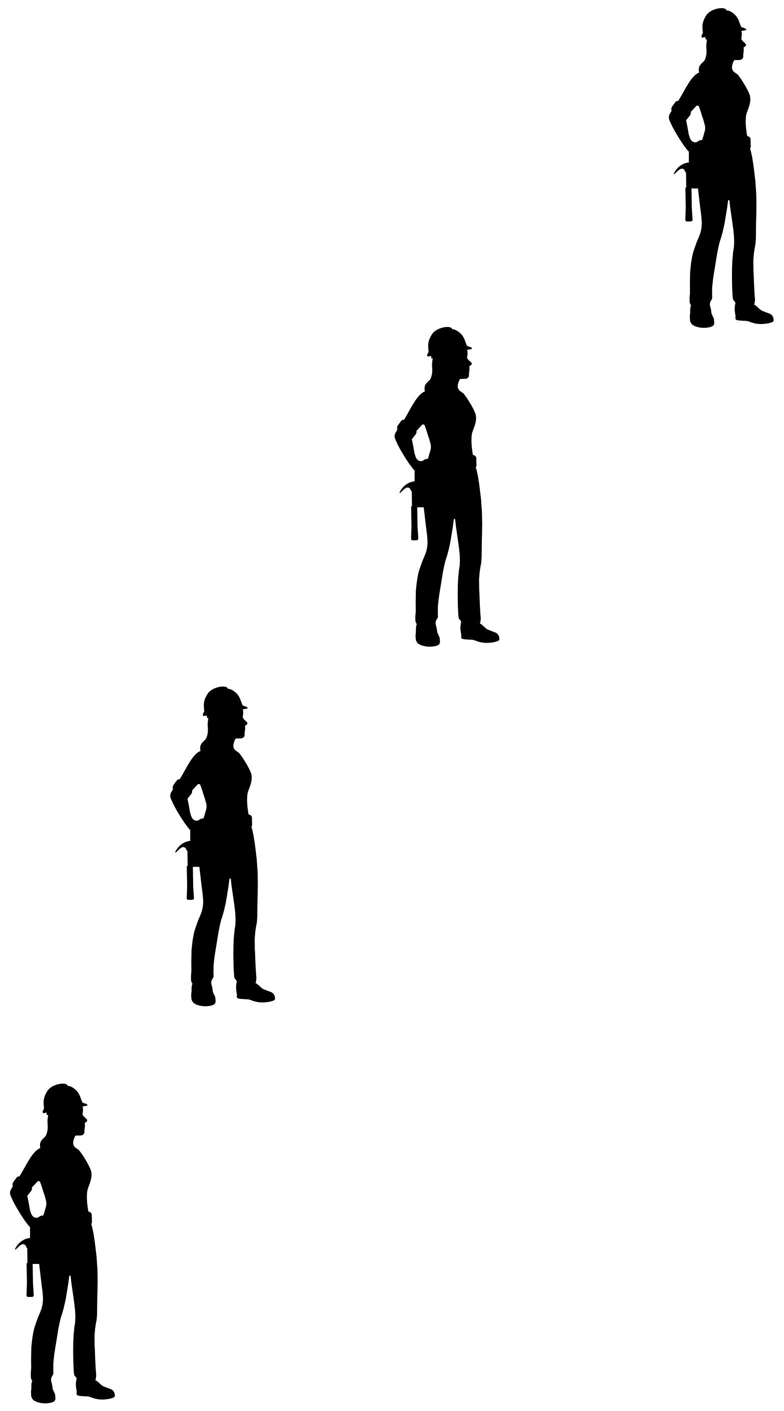

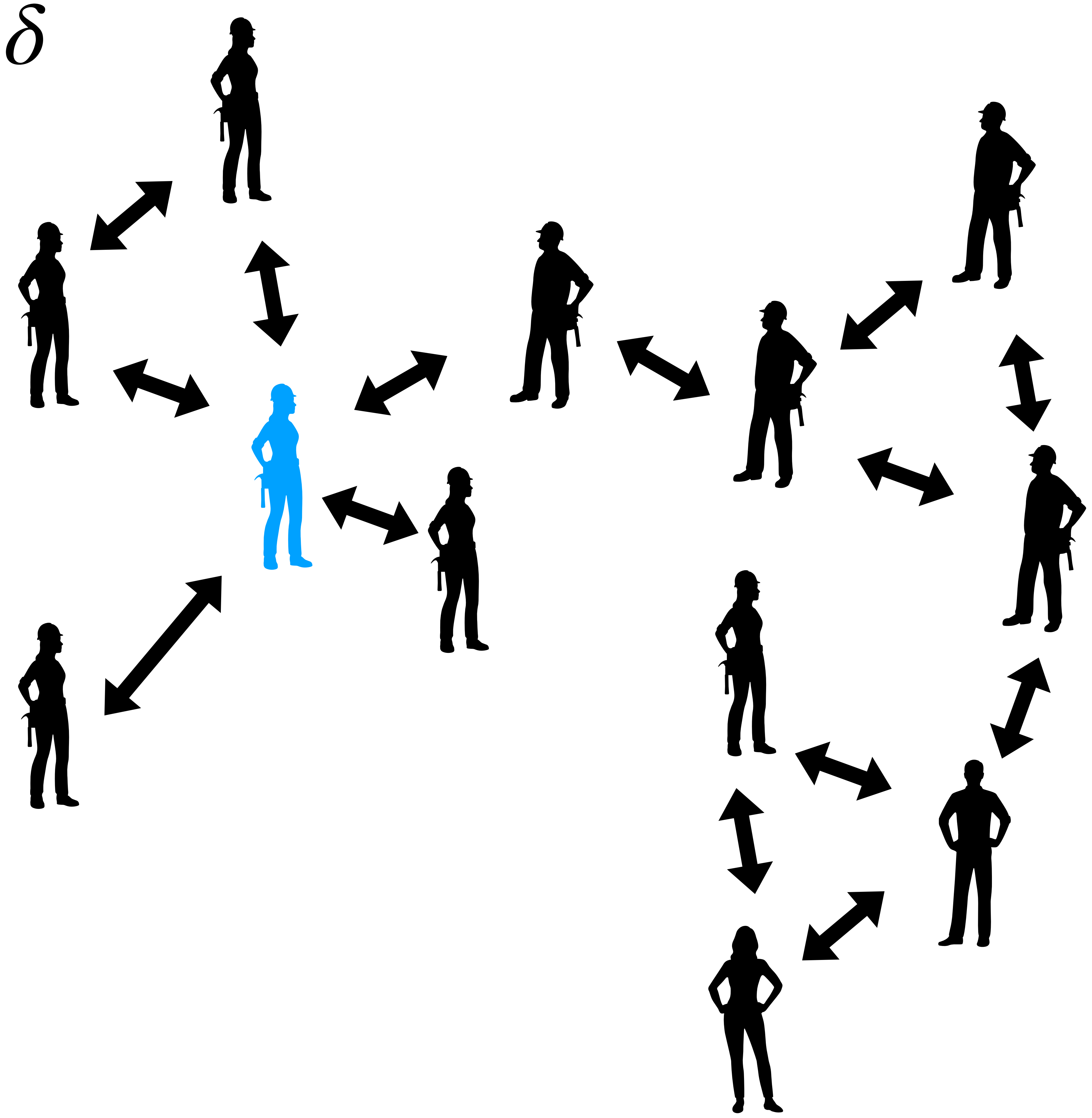

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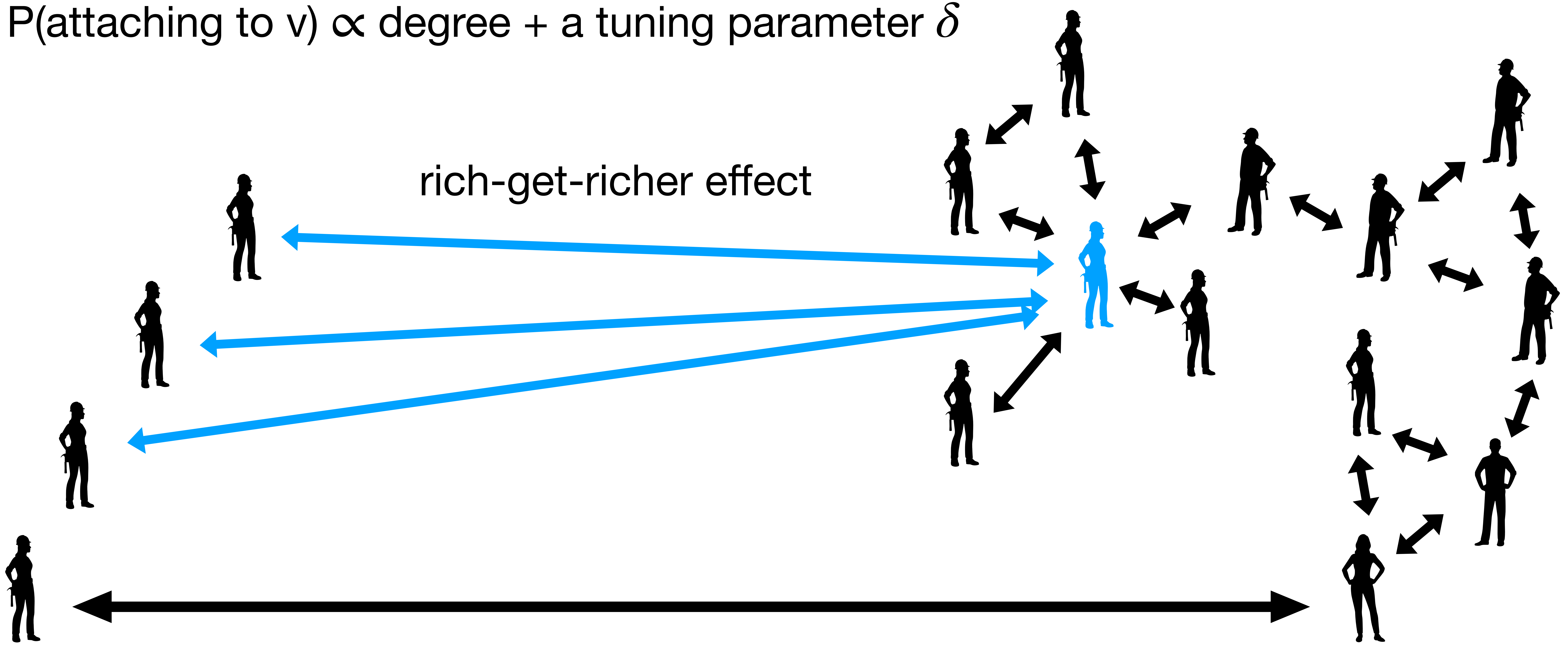


What do we know?

What do we know?

- Scale-freeness and Degree distribution
  [Barabasi and Albert 1999; Dorogovtsev, Mendes and Samukhin 2000; Krapivsky, Redner and Leyvraz 2000]

pmf of degree distribution

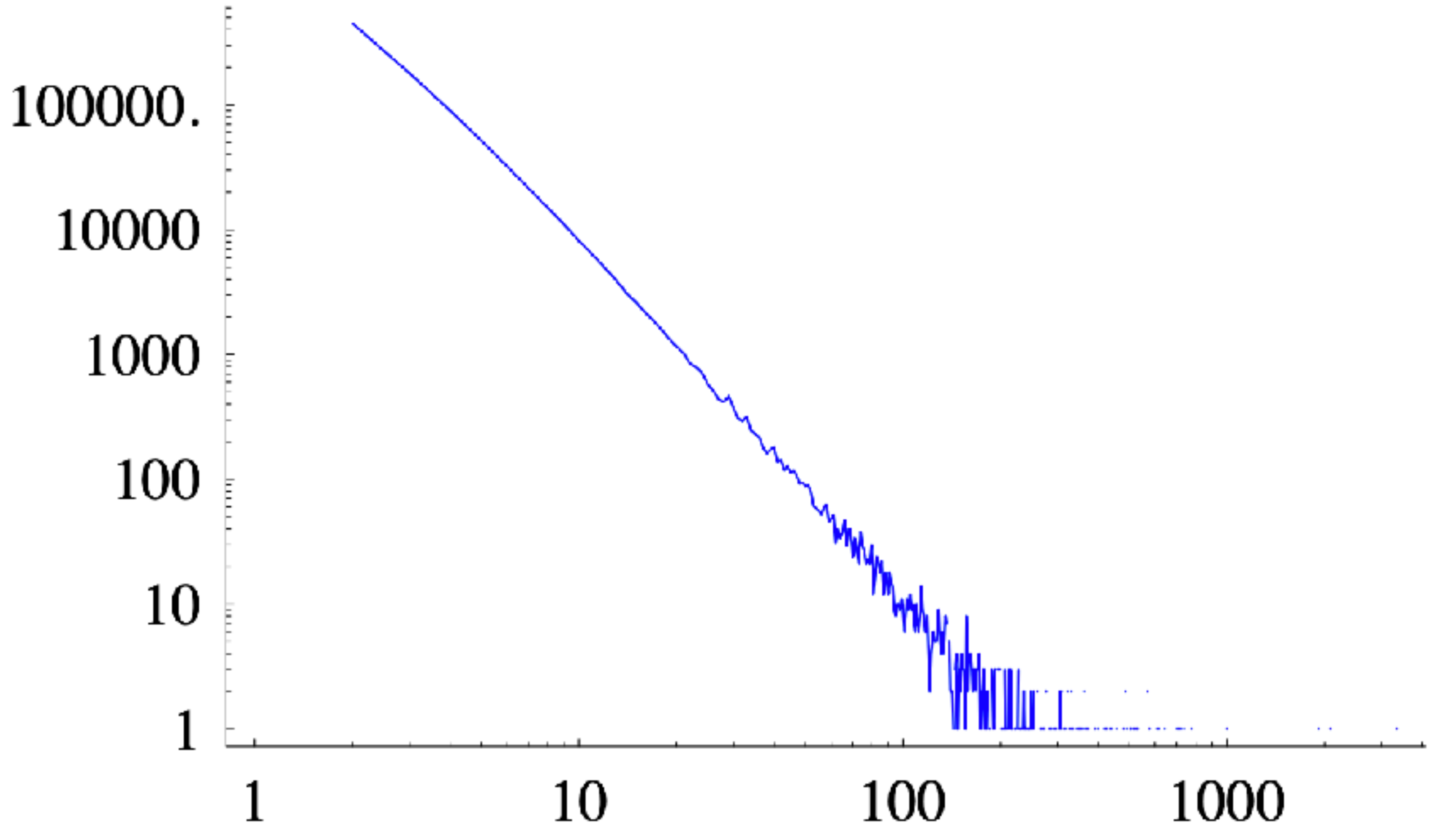
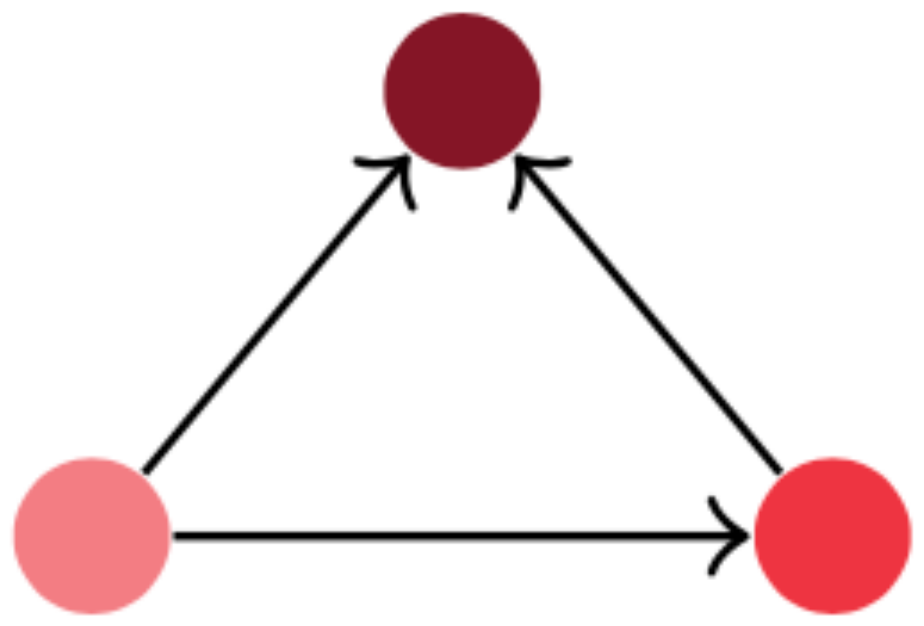


Fig 8.3 of R. Hofstad (2013).
Random Graphs and Complex Networks.
https://doi.org/10.1017/9781316779422

What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013, Garavaglia and Stegehuis 2019]



(a) $t^{(3-\tau)/(\tau-1)} \log(t)$

Fig 2 of A. Garavaglia and C. Stegehuis (2019). Subgraphs in Preferential Attachment Models. https://doi.org/10.1017/apr.2019.36

What do we know?

- subgraph counts [Garavaglia and Stegehuis 2019]

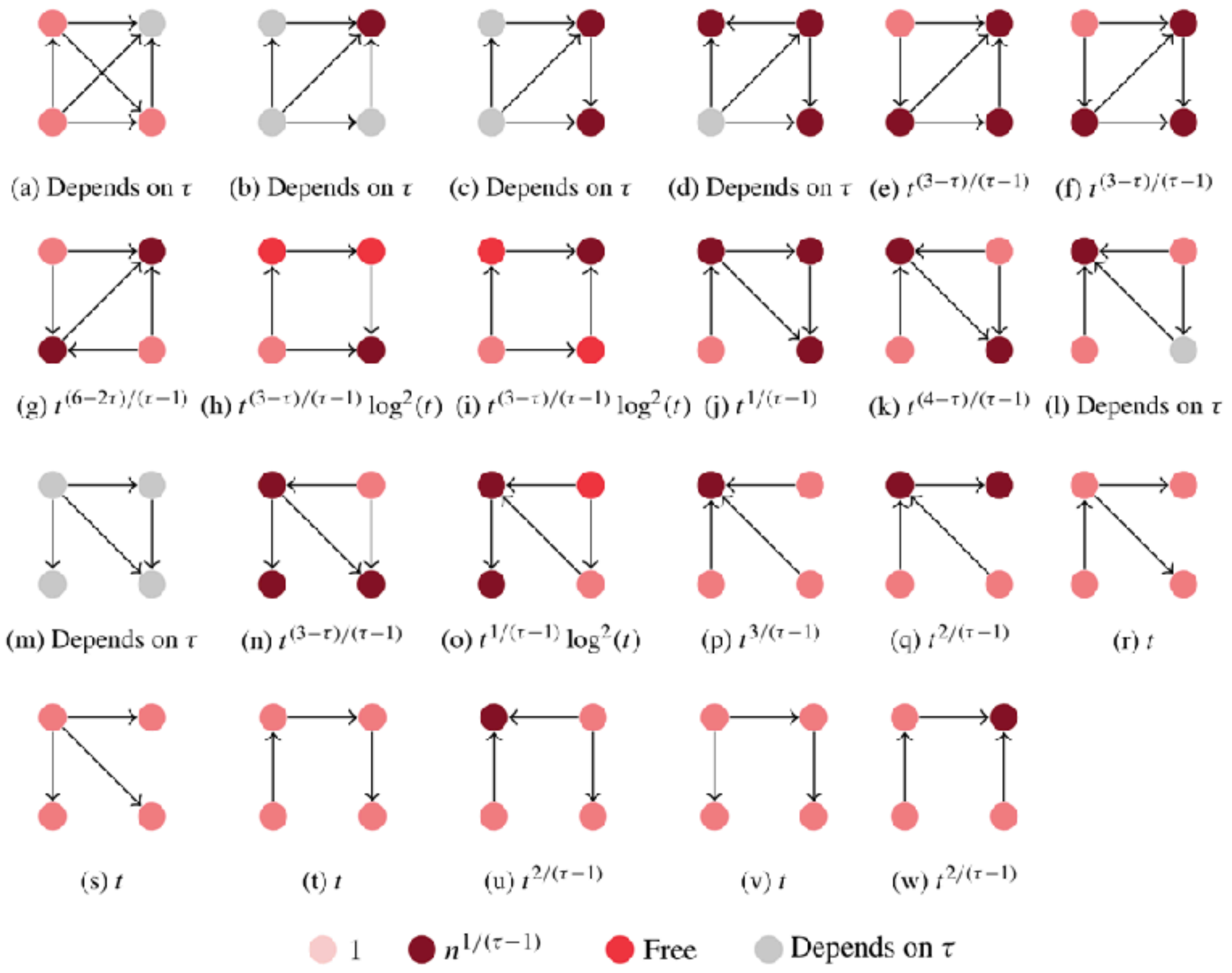
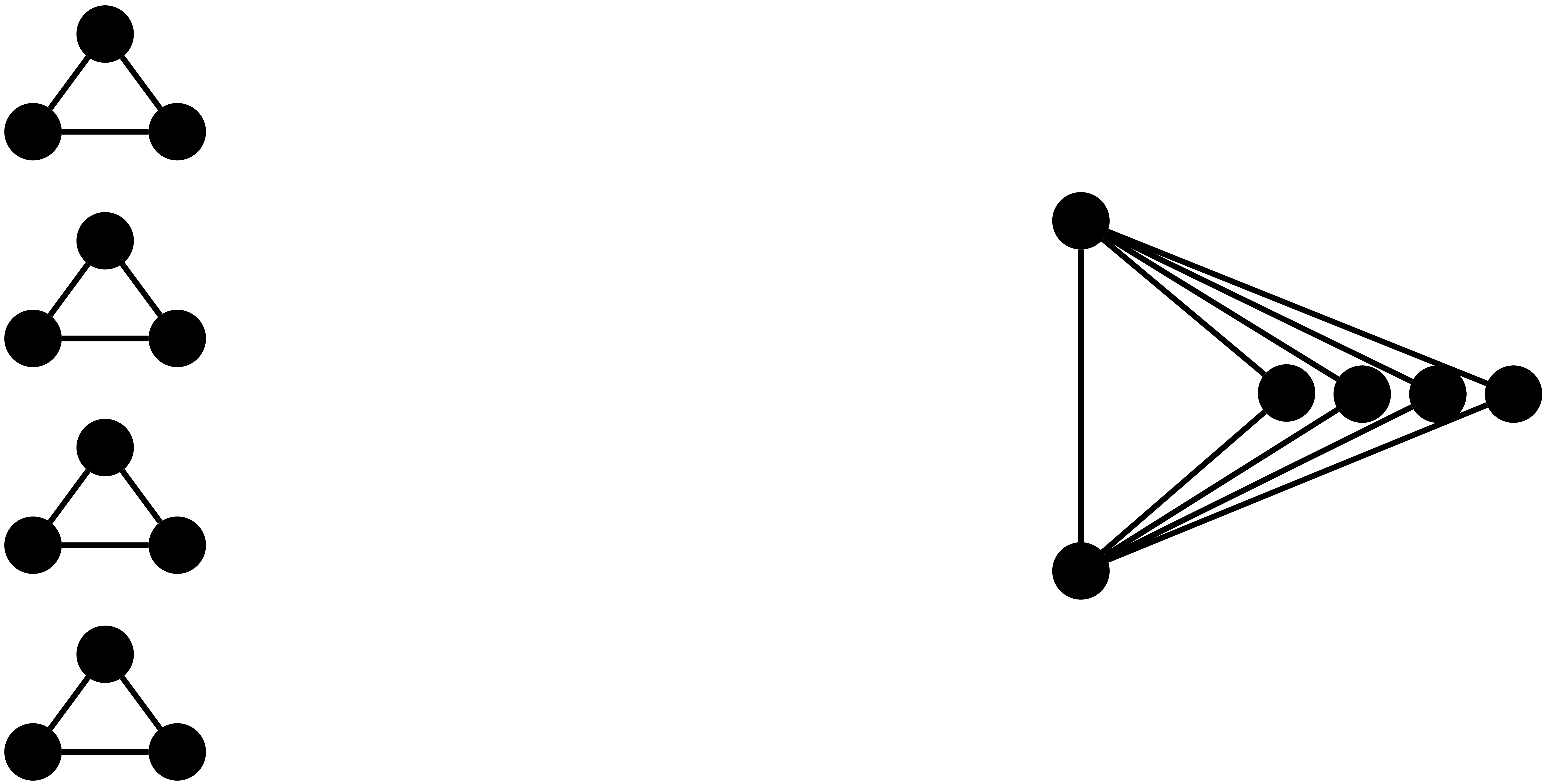


Fig 3 of A. Garavaglia and C. Stegehuis (2019). Subgraphs in Preferential Attachment Models. https://doi.org/10.1017/apr.2019.36

Higer-Order Connectivity?



II. Into Topology

Counting everything in every dimension all at once

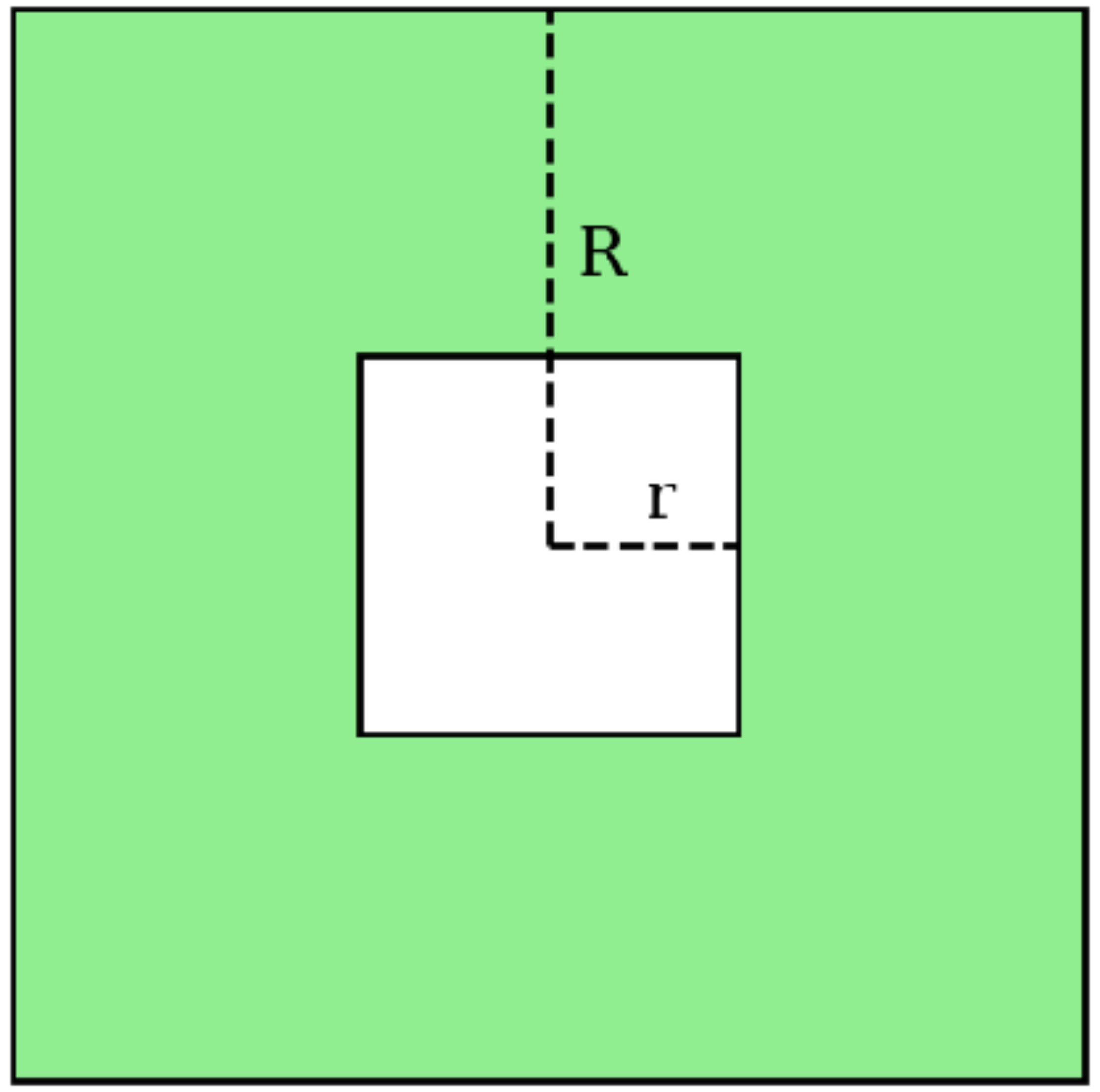
k-dim Betti number β_k

= count of k-dim holes

= count of k-dim repeated connections

Betti numbers β_k

Count of Holes

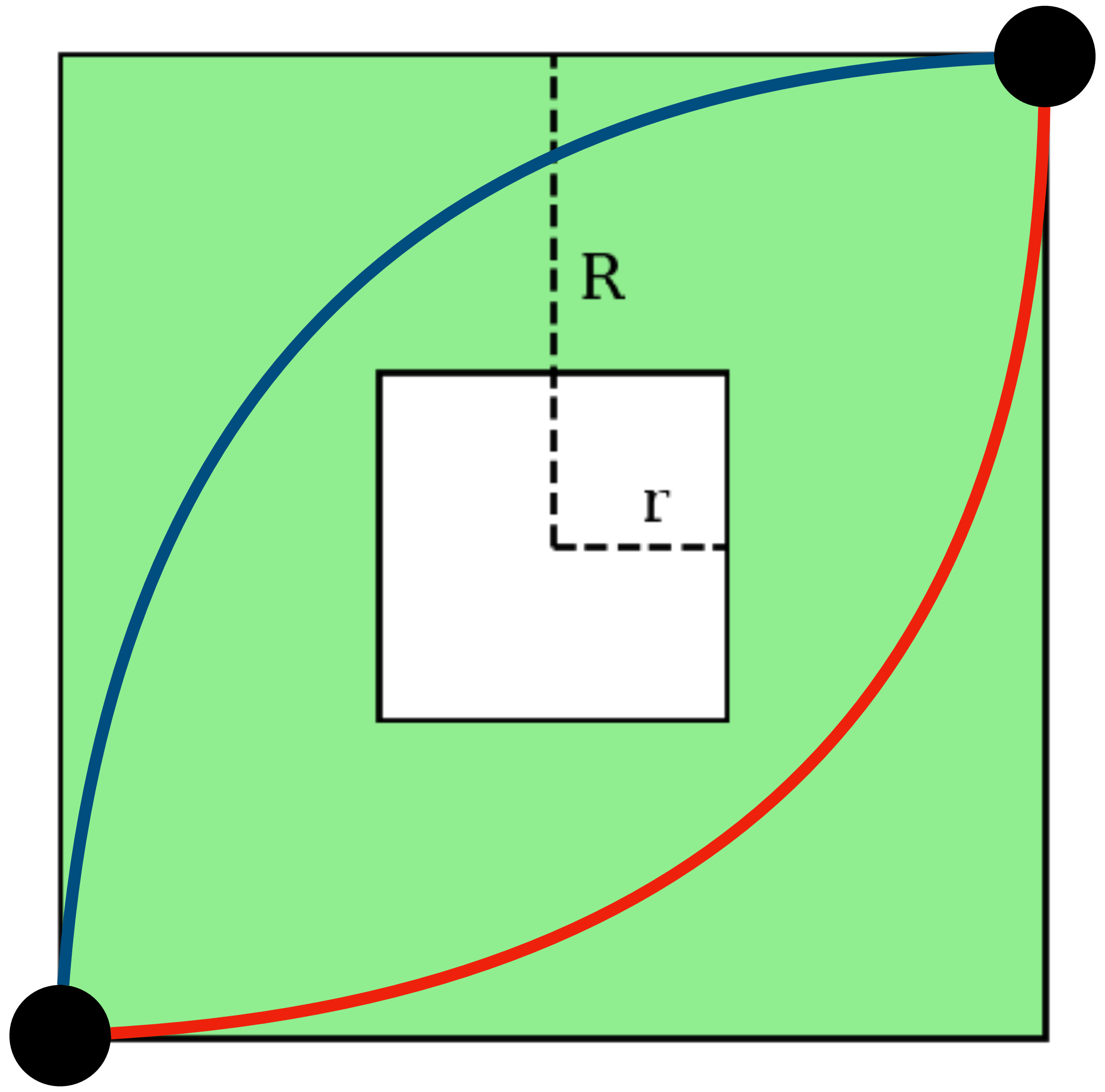


$\beta_1 = 1$

1 loop

Betti numbers β_k

Count of Repeated Connections



$\beta_1 = 1$

1 loop

1 alternative path

k-dim Betti number β_k

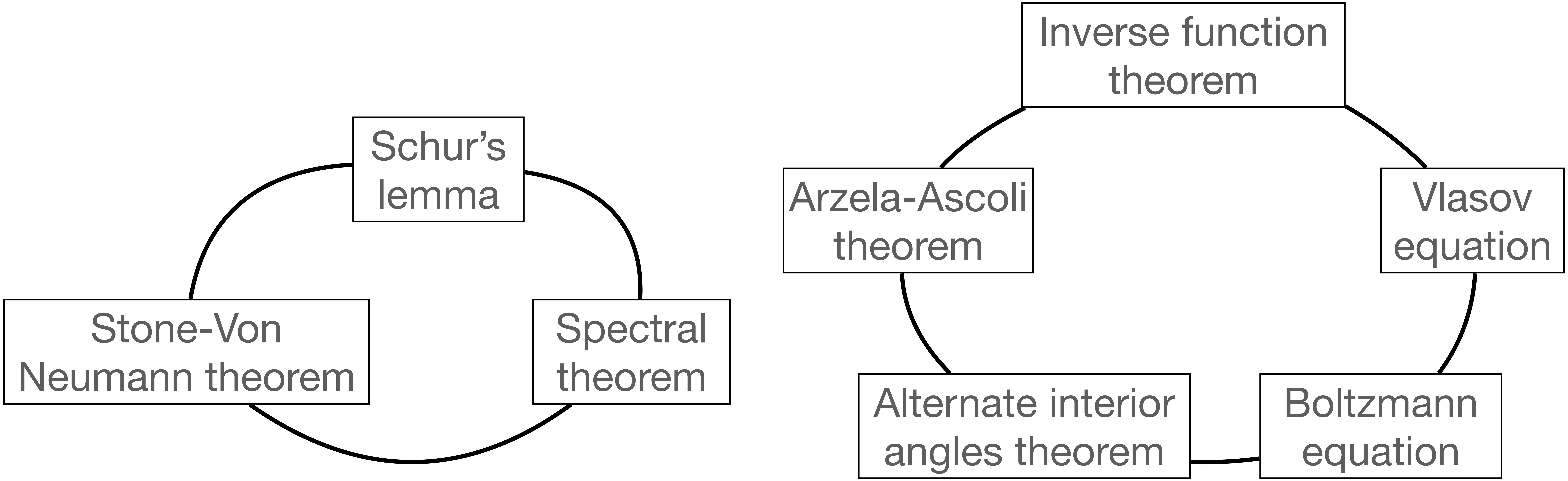
= count of k-dim holes

= count of k-dim repeated connections

Research Network
[Salikov et al, 2018]

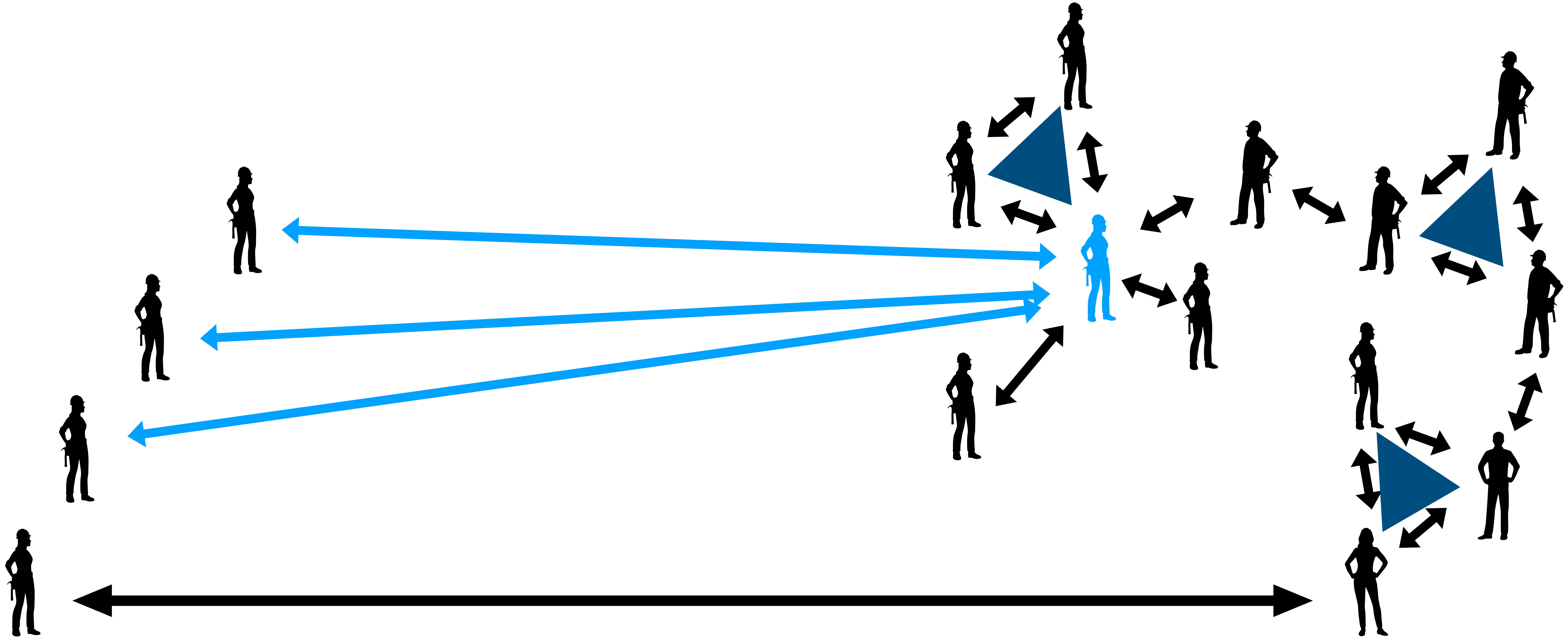
- Co-occurence complex in Math research paper

Gap in Understanding



Clique Complex

aka Flag Complex



III Topology of Preferential Attachment

Expected Betti Number $E[\beta_q]$

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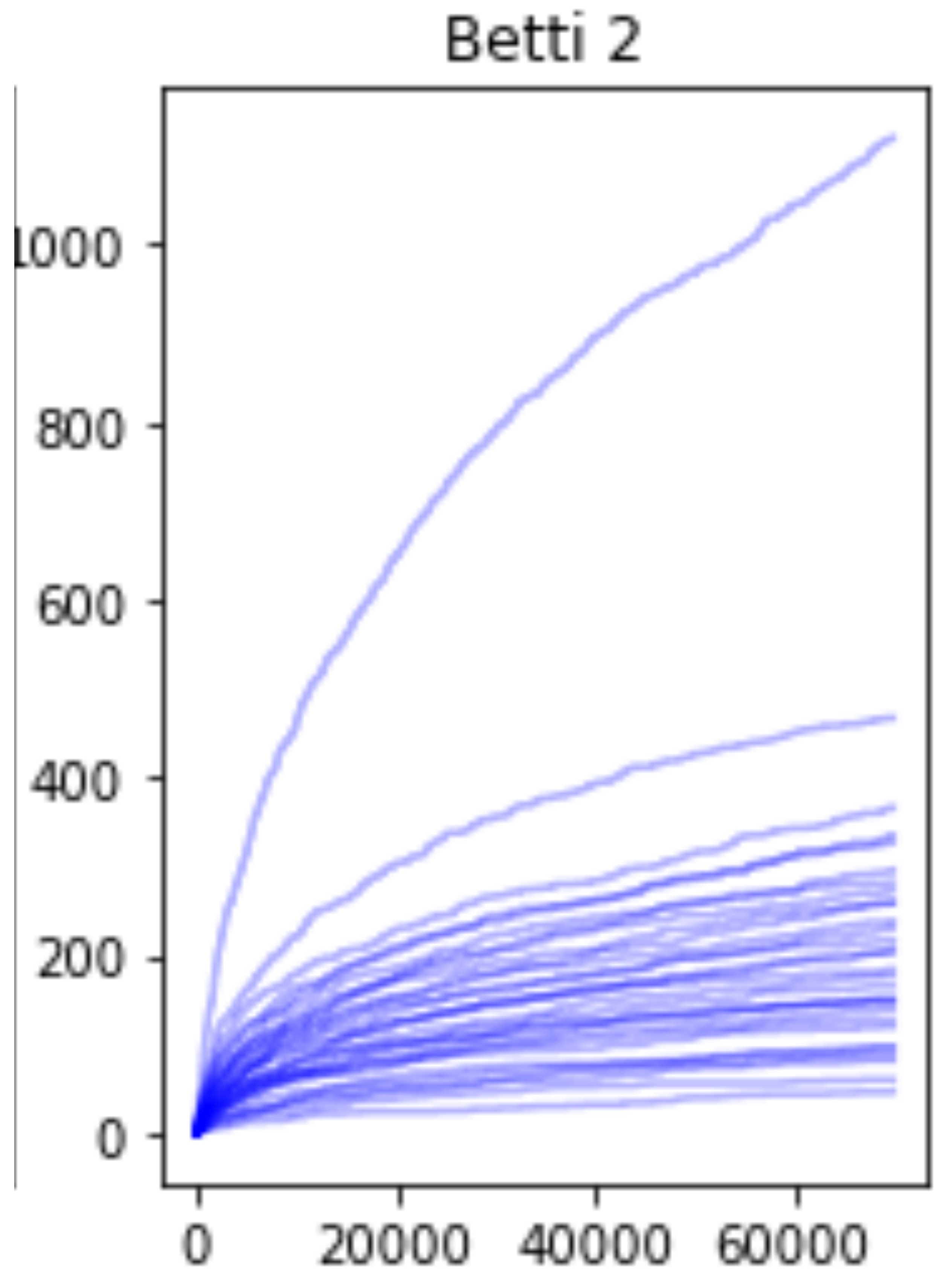
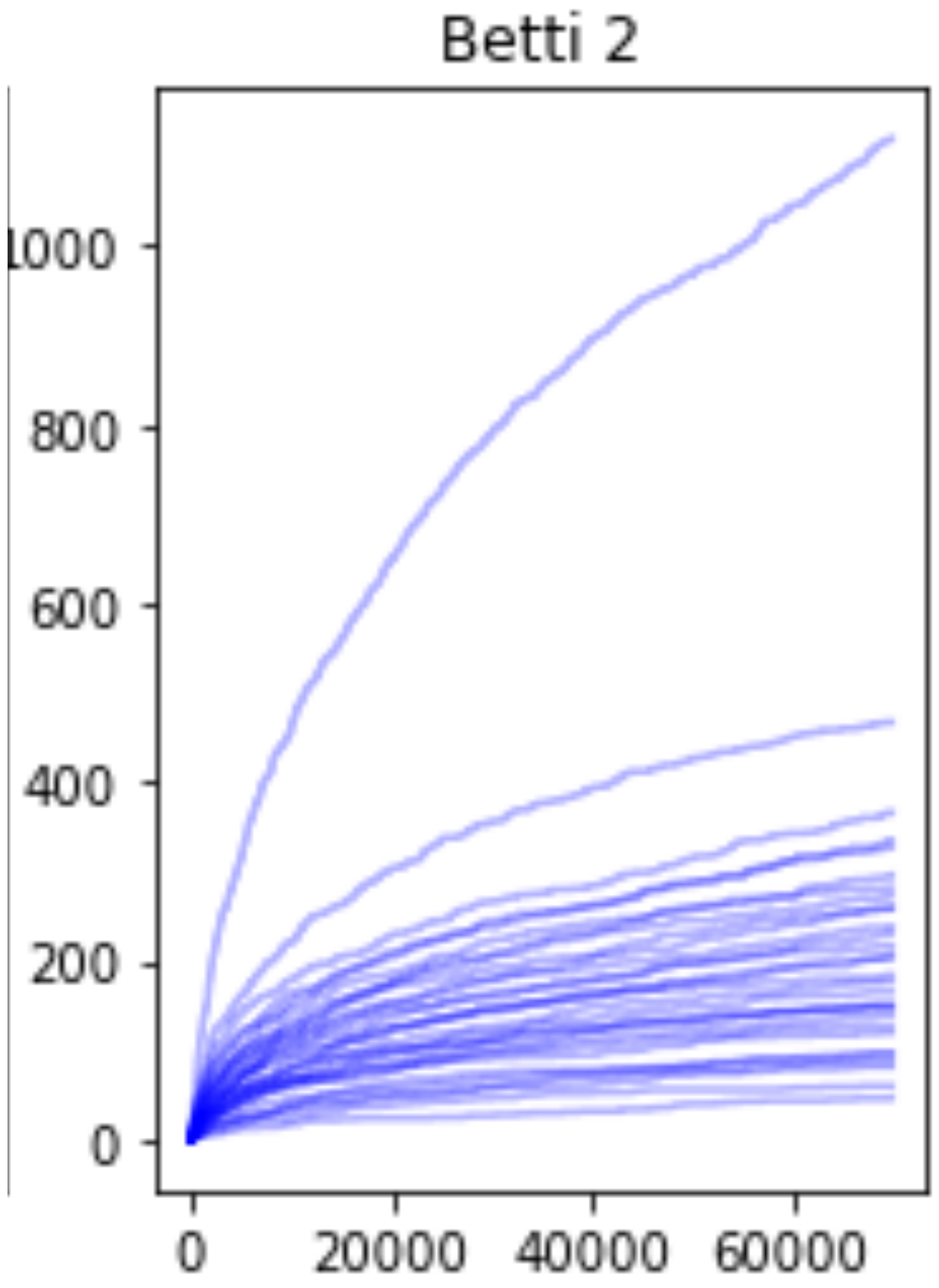


Different curves, different random seeds.
All curves have the same model parameters.

Expected Betti Number $E[\beta_q]$

- increasing trend



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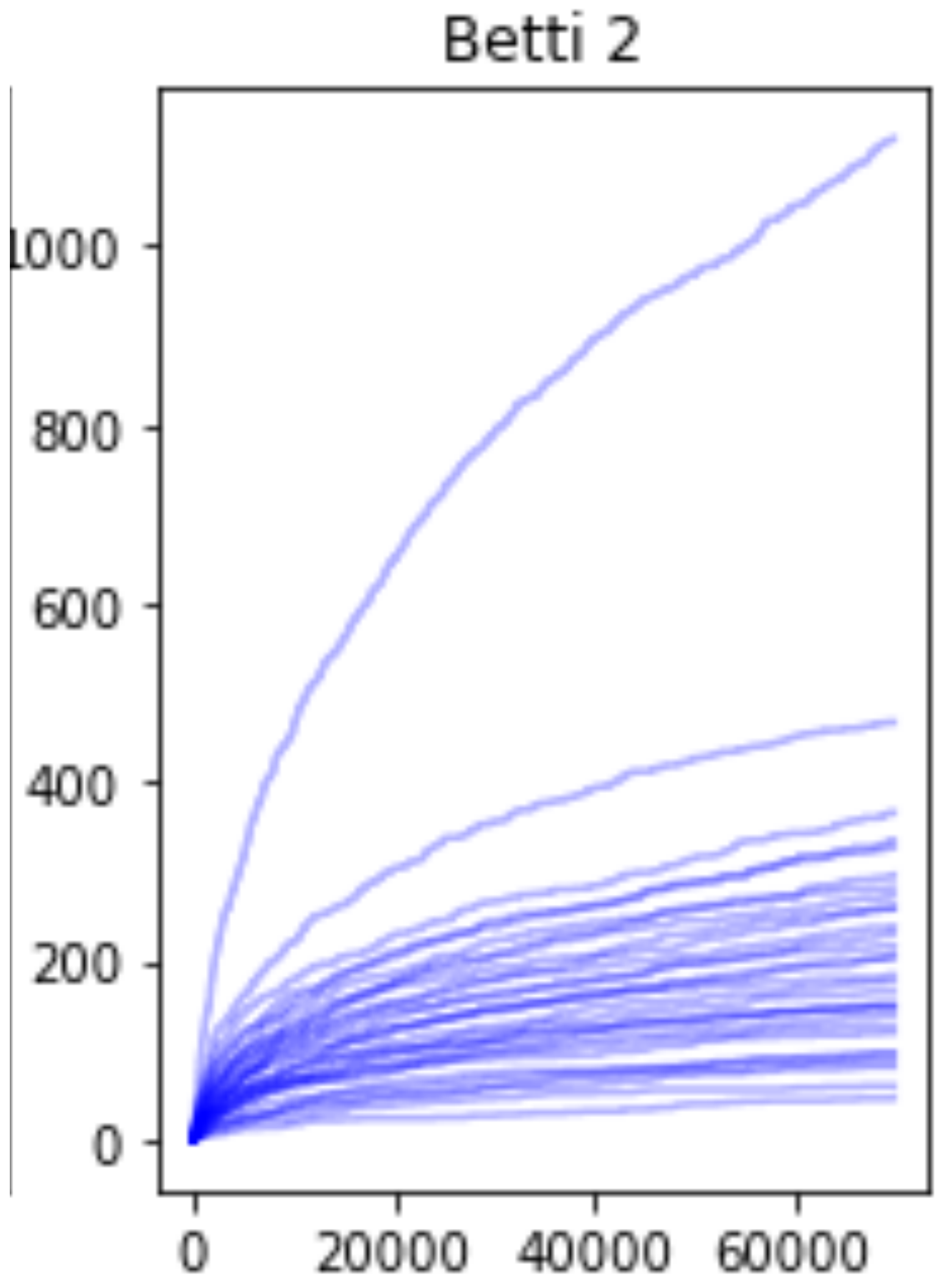
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth



Different curves, different random seeds.
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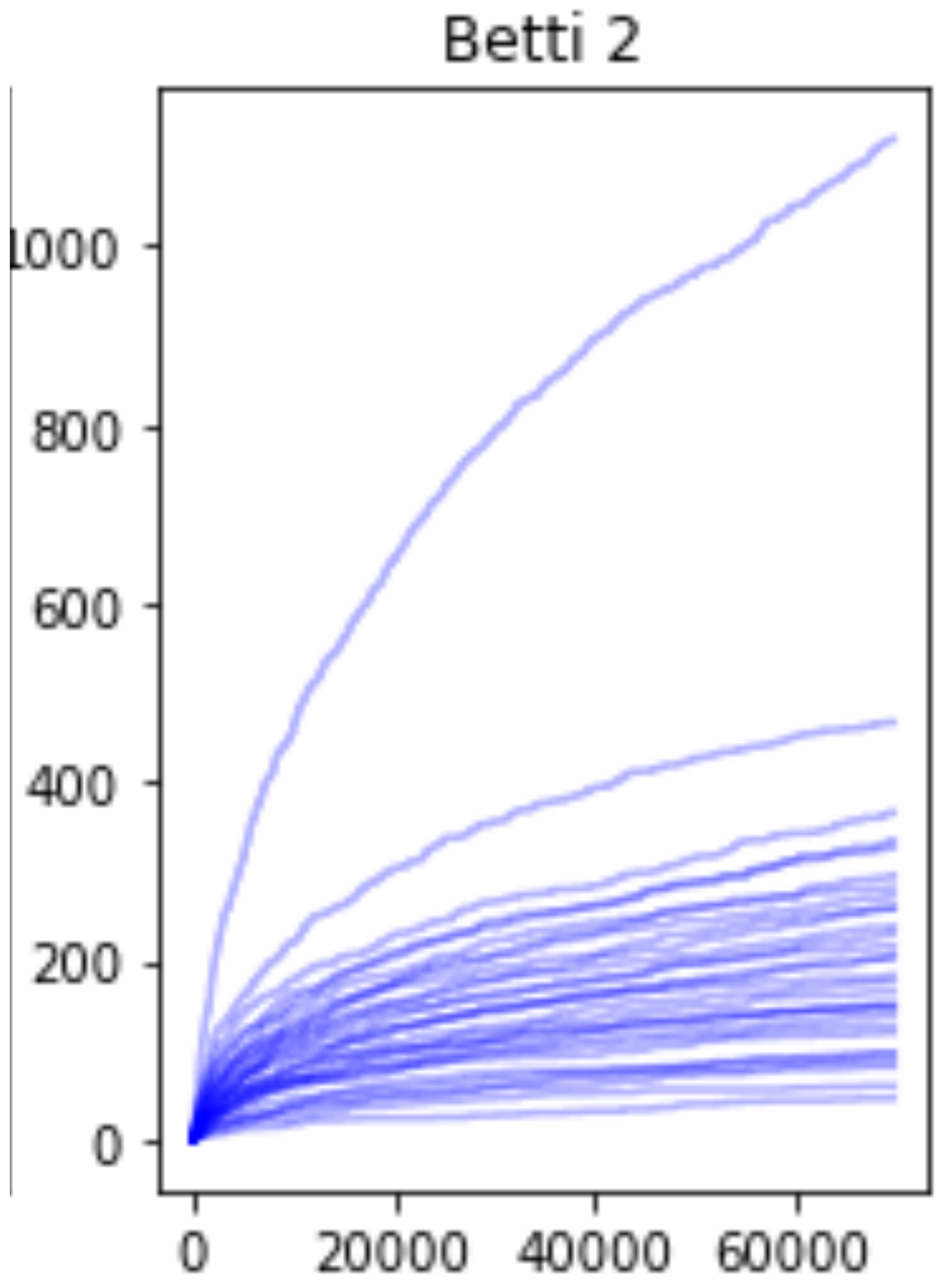
Expected Betti Number $E[\beta_q]$

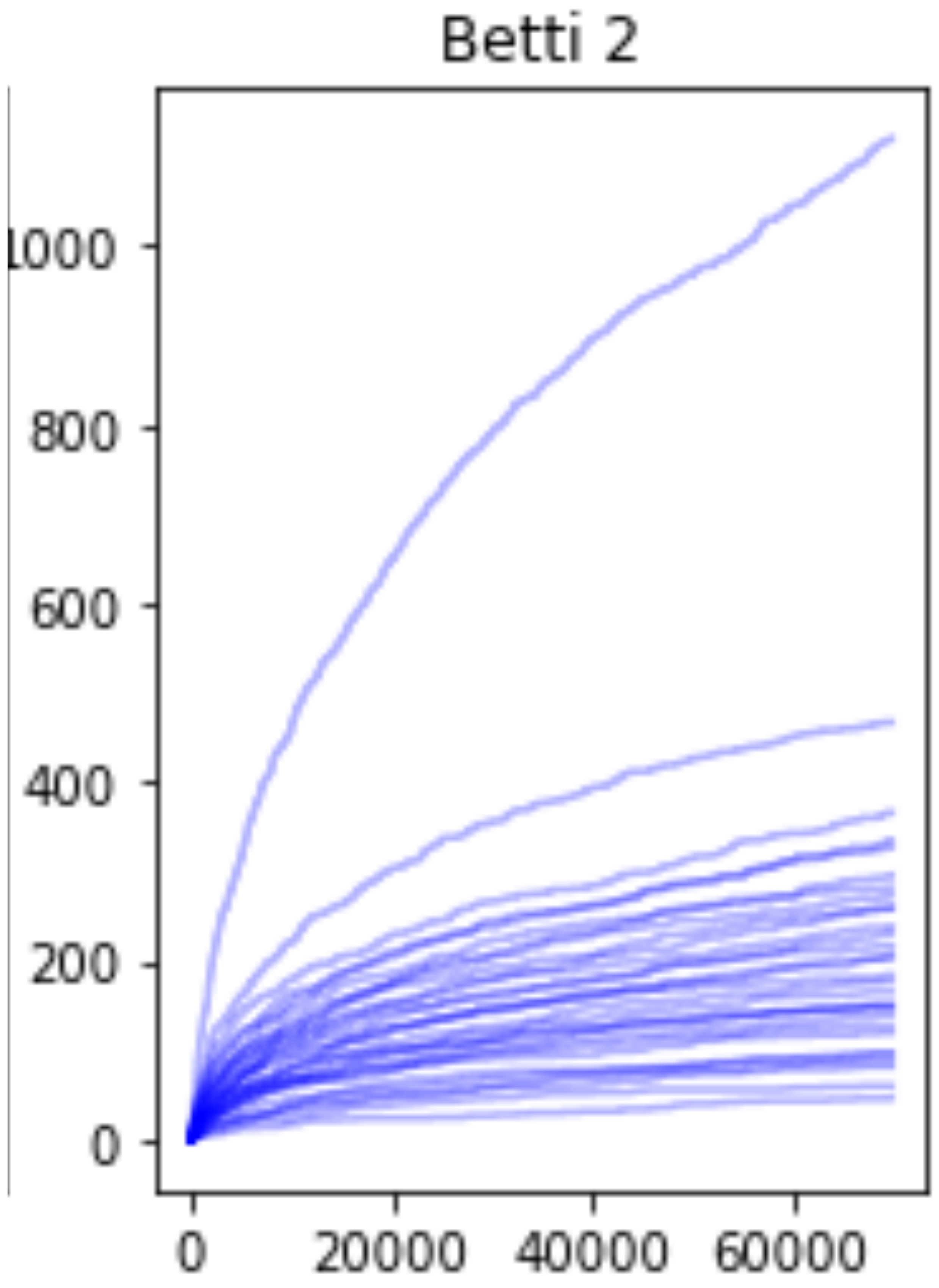
- increasing trend
- concave growth
- outlier



Different curves, different random seeds.
All curves have the same model parameters.

Expected Betti Number $E[\beta_q]$

- $c(\text{num of nodes}^{1-4x}) \leq E[\beta_2] \leq C(\text{num of nodes}^{1-4x})$
  under mild assumptions
  - $x \in (0,1/2)$ depends on pref. attachment strength



Different curves, different random seeds.
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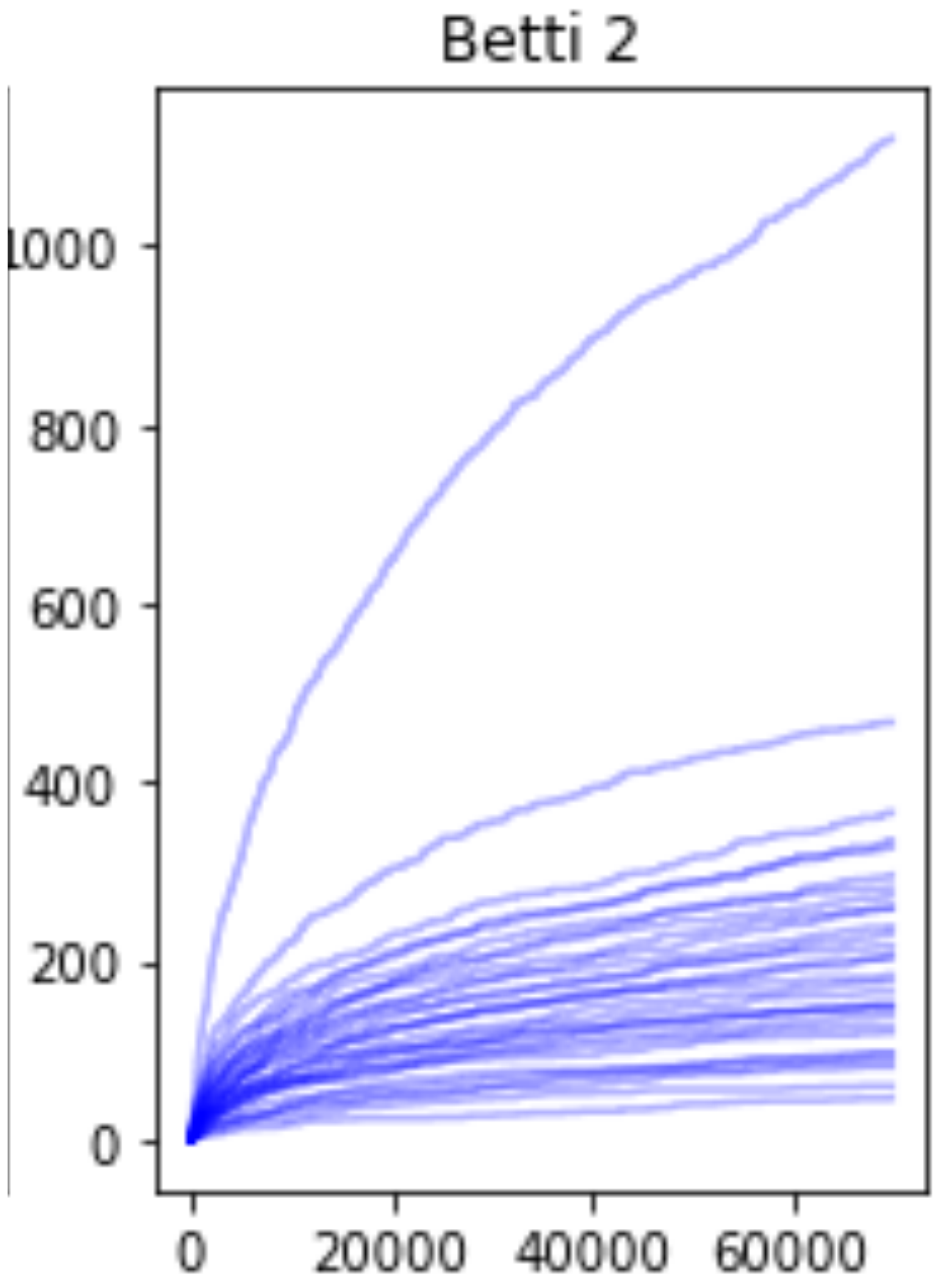
Expected Betti Number $E[\beta_q]$

- c(num of nodes^{1-4x}) $\leq E[\beta_2] \leq C$(num of nodes^{1-4x})
  under mild assumptions
  - $x \in (0,1/2)$ depends on pref. attachment strength
  - If $1 - 4x < 0$, then $E[\beta_2] \leq C$.



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Expected Betti Number $E[\beta_q]$

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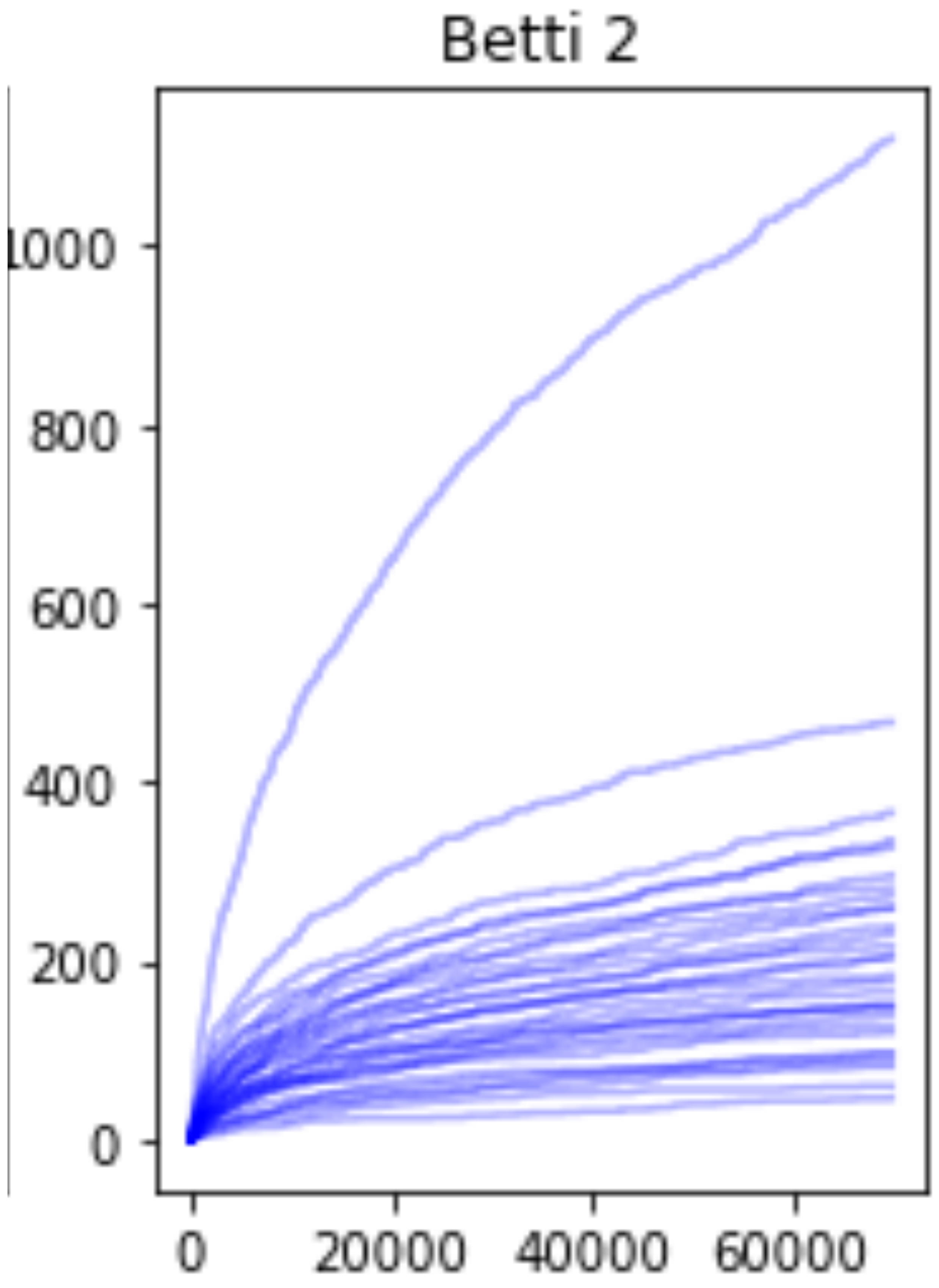
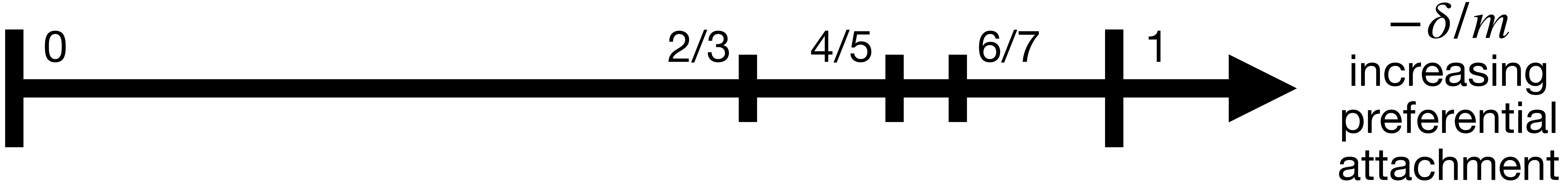
  under mild assumptions
  - $x \in (0,1/2)$ depends on pref. attachment strength
  - If $1 - 4x < 0$, then $E[\beta_2] \leq C$.
- $c(\text{num of nodes}^{1-2qx}) \leq E[\beta_q] \leq C(\text{num of nodes}^{1-2qx})$ for $q \geq 2$ if $1 - 2qx > 0$



Different curves, different random seeds.
All curves have the same model parameters.

Phase transition

Recall
P(attaching to v) ∝ degree + δ
m = number of edges per new node



Phase transition

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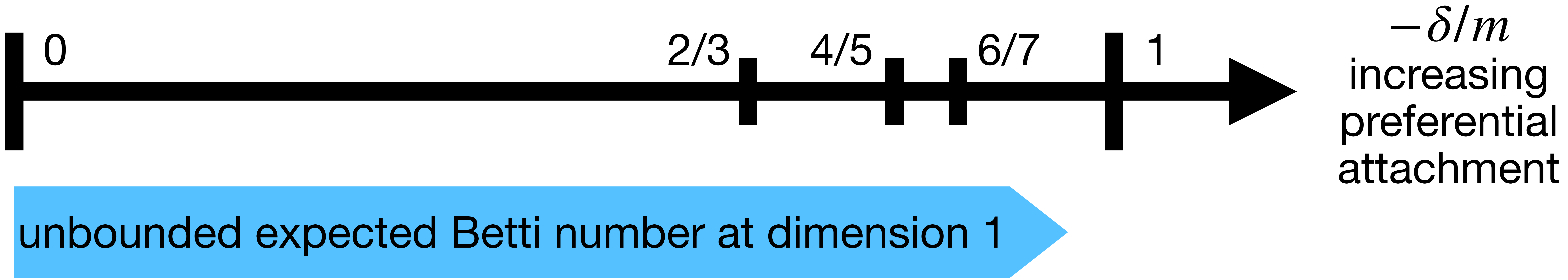

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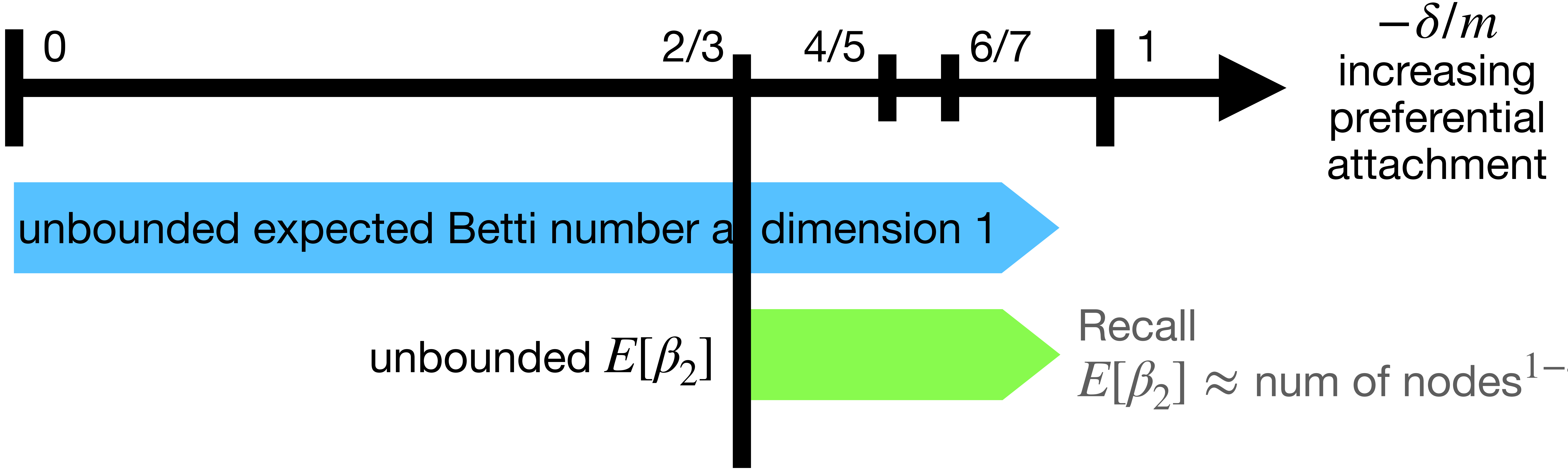

Recall

$E[\beta_2] \approx$ num of nodes^{1-4x}

Phase transition

Recall

P(attaching to v) ∝ degree + δ

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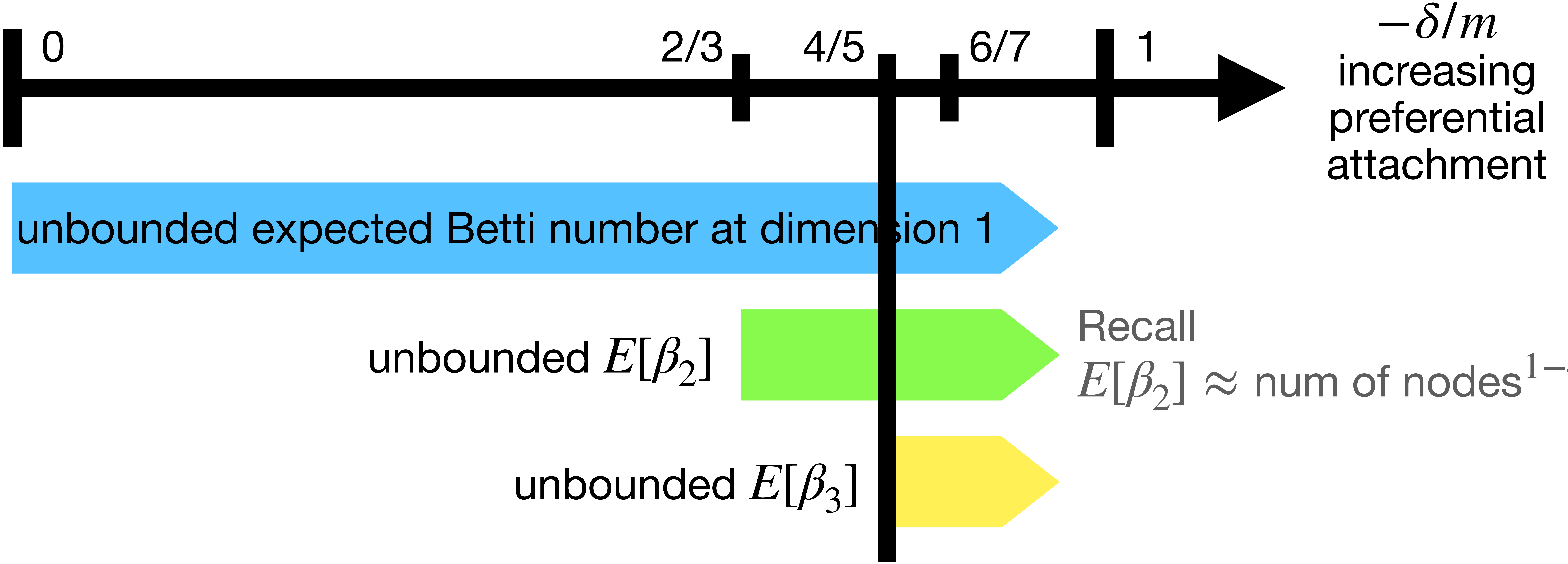

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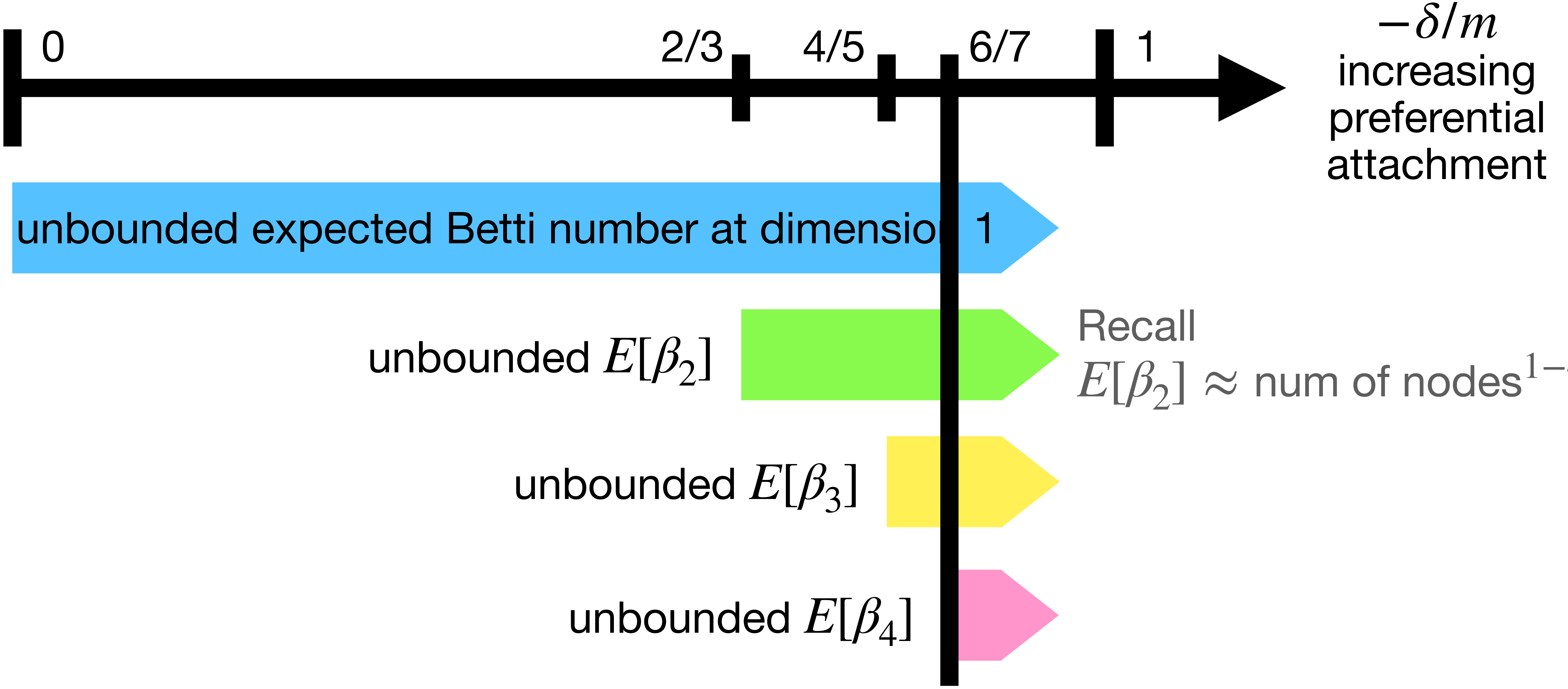

Recall
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Phase transition

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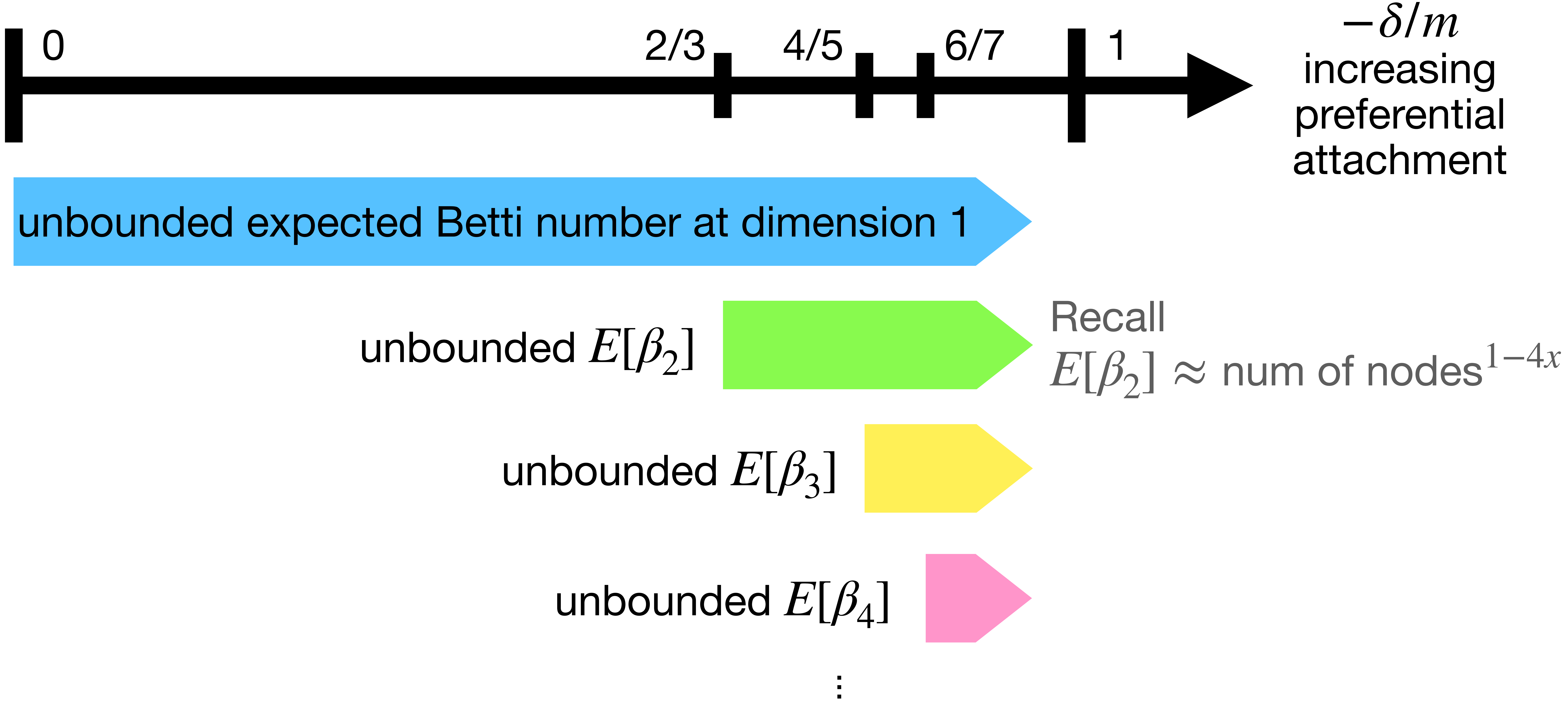
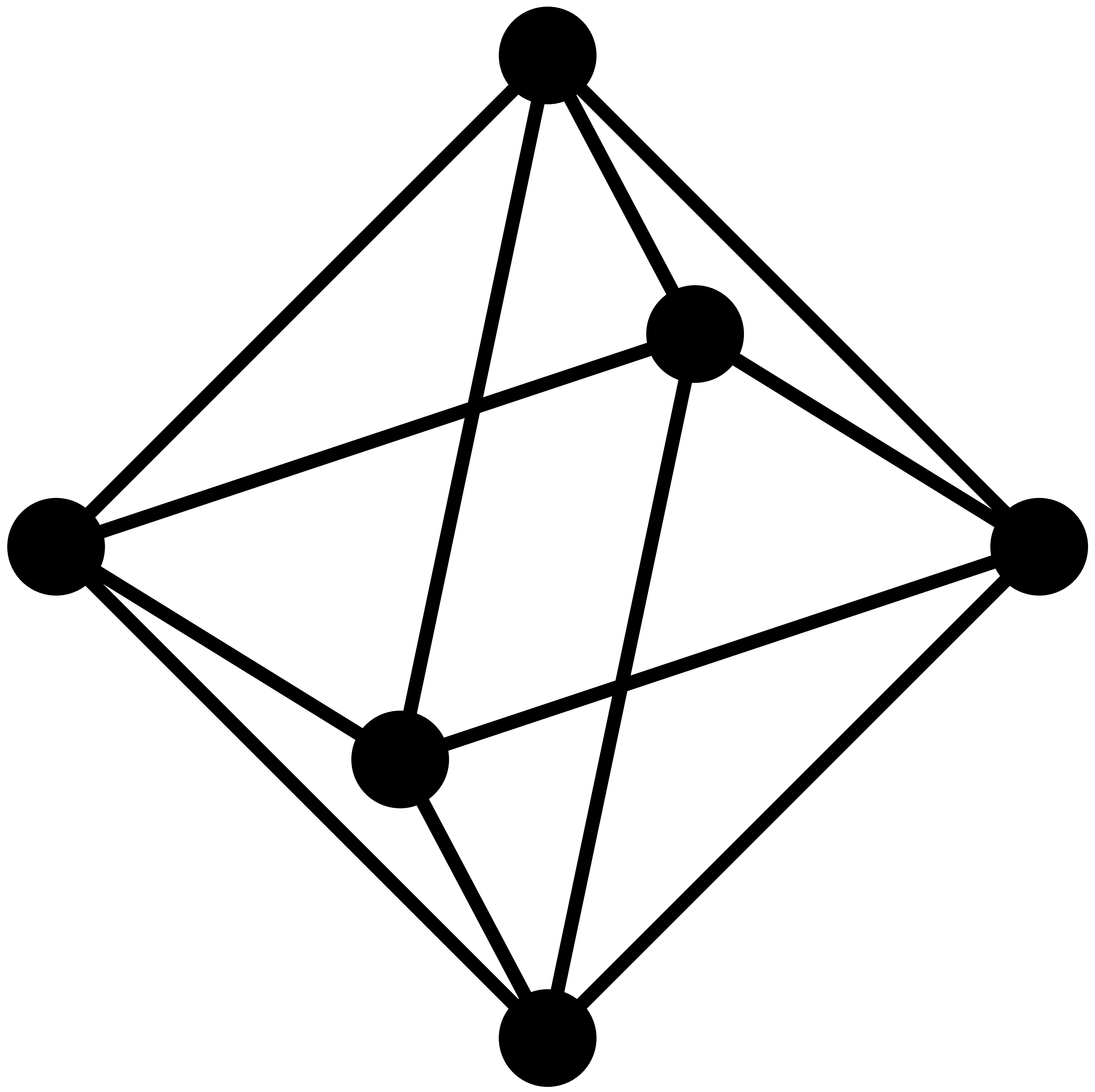


Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

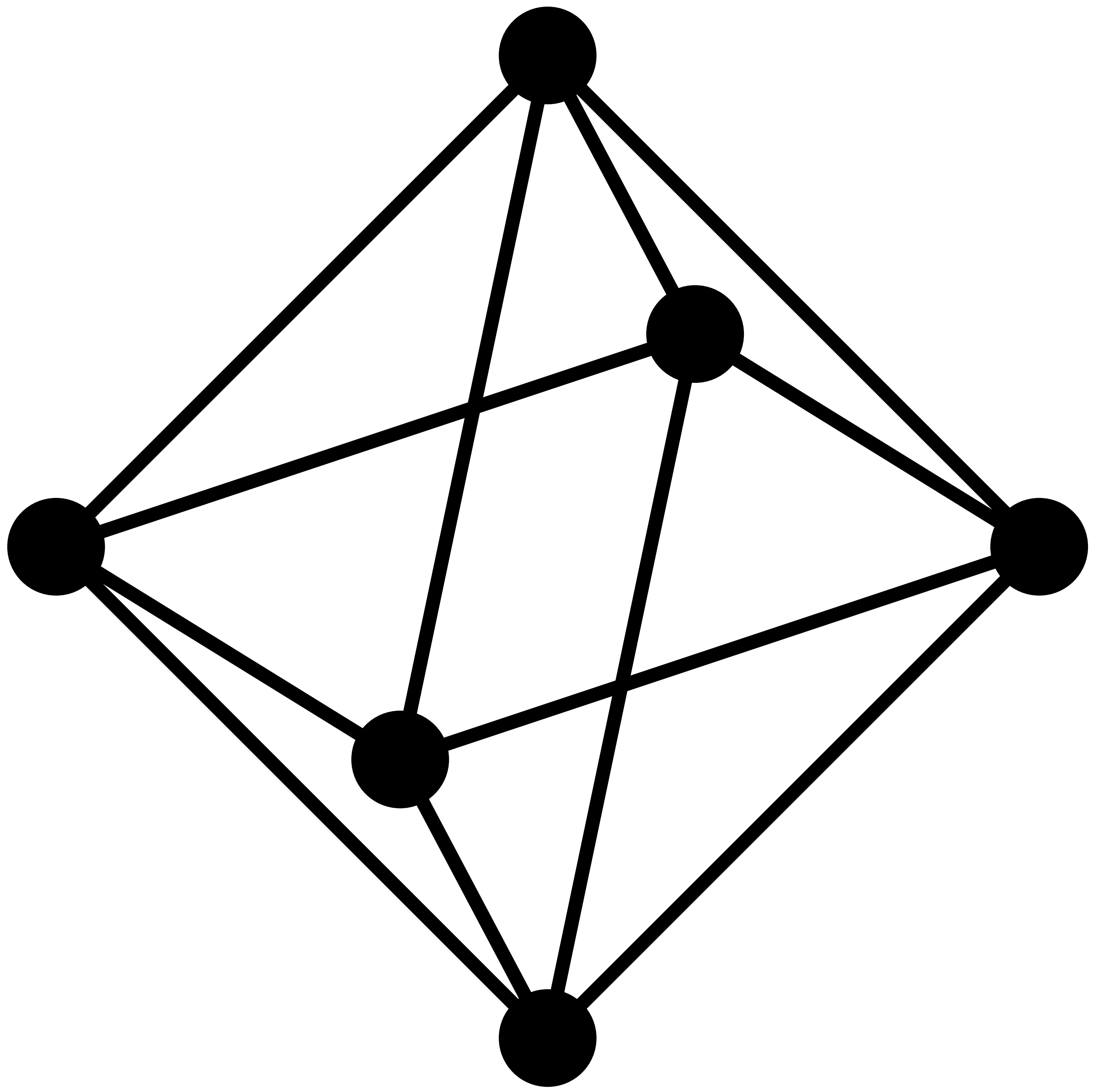
Proof?

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}

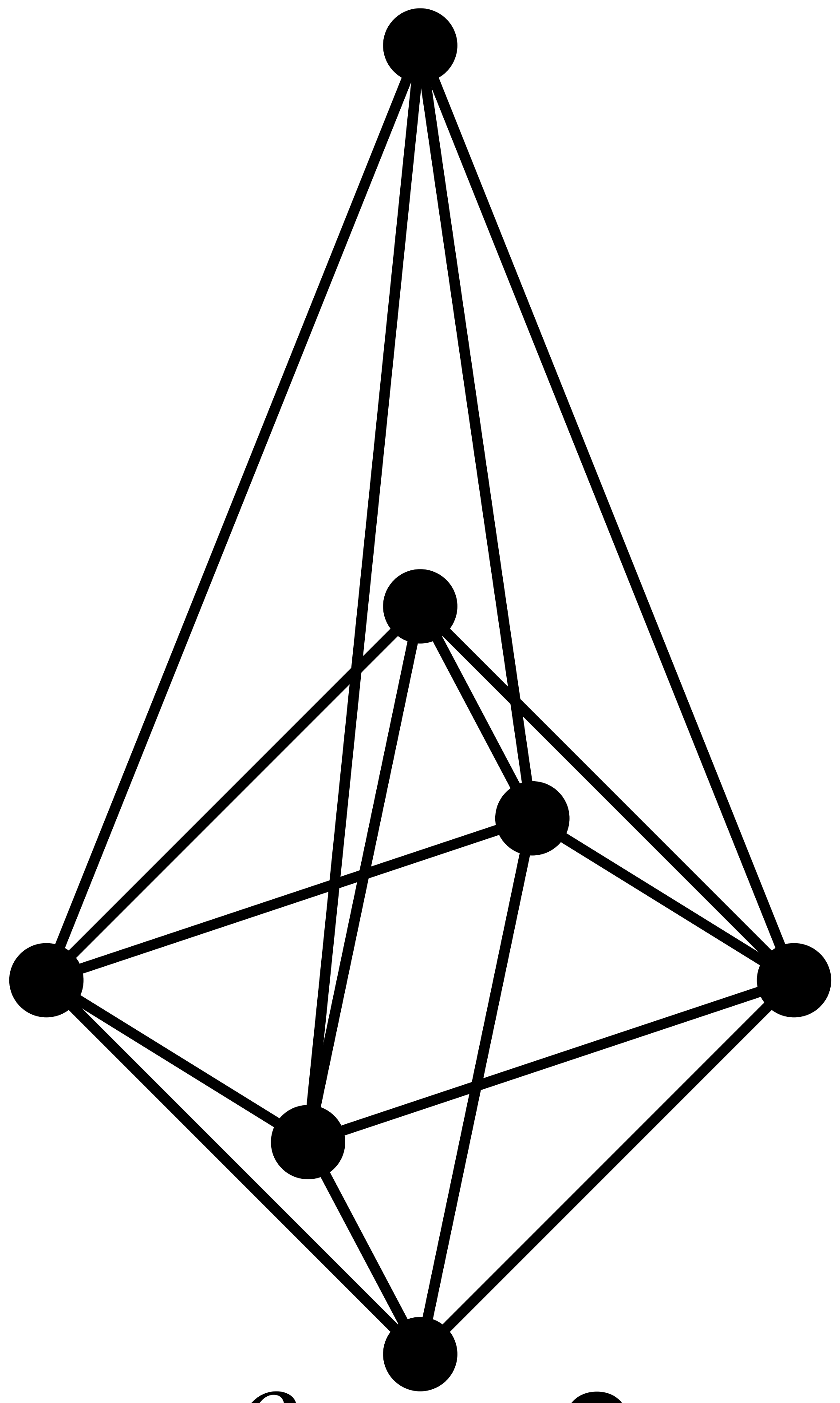


$\beta_2 = 1$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}

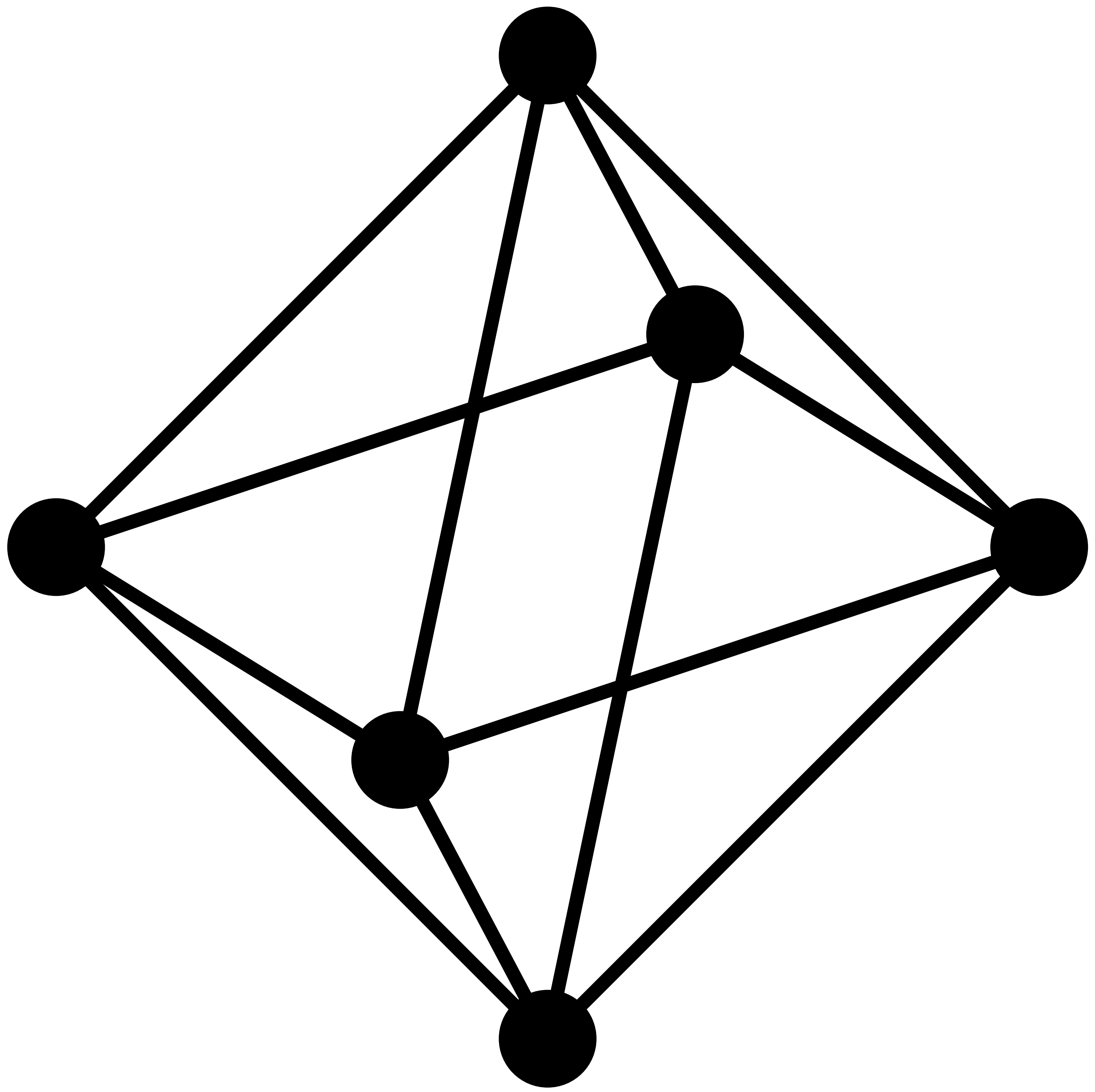


$\beta_2 = 1$

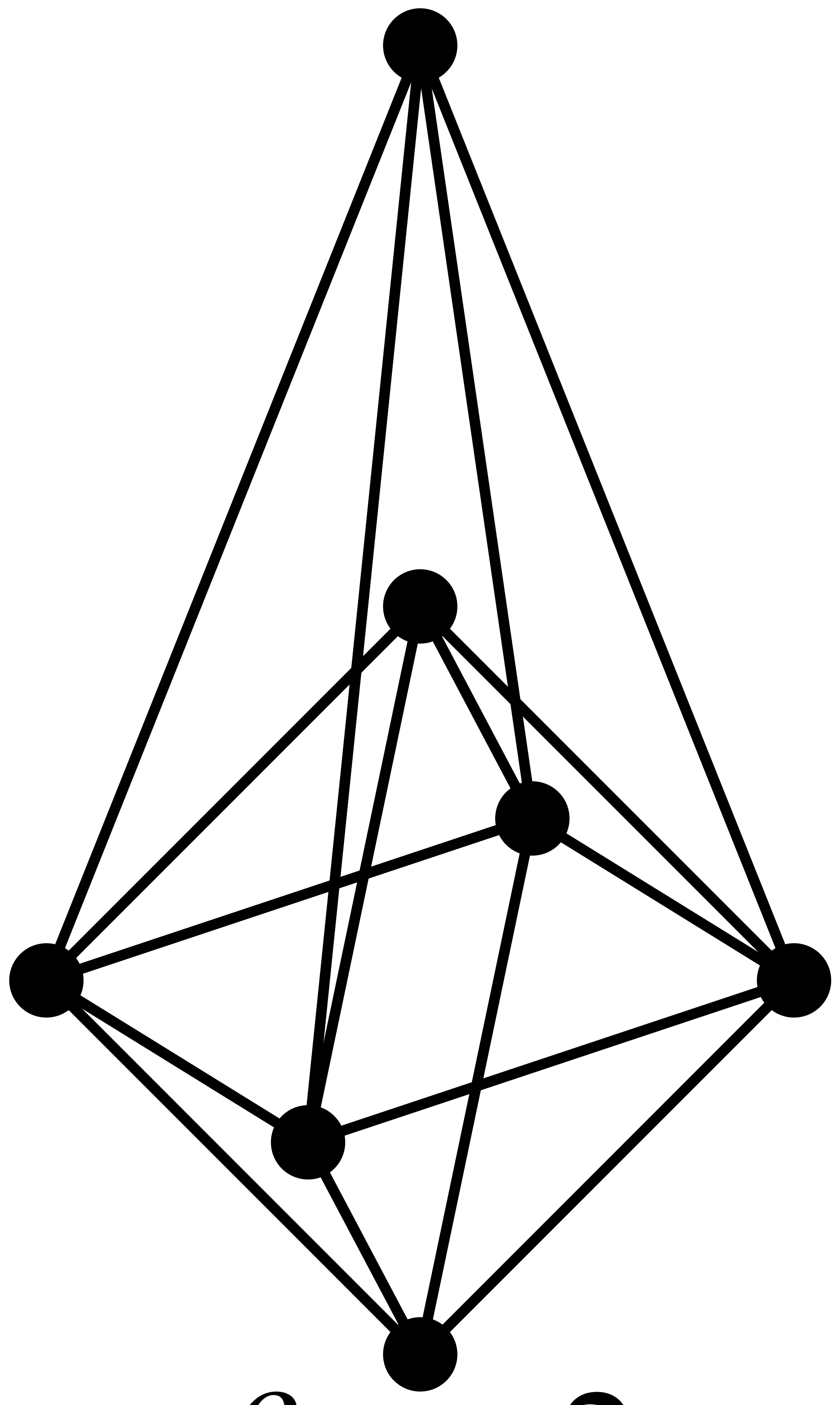


$\beta_2 = 2$

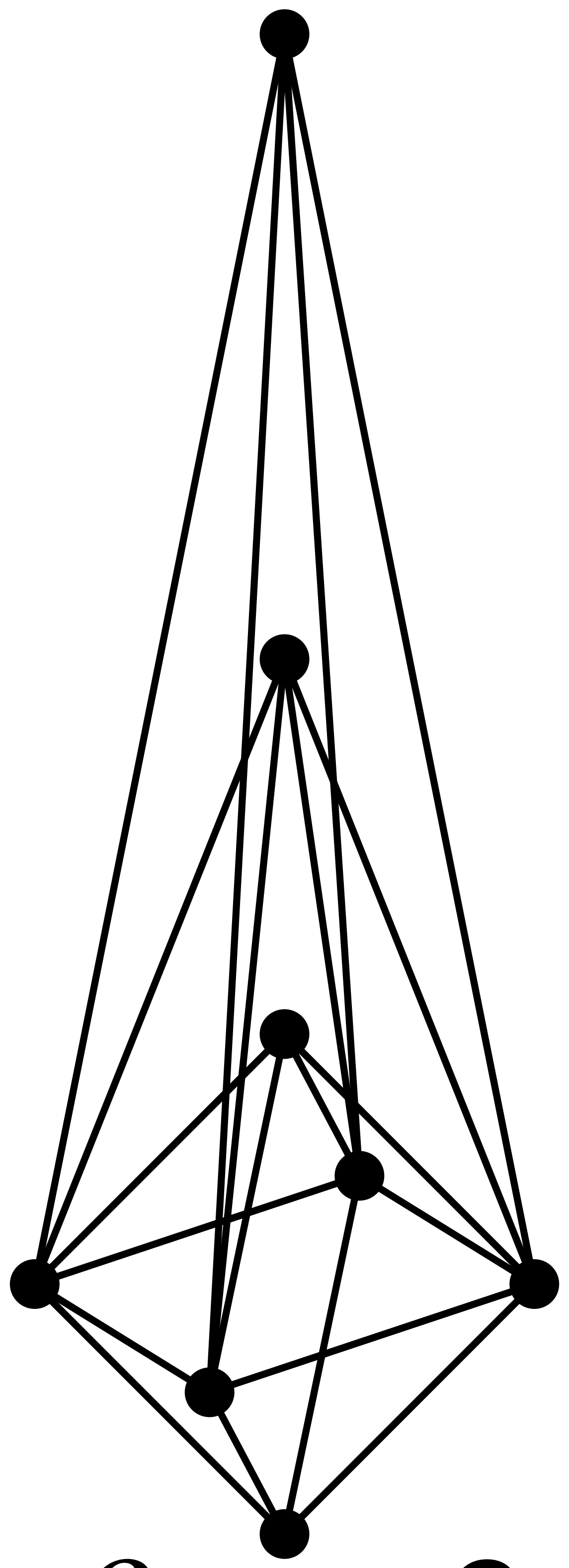
Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



$\beta_2 = 1$



$\beta_2 = 2$



$\beta_2 = 3$

IV. What lies ahead

order of magnitude of
expected Betti numbers

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

simplicial preferential attachment?

parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

simplicial preferential attachment?

other non-homogeneous complexes?

What did we learn today?

- Random topology is cool.

What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.

What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

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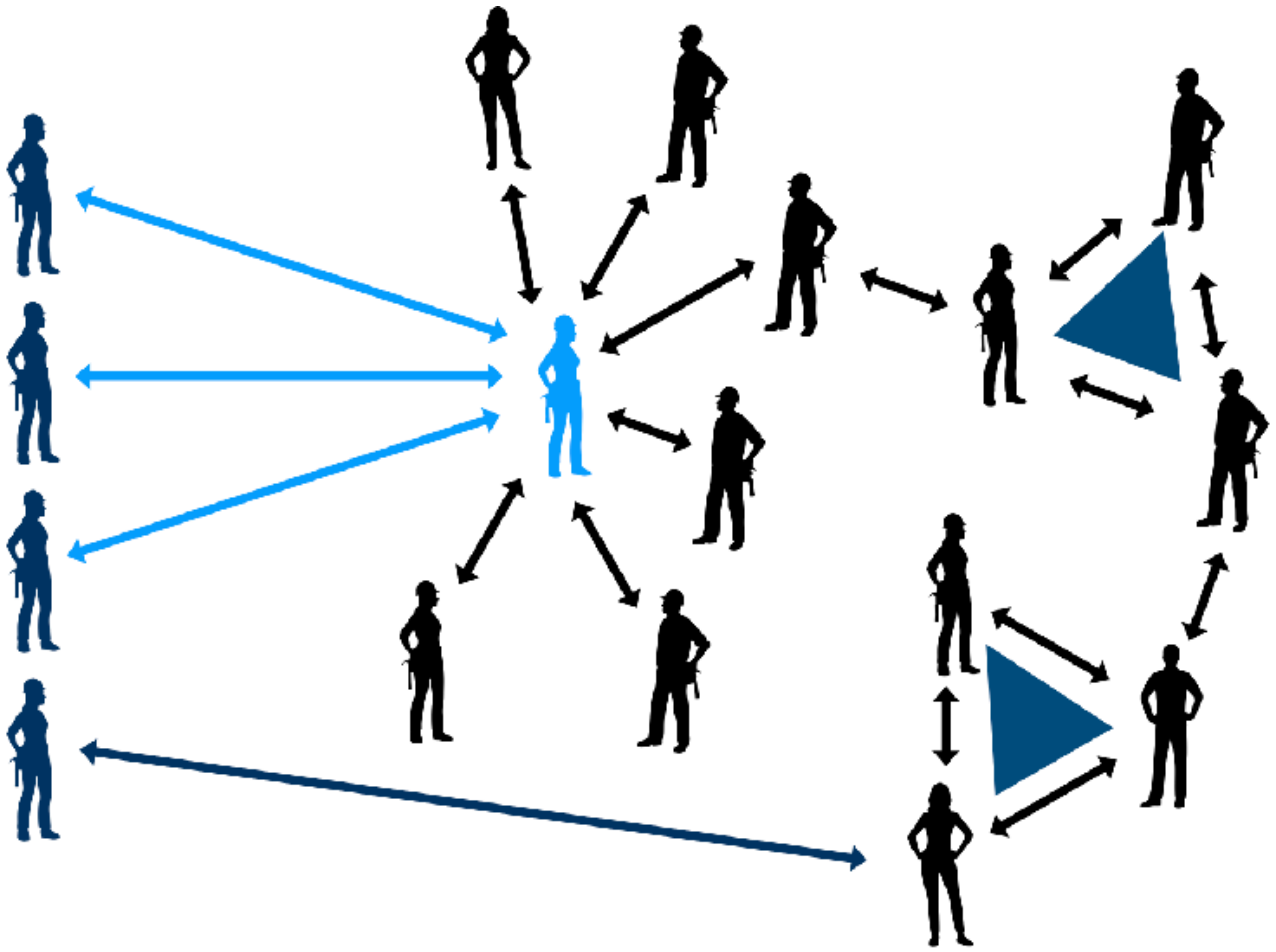

arxiv paper

Thank you!

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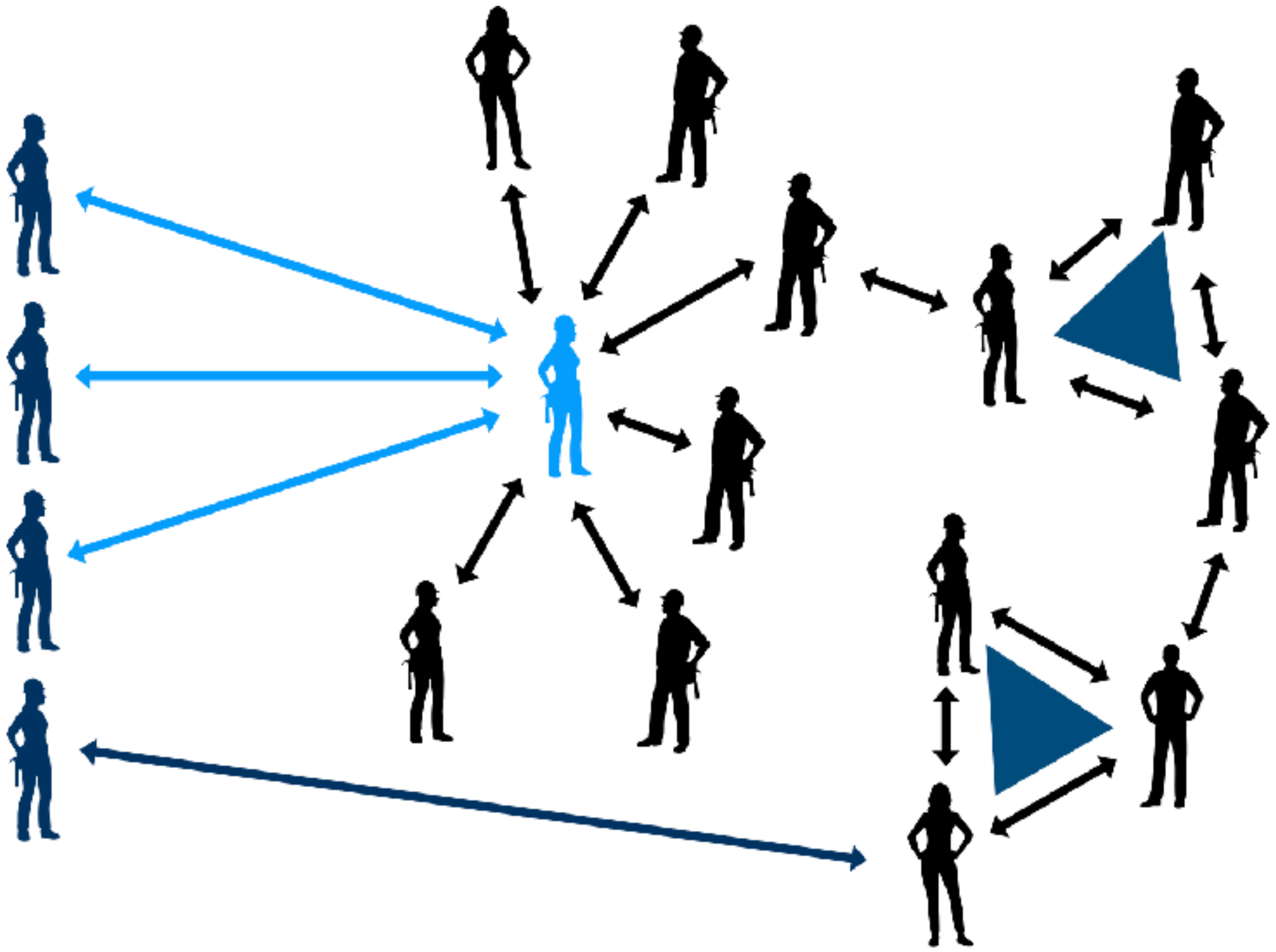

arxiv paper