Topology of Scale-Free Graphs

How Random Interaction Begets Holes

Chunyin Siu
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Topology of Scale-Free Graphs — Homology and Homotopy

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So, preferential attachment…

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

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- topological properties

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- random fluctuation?

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So, preferential attachment…

- topological properties
- random fluctuation?
- —> random topology

(Stephen Coast
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Agenda

random topology

Agenda

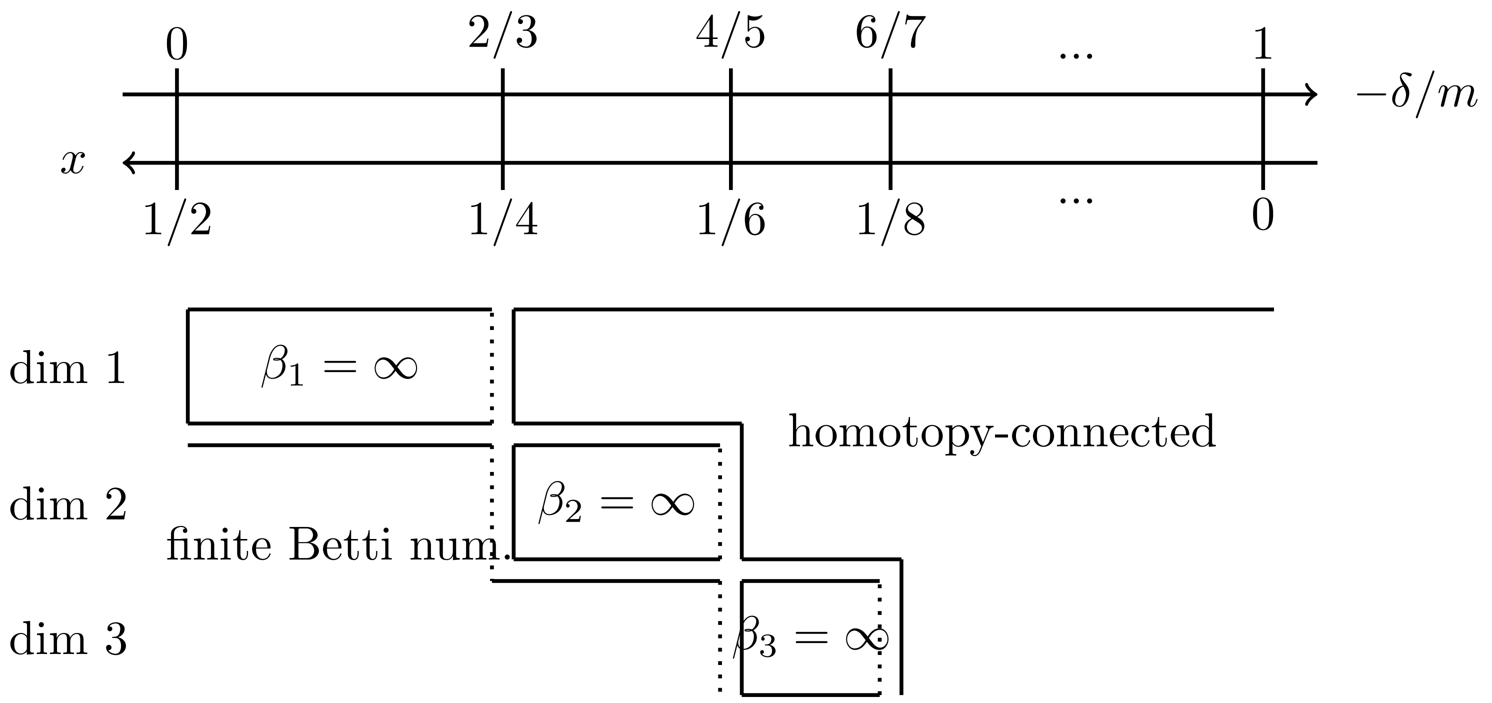
random topology

preferential attachment

Agenda

random topology

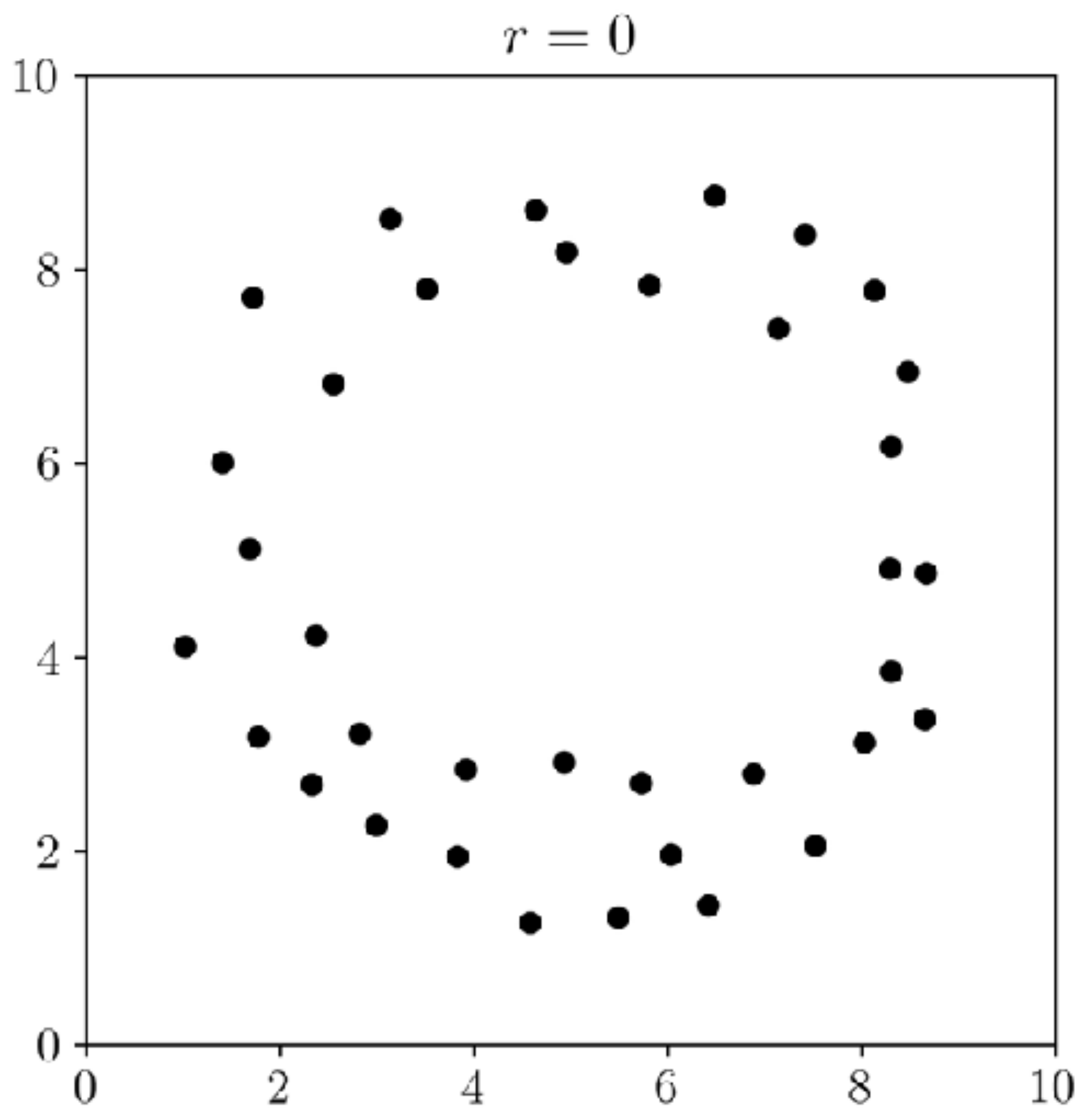
preferential attachment



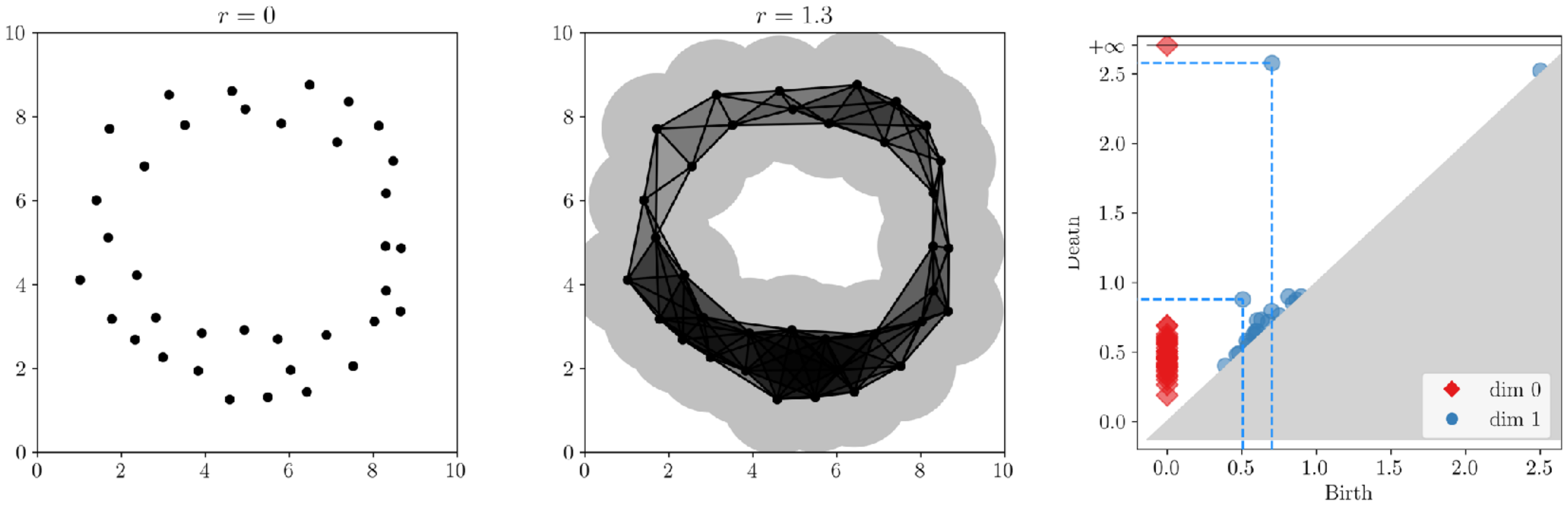
our result

I. A Probabilist’s Apology

Why Random Topology and What we Know

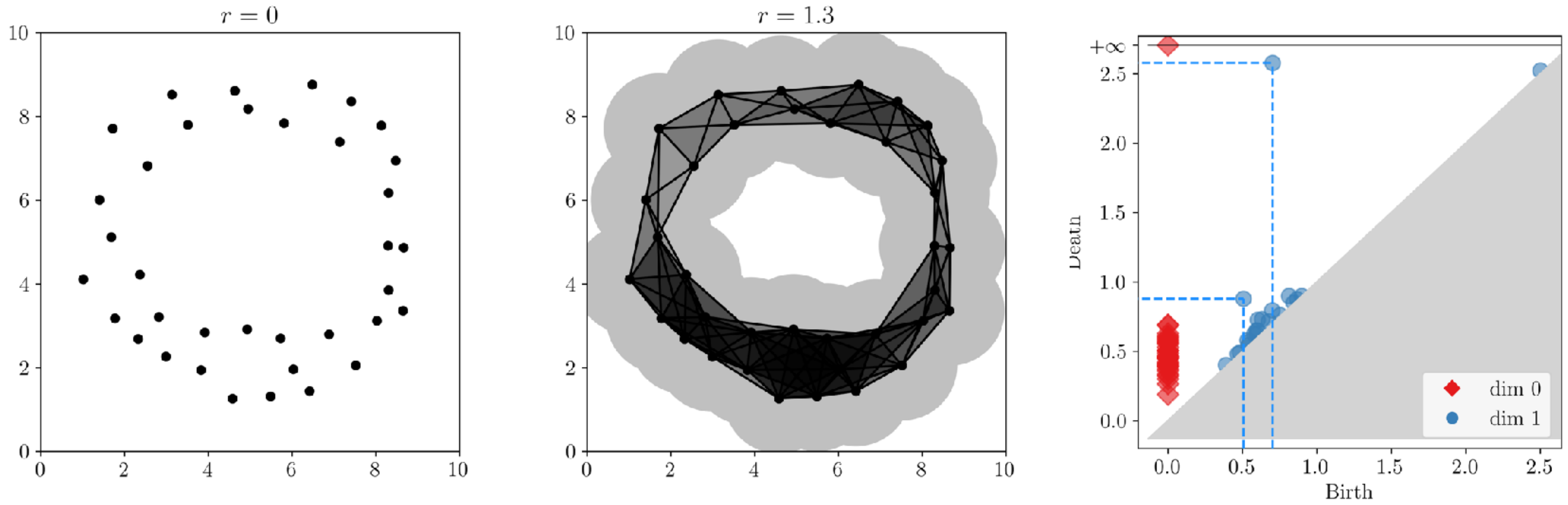


$r = 0$
0
2
4
6
8
10
0
2
4
6
8
10

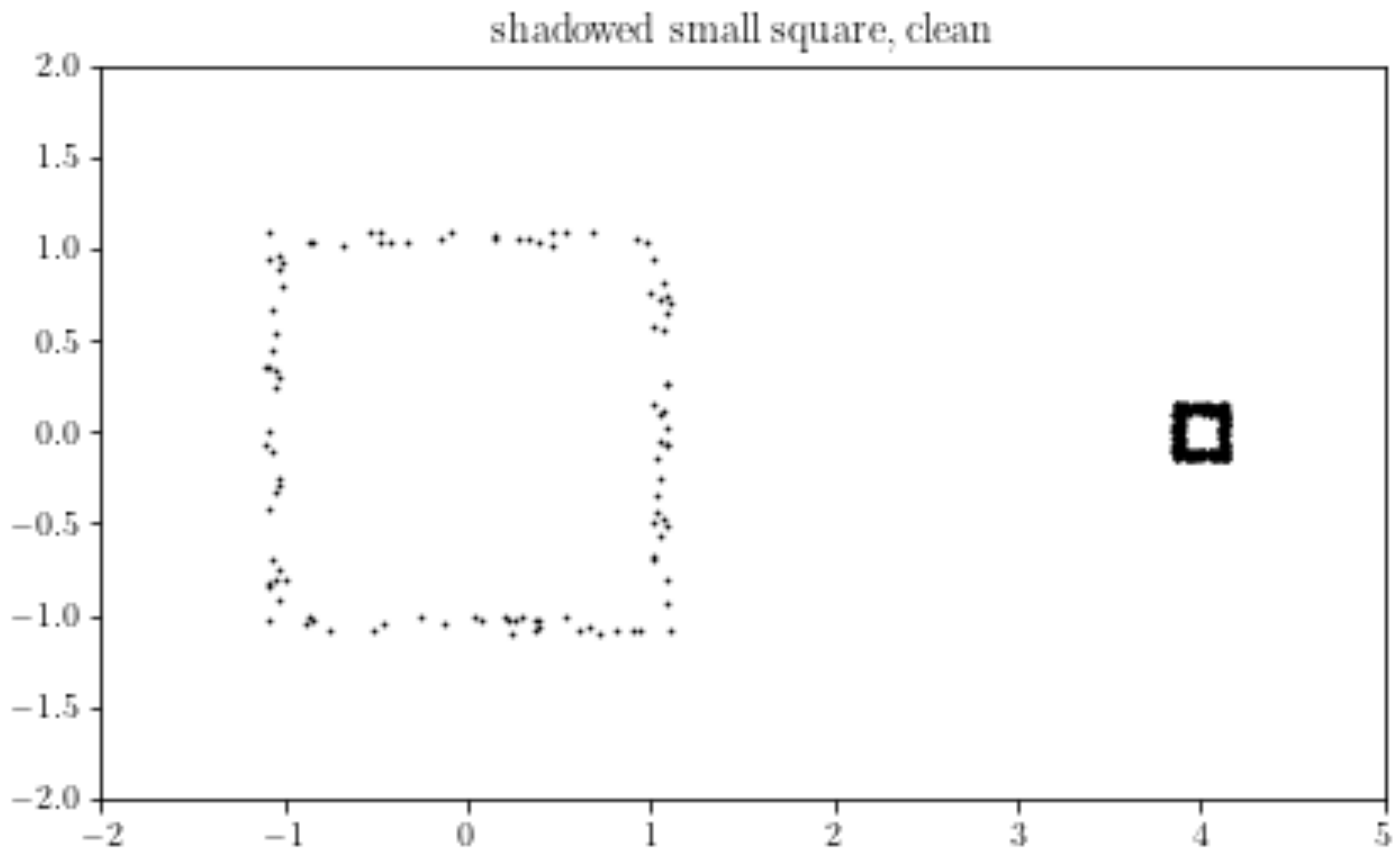


plots generated by Andrey Yao

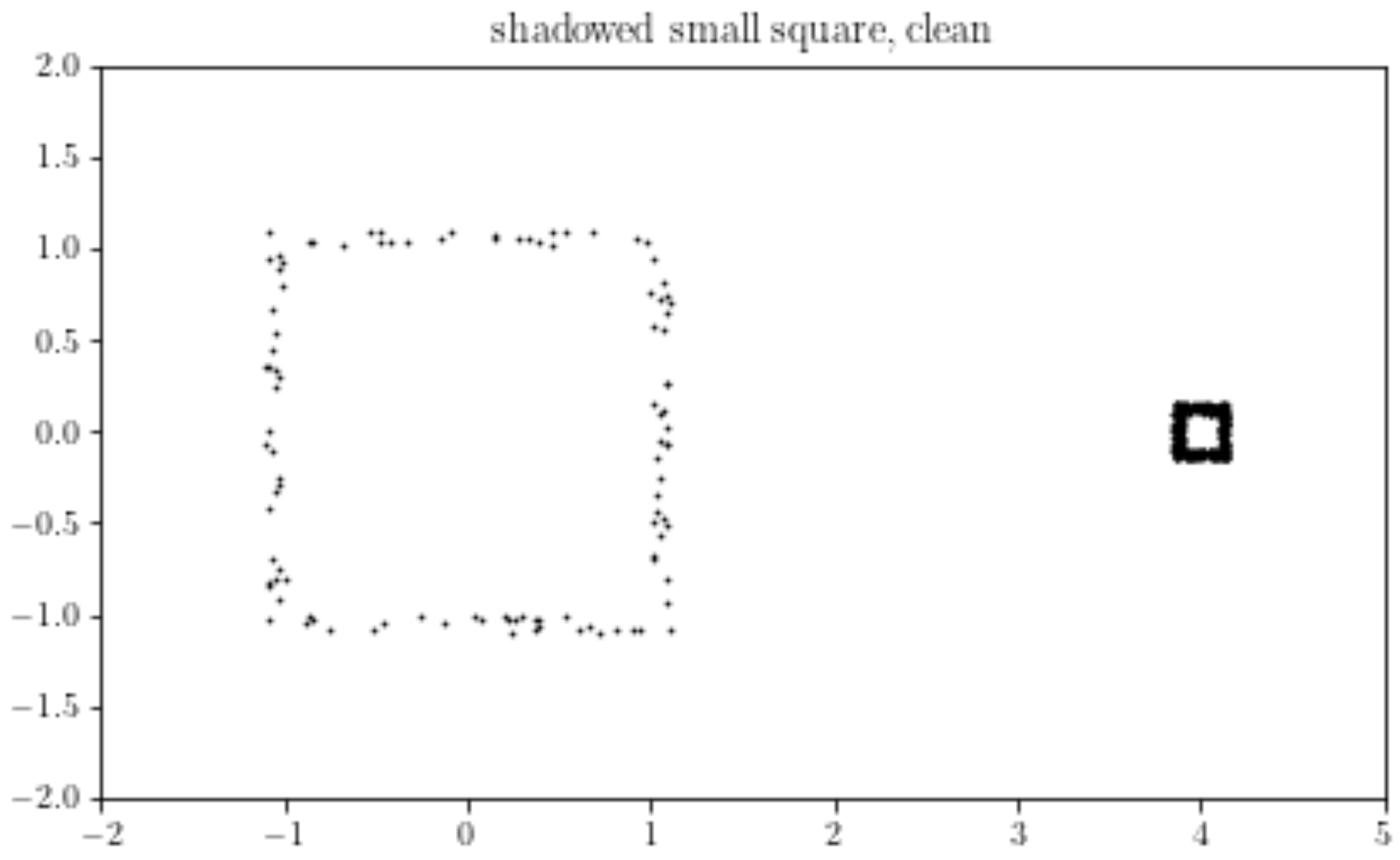
Size is Signal



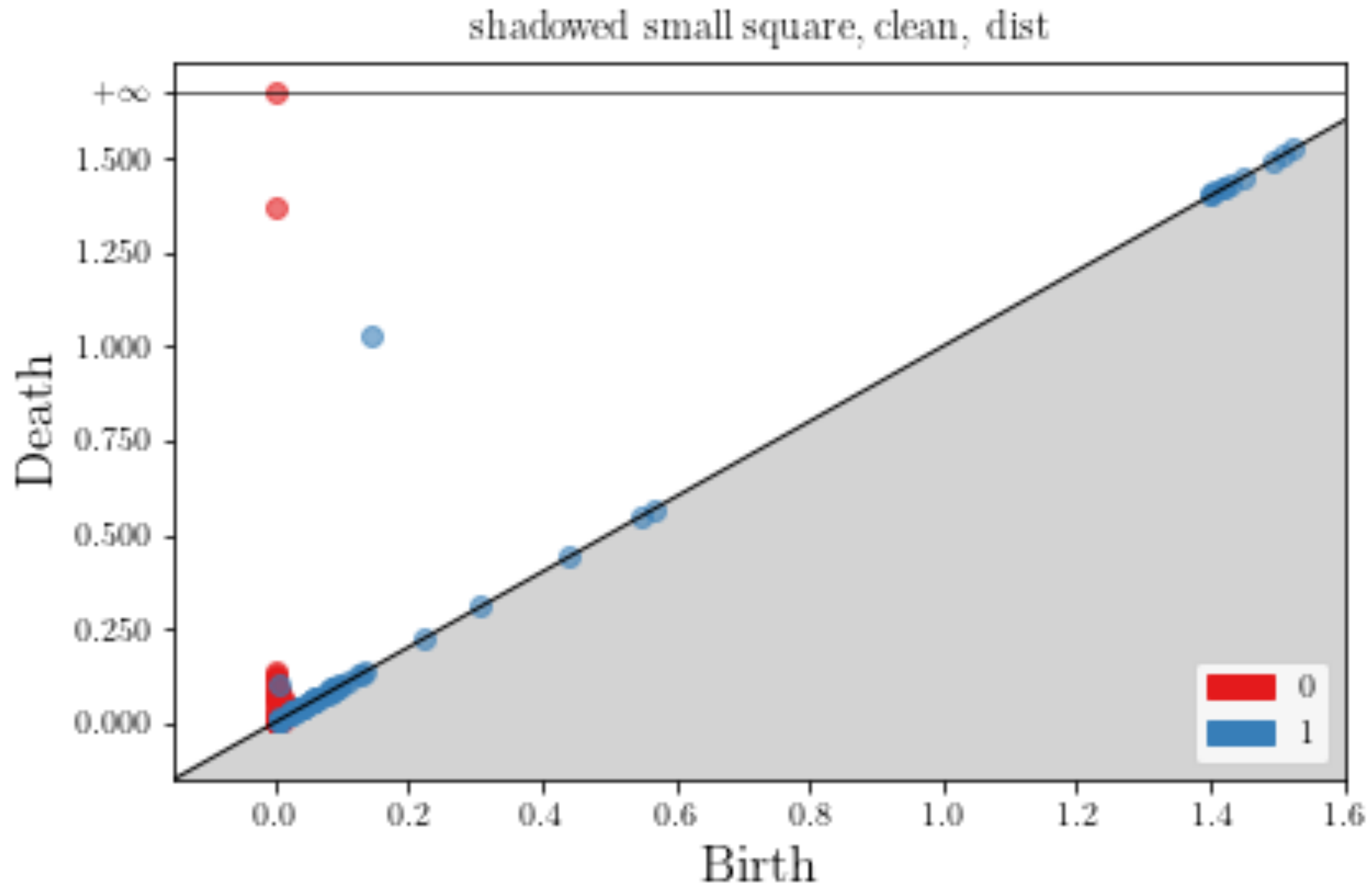
Or is it?


shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5

Or is it?



shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5



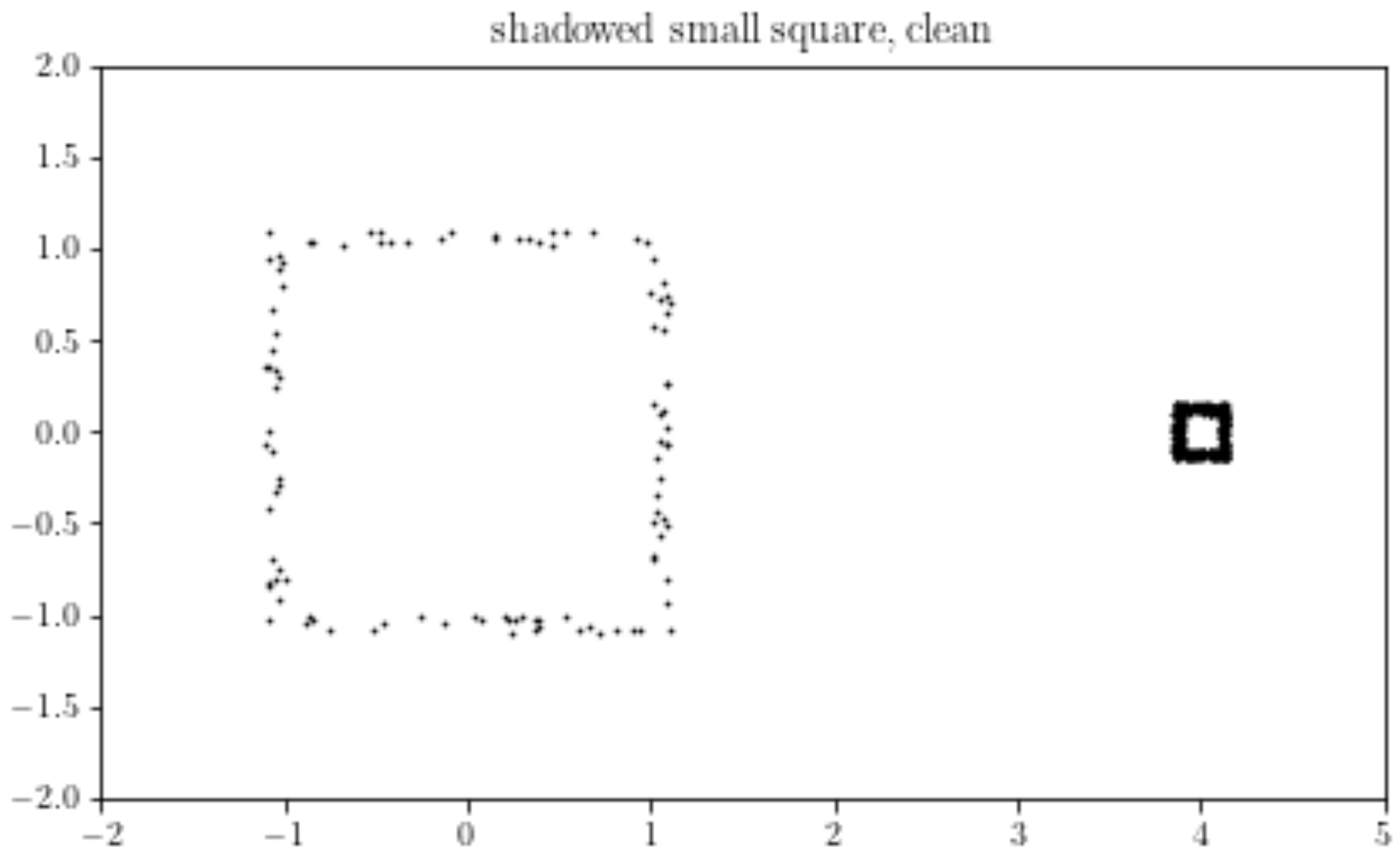
shadowed small square, clean, dist
+∞
1.500
1.250
1.000
0.750
0.500
0.250
0.000
Death
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Birth
0
1

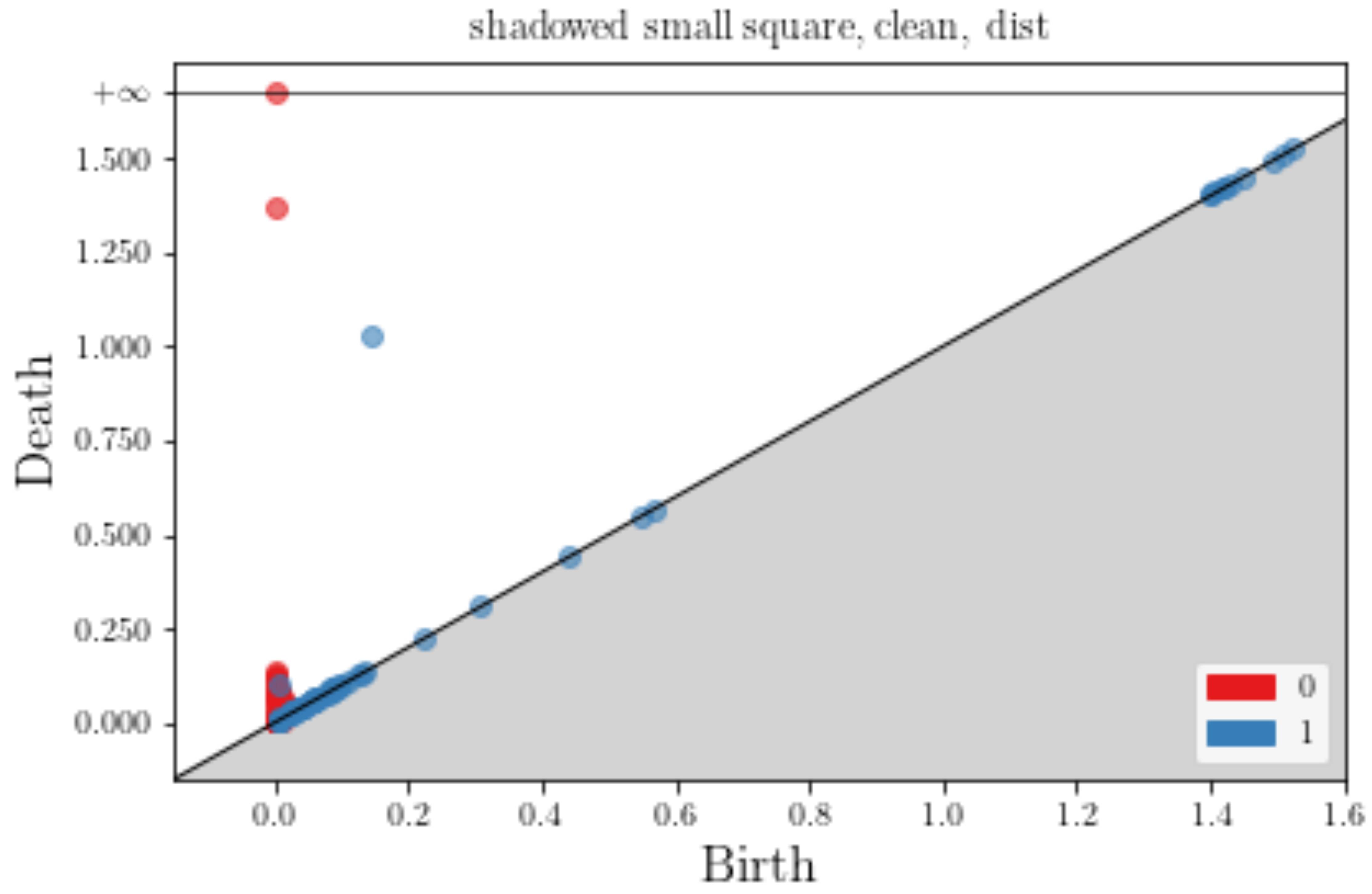
Size is Signal?

Surprise

~~Size~~ is Signal.

Random points don’t do that.

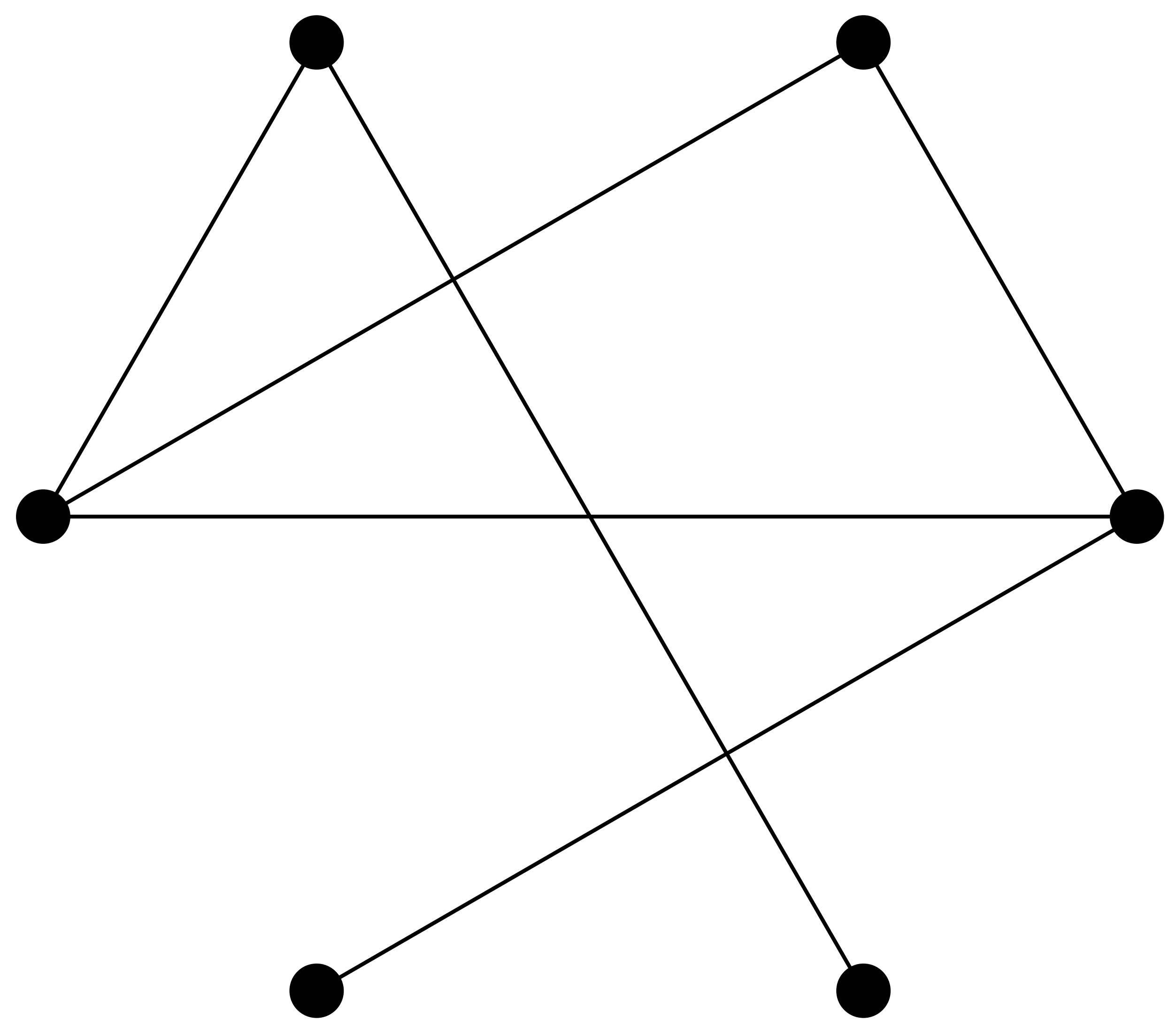




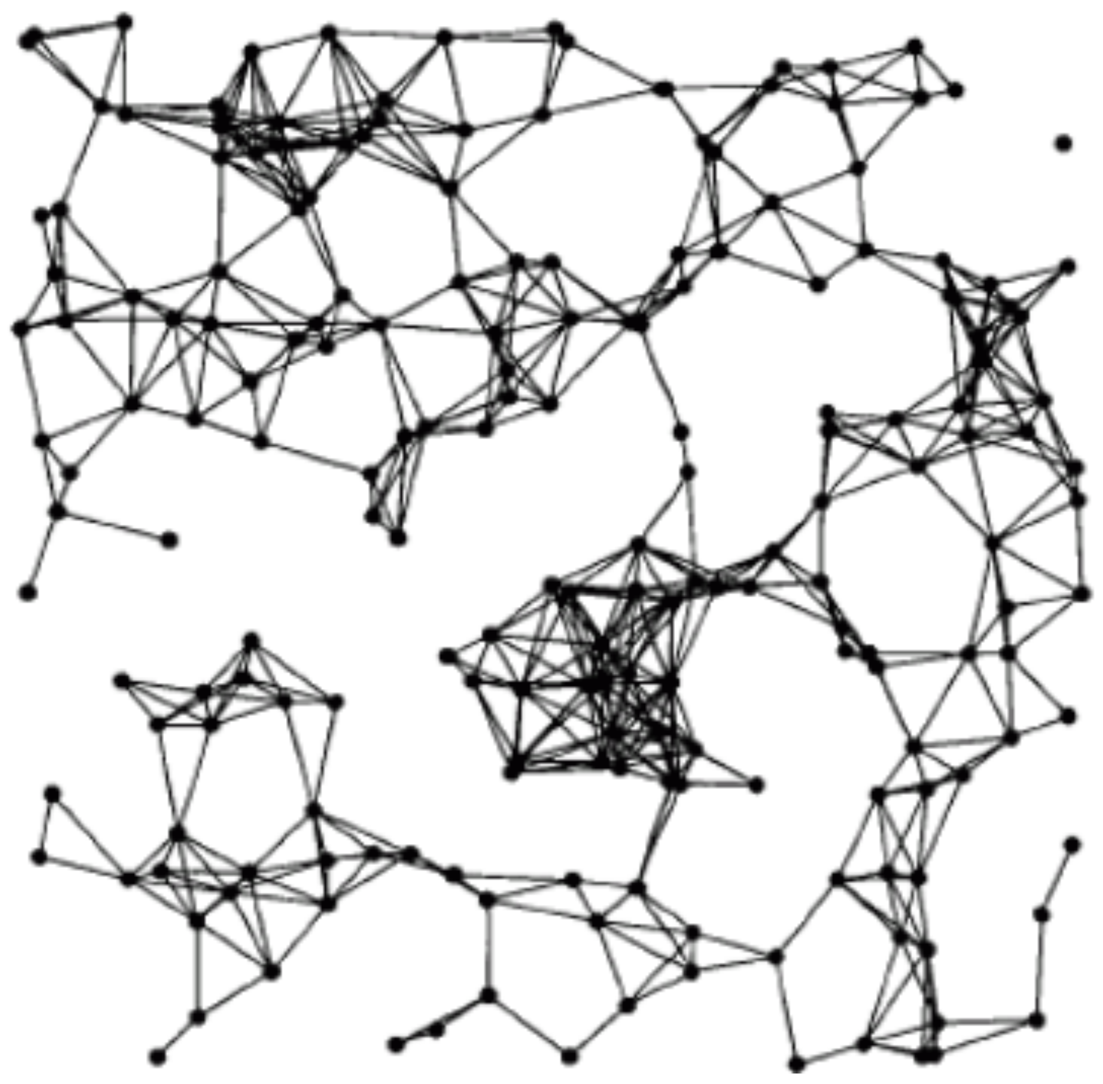
Signal is what is not random.

Signal is what is not **random.**
So what is **random?**

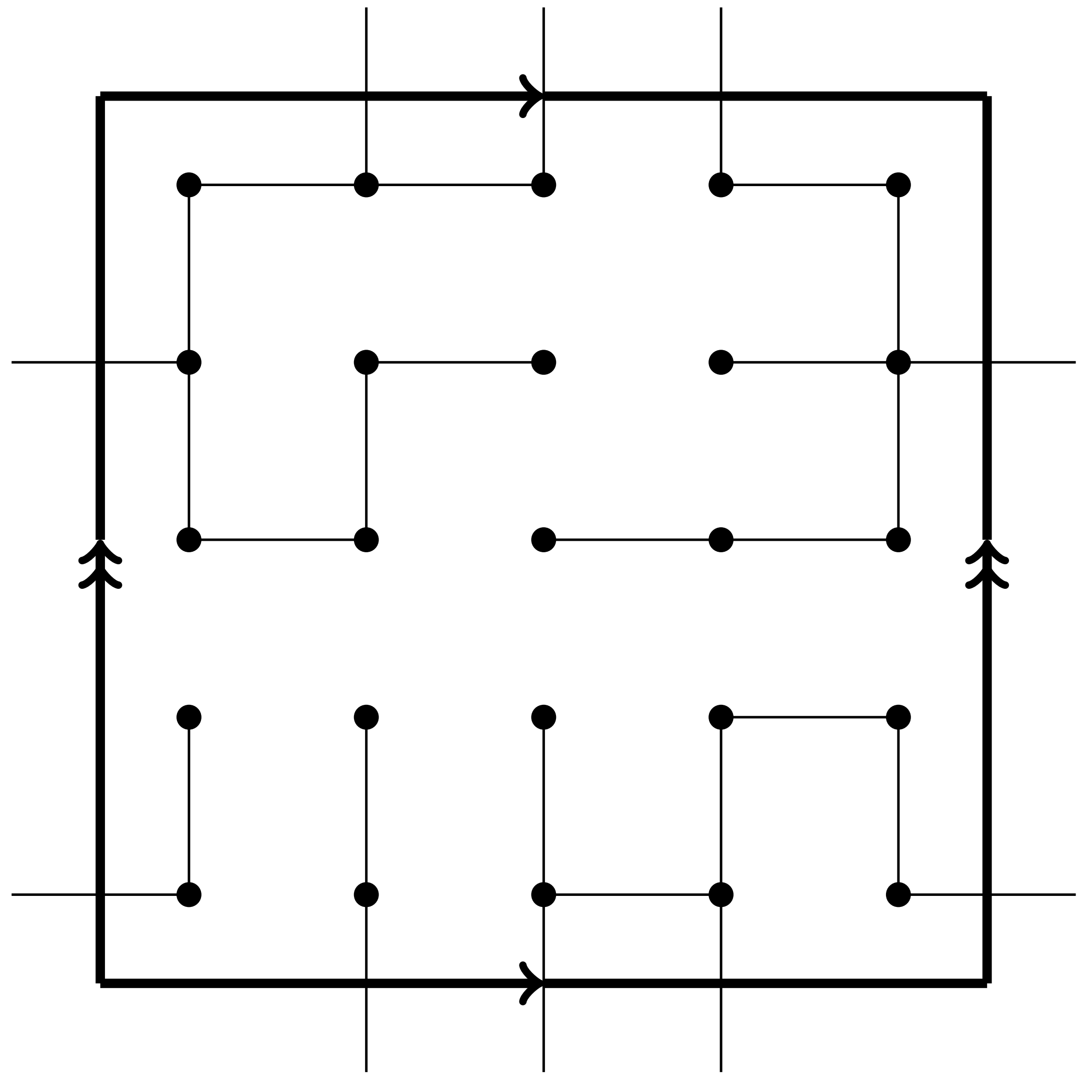
Tea with Random Topology



Erdo-Renyi Complexes

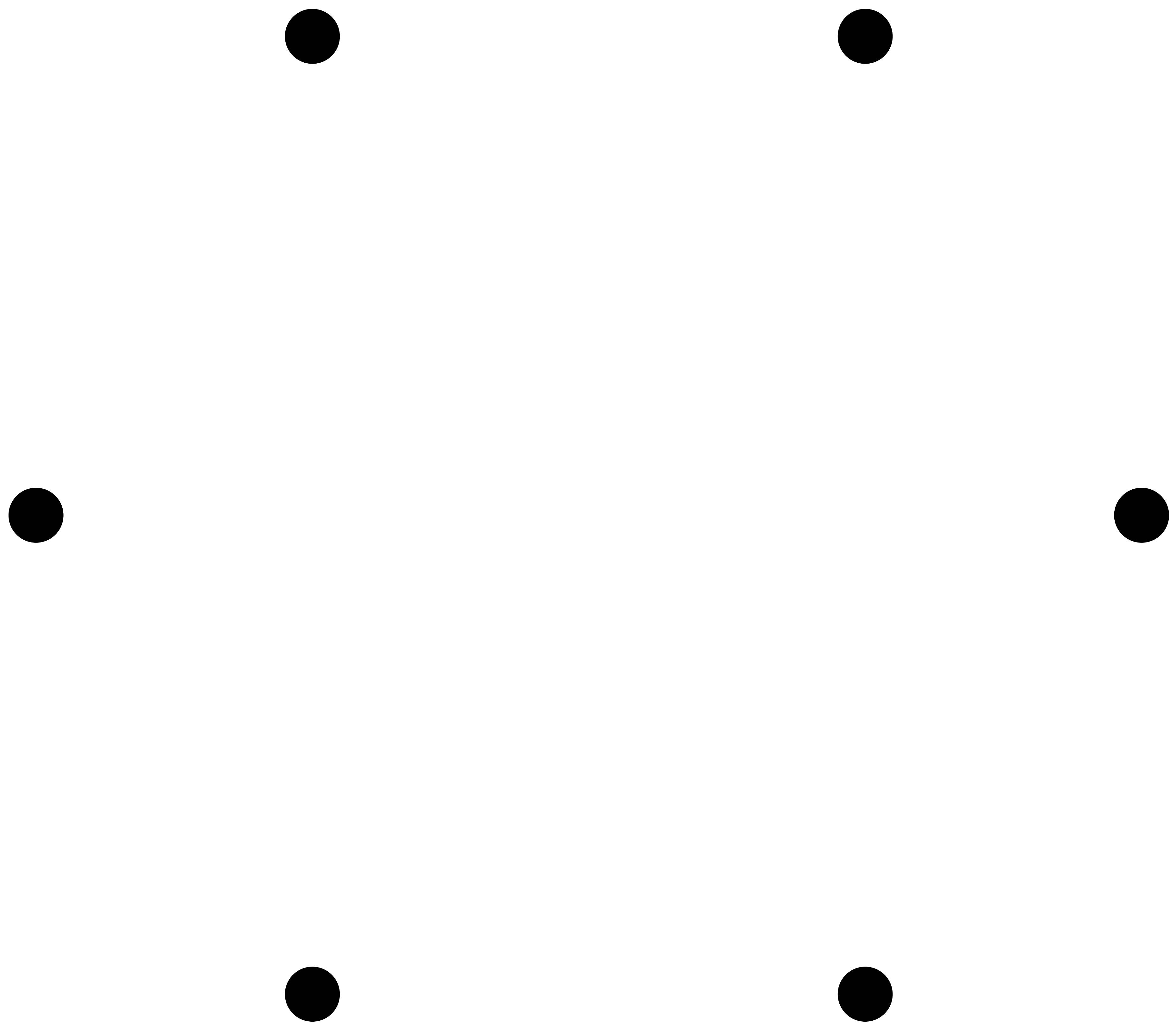


Geometric Complexes

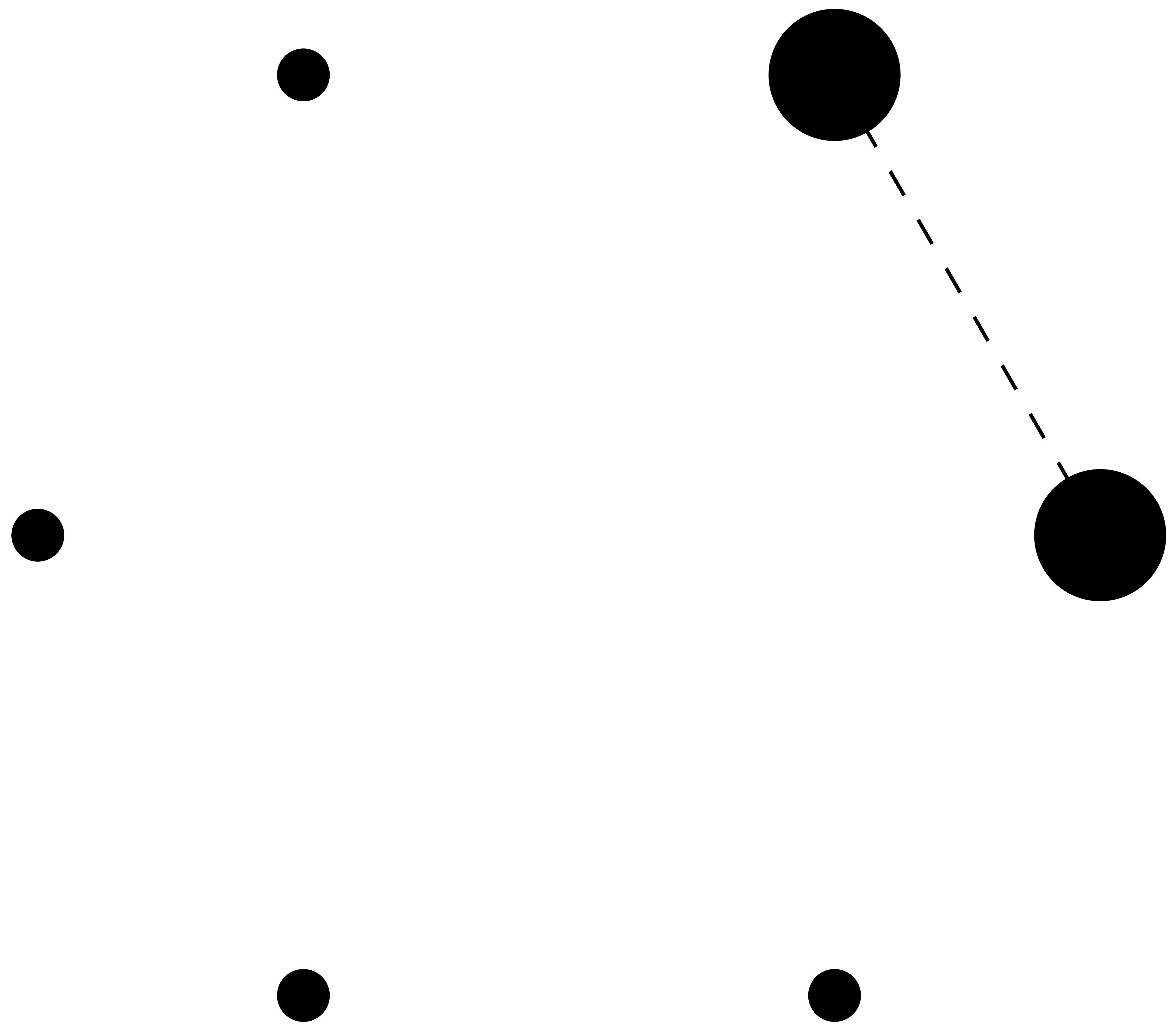


Topological Percolation

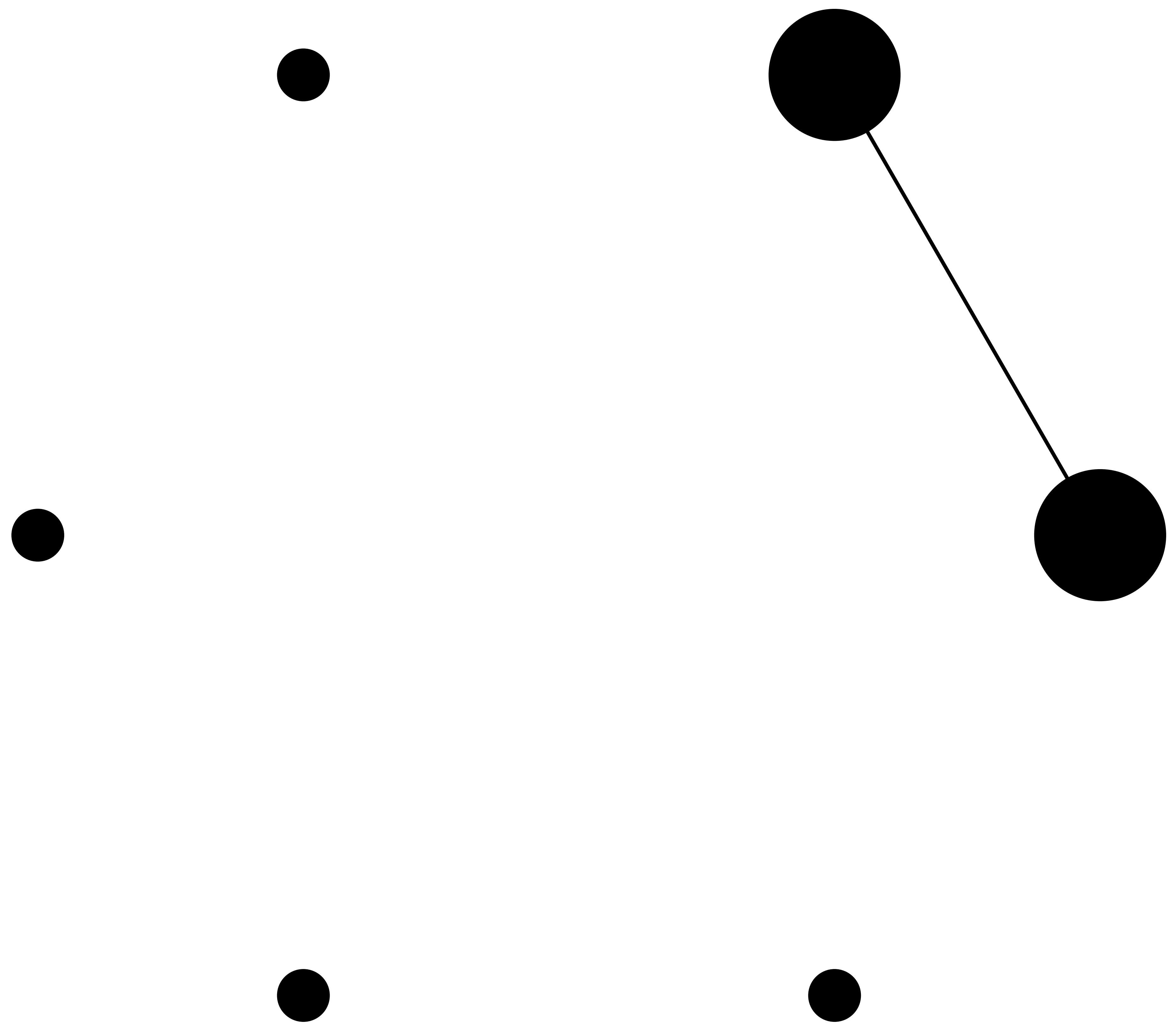
Erdos-Renyi graphs



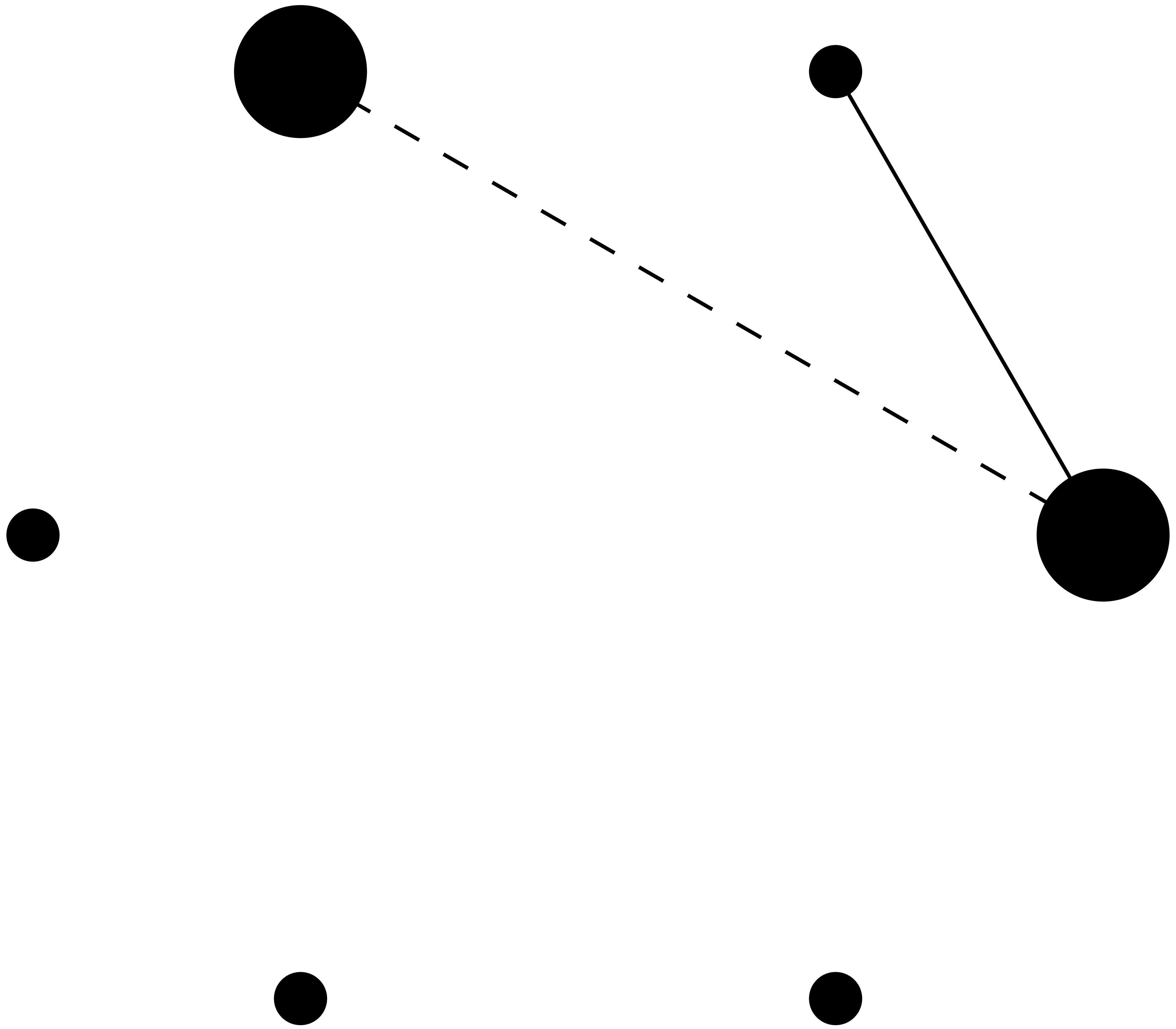
Erdos-Renyi graphs



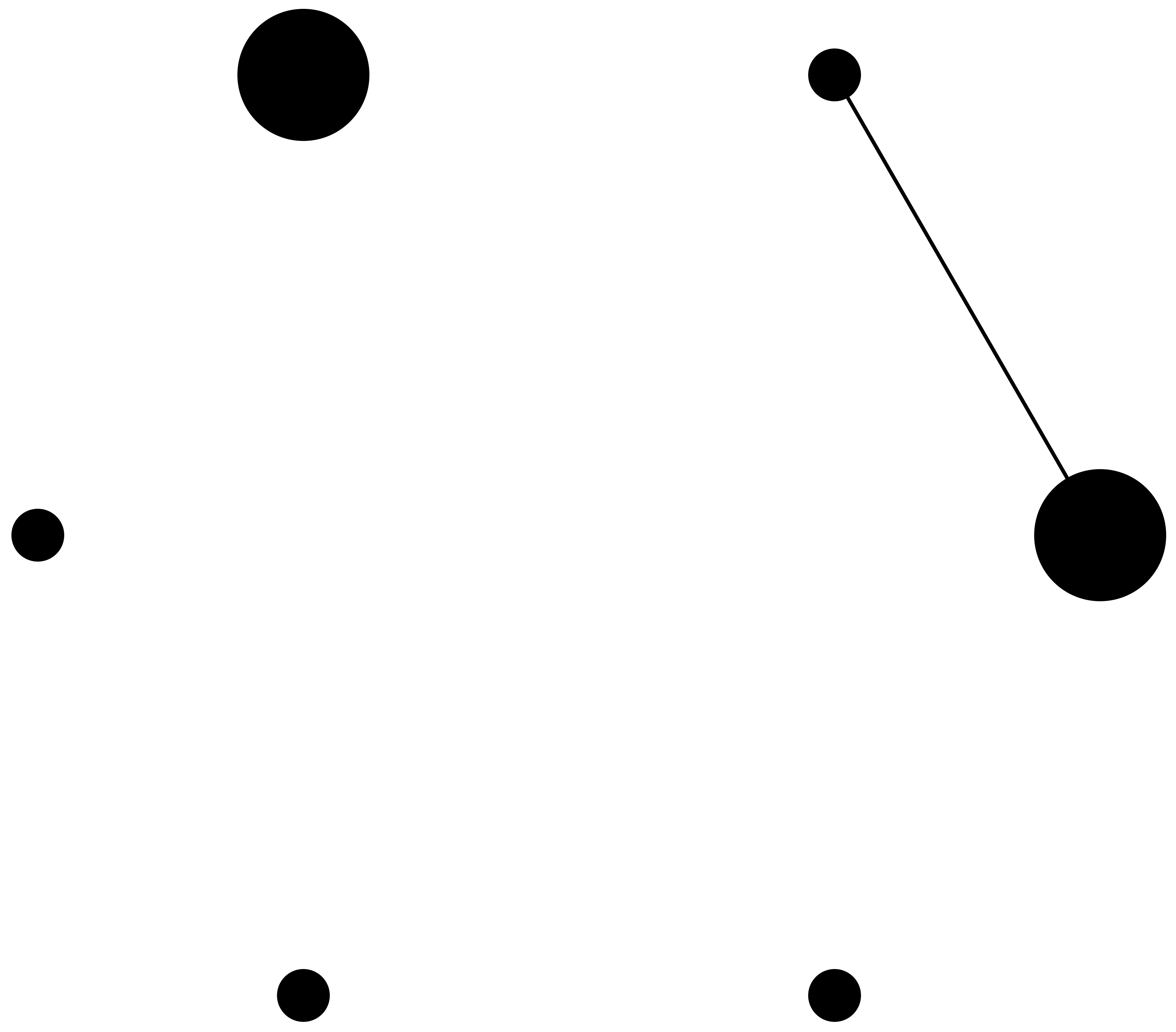
Erdos-Renyi graphs



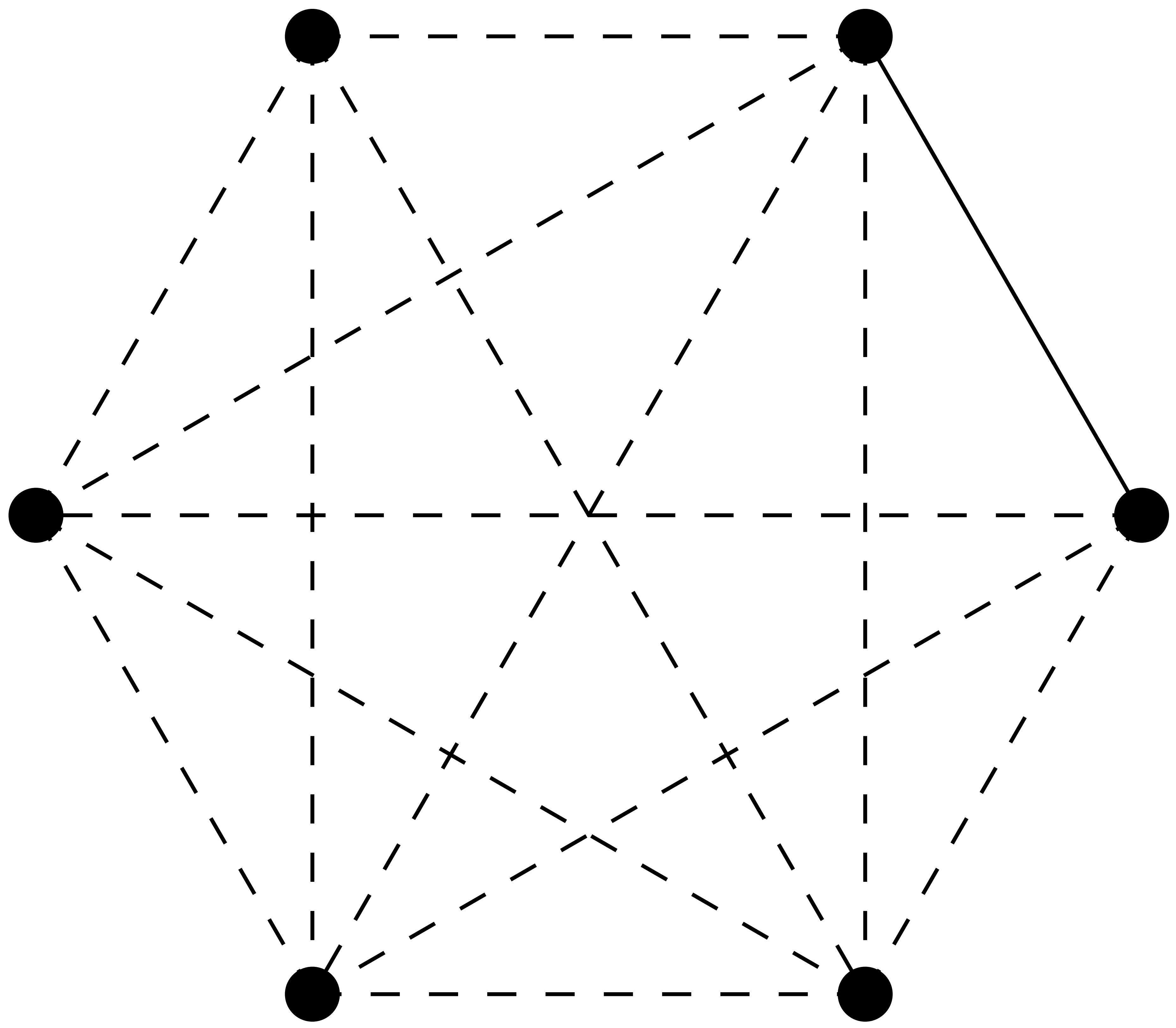
Erdos-Renyi graphs



Erdos-Renyi graphs



Erdos-Renyi graphs



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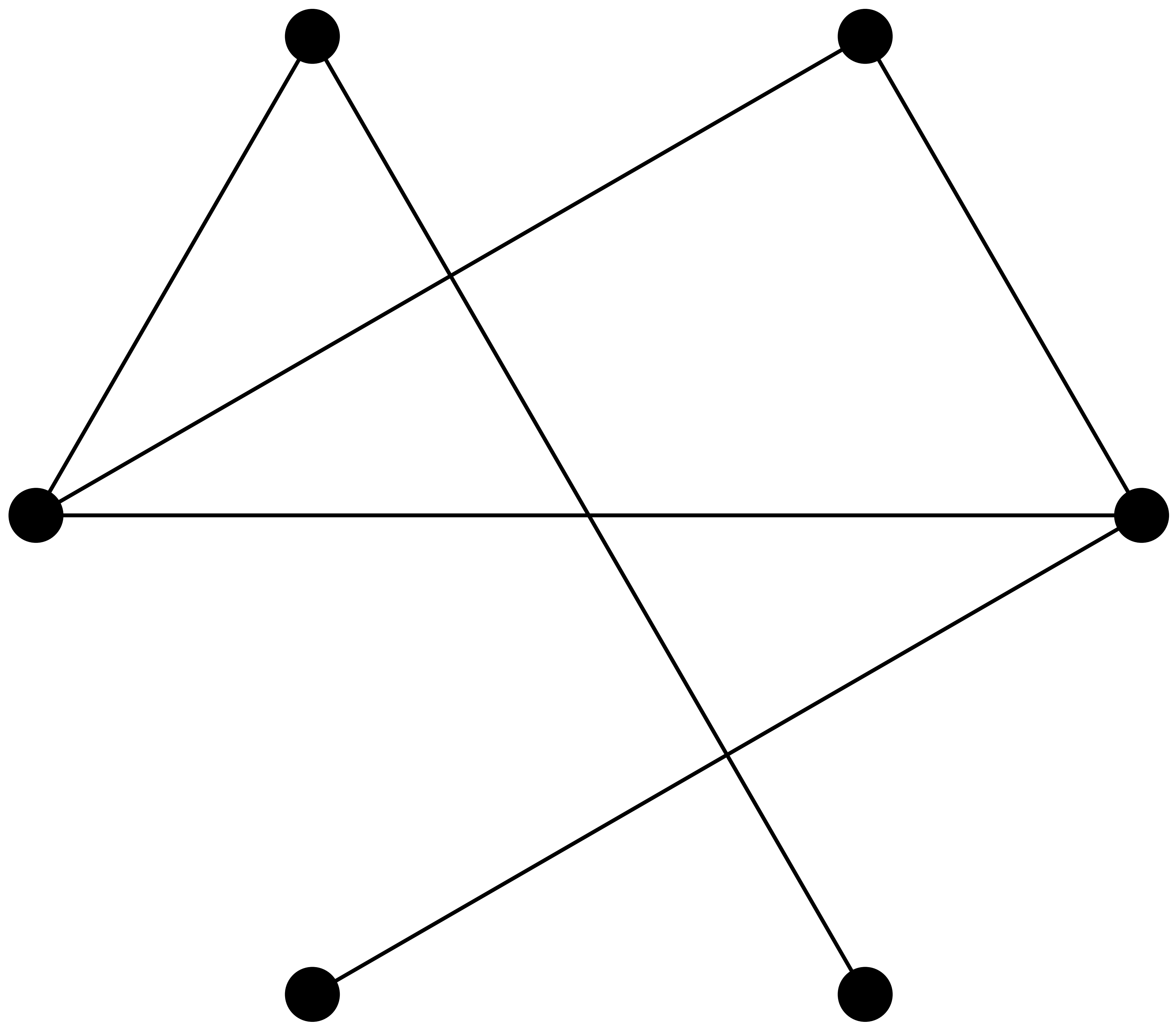
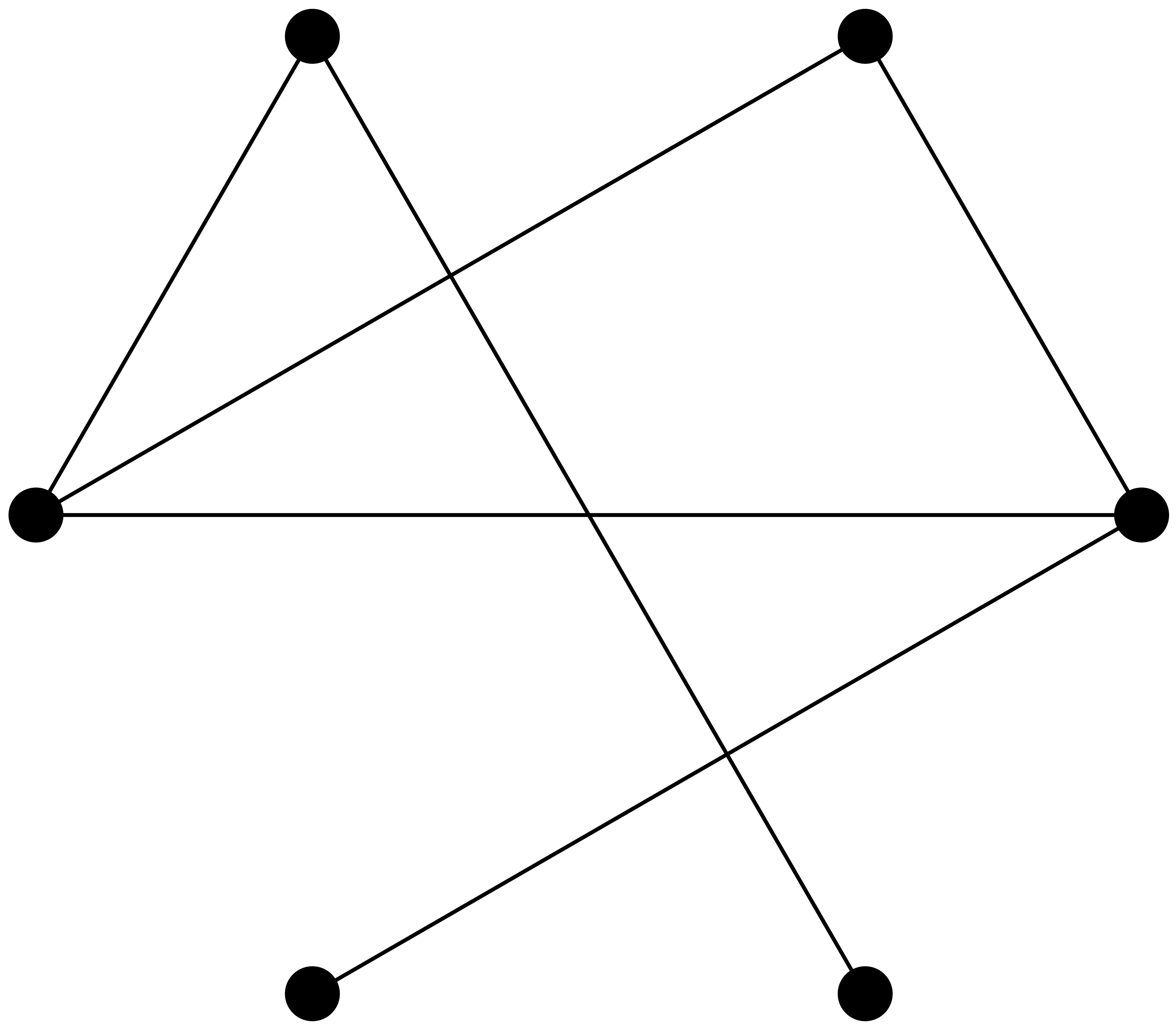
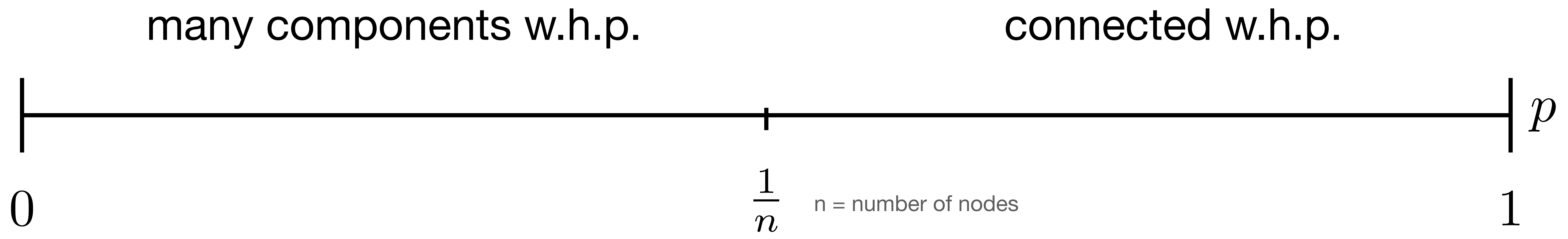


Erdos-Renyi graphs



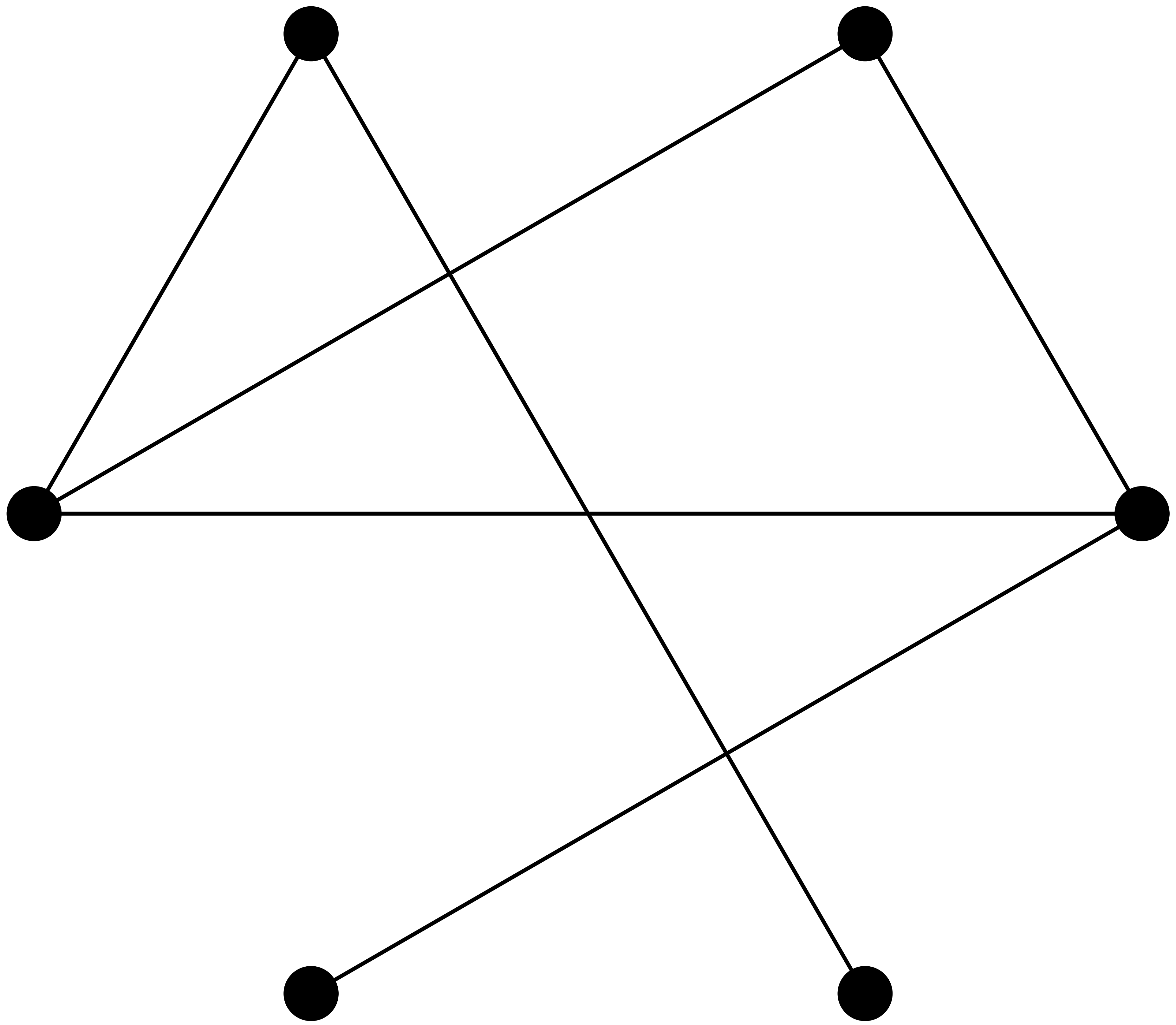
Phase Transition

[Erdos-Renyi 1960]

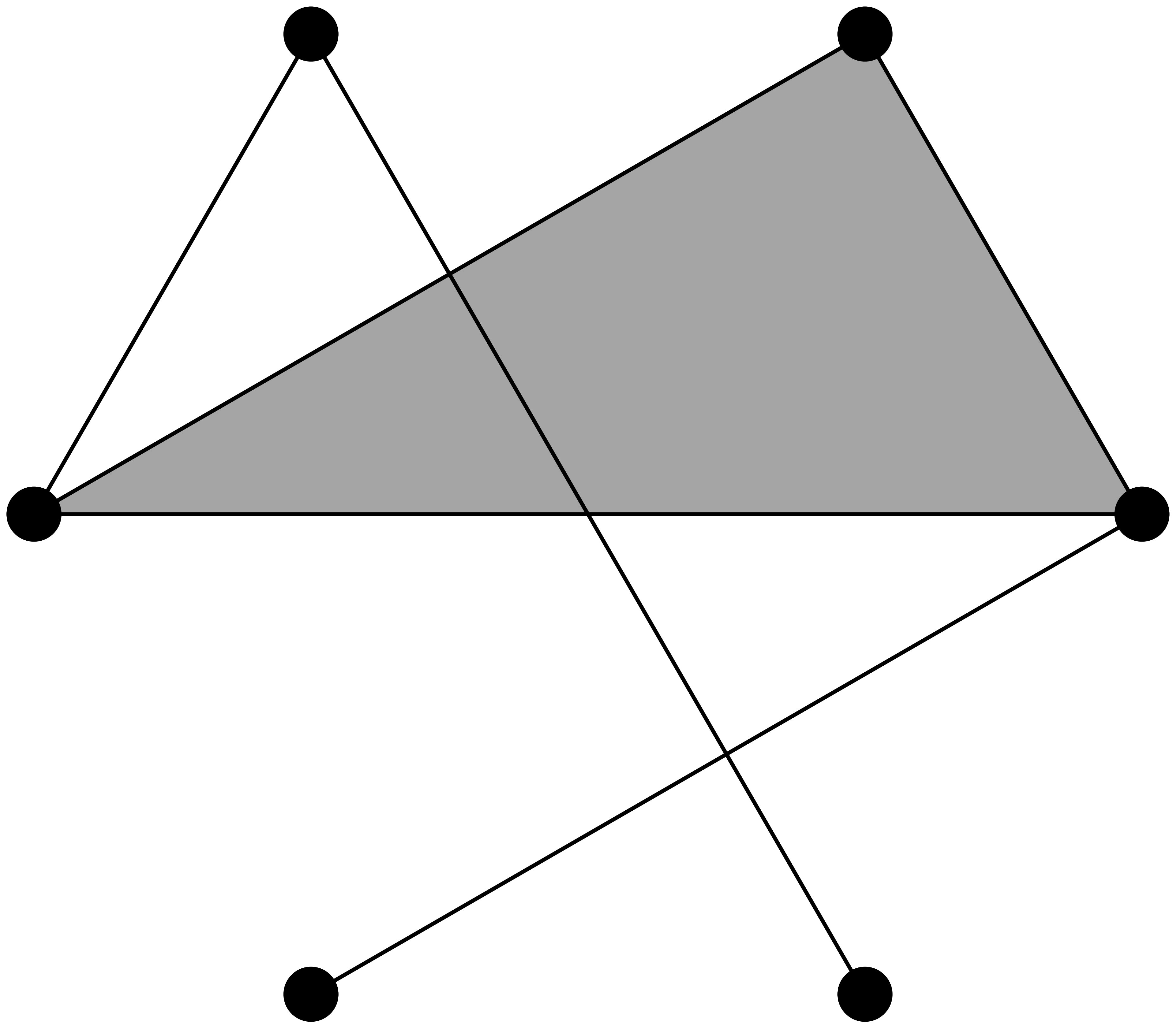


all log terms and constants forgone

Erdos-Renyi Clique Complex

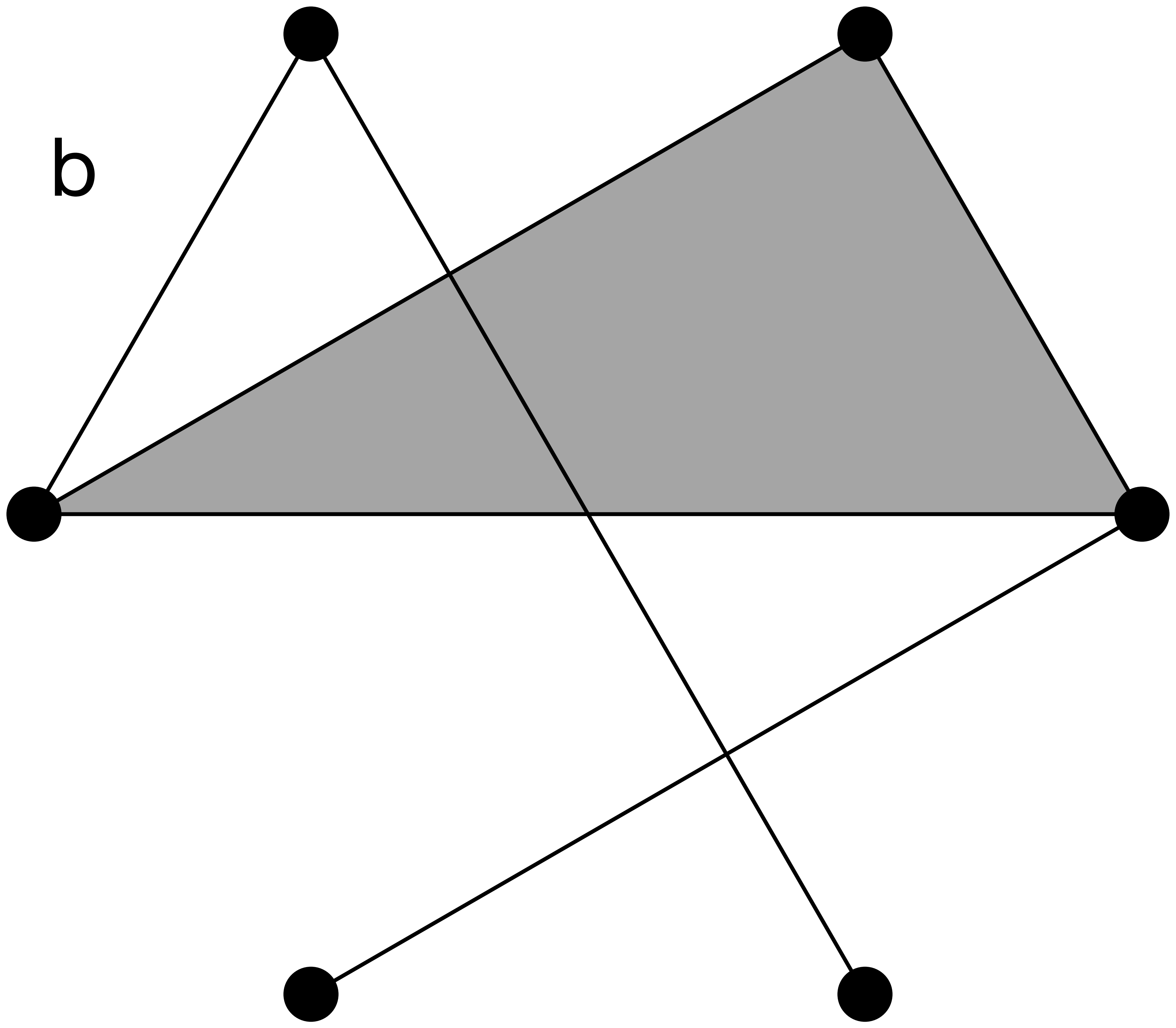


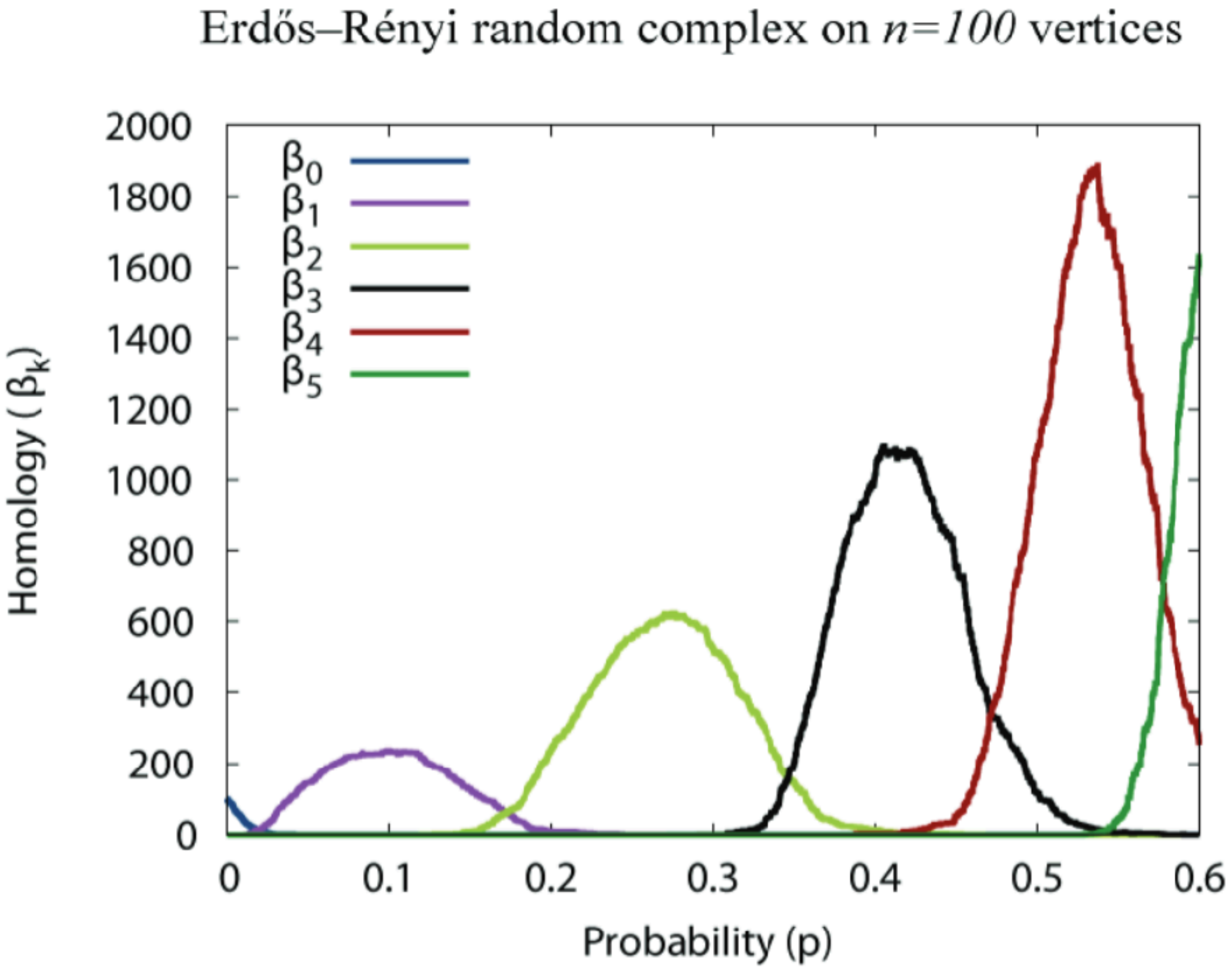
Erdos-Renyi Clique Complex



Betti Numbers

- b

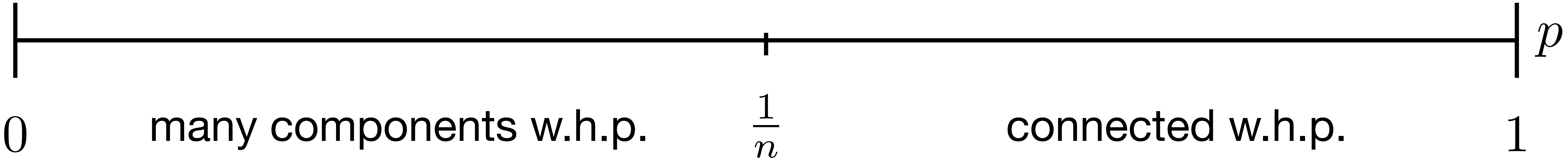




computation and plotting done by Zomorodian

Phase Transition

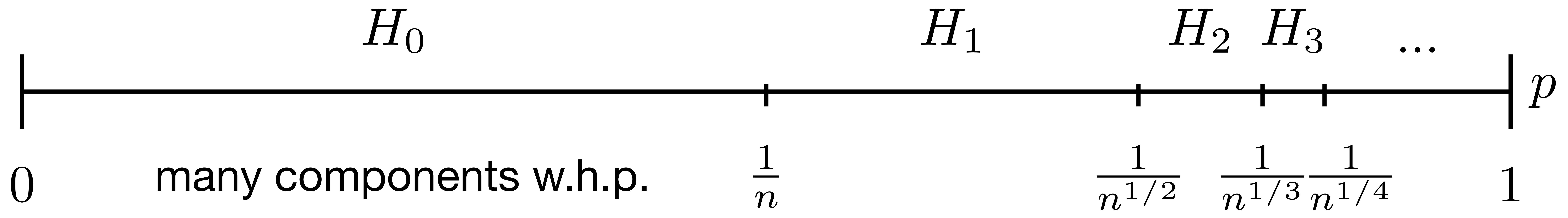
[Erdos-Renyi 1960]



n = number of nodes
all log terms and constants forgone

Phase Transition

[Kahle 2009, 2014]



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Phase Transition
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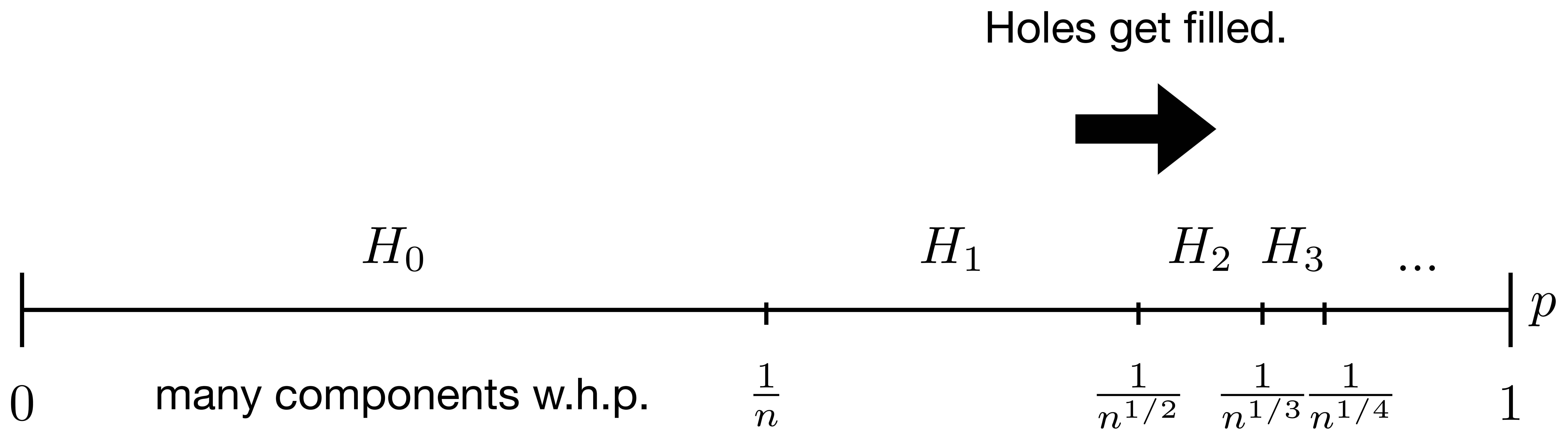

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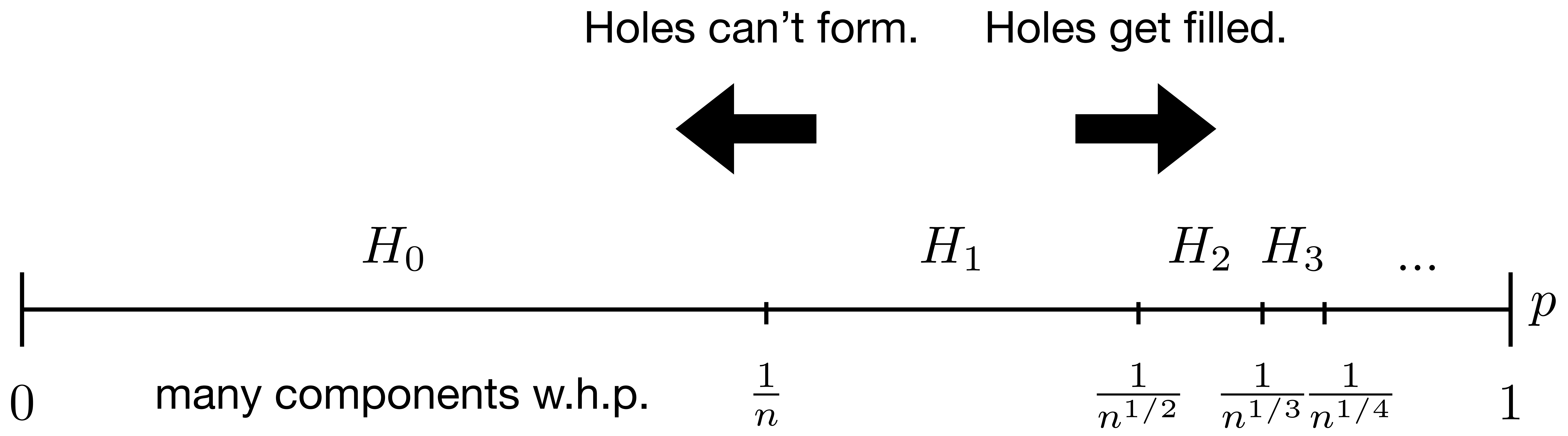
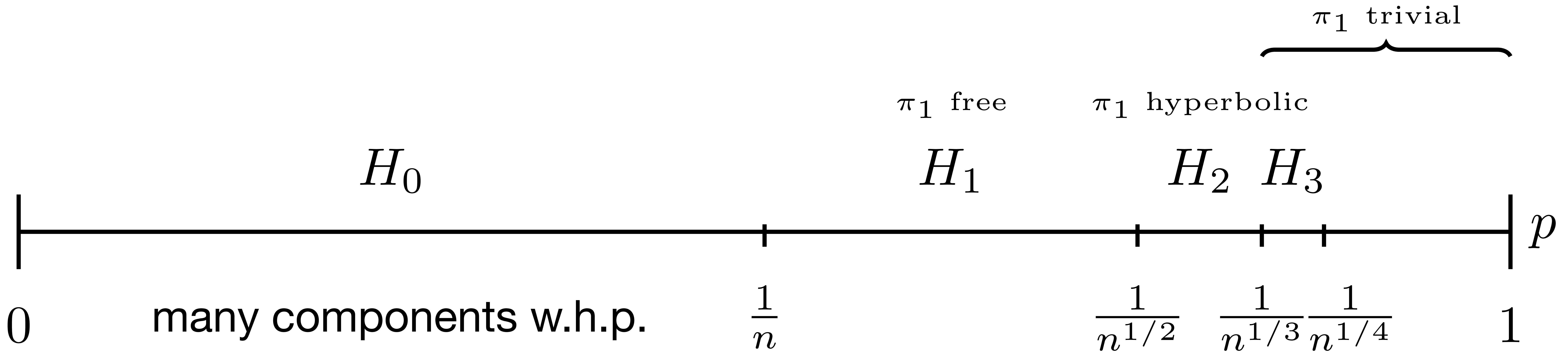


n = number of nodes
all log terms and constants forgone

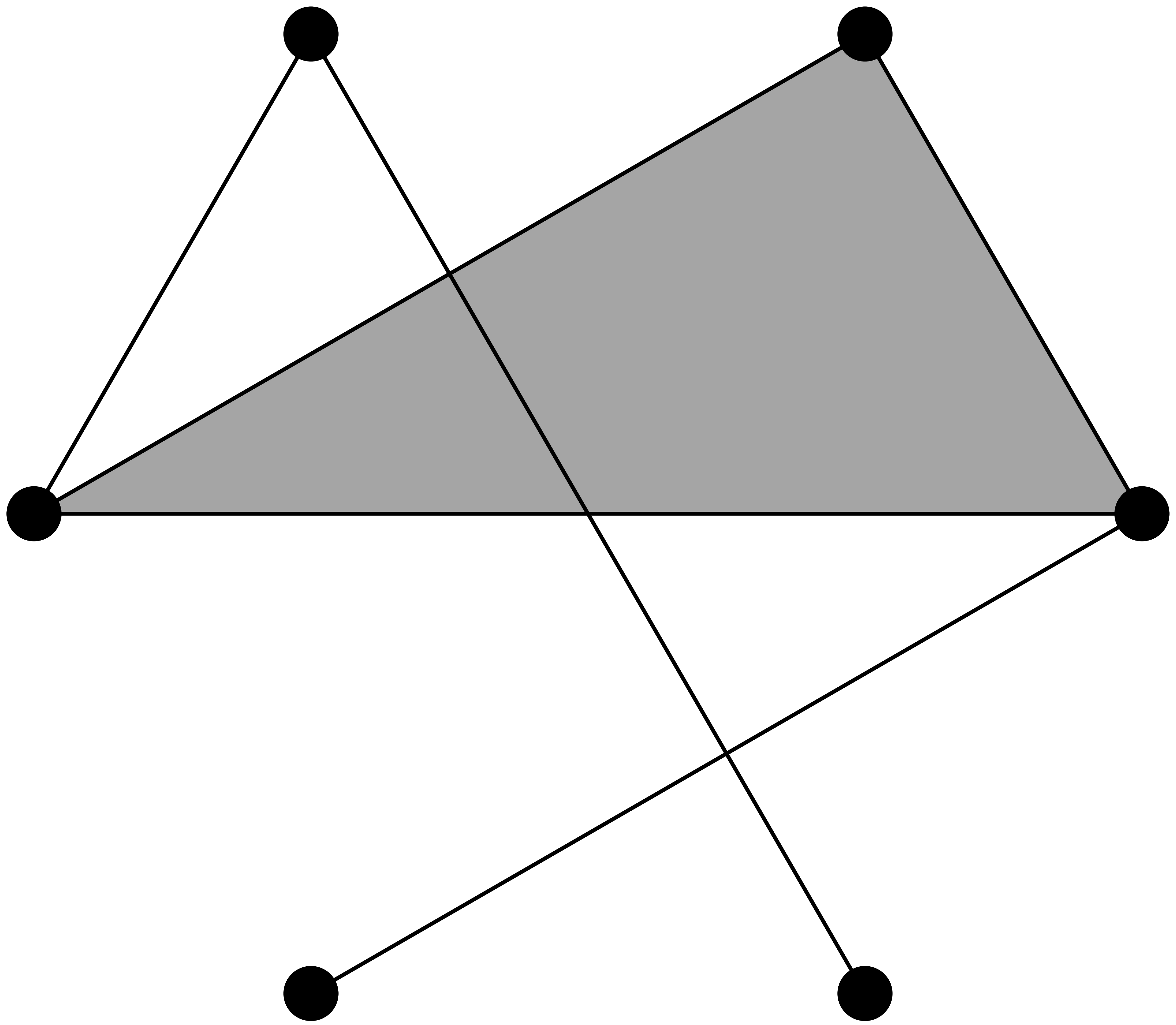
Fundamental Group

[Kahle 2009, Babson 2012, Costa-Farber-Horak 2015]



all log terms and constants forgone

Erdos-Renyi Clique Complex



Geometric Complexes

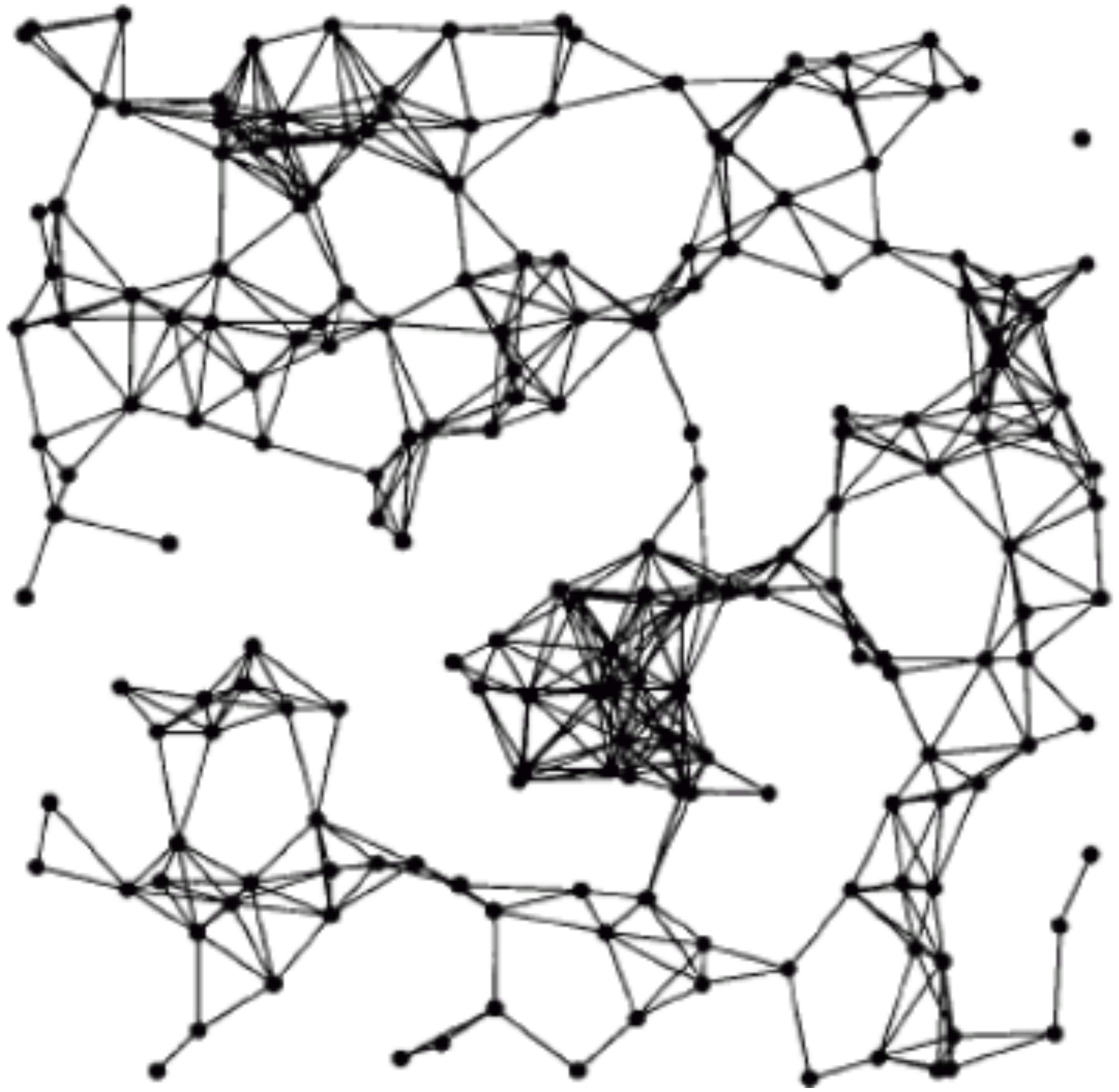


image credit: Penrose

Expected Betti numbers at dimension k
[Kahle 2011]

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- n, the number of points

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- $\omega = nr^D$, where D is the ambient dimension

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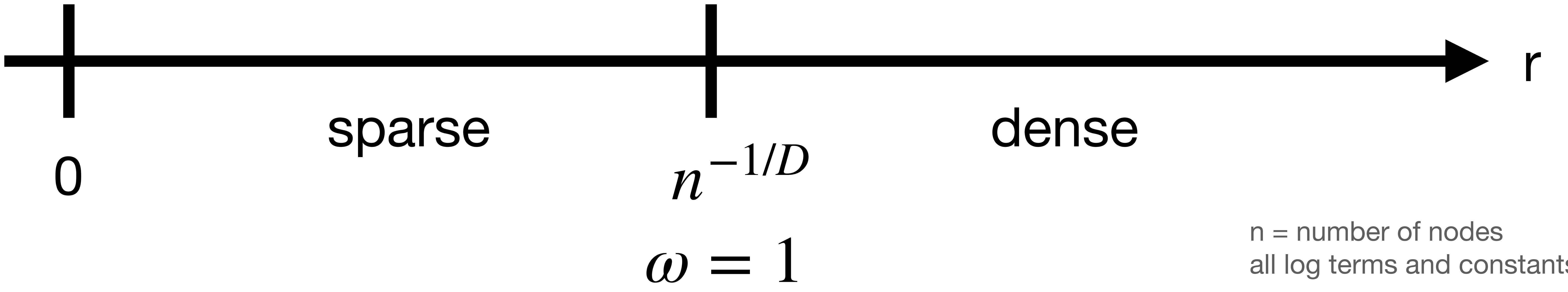

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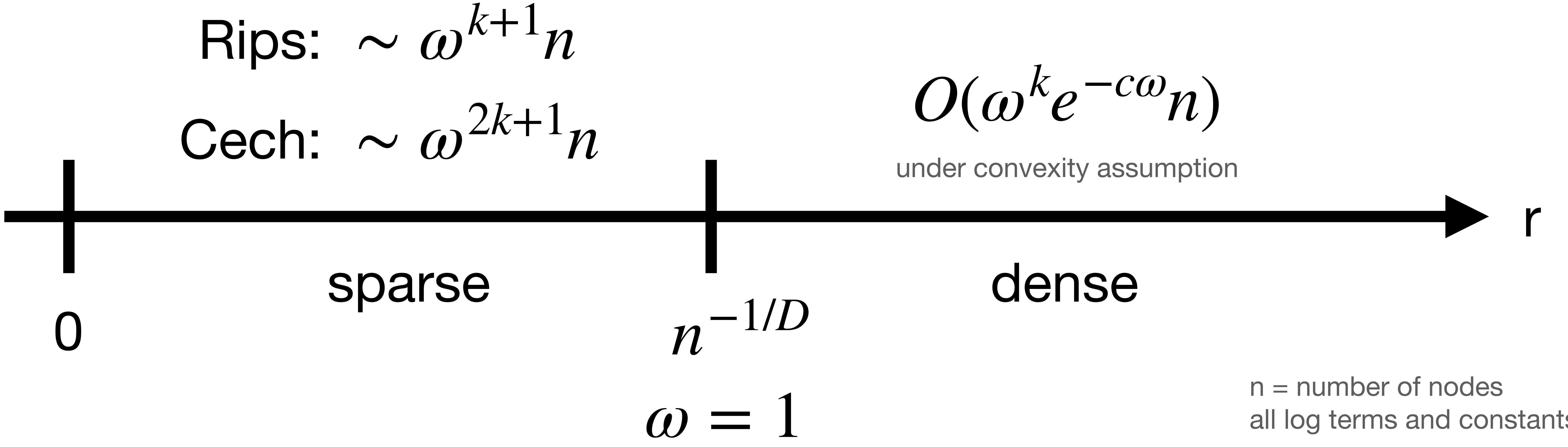

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Expected Betti numbers at dimension k
[Kahle 2011]

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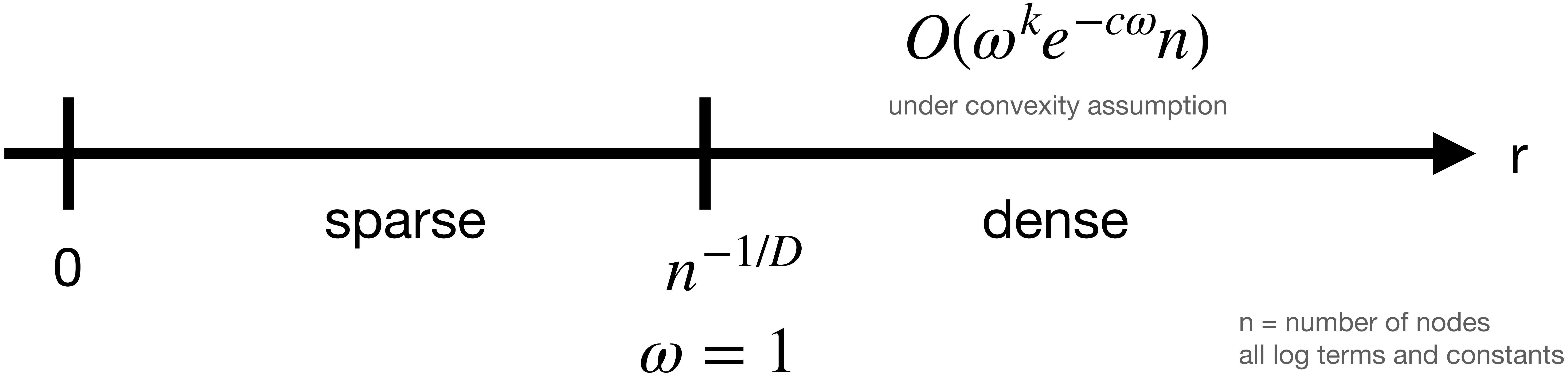
- $E\beta_k(\text{Cech}) \sim \omega^{2k+1}n$



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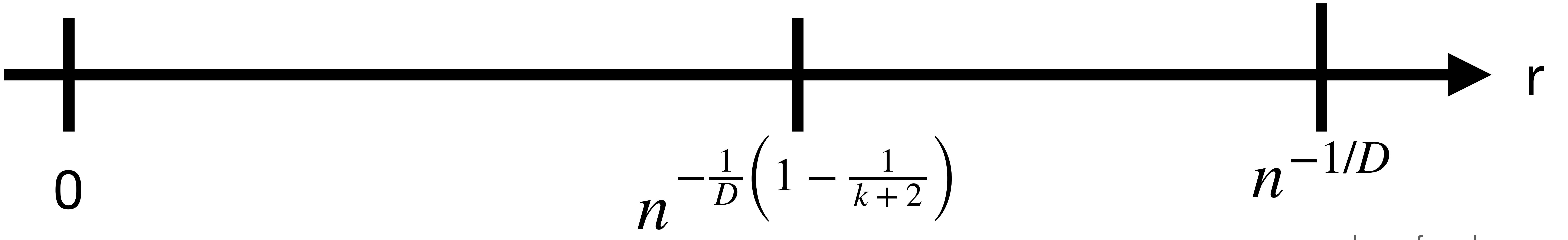

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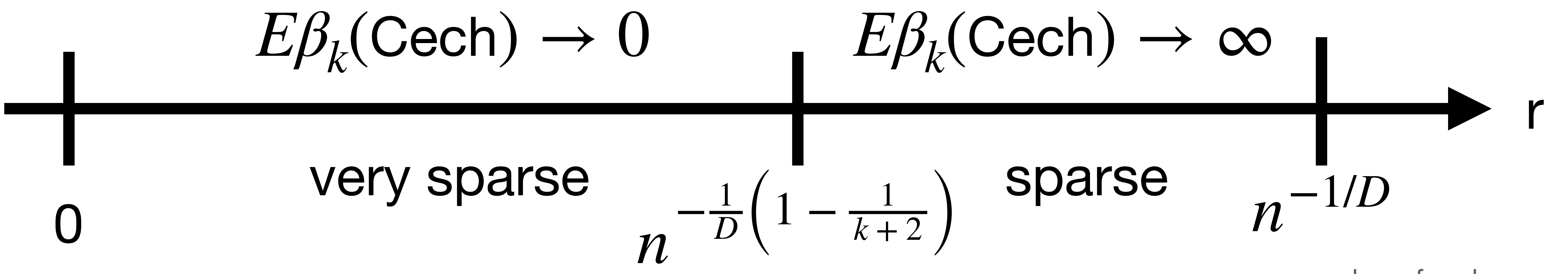

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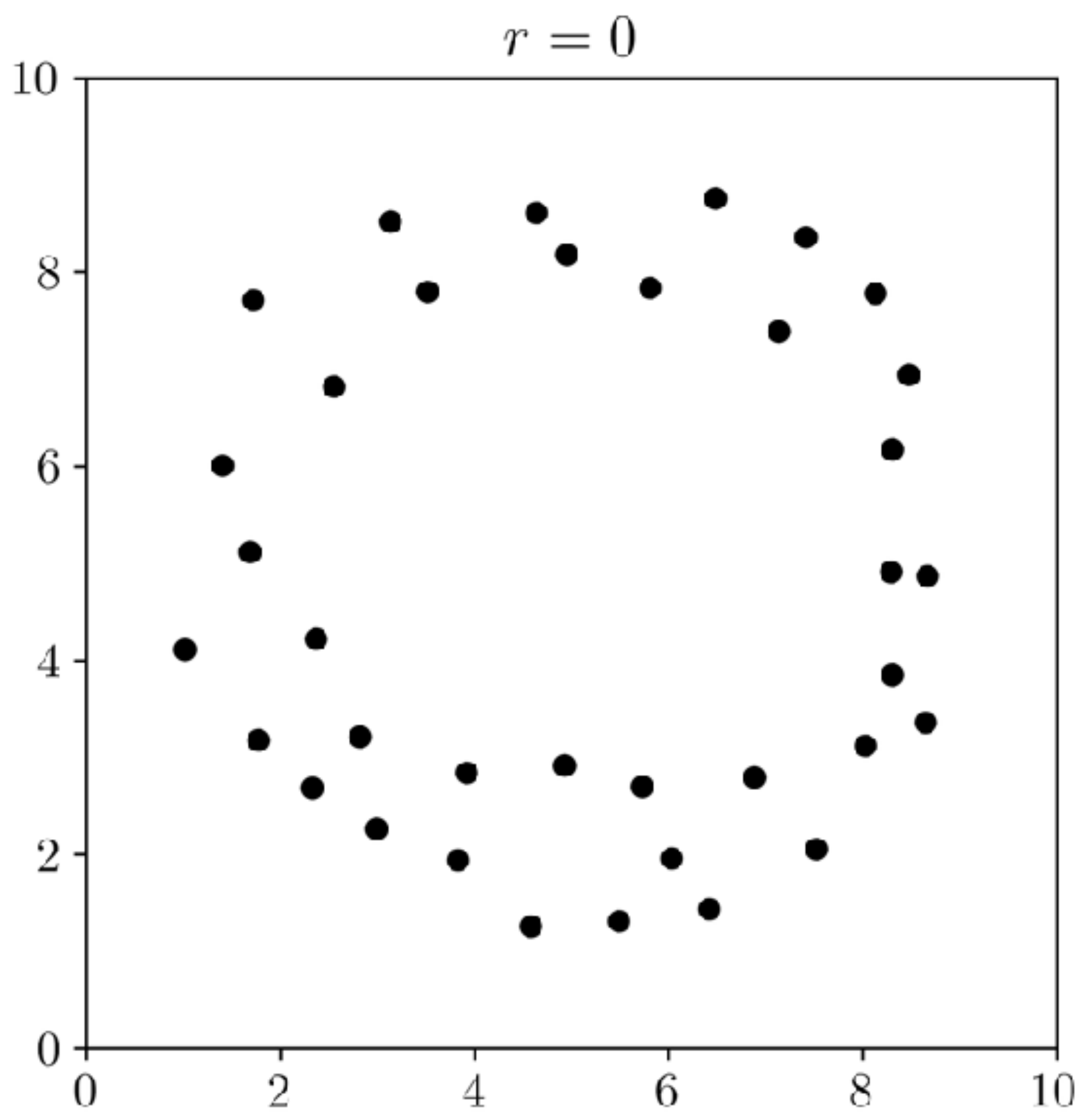
Maximally Persistent Cycles



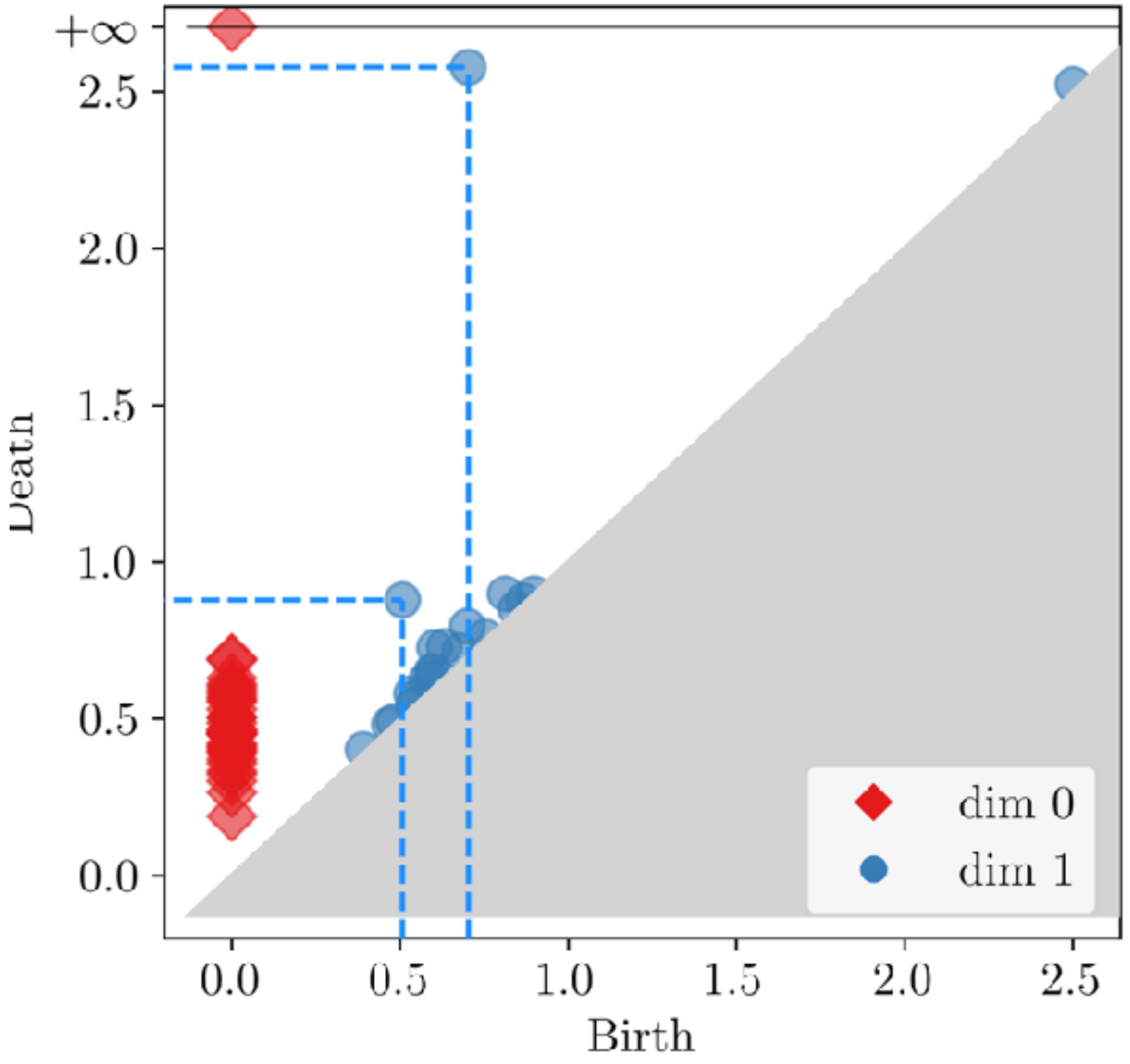


image credit: Andrey Yao

Maximally Persistent Cycles

n points in expectation

k-cycle

Maximally Persistent Cycles

[Bobrowski-Kahle-Skraba 2017]

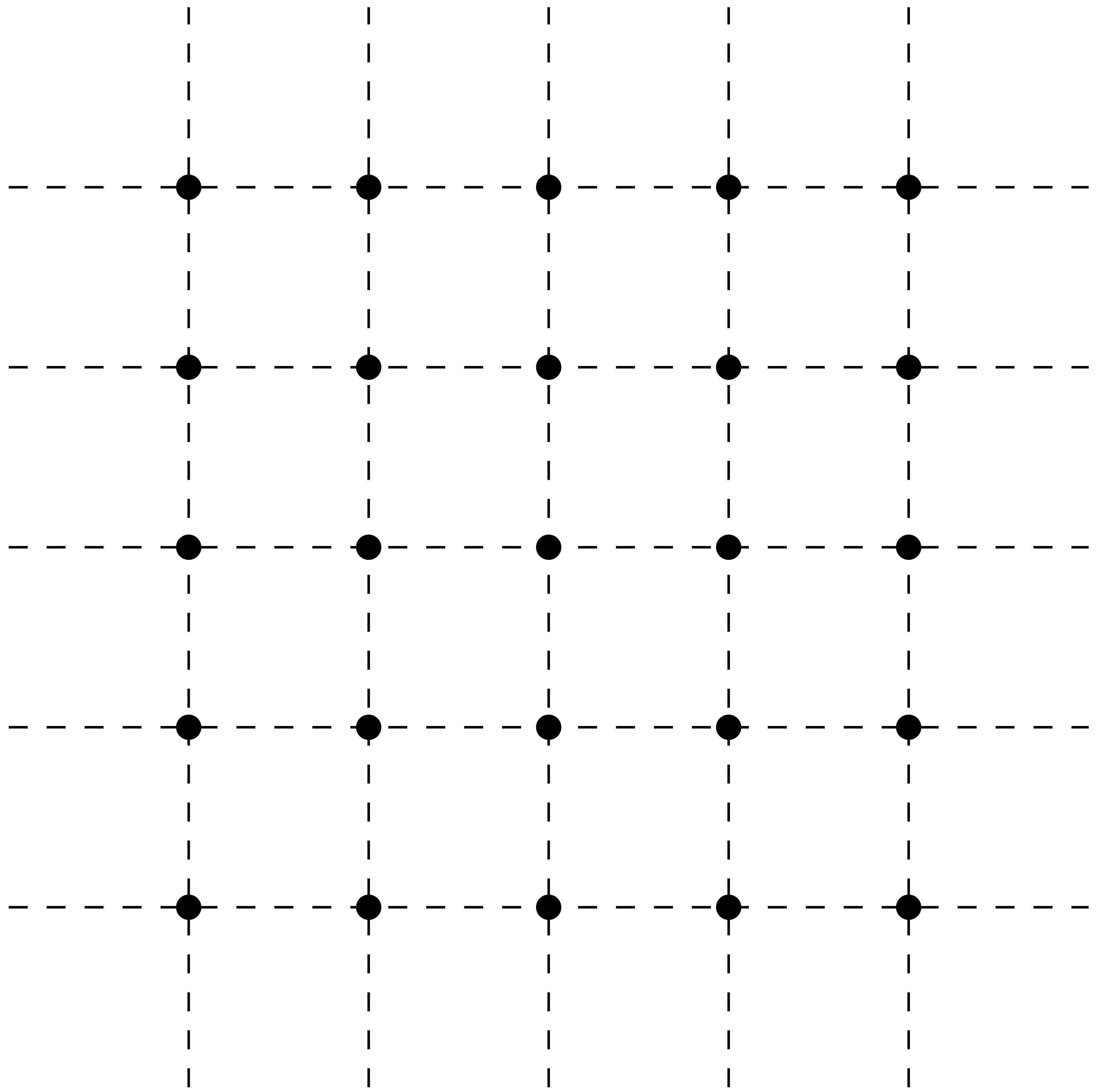
n points in expectation

k-cycle

$$c\left(\frac{\log n}{\log\log n}\right)^{1/k} \leq \text{max persistence} \leq C\left(\frac{\log n}{\log\log n}\right)^{1/k}$$

a.a.s.

Bernoulli Bond Percolation



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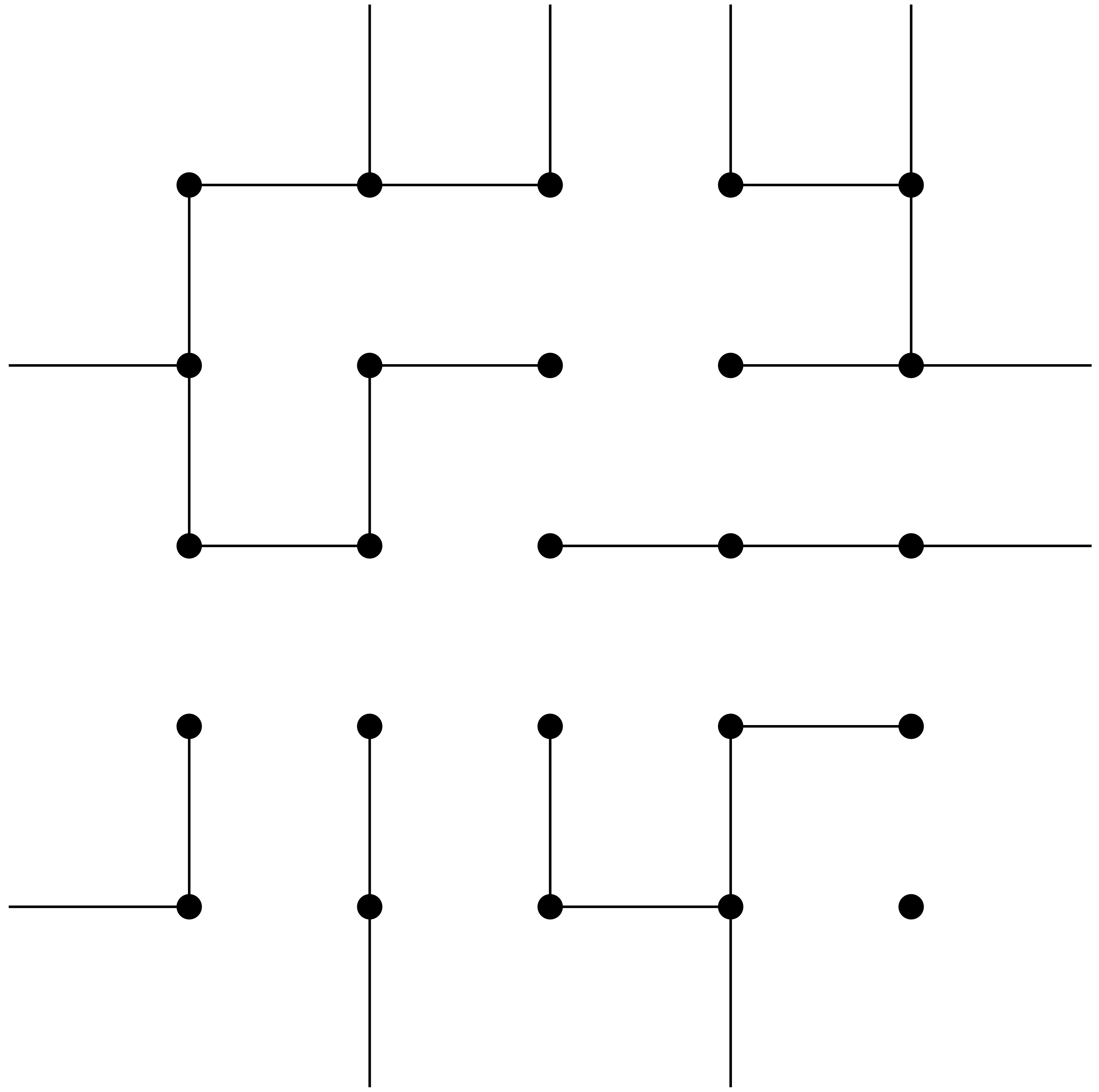
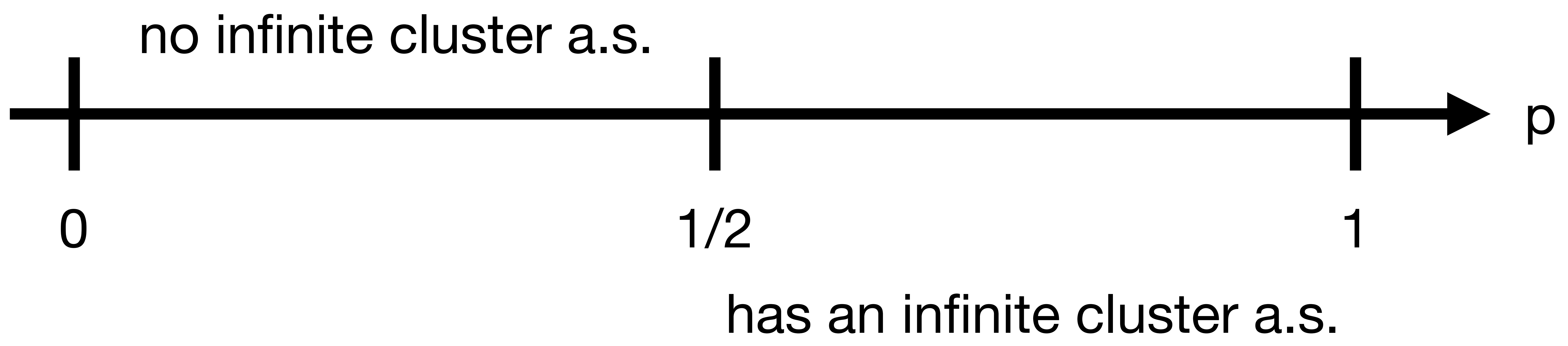


Phase Transition
[Harris 1960, Kesten 1980]



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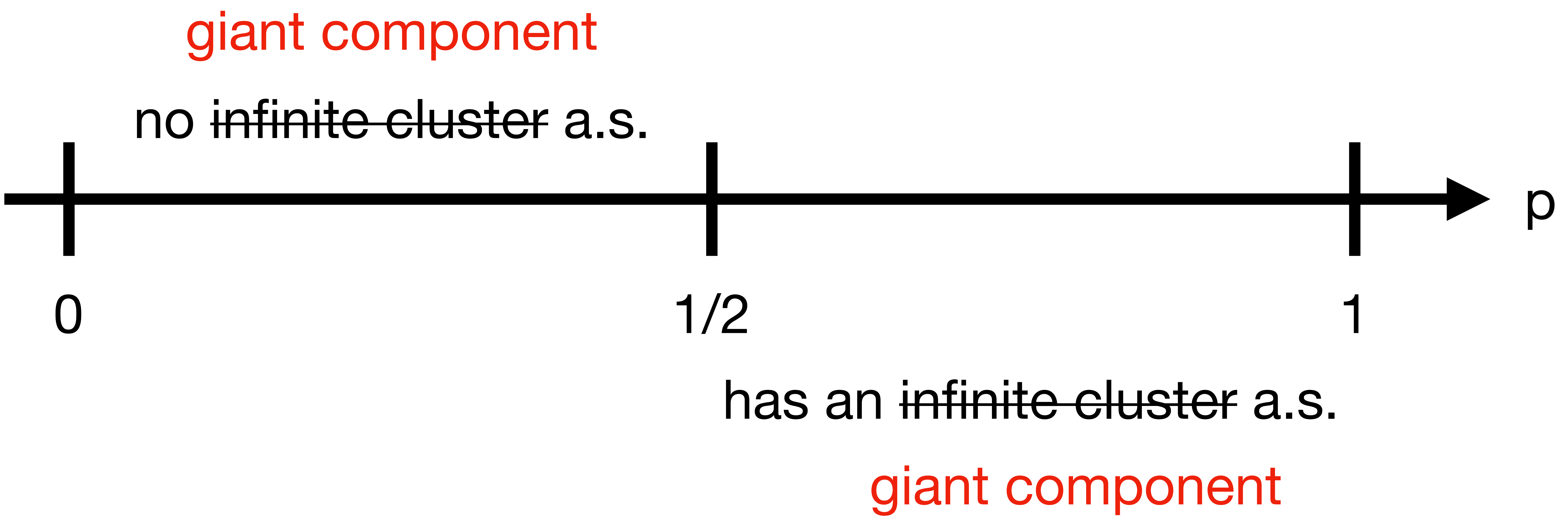
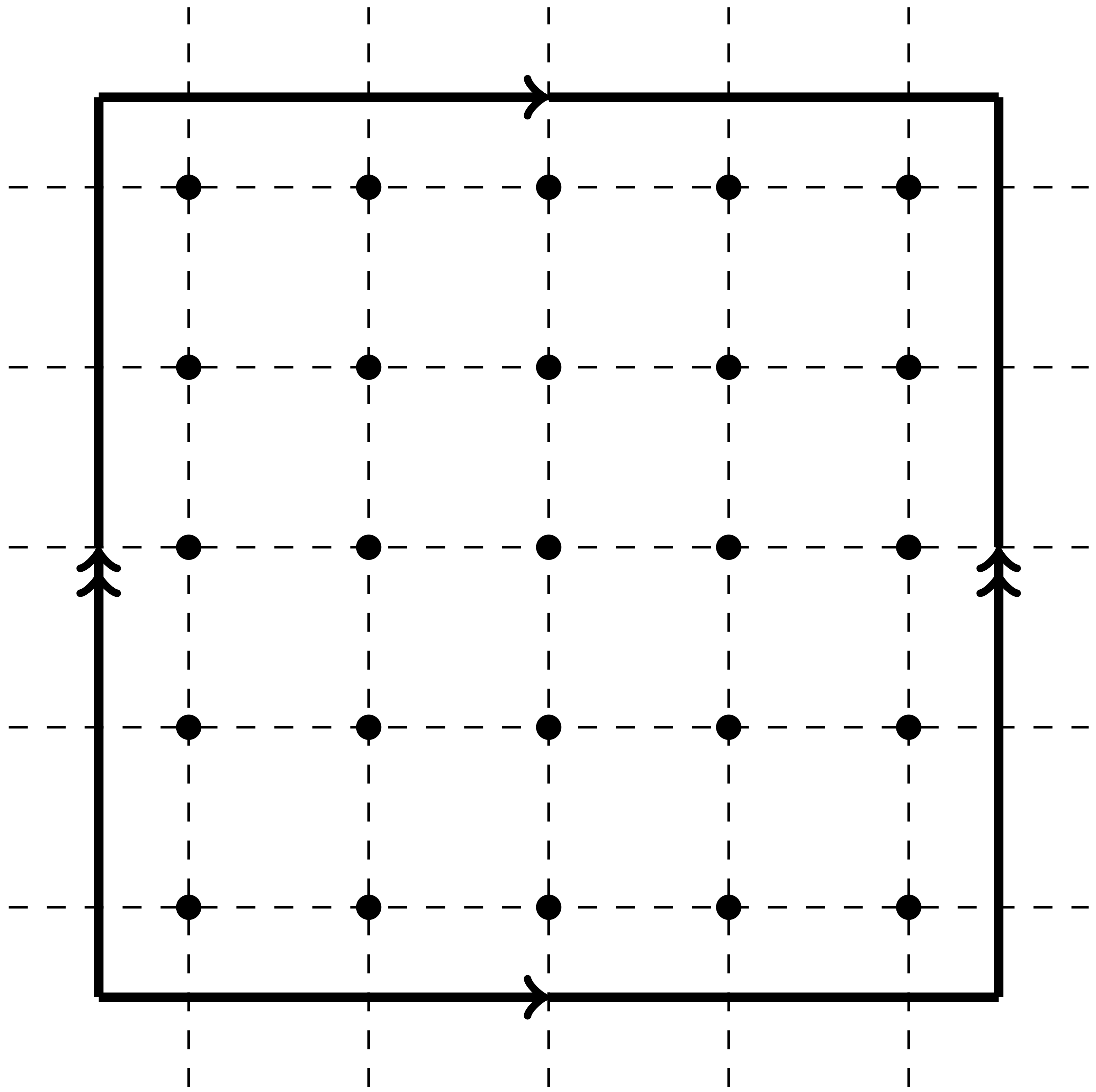
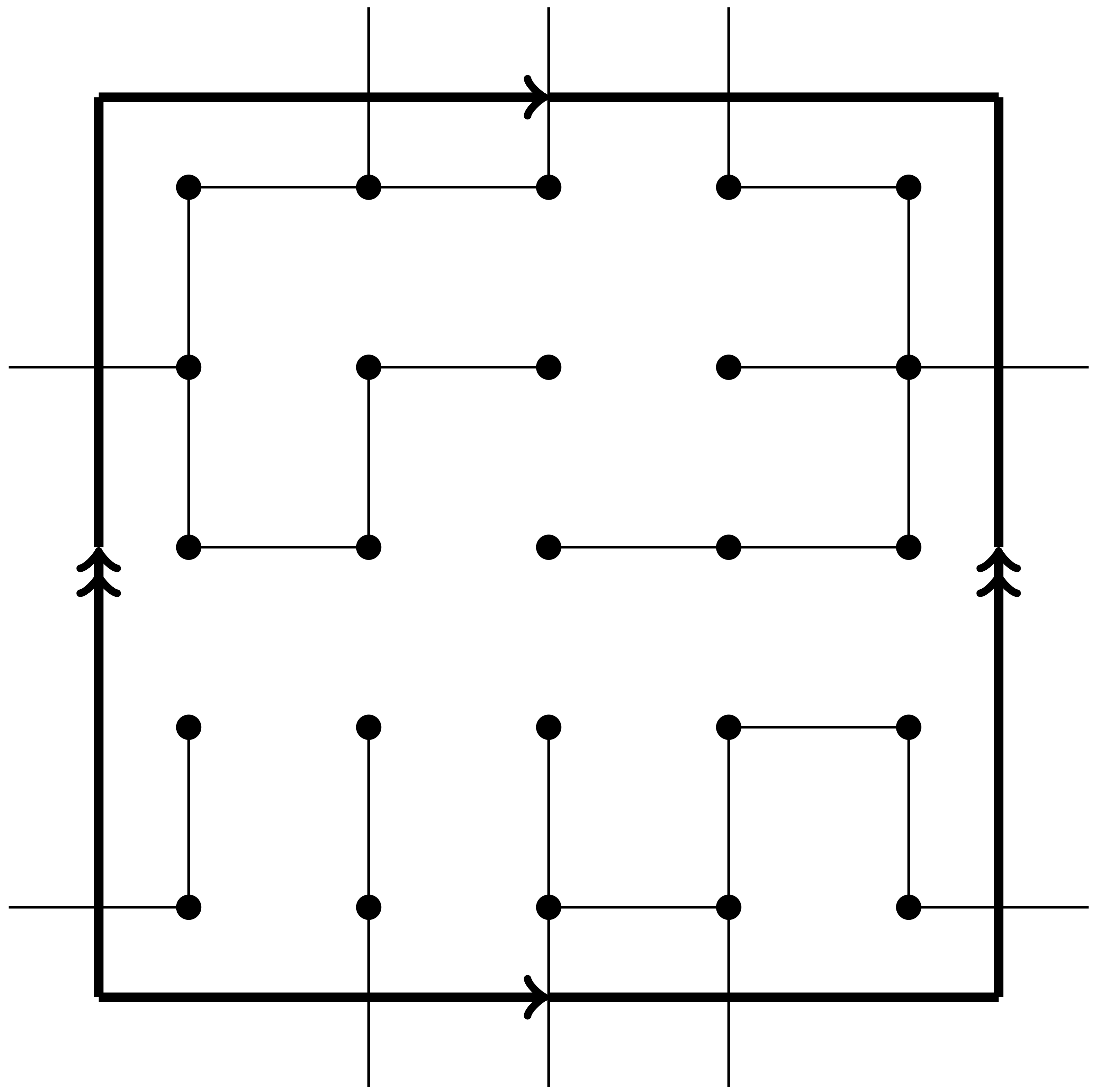


Giant Cycles?

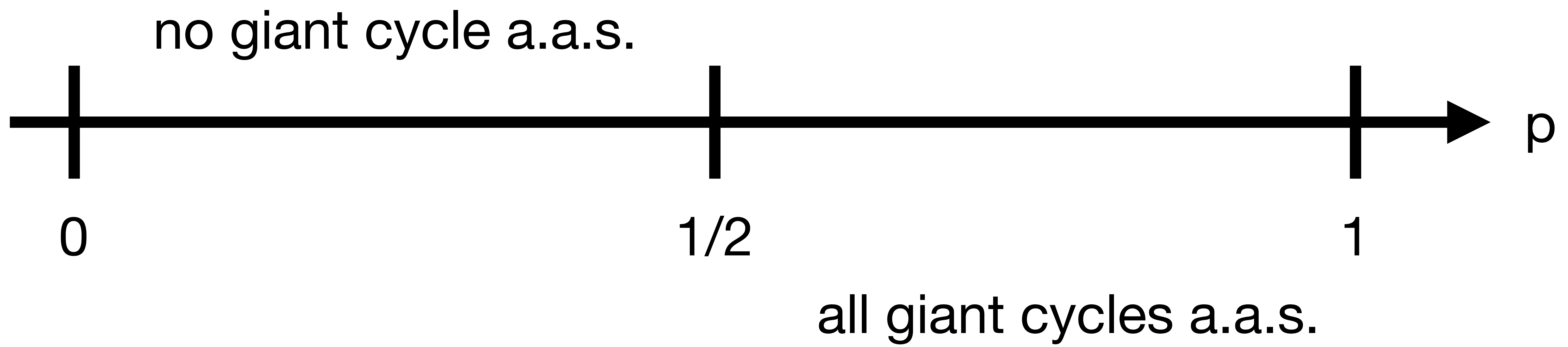
Bernoulli Bond Percolation



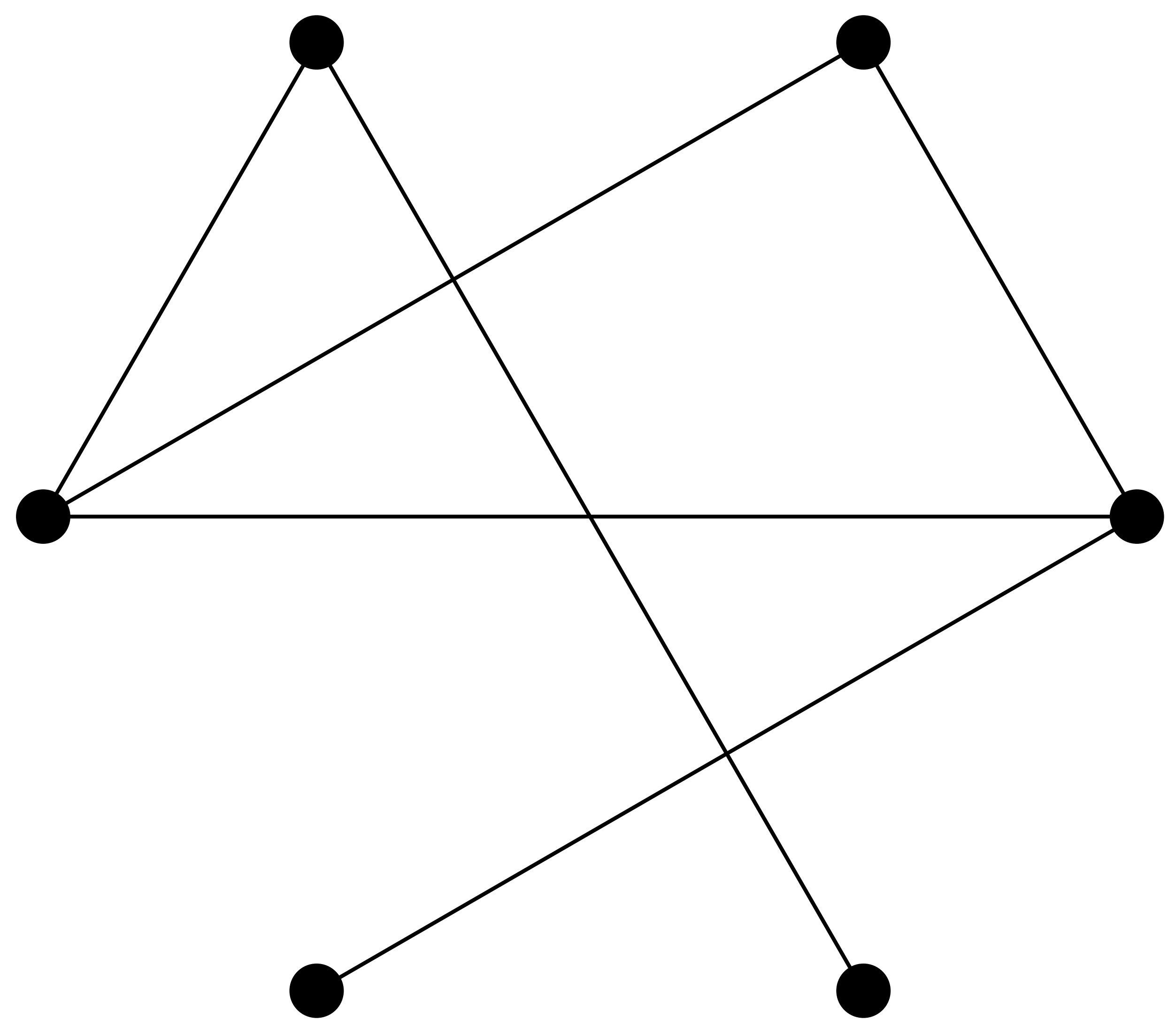


Phase Transition

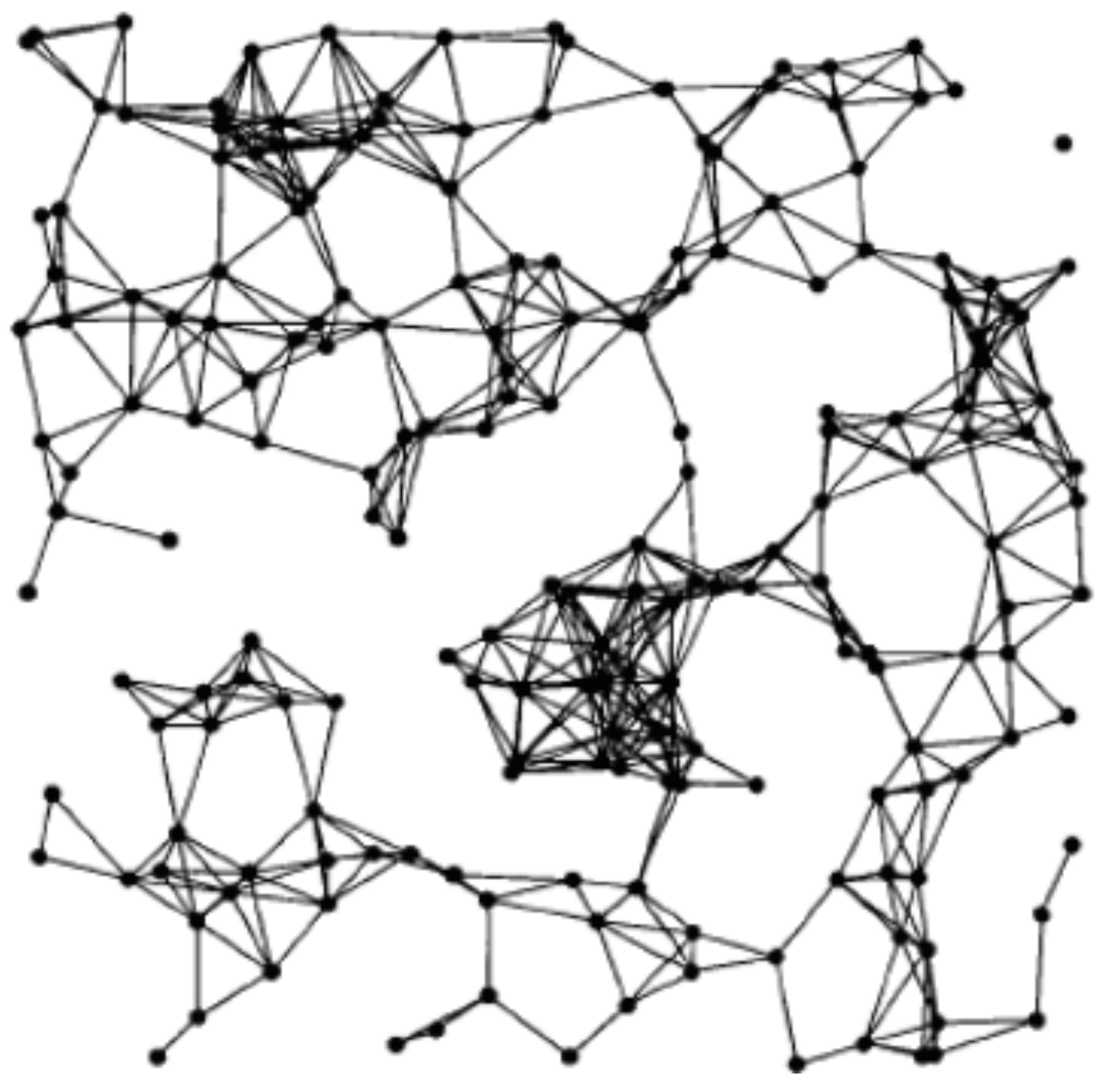
[Duncan-Kahle-Schweinhart, 2021]



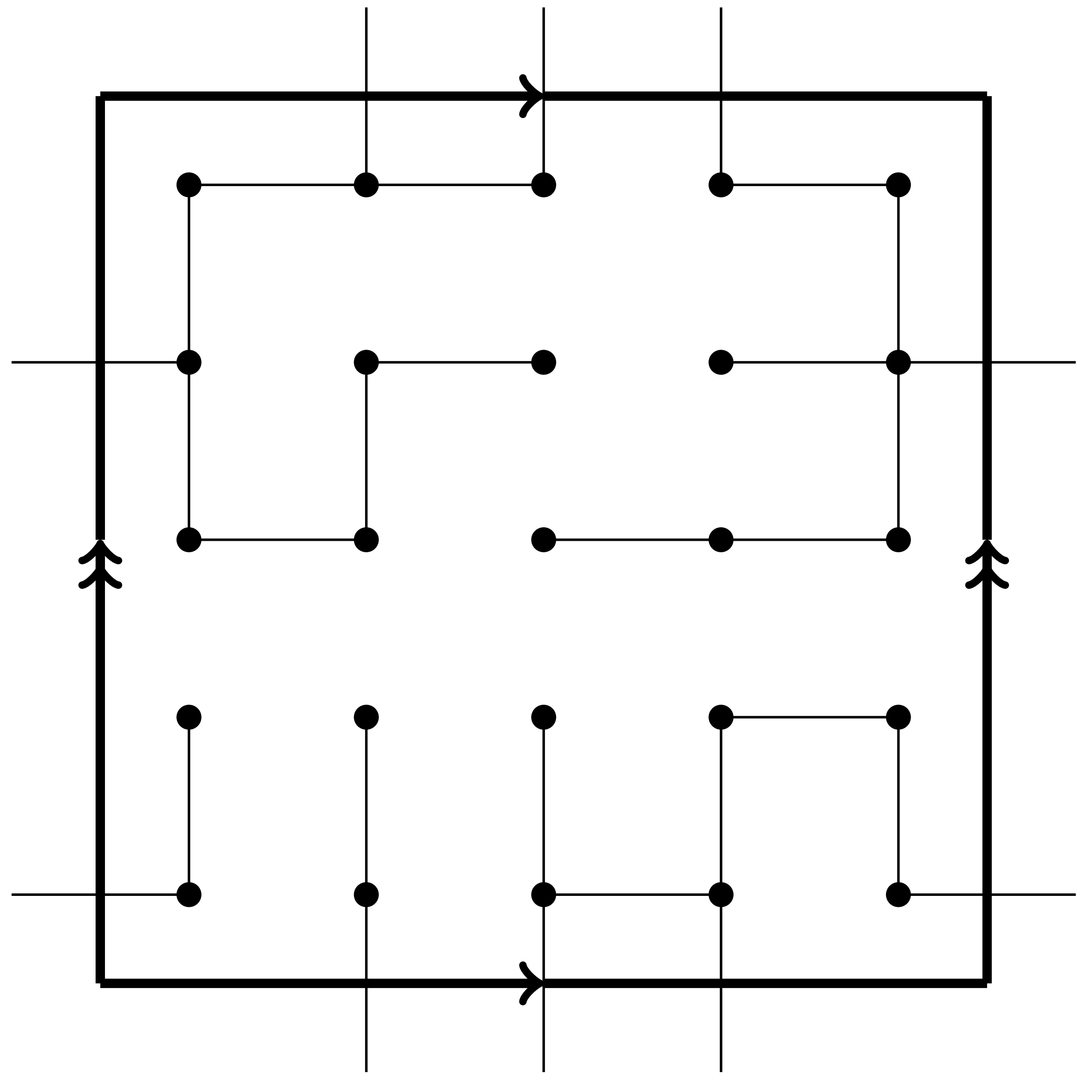
Tea with Random Topology



Erdo-Renyi Complexes



Geometric Complexes



Topological Percolation

II. Preferential Attachment

Beyond independence and homogeneity

Independent and identically distributed?

Independent and identically distributed?

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Preferential Attachment

[Albert and Barabasi 1999]

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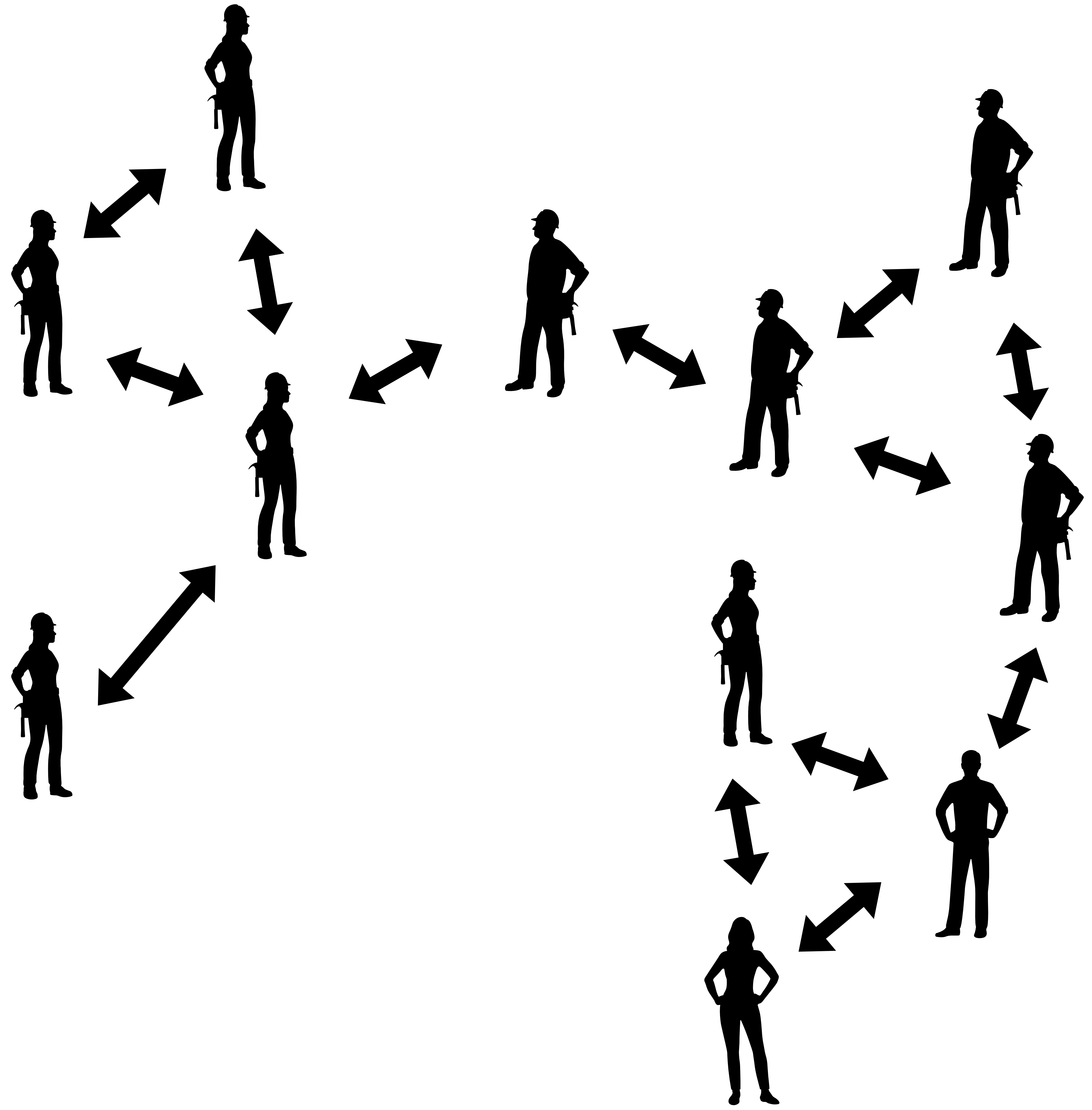
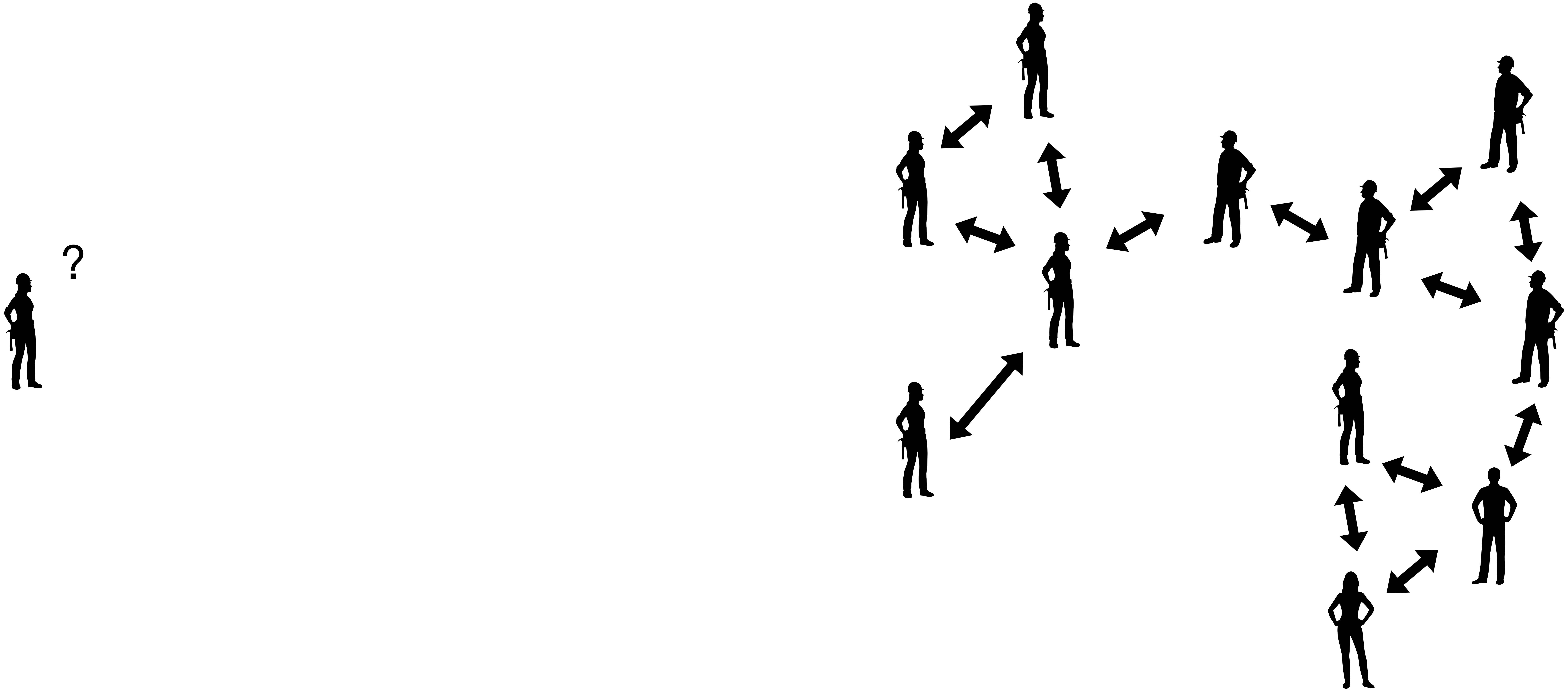


Preferential Attachment
[Albert and Barabasi 1999]



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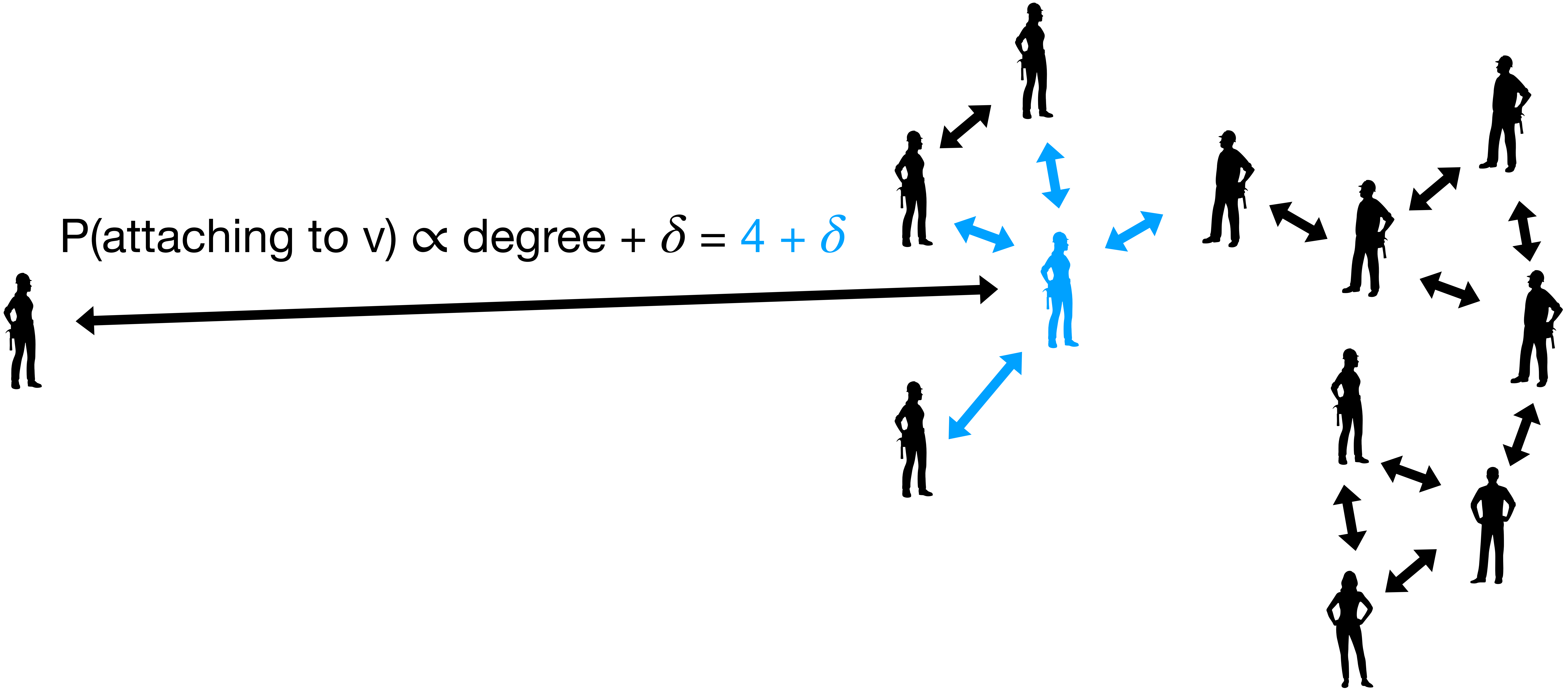
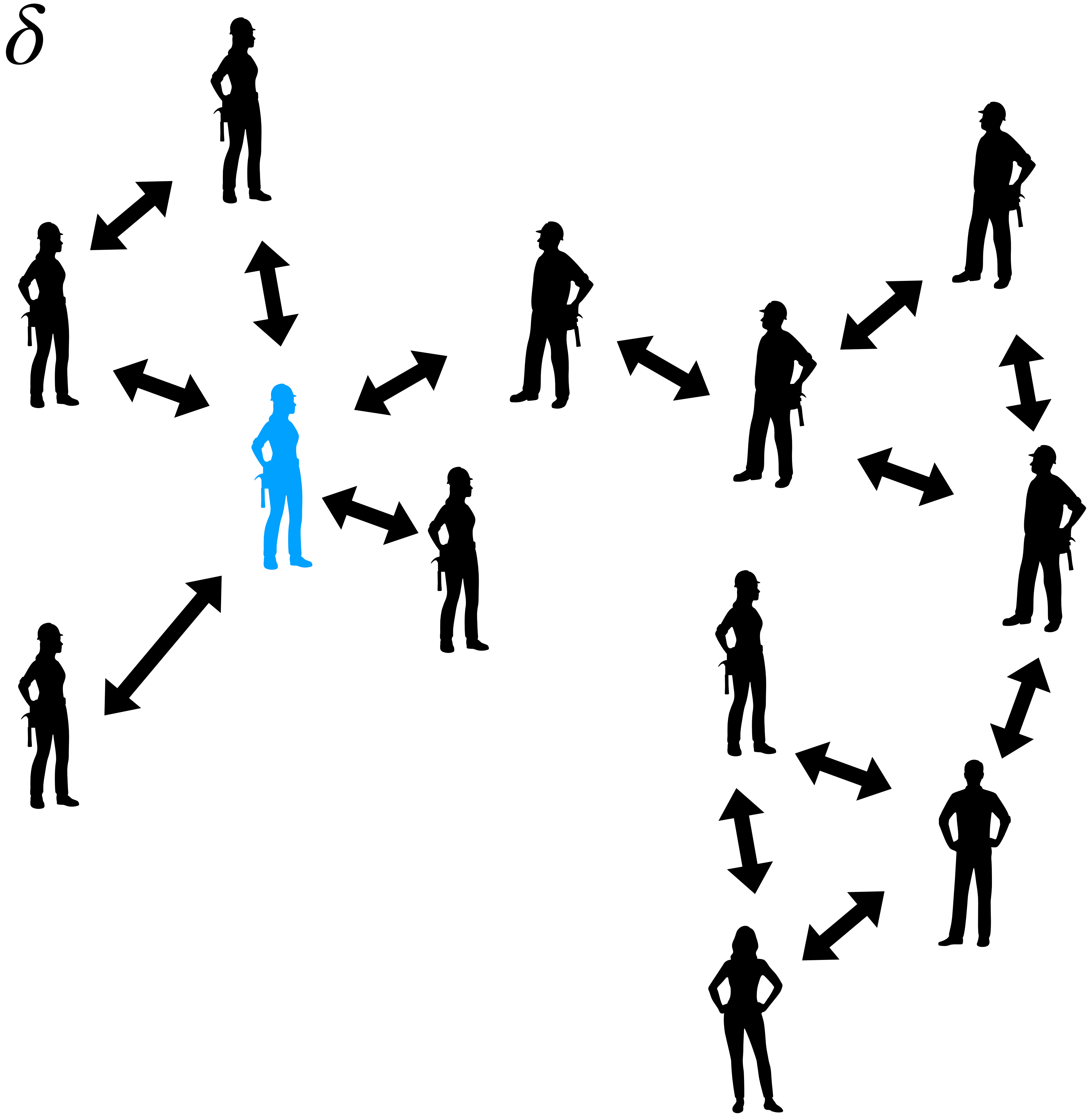


Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) $\propto$ degree + a tuning parameter δ



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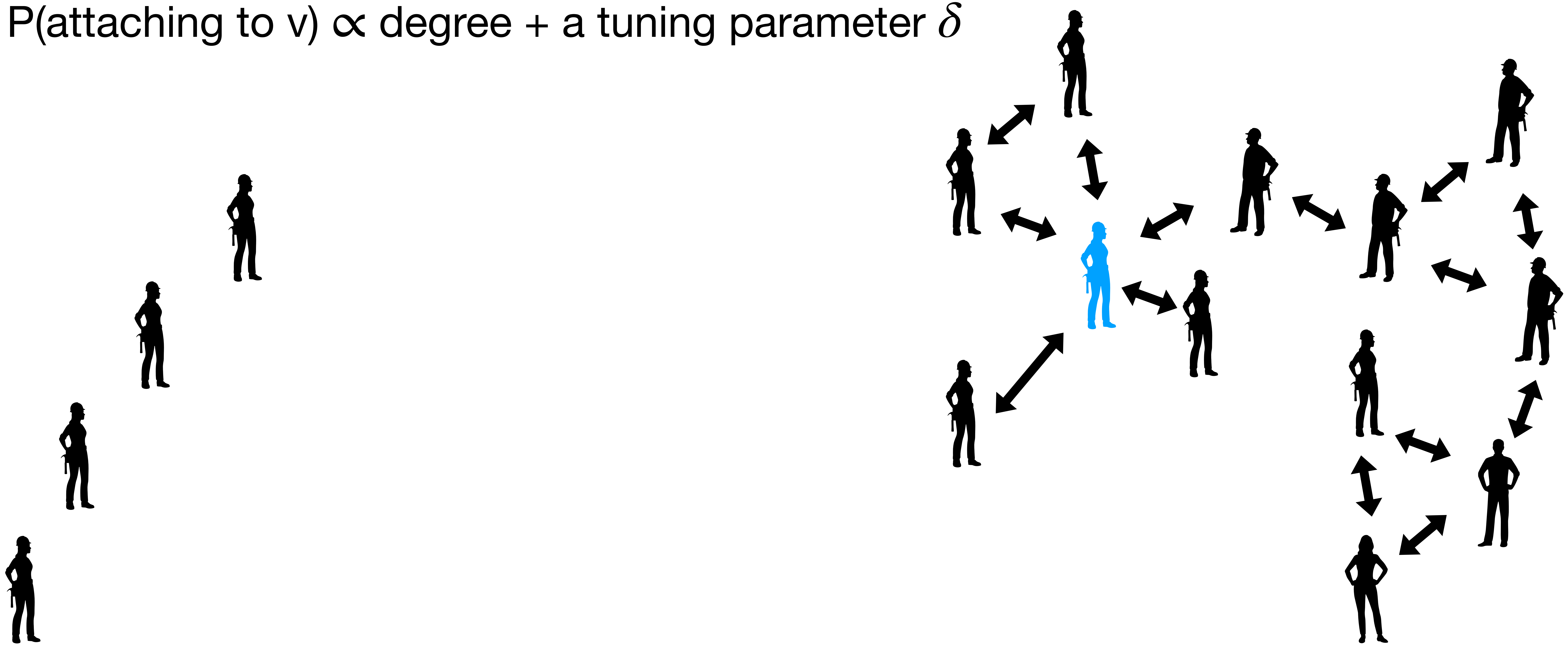

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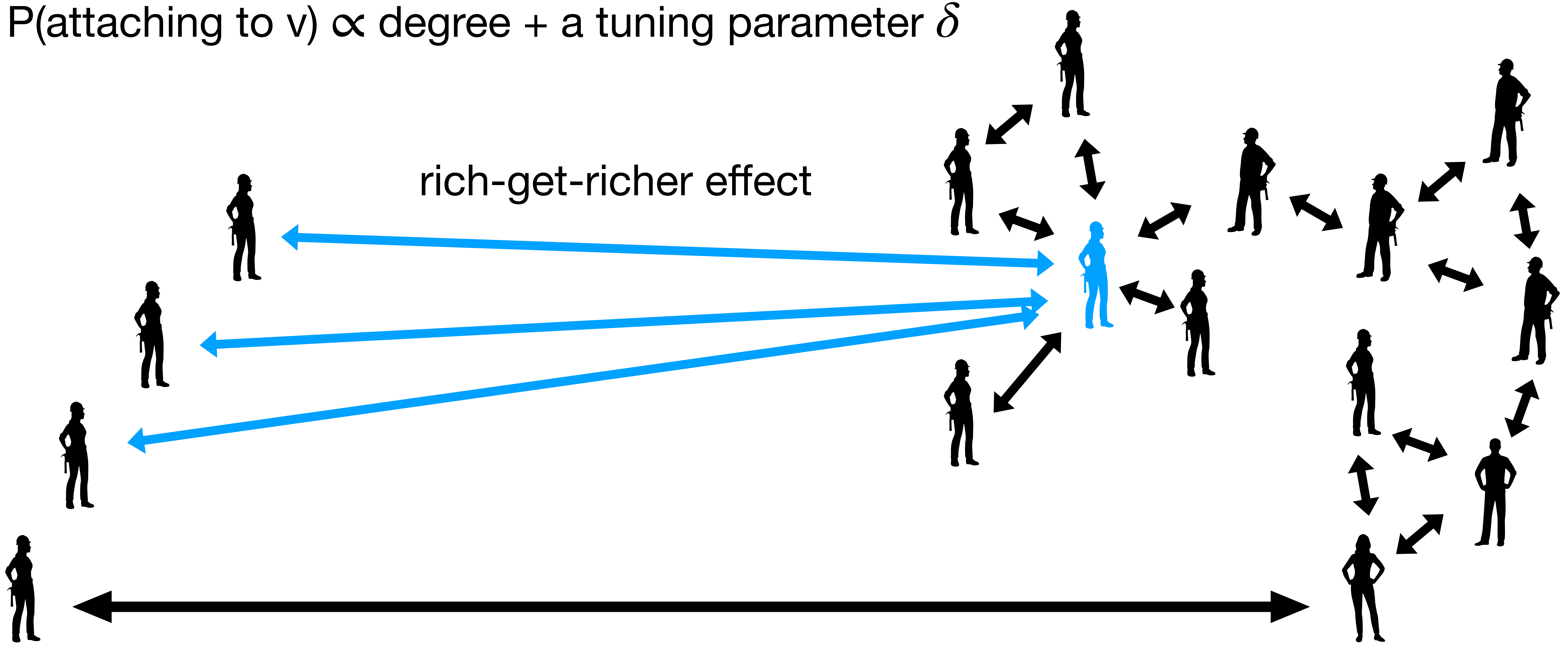

What do we know?

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- degree distribution [Albert and Barabasi 1999]

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- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]

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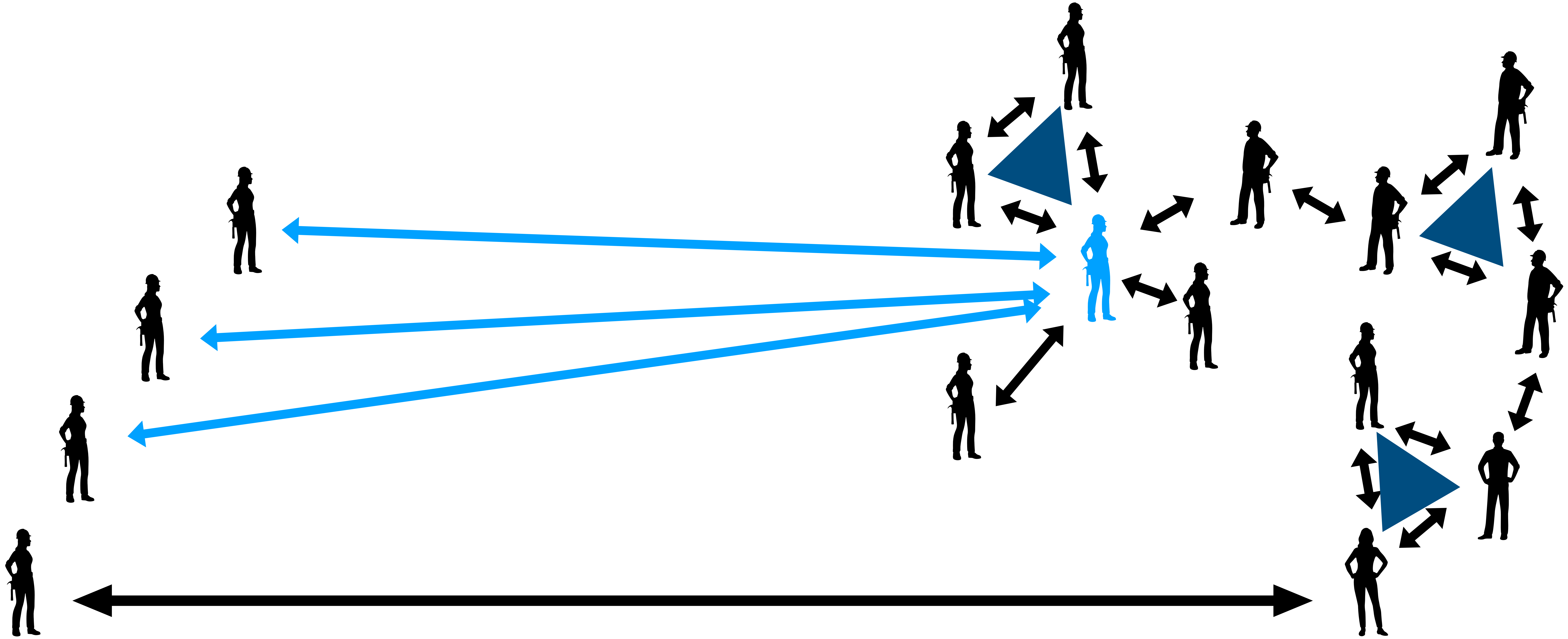
- subgraph counts [Garavaglia and Steghuis 2019]

What do we know?

- degree distribution [Albert and Barabasi 1999]
- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]
- and more…

Clique Complex

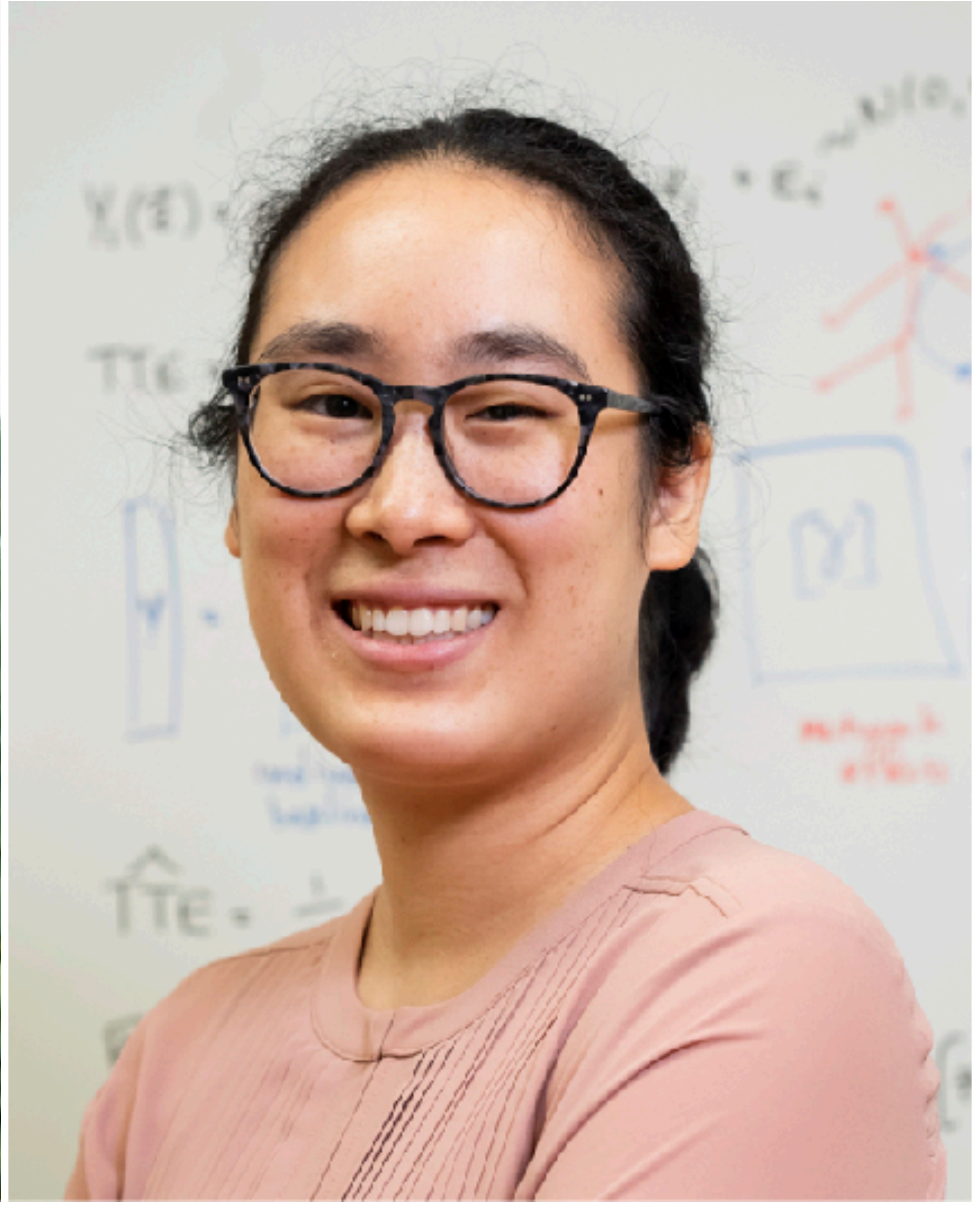
aka Flag Complex



III Topology of Preferential Attachment

My Lovely Collaborators

Avhan Misra



Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

Betti Number β_q

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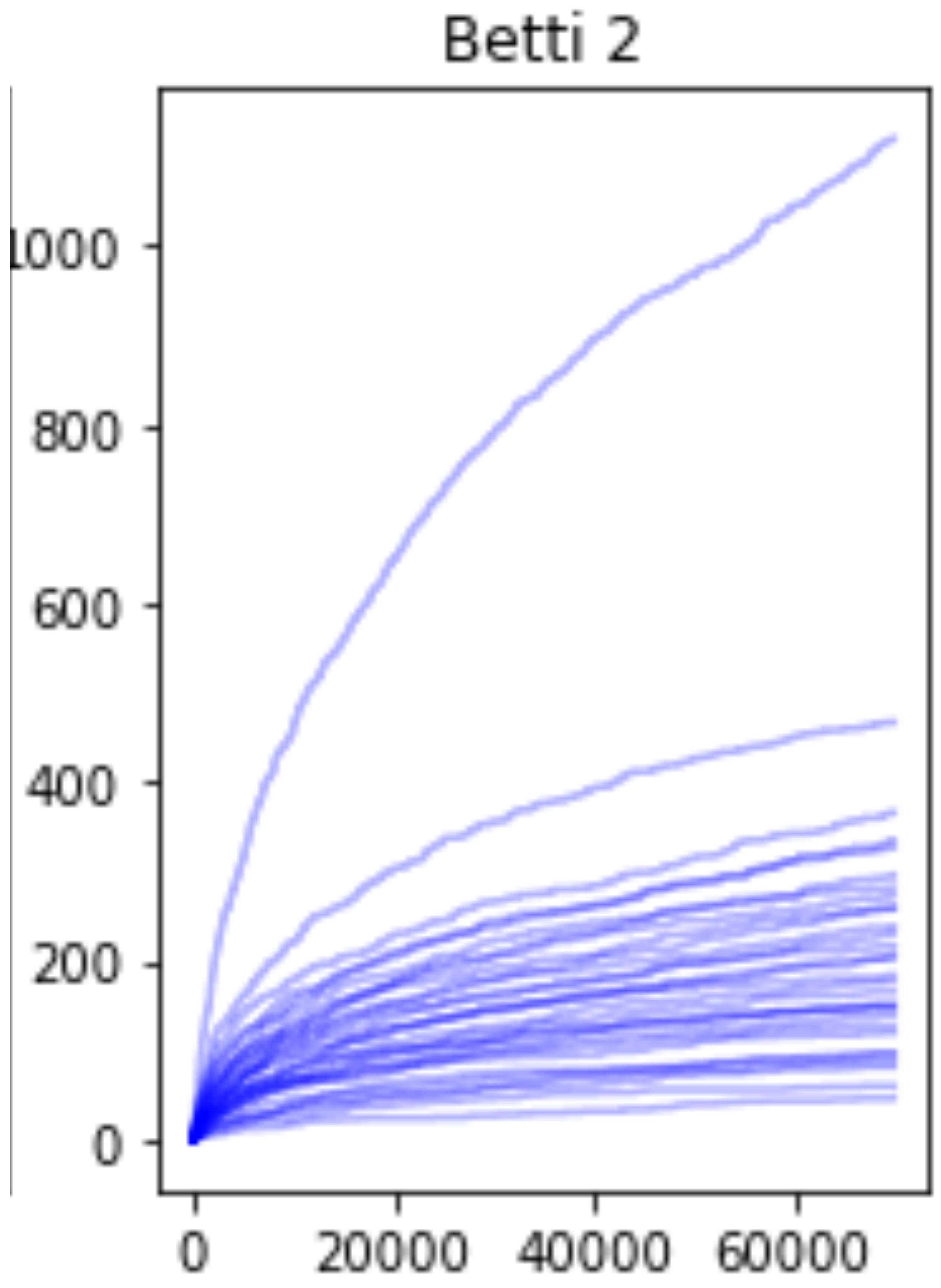
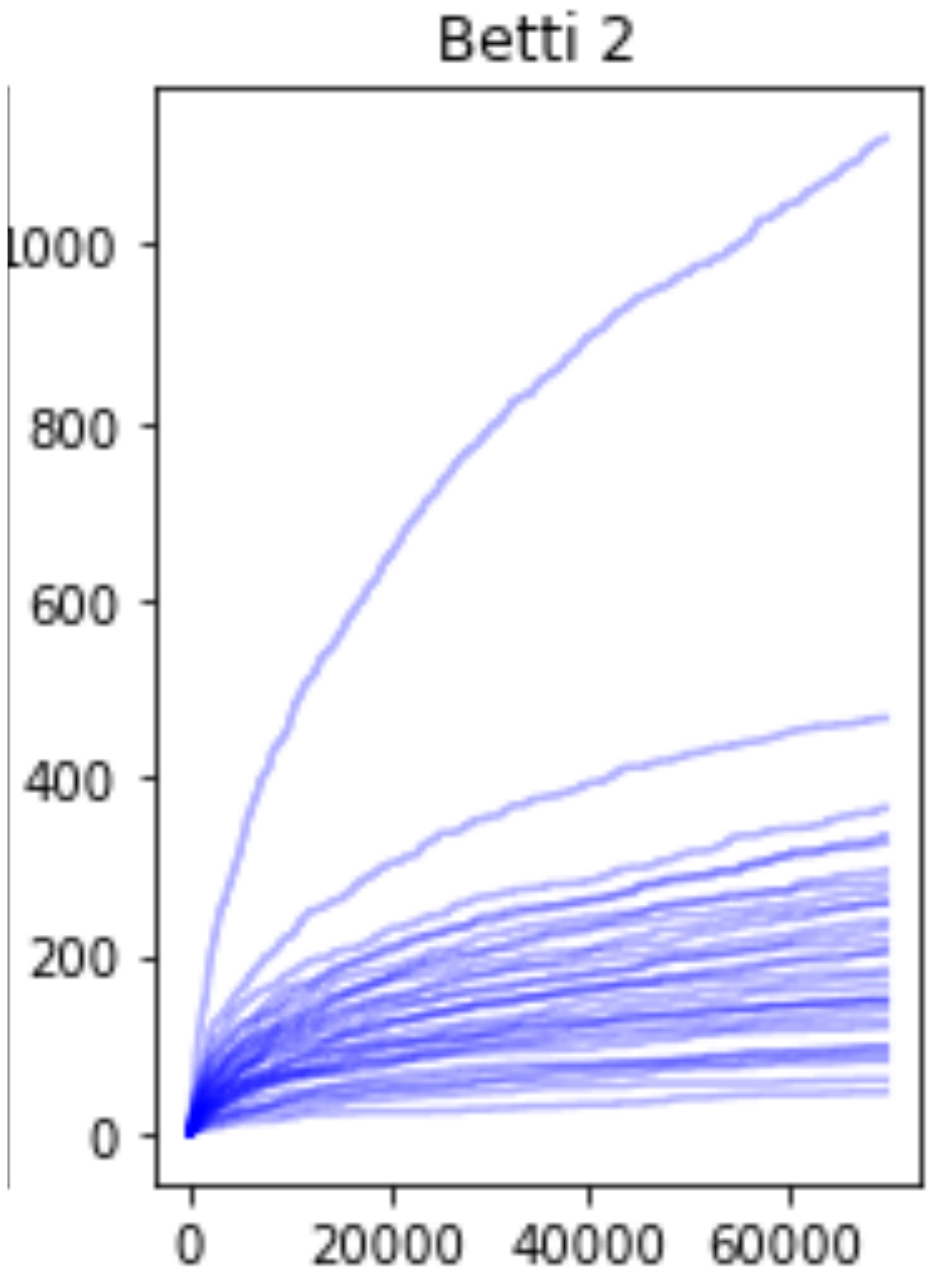


Different curves, different random seeds.
All curves have the same model parameters.

Betti Number β_q

- increasing trend



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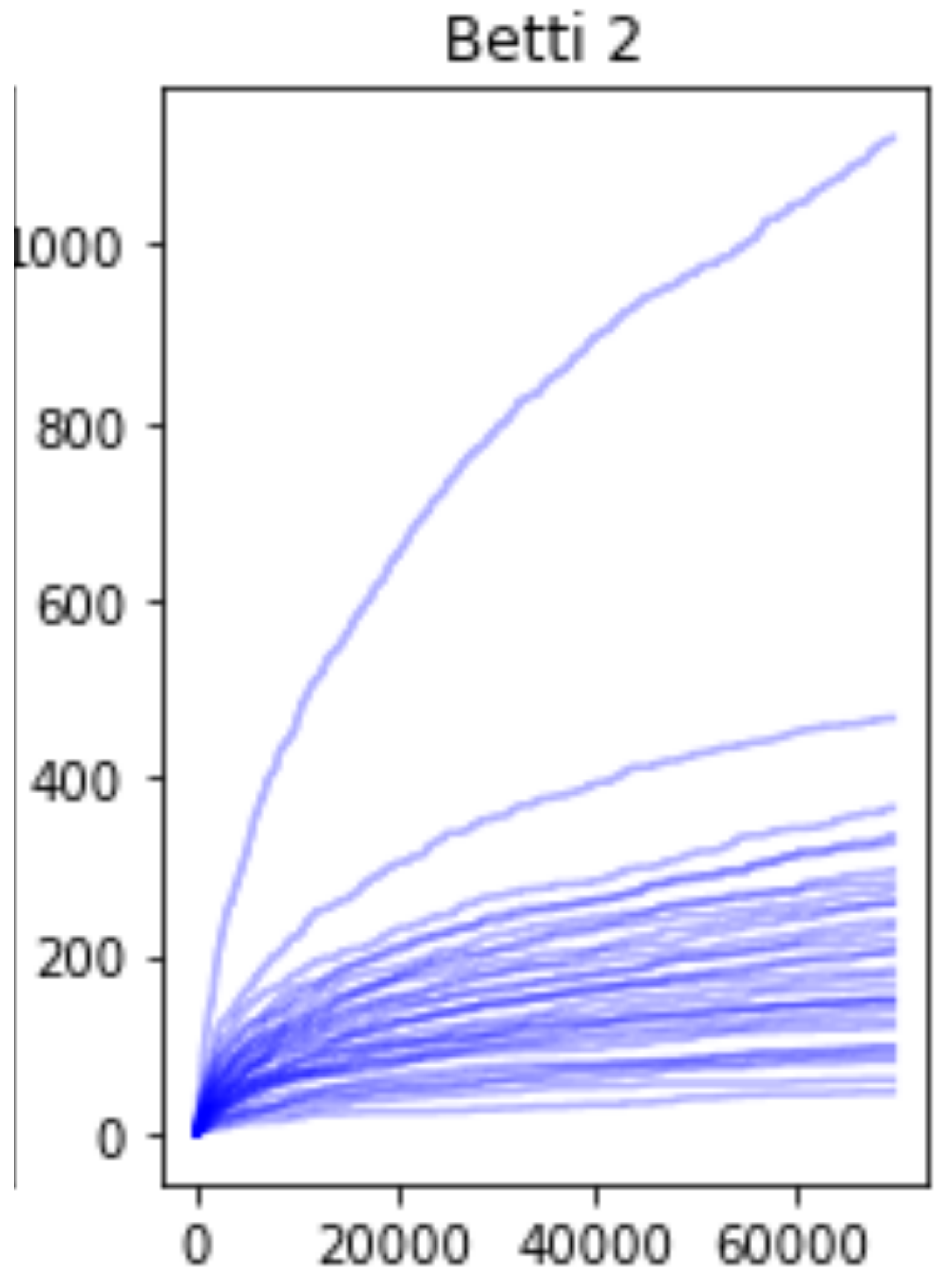

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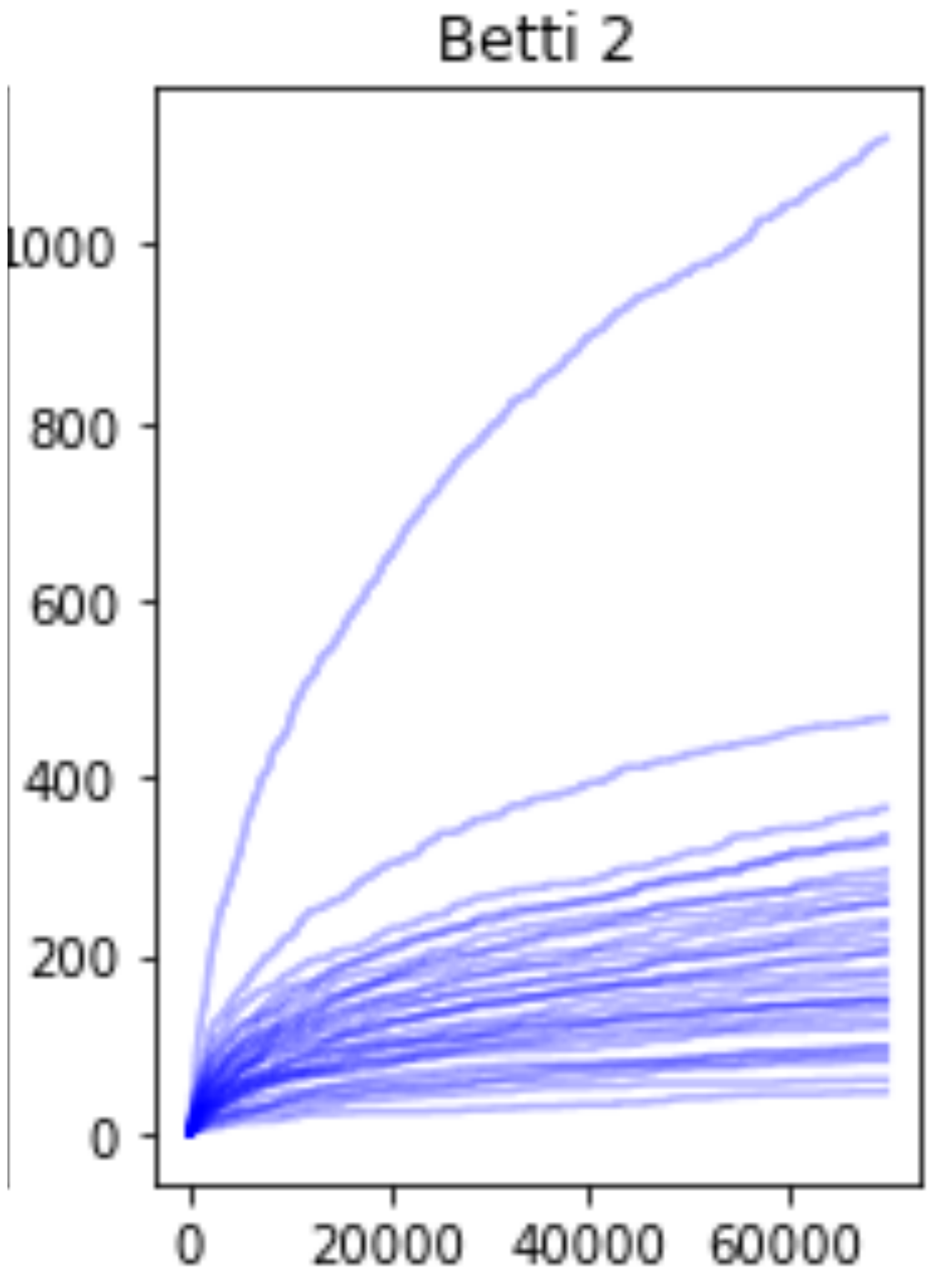
Betti Number β_q

- increasing trend
- concave growth
- outlier



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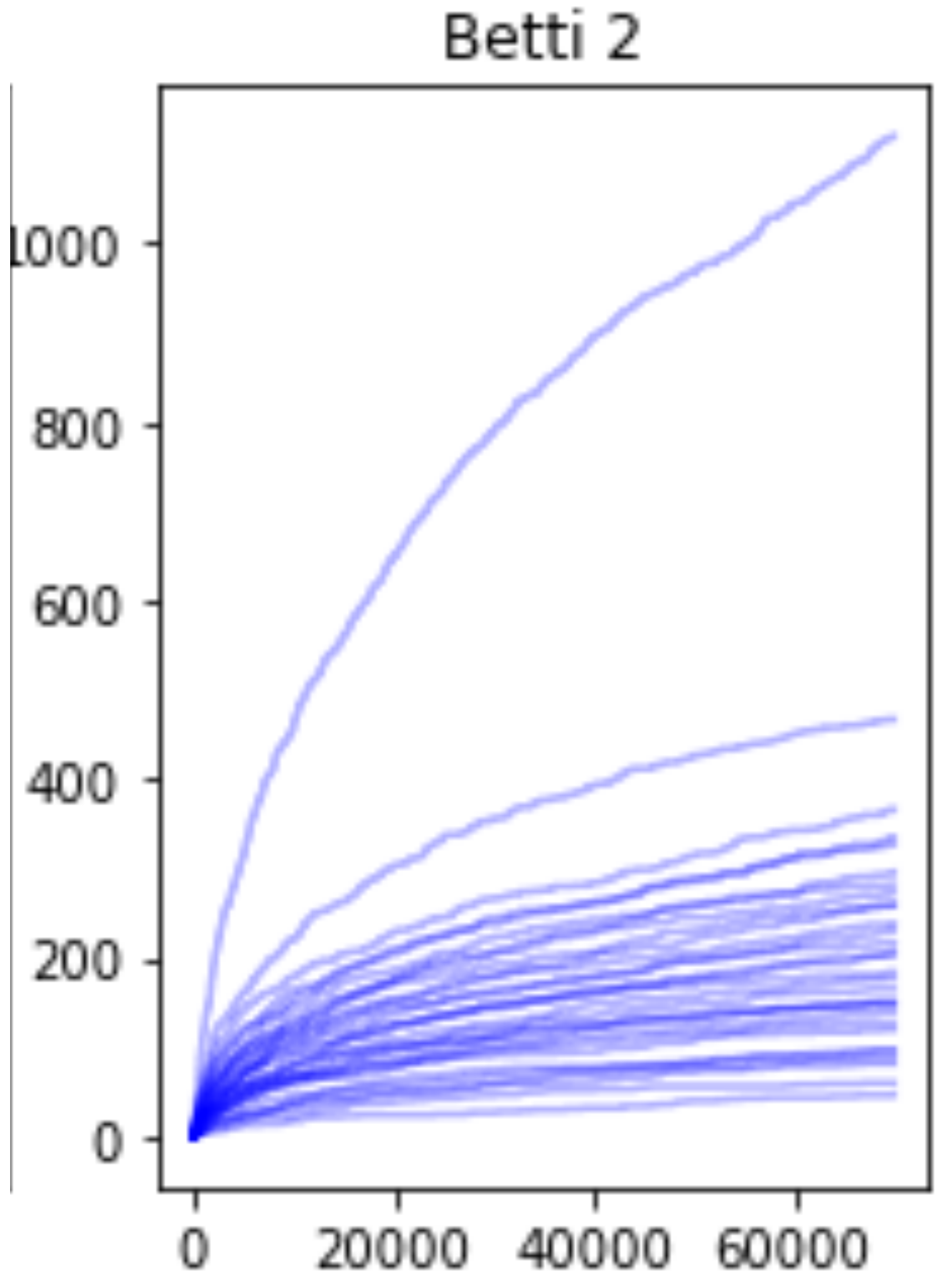
- With probability at least $1 - \varepsilon$,
- $c_\varepsilon(\text{num of nodes}^{1-4x}) \leq \beta_2 \leq C_\varepsilon(\text{num of nodes}^{1-4x})$
  - $x \in (0,1/2)$ decreases with the preferential attachment strength
    - $P[T \text{ attaches to } i] \propto T^{-x}$



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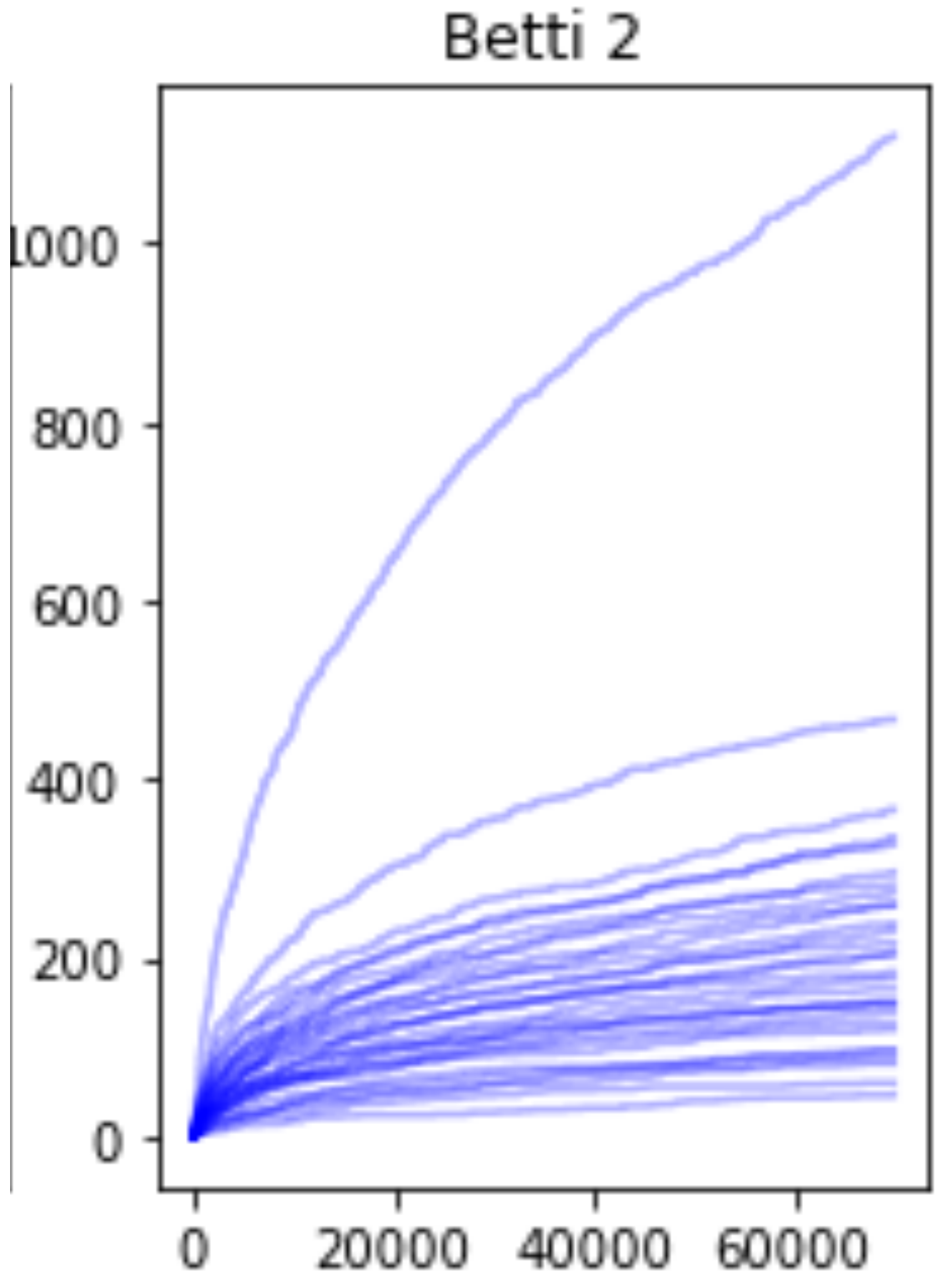

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- With probability at least $1 - \varepsilon$,
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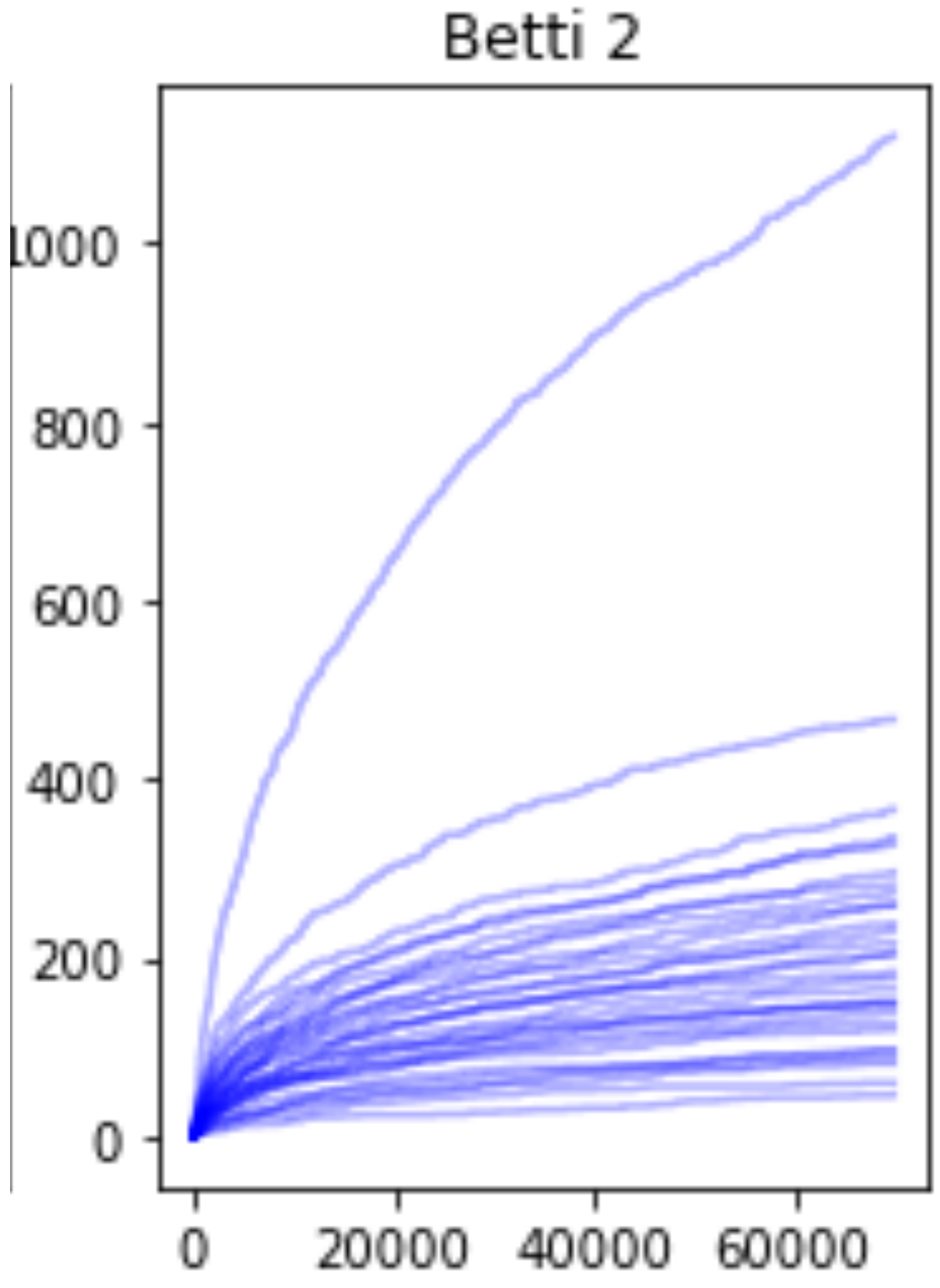
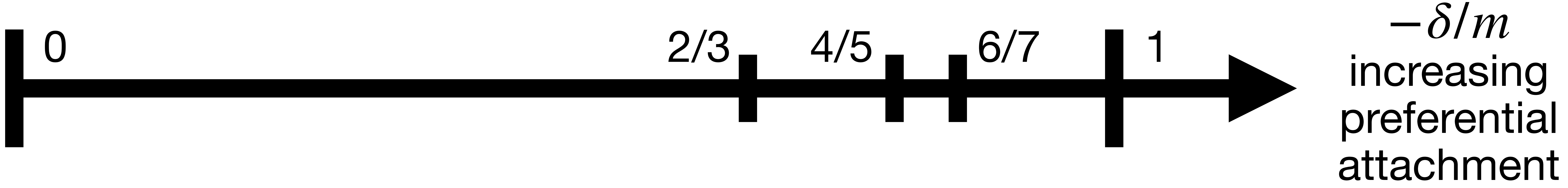


Phase transition

Recall
P(attaching to v) ∝ degree + δ
m = number of edges per new node



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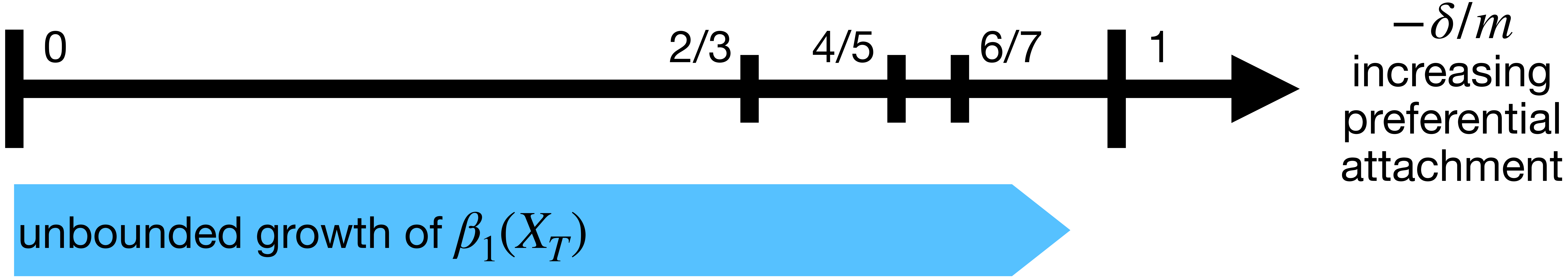

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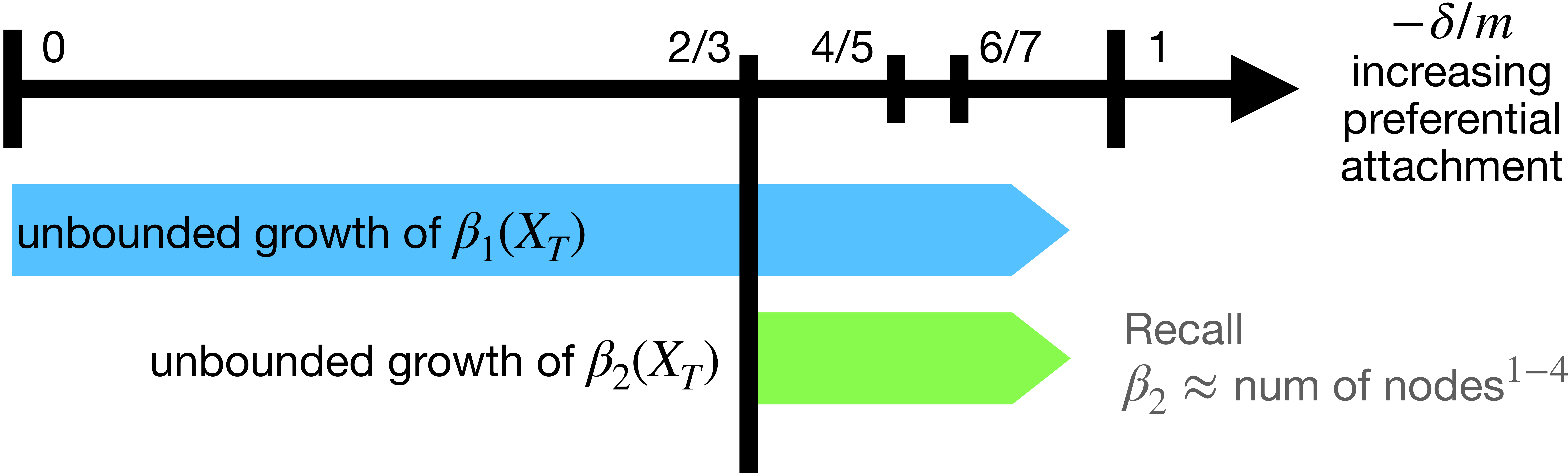

Recall

$\beta_2 \approx$ num of nodes^{1-4x}

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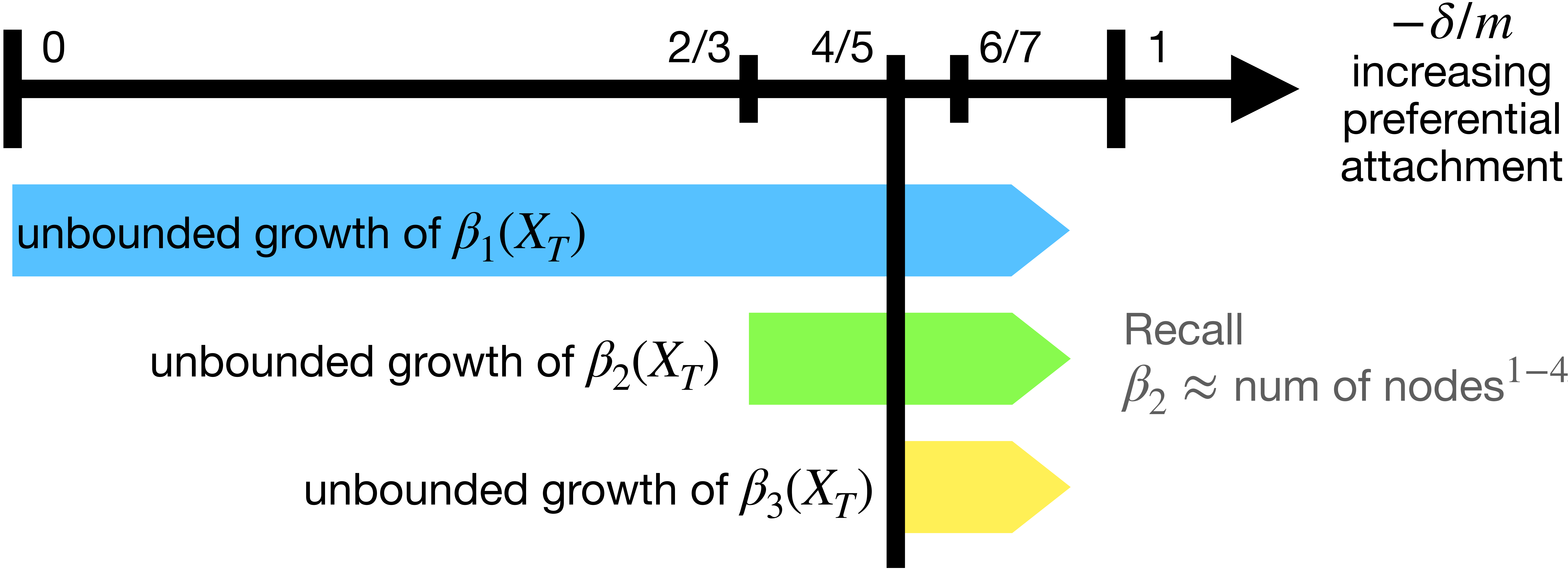

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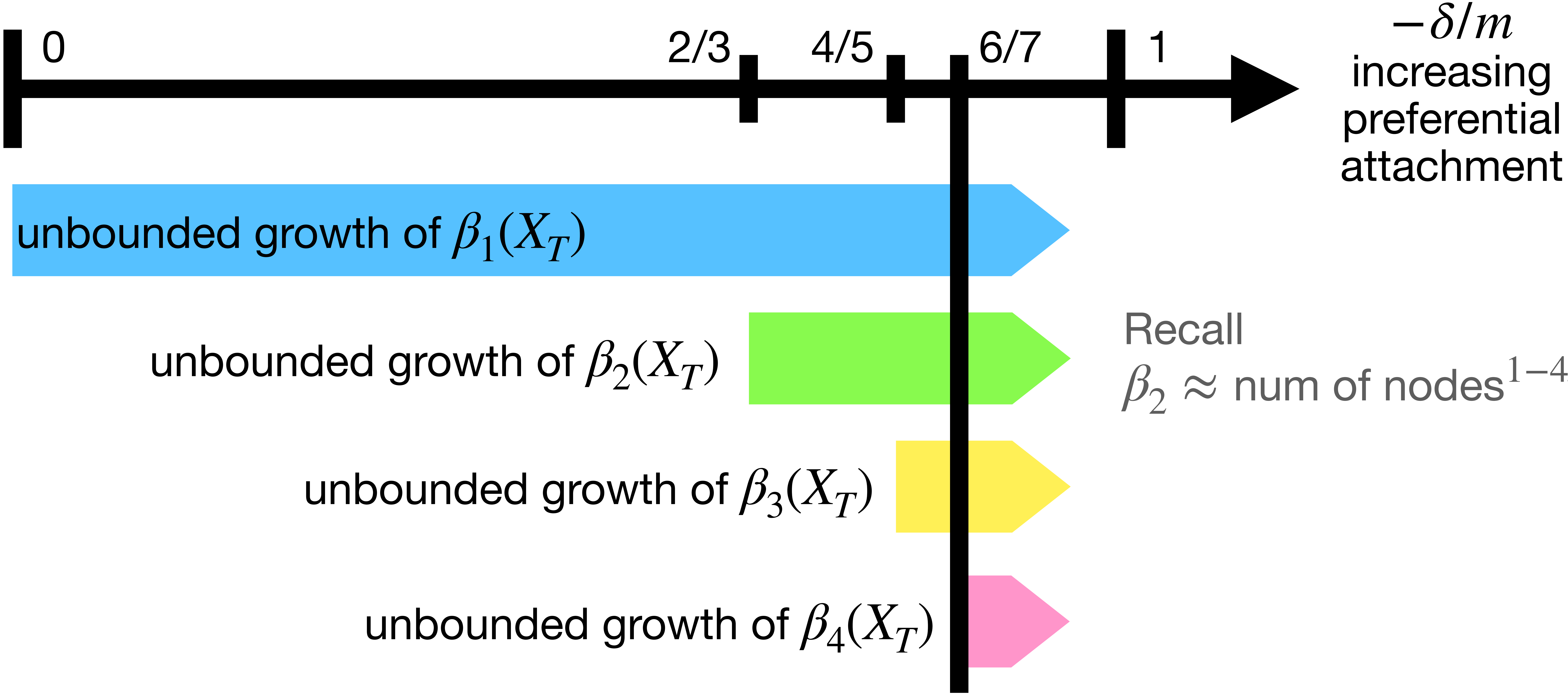

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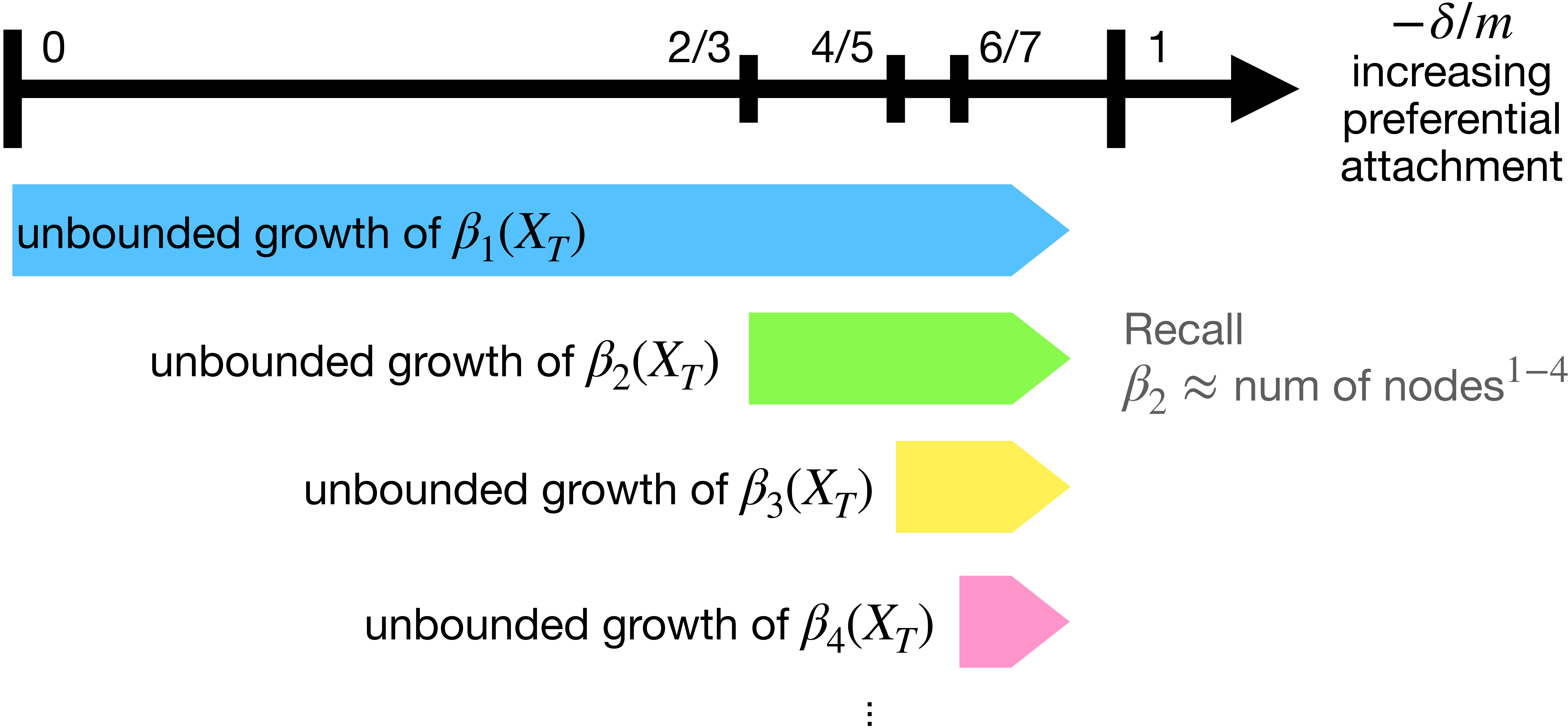
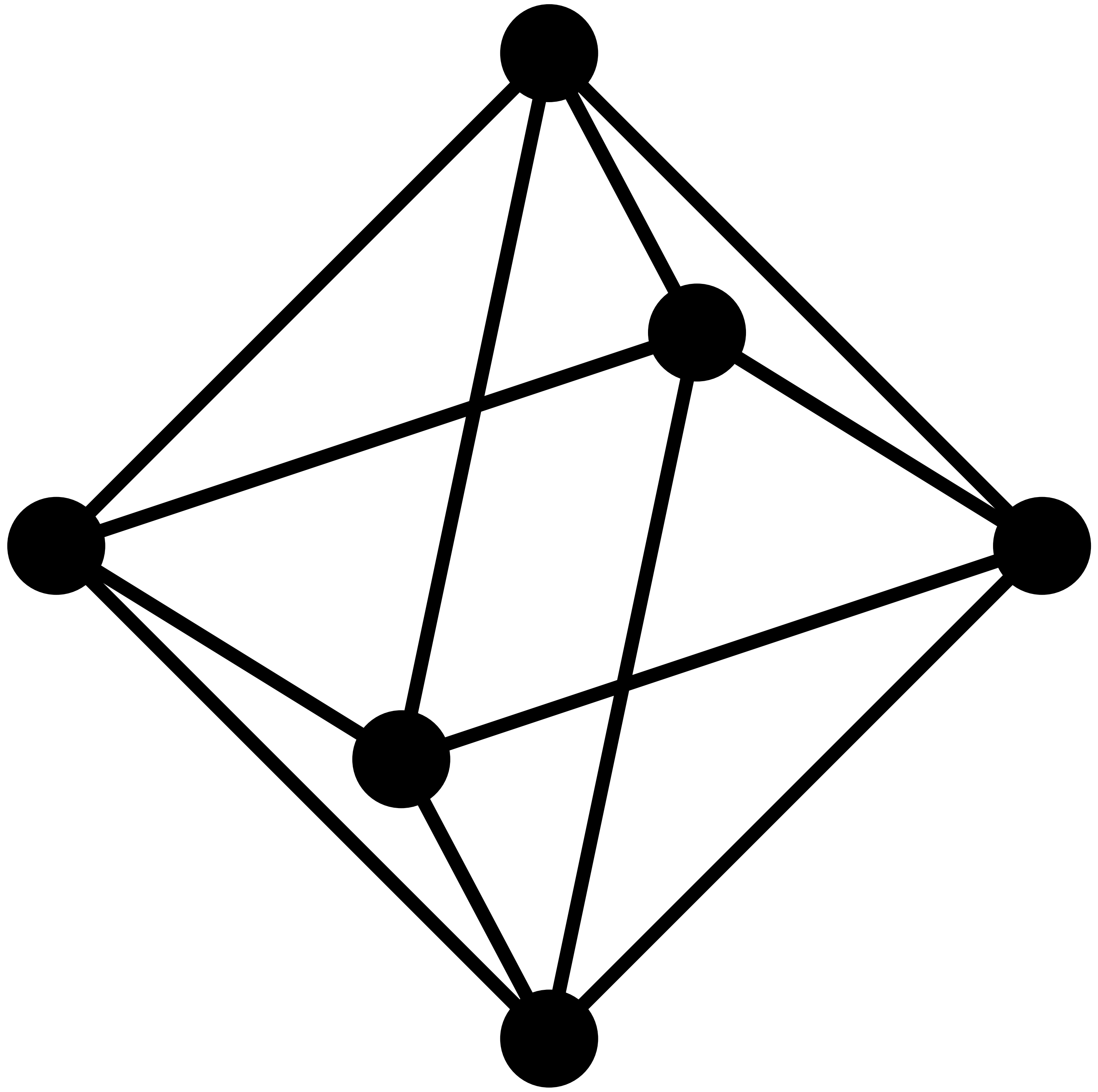


Recall

$\beta_2 \approx$ num of nodes^{1-4x}

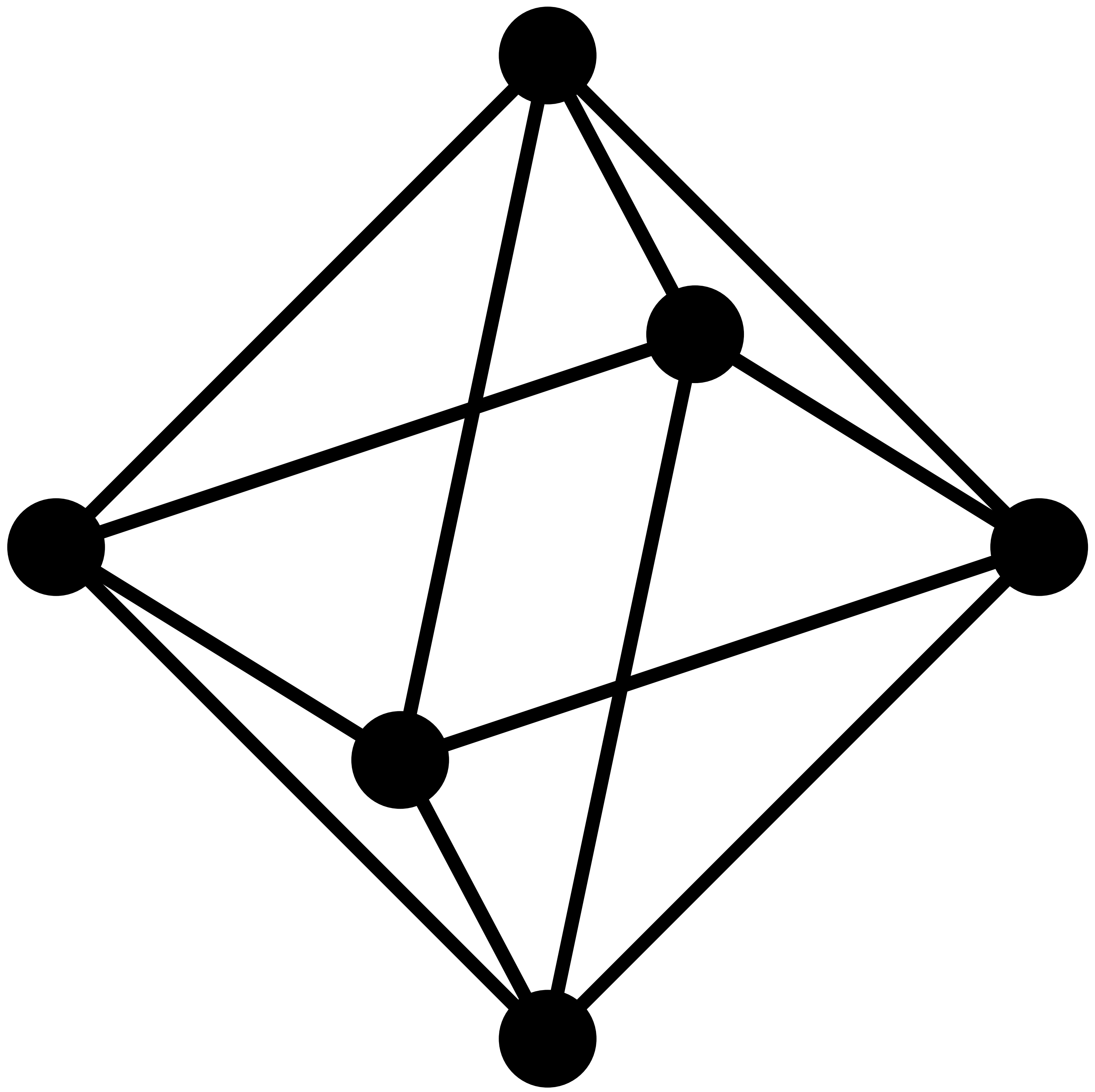
Theorem: $\beta_2 \approx$ num of nodes^{1-4x}
Proof?

Proof of $\beta_2 \approx$ num of nodes^{1-4x}



$\beta_2 = 1$

Proof of $\beta_2 \approx$ num of nodes^{1-4x}



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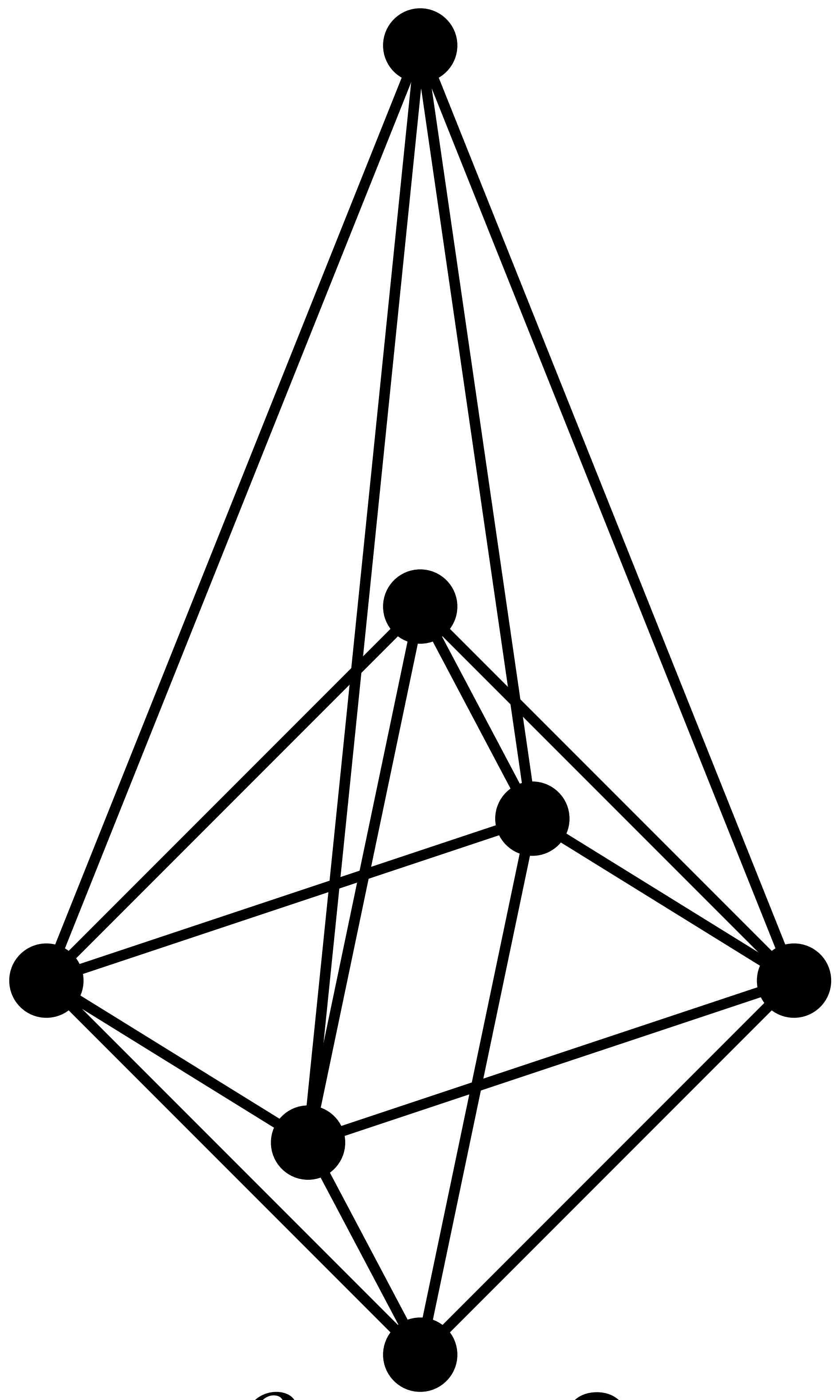
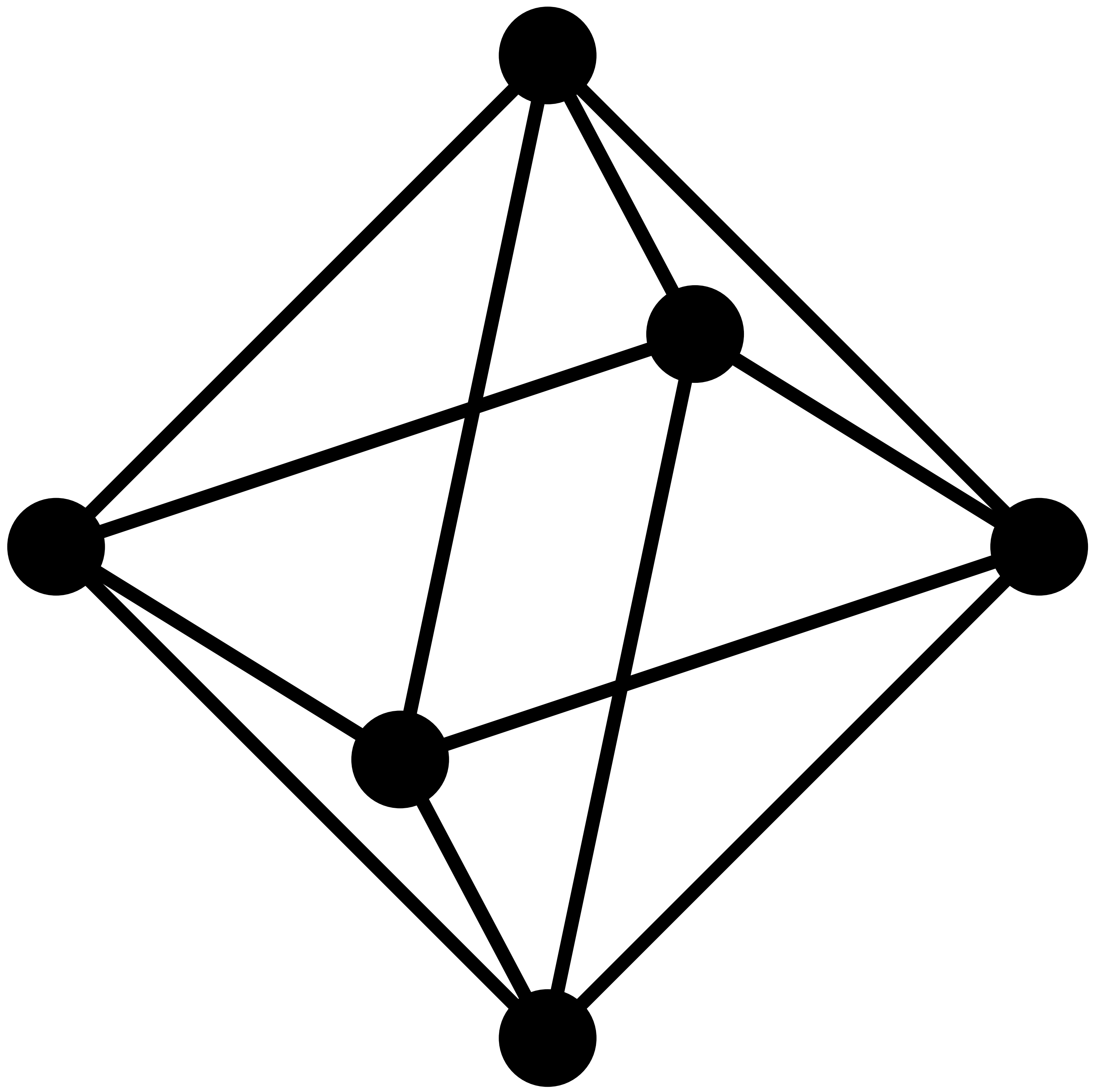


$\beta_2 = 2$

Proof of $\beta_2 \approx$ num of nodes^{1-4x}



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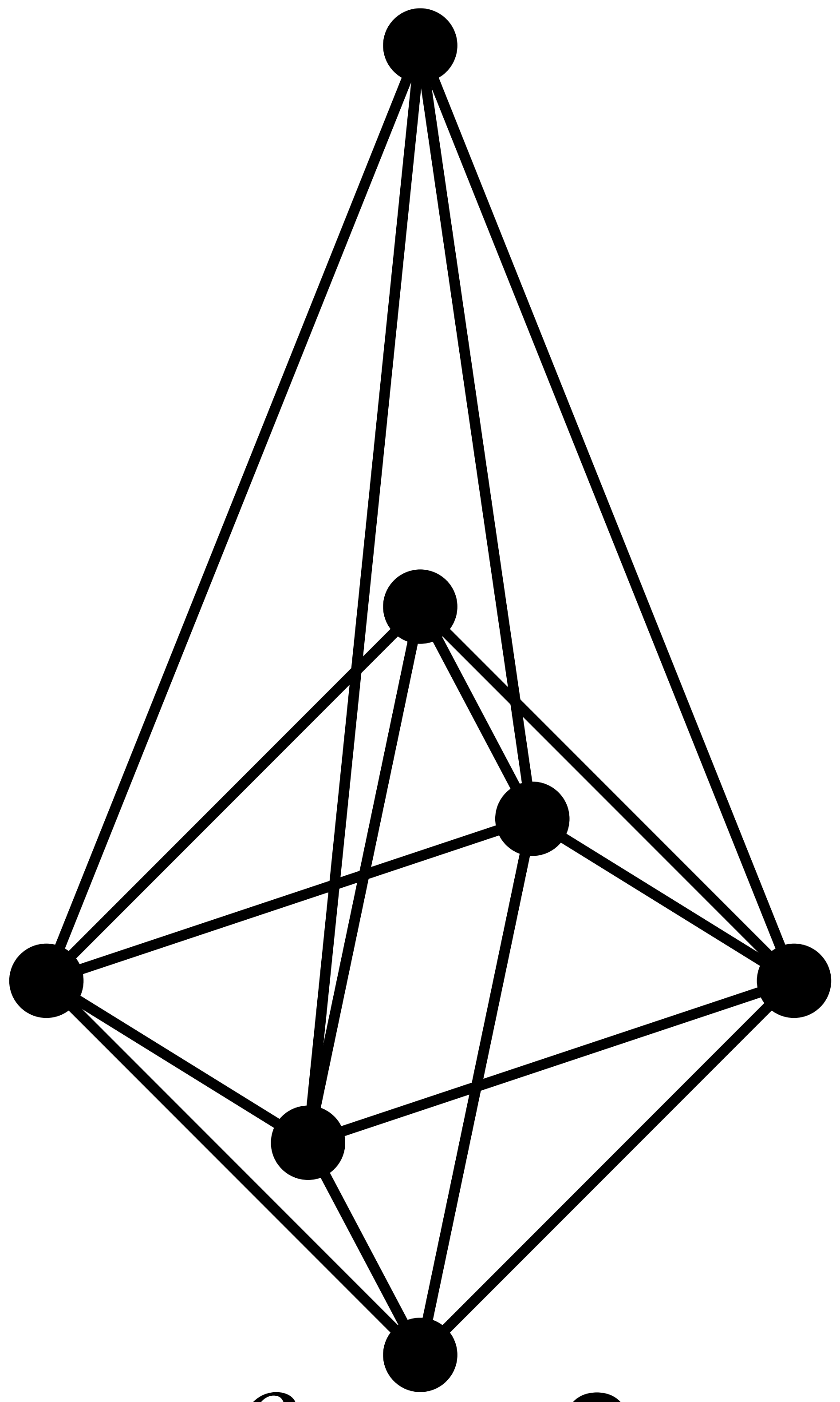

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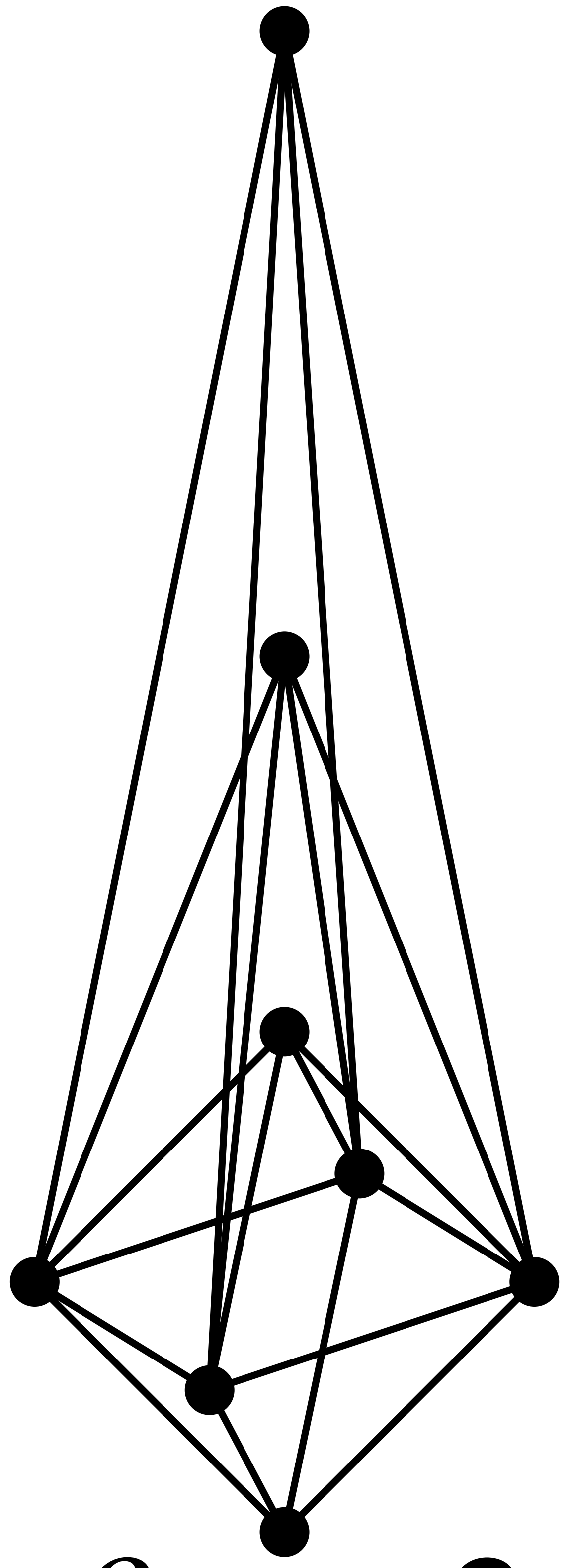


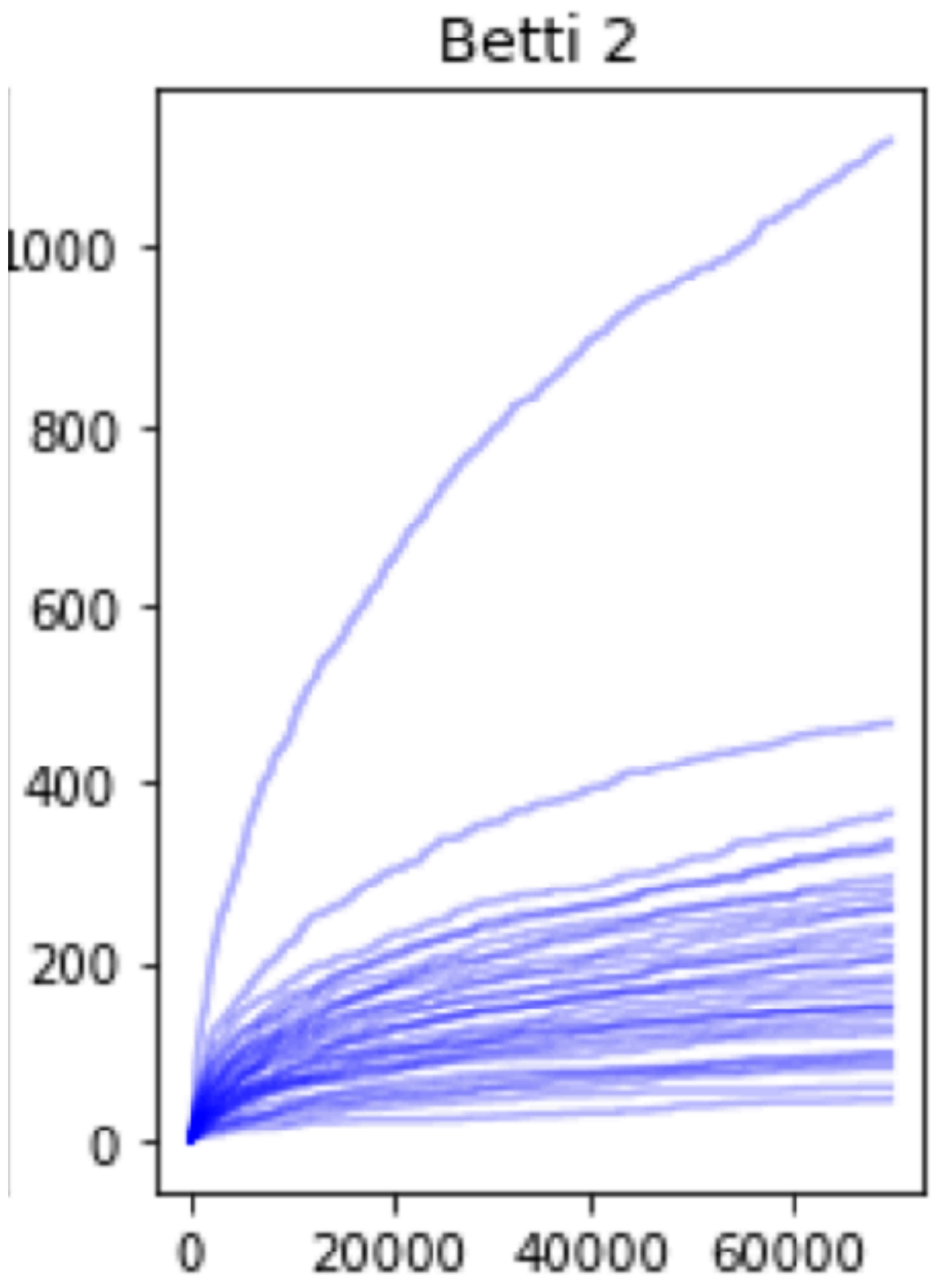
$\beta_2 = 3$

Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

In practice???

$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

$$\log E[\beta_2] \approx (1-4x)\log(\textbf{num of nodes})$$



$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

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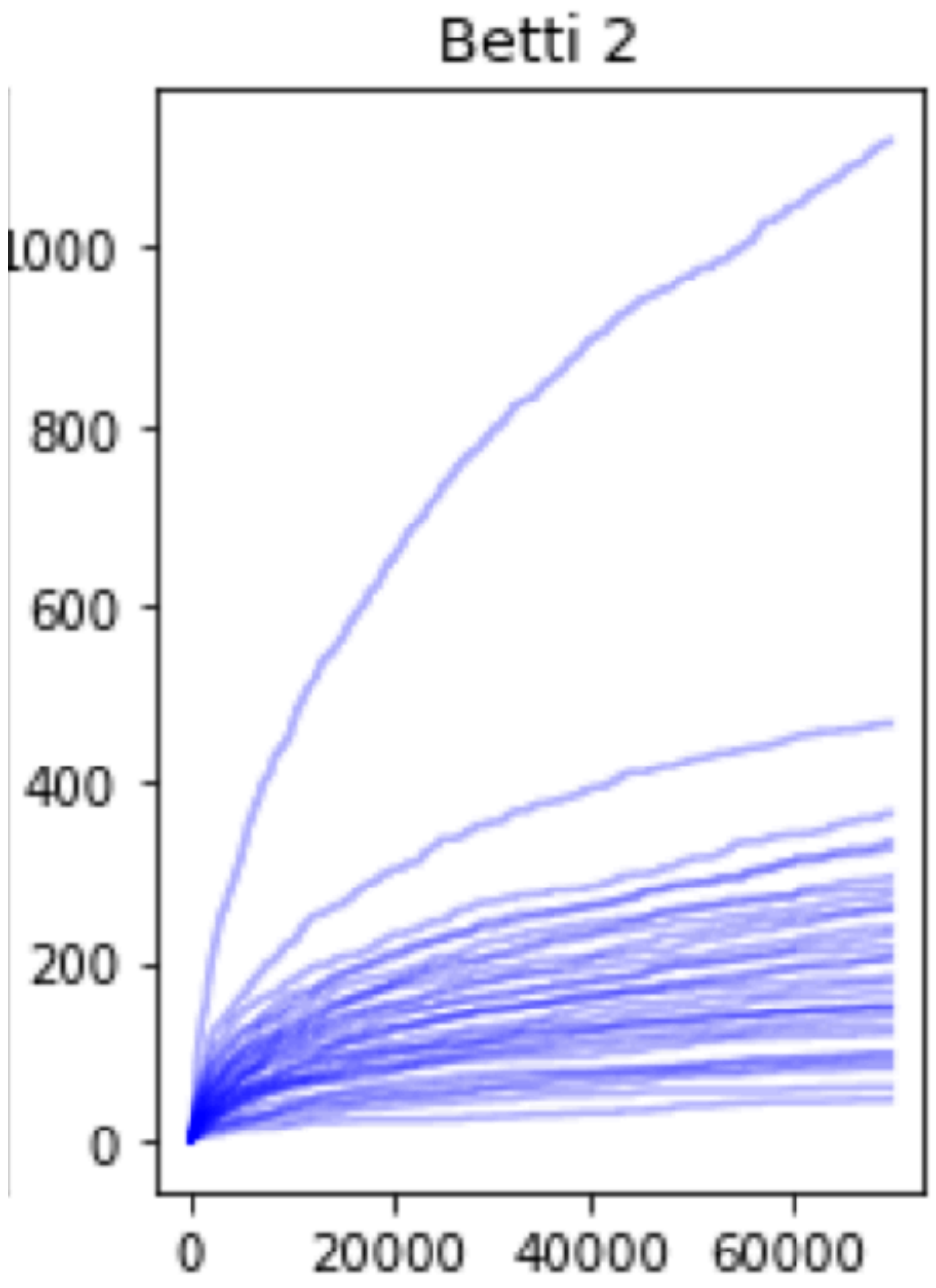
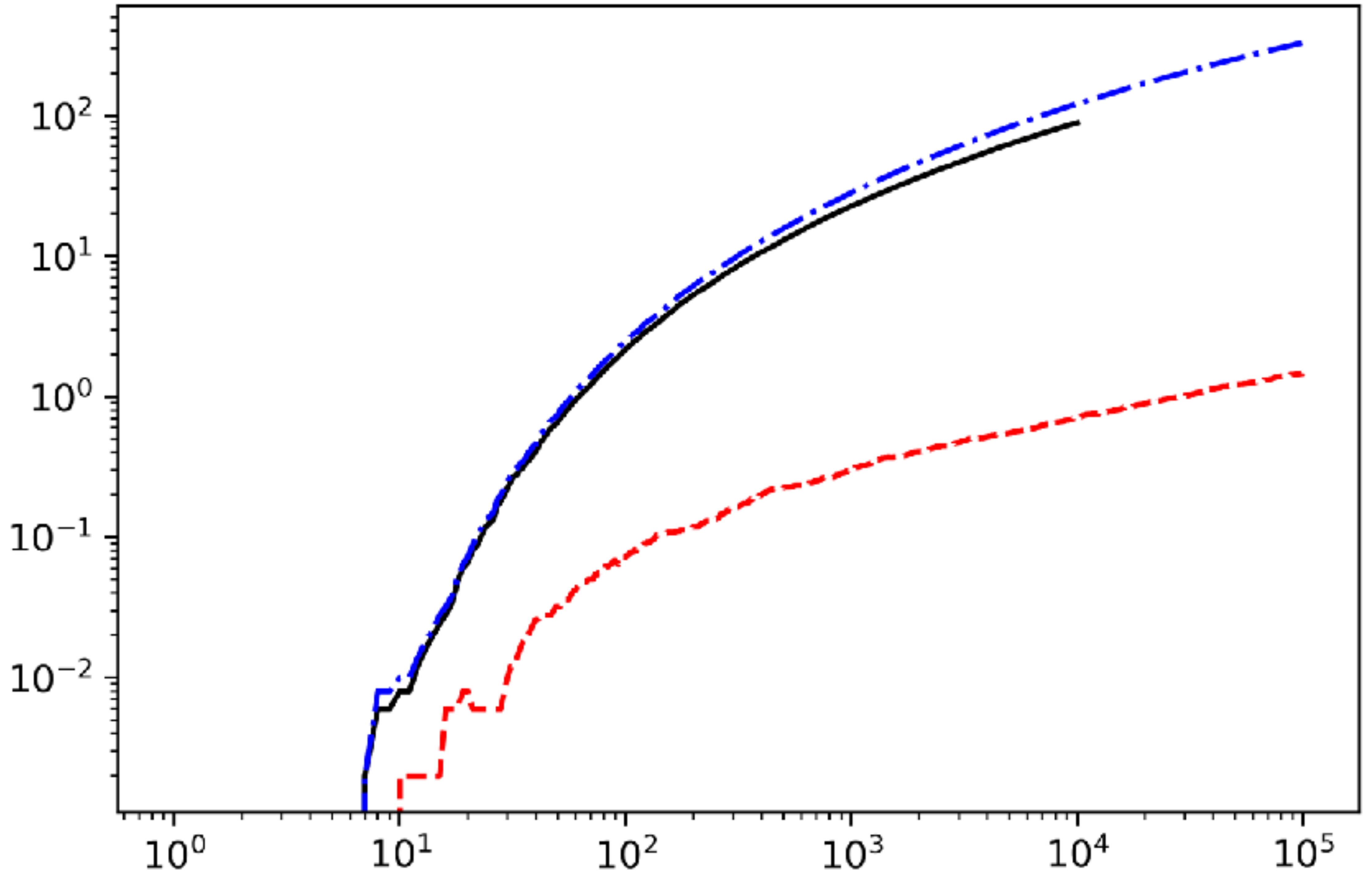
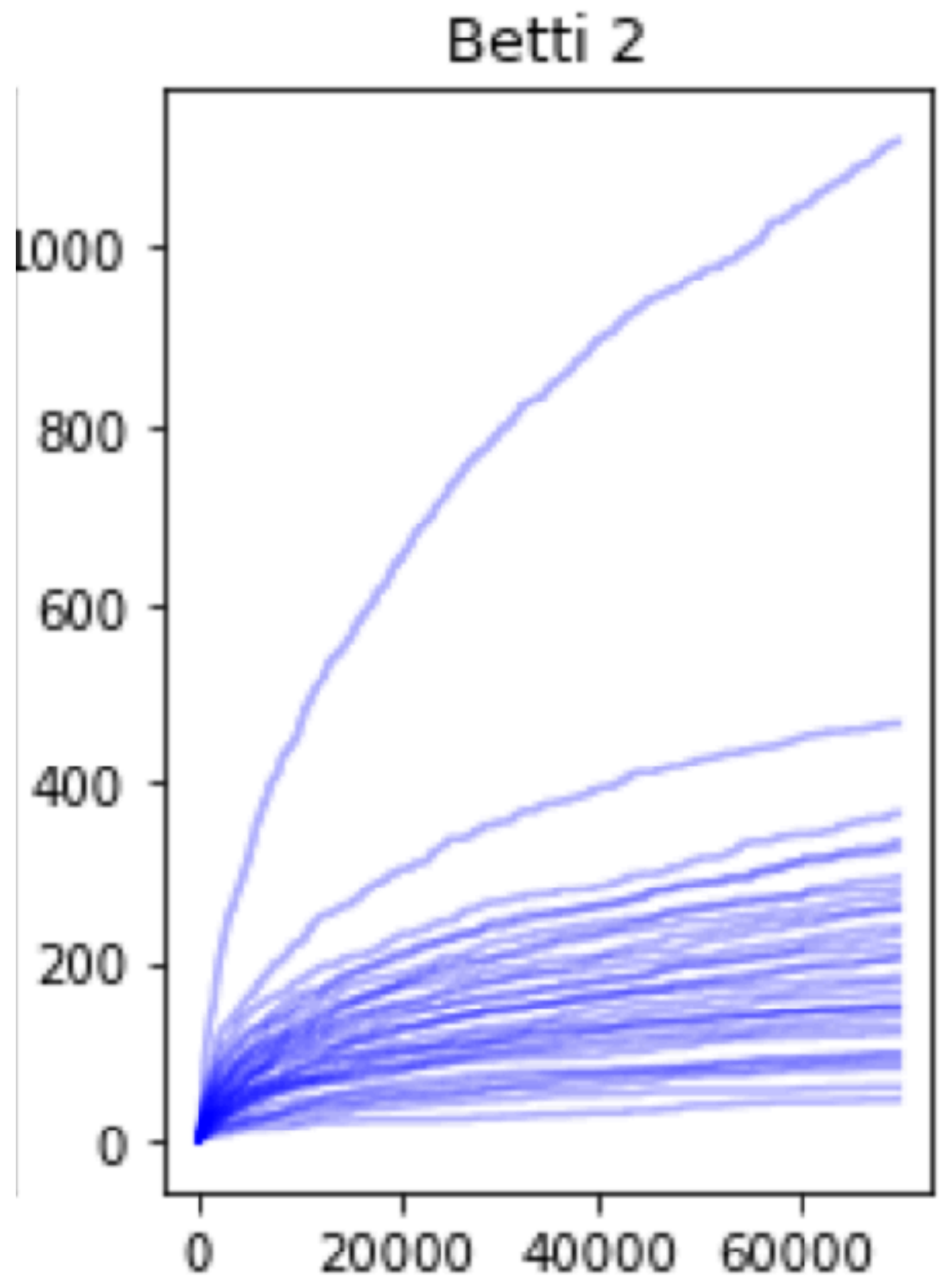
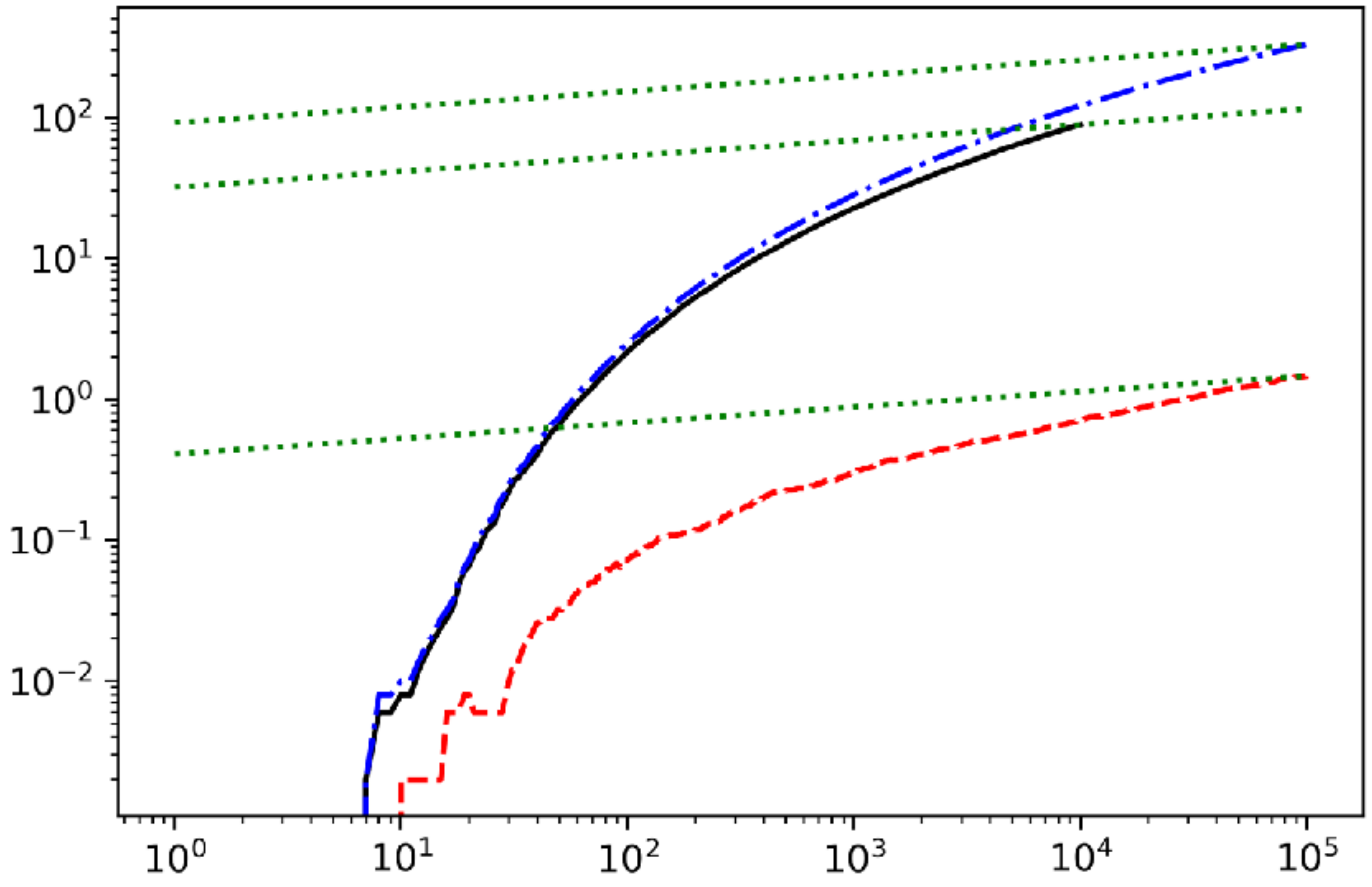




$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$





Homotopy-Connectivity?

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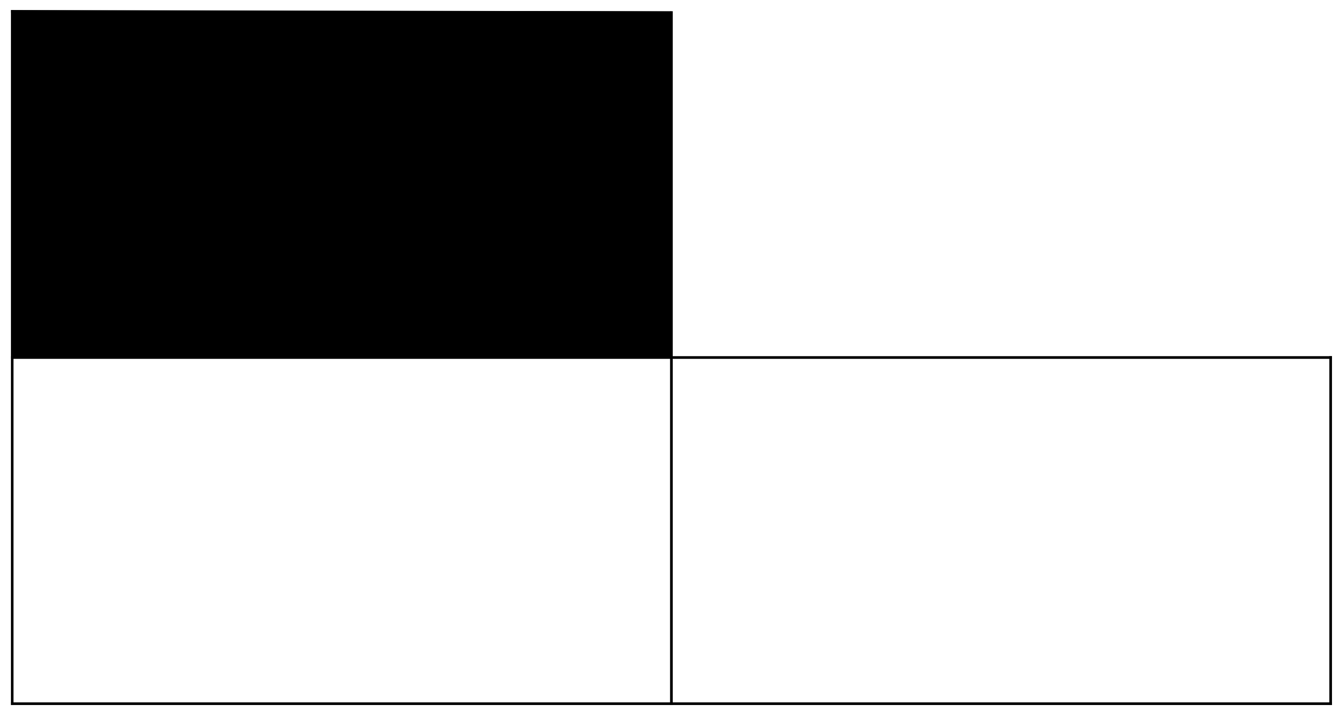
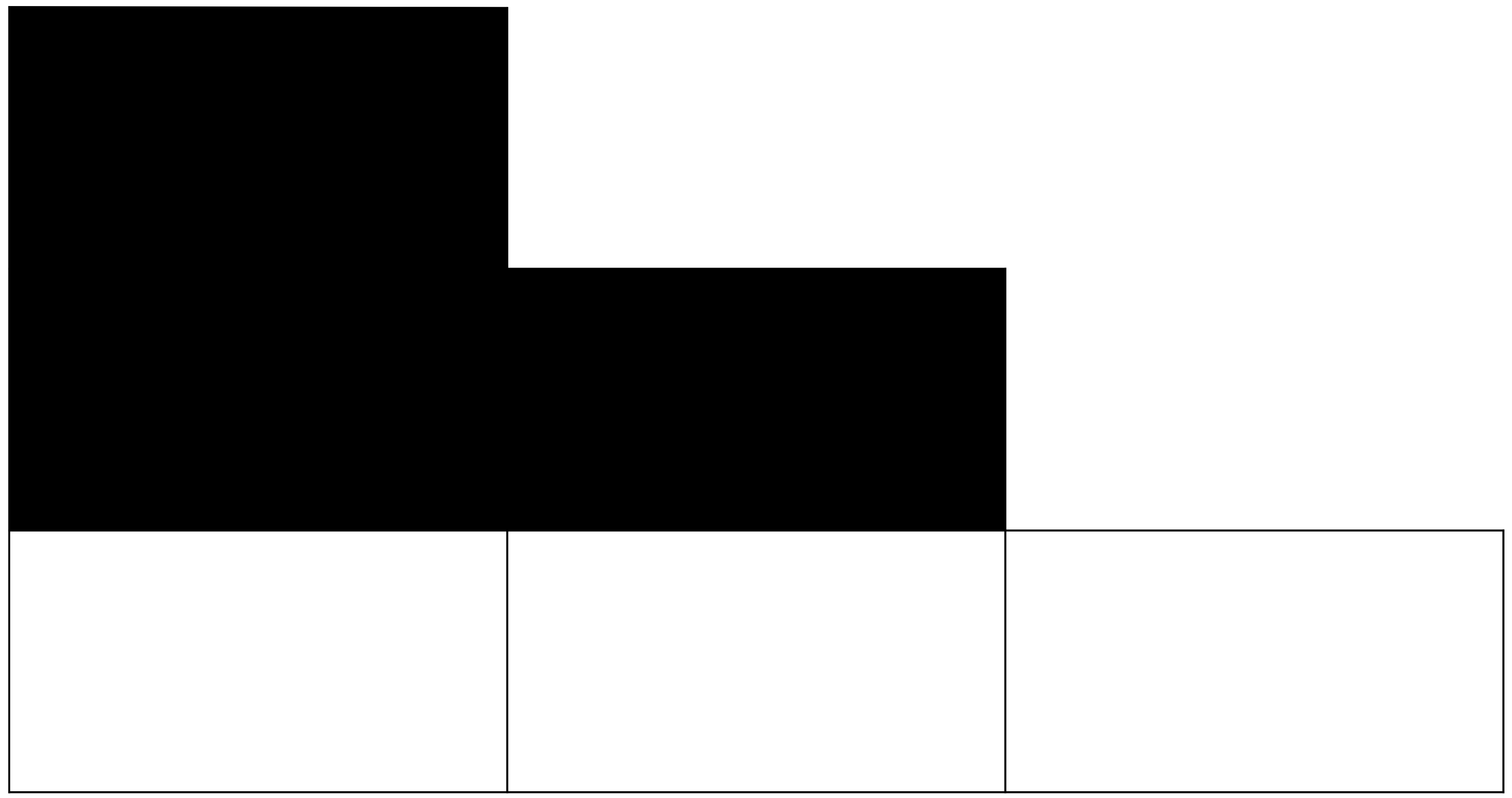
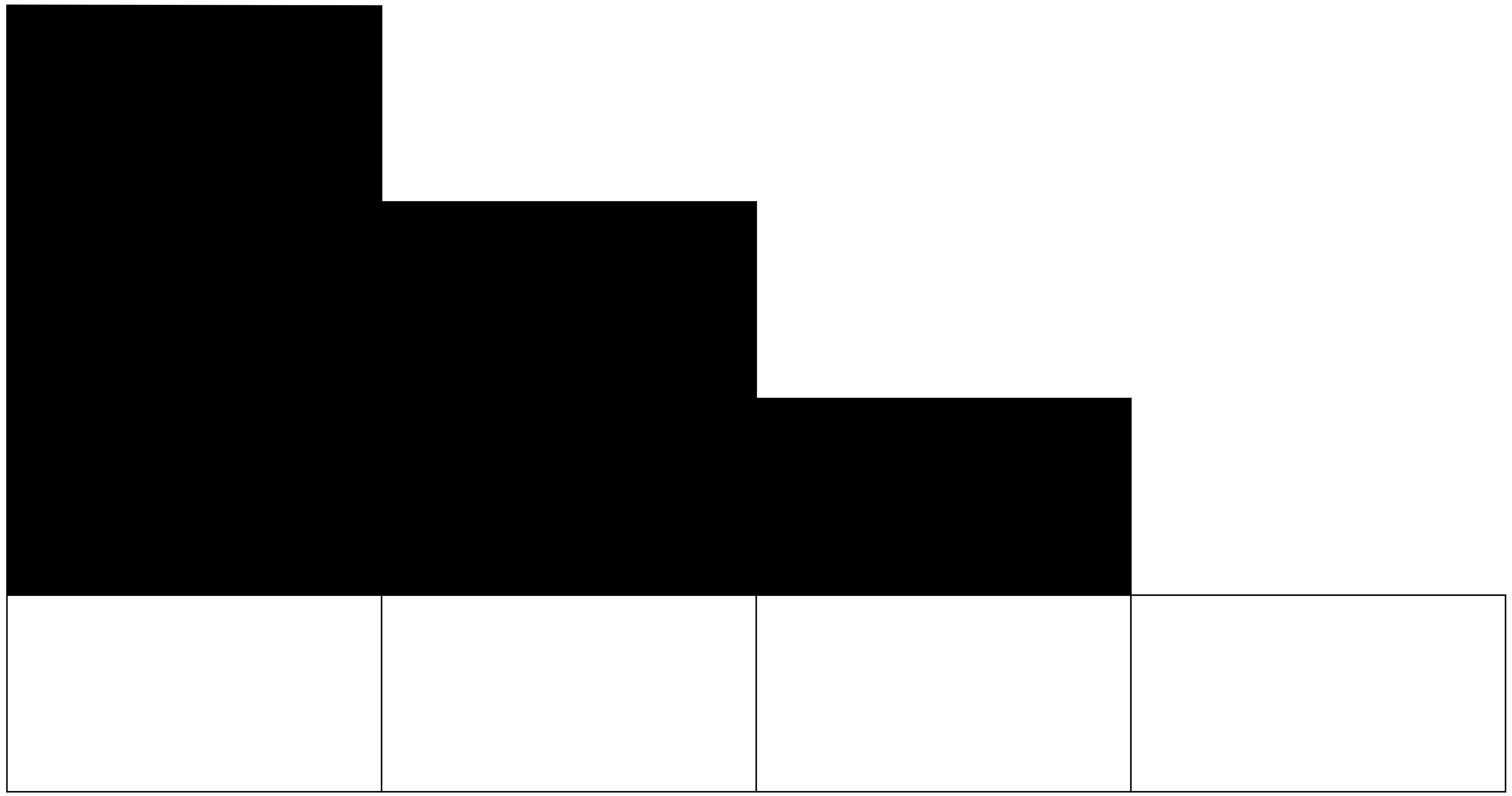
$$\beta_2 \approx \text{num of nodes}^{1-4x}$$

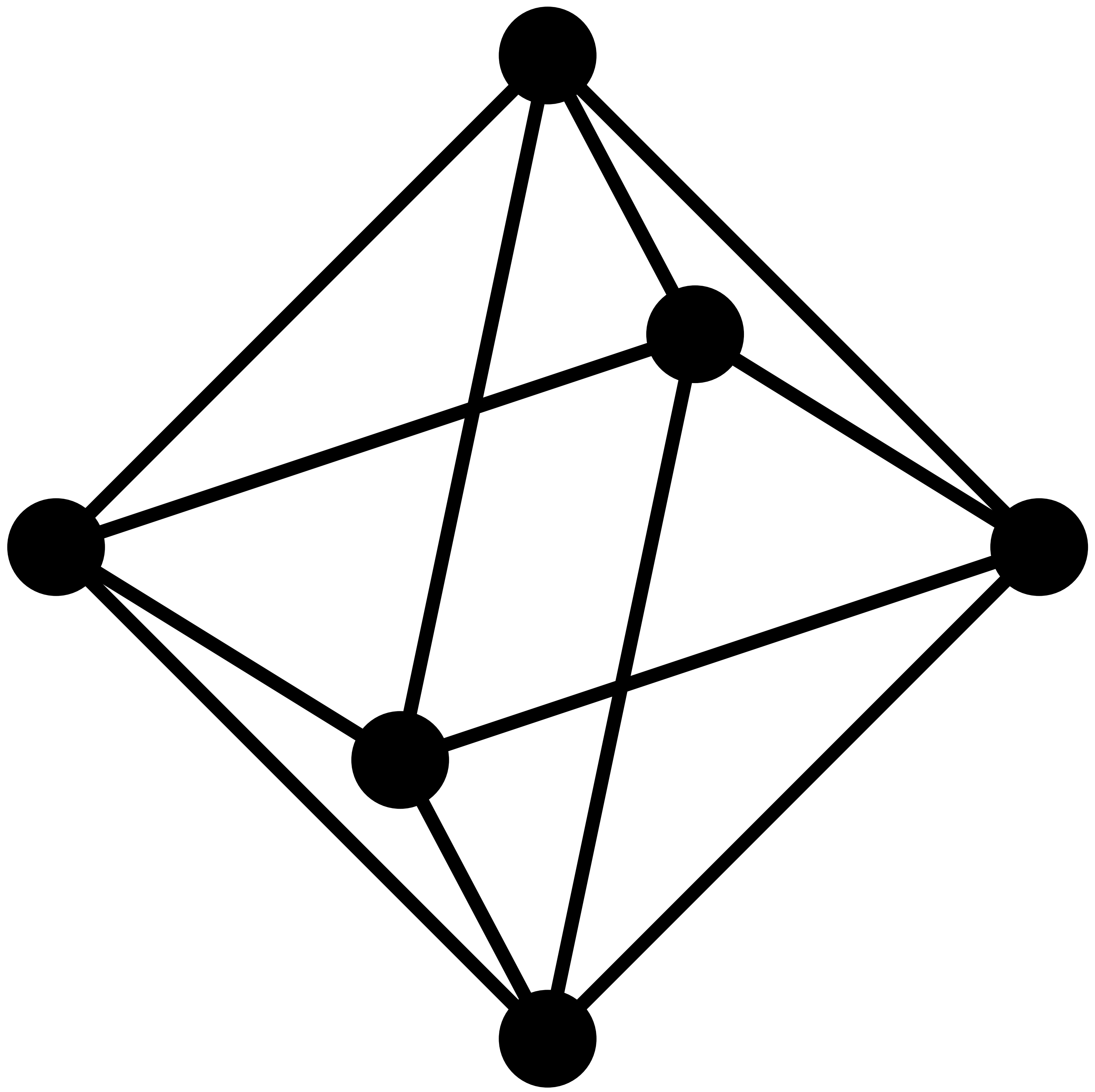
Pass to infinity







Will all of these be filled in at infinity?



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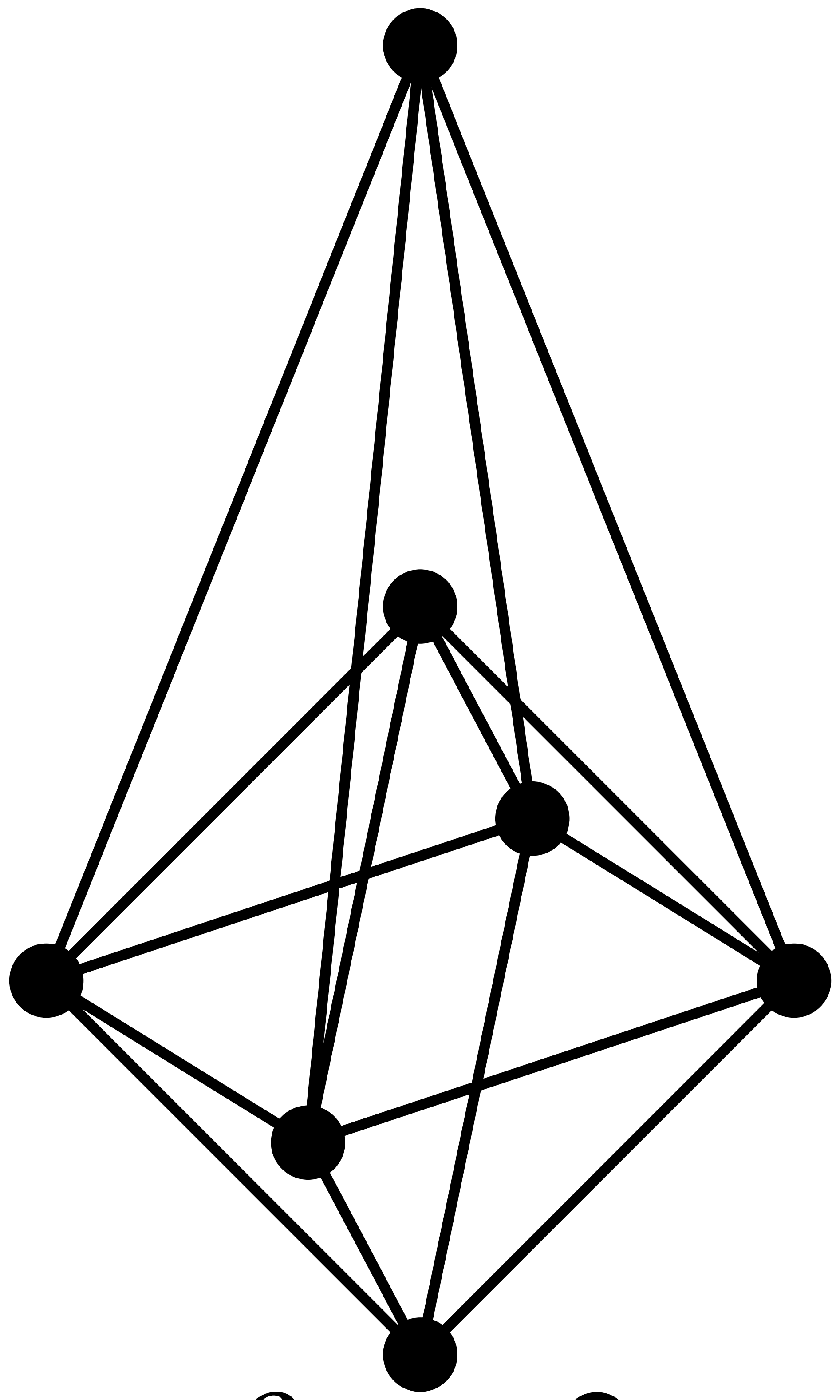

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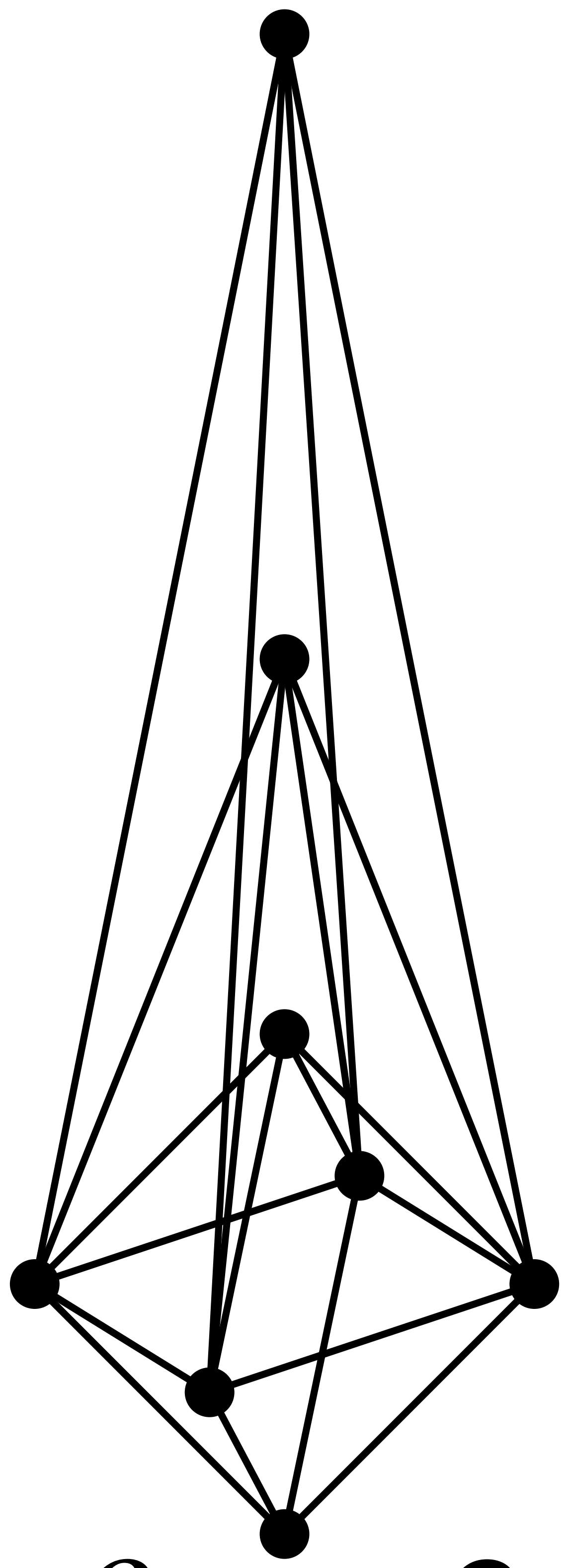

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[Barmak 2023]

- A clique complex is q-homotopy-connected
- if every collection of 2(q + 1) nodes has a common neighbor.

[Barmak 2023]

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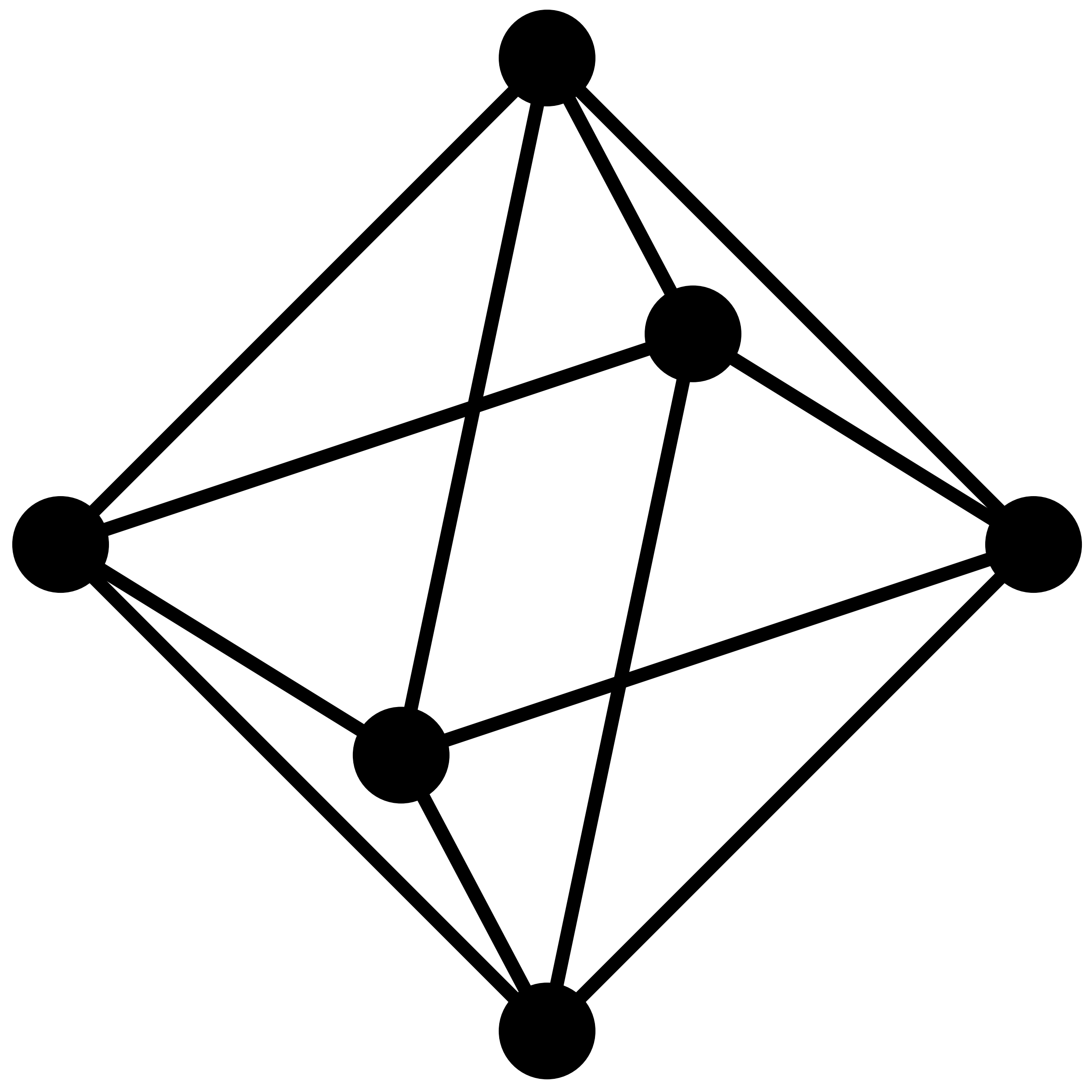

Homotopy-Connected

- Almost surely, the infinite preferential attachment complex
- is q-homotopy-connected if $x \leq \dfrac{1}{2(q+1)}$

Recall:
$x \in (0,1/2)$ decreases with
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$P[T \text{ attaches to } i] \propto T^{-x}$

Homotopy-Connected

- Almost surely, the infinite preferential attachment complex
- is q-homotopy-connected if $x \leq \dfrac{1}{2(q+1)}$
- has infinite Betti number at dimension q if $\dfrac{1}{2(q+1)} < x \leq \dfrac{1}{2q}$

Recall:
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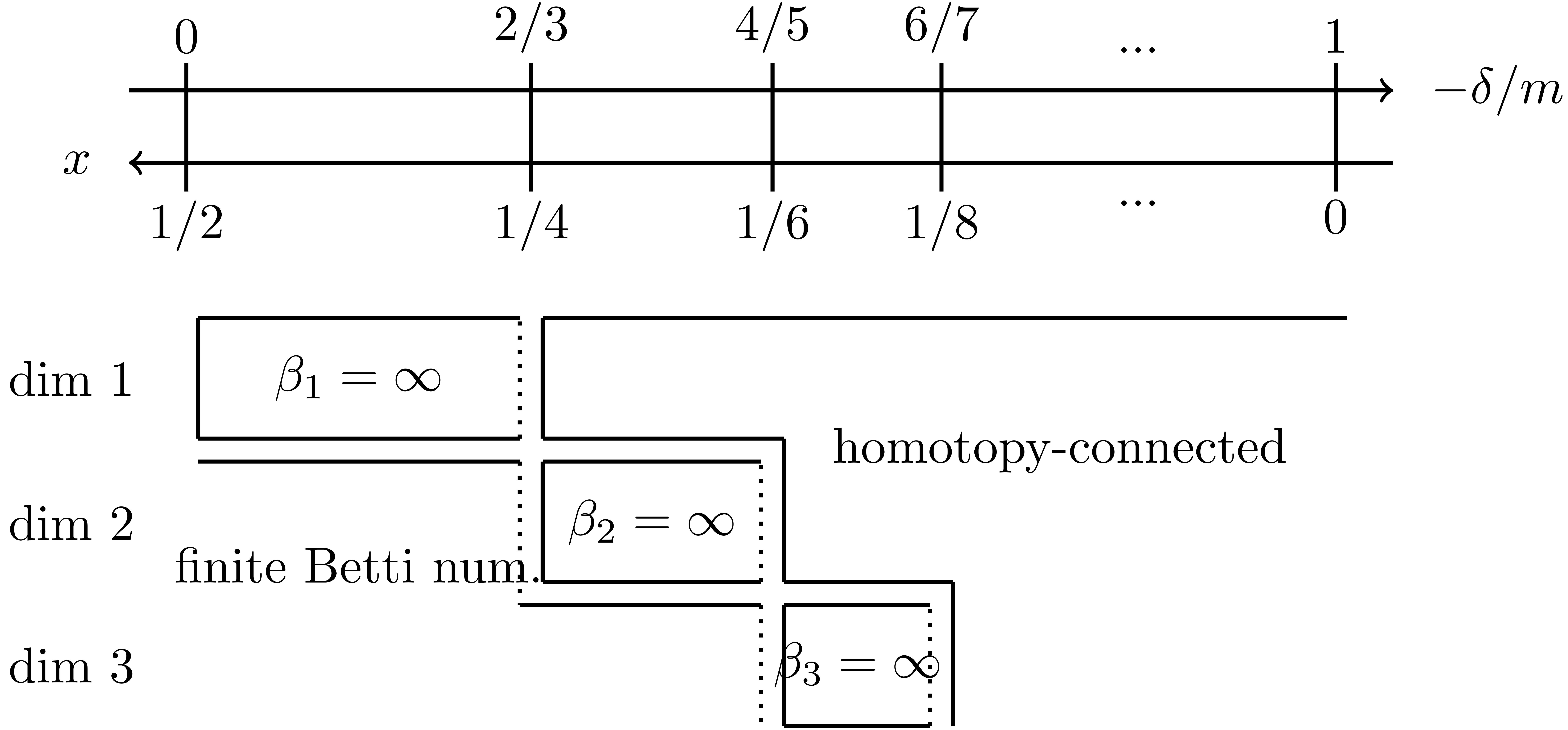
$P[T$ attaches to $i] \propto T^{-x}$

Phase Transition



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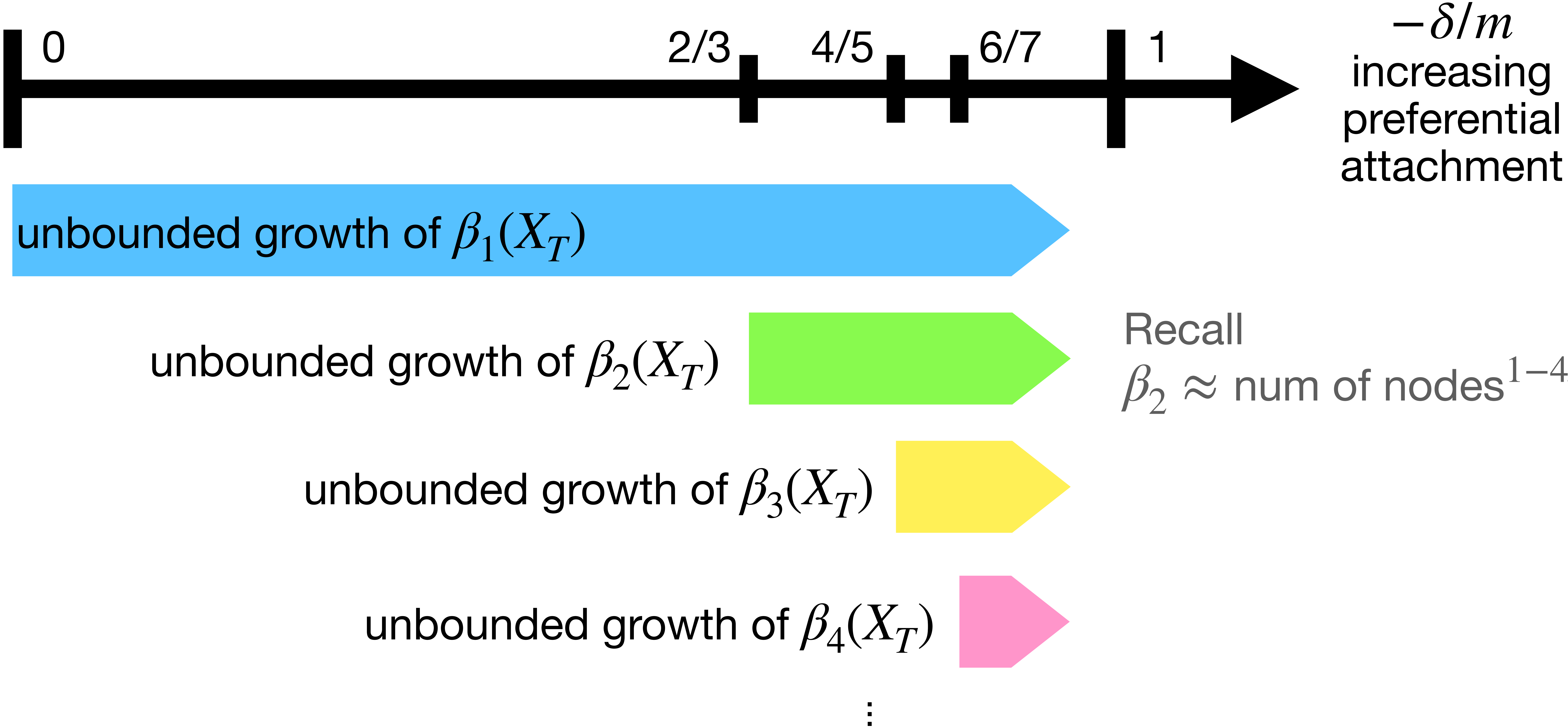


Recall

$\beta_2 \approx$ num of nodes^{1-4x}

- If the preferential attachment effect is strong enough,
- $\beta_q(X_T)$ grows sublinearly with high probability
- $\pi_q(X_\infty) \cong 0$ almost surely

V. What lies ahead

orders of magnitude of
Betti numbers

homotopy connectedness

orders of magnitude of Betti numbers

homotopy connectedness

parameter estimation?

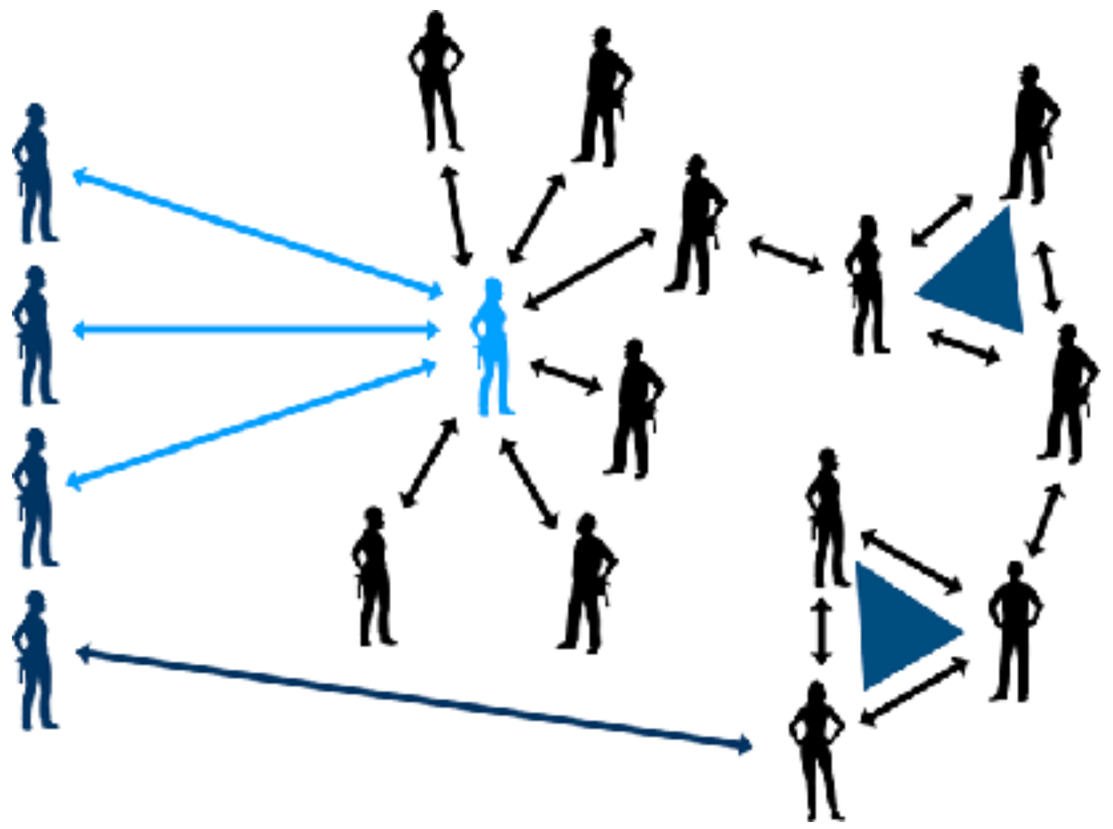
simplicial preferential attachment?

other non-homogeneous complexes?

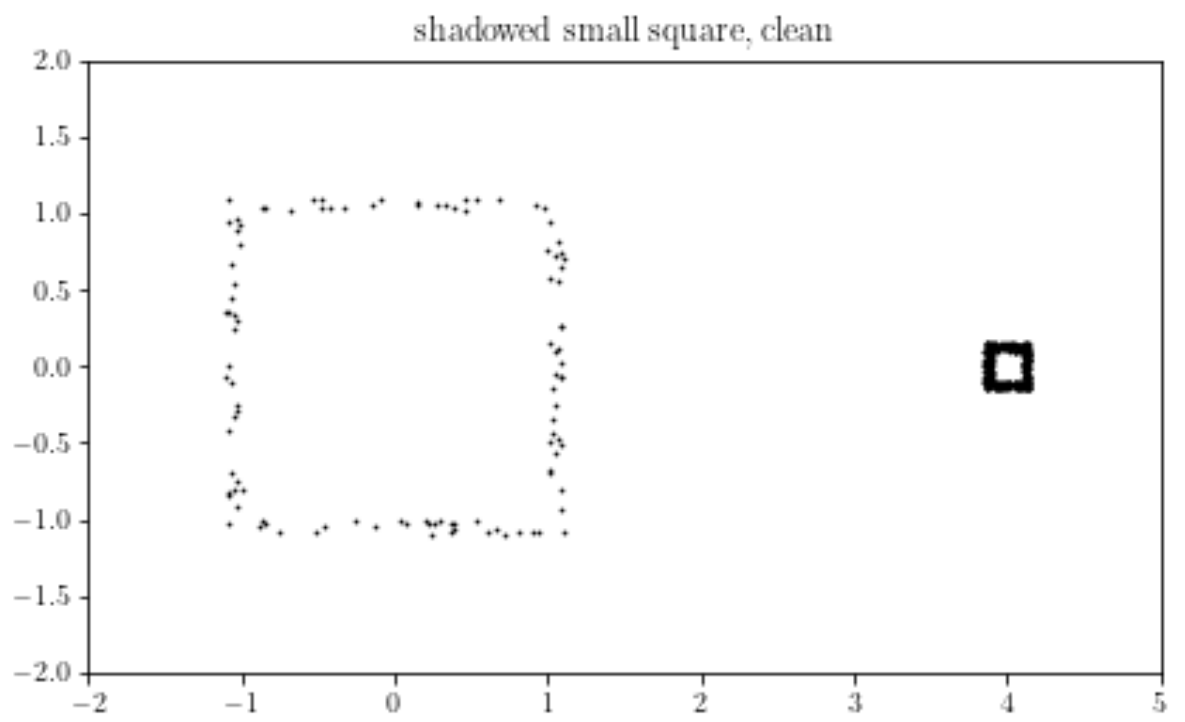
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

Chunyin Siu **cs2323@cornell.edu**
Cornell University



arxiv paper



my video about small holes

Thank you!

Chunyin Siu cs2323@cornell.edu
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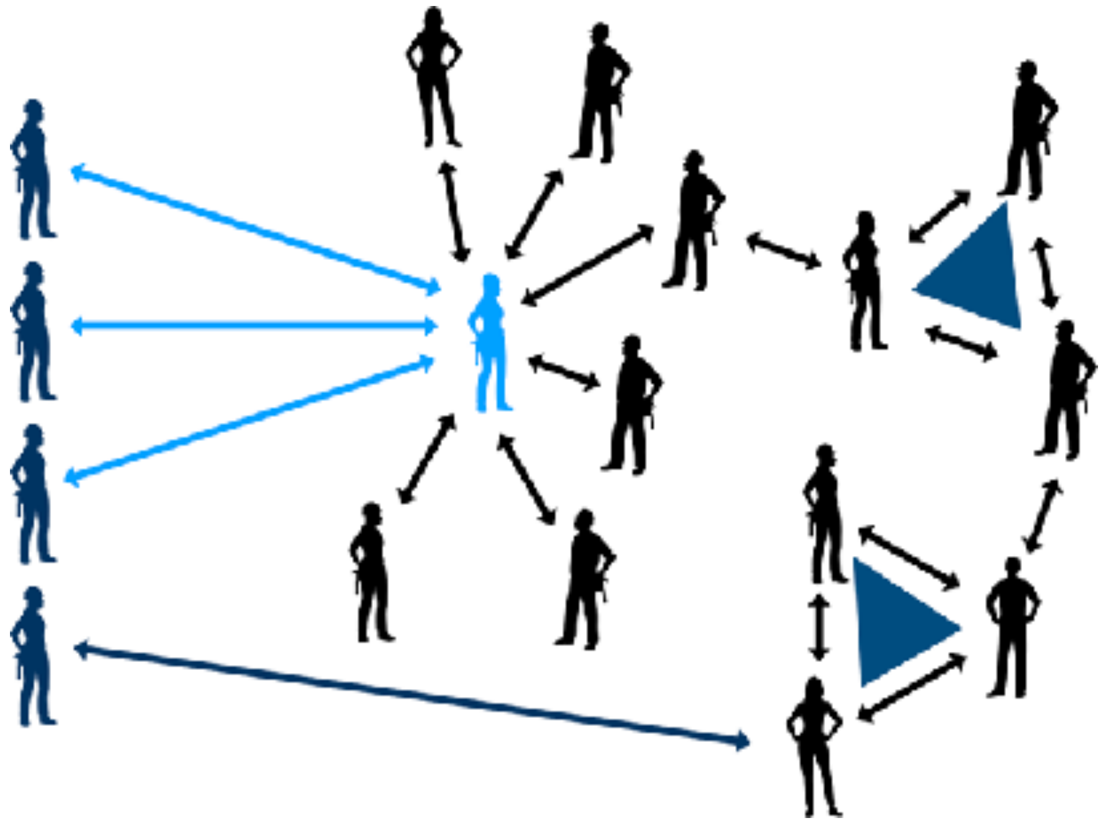

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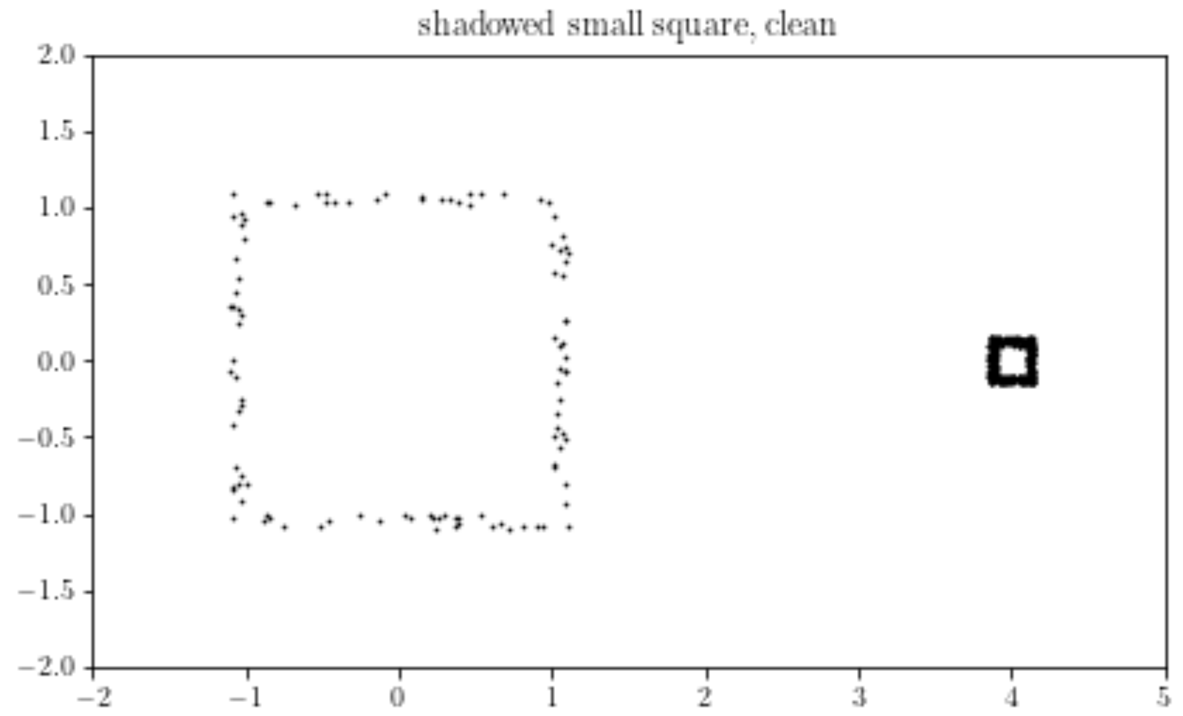

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What we know
[not meant to be complete]

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  - etc

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  - Yogeshwaran and Adler 2015
  - Bobrowski et al 2017
  - Hiraoka et al 2018
  - Thomas and Owada 2021a, b
  - Owada and Wei 2022
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Functional Convergence at dimension k?

[Thomas and Owada 2020]

n = number of nodes
all log terms and constants forgone

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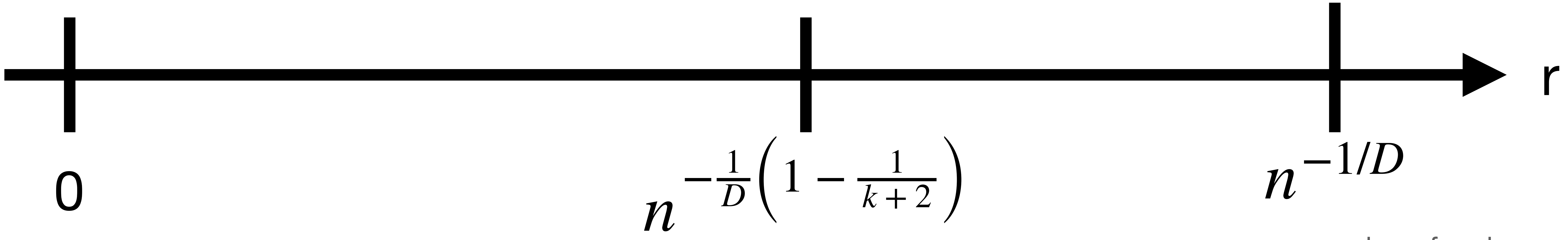
- Cech: weak convergence in finite-dimensional sense



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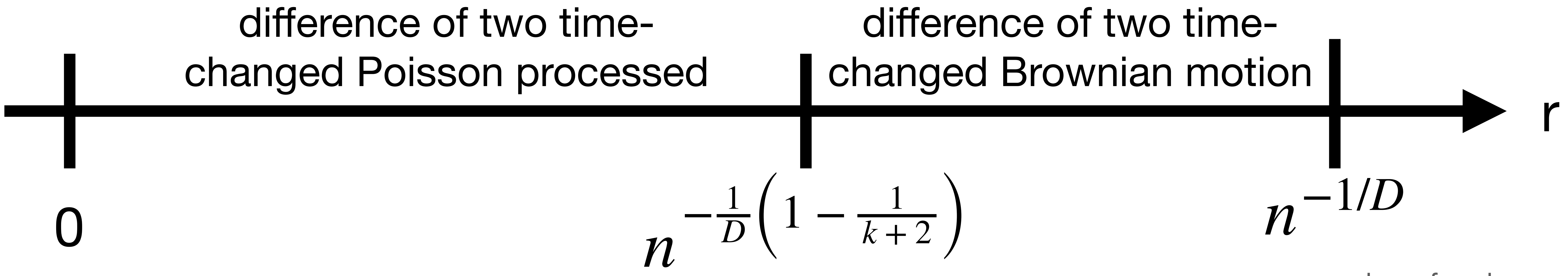

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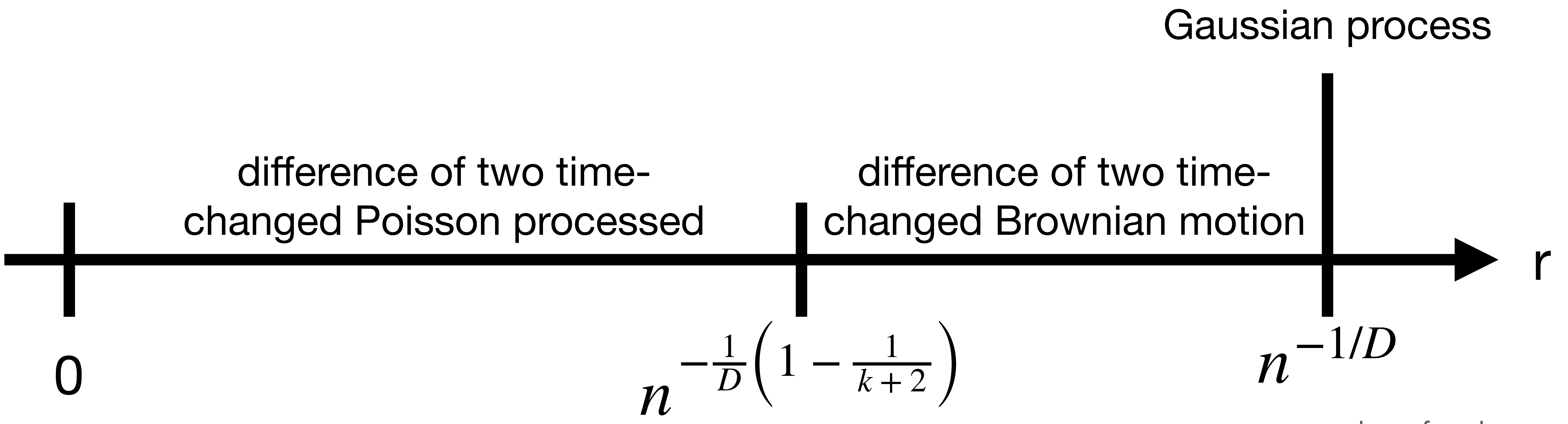

n = number of nodes
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- 4 CPU cores
- 40 minutes for the Betti numbers
- 7.5 hours for bounds
- memory issues for larger graphs

Subtleties

- Need homological algebra to relate Betti numbers with counts

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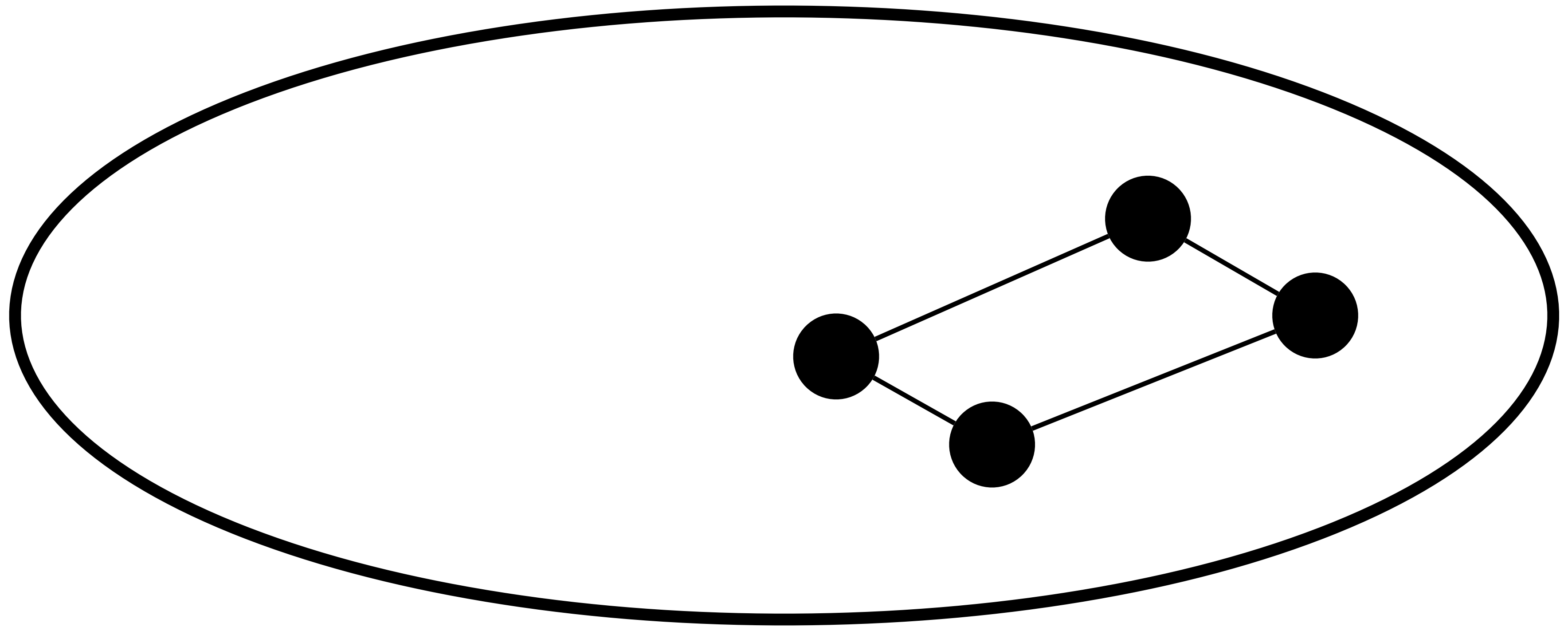

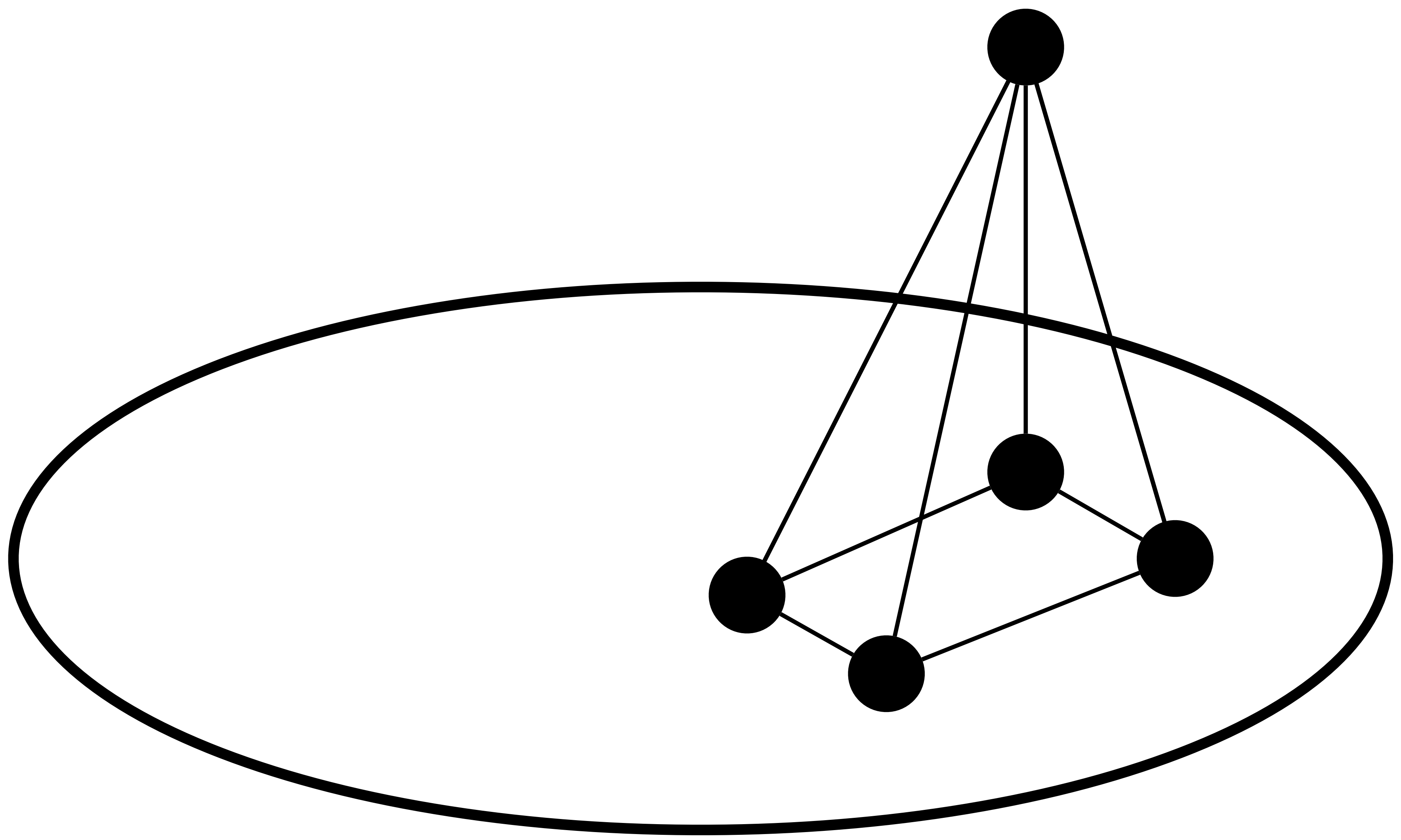

Subtleties

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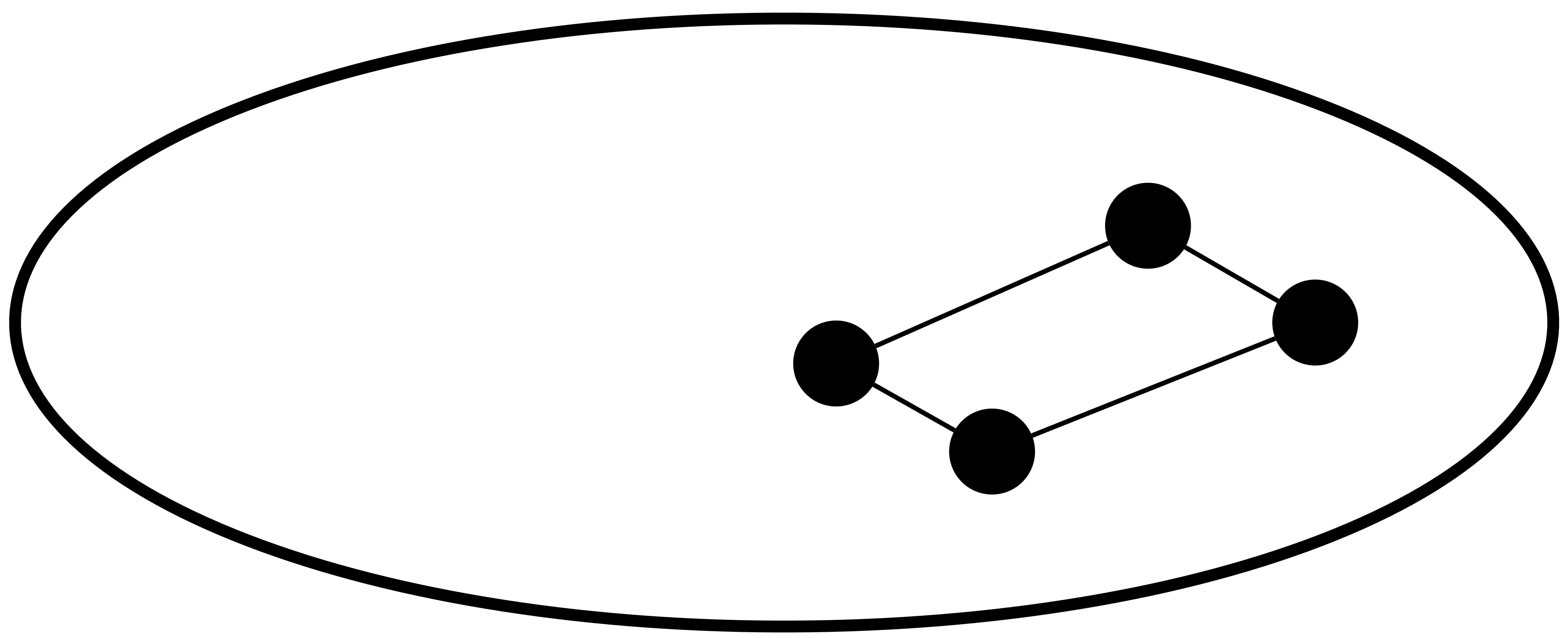
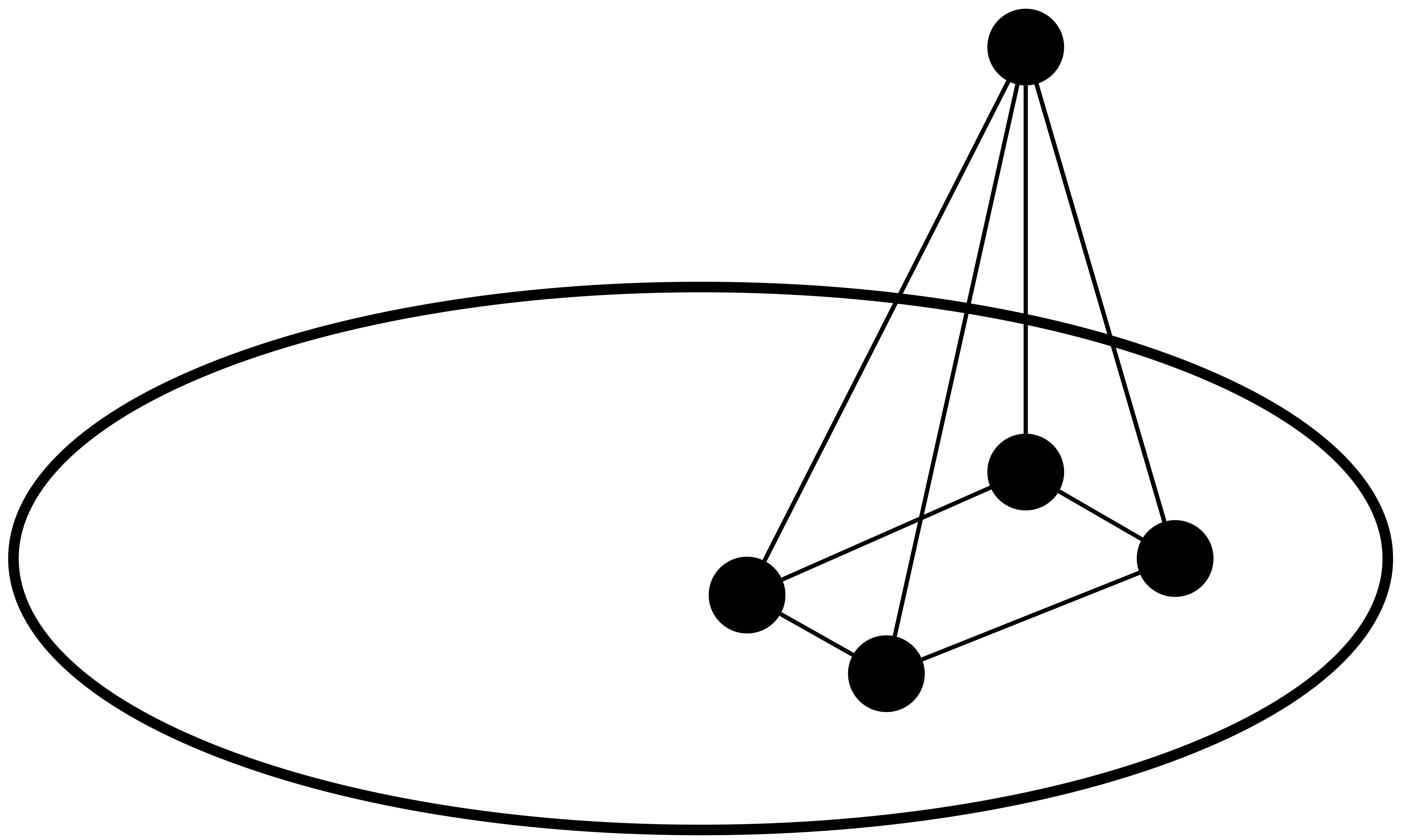
  - adding a vertex = construct mapping cone
  - $\beta_q(\text{new}) \leq \beta_q(\text{old}) + \beta_{q-1}(\text{link})$





Subtleties

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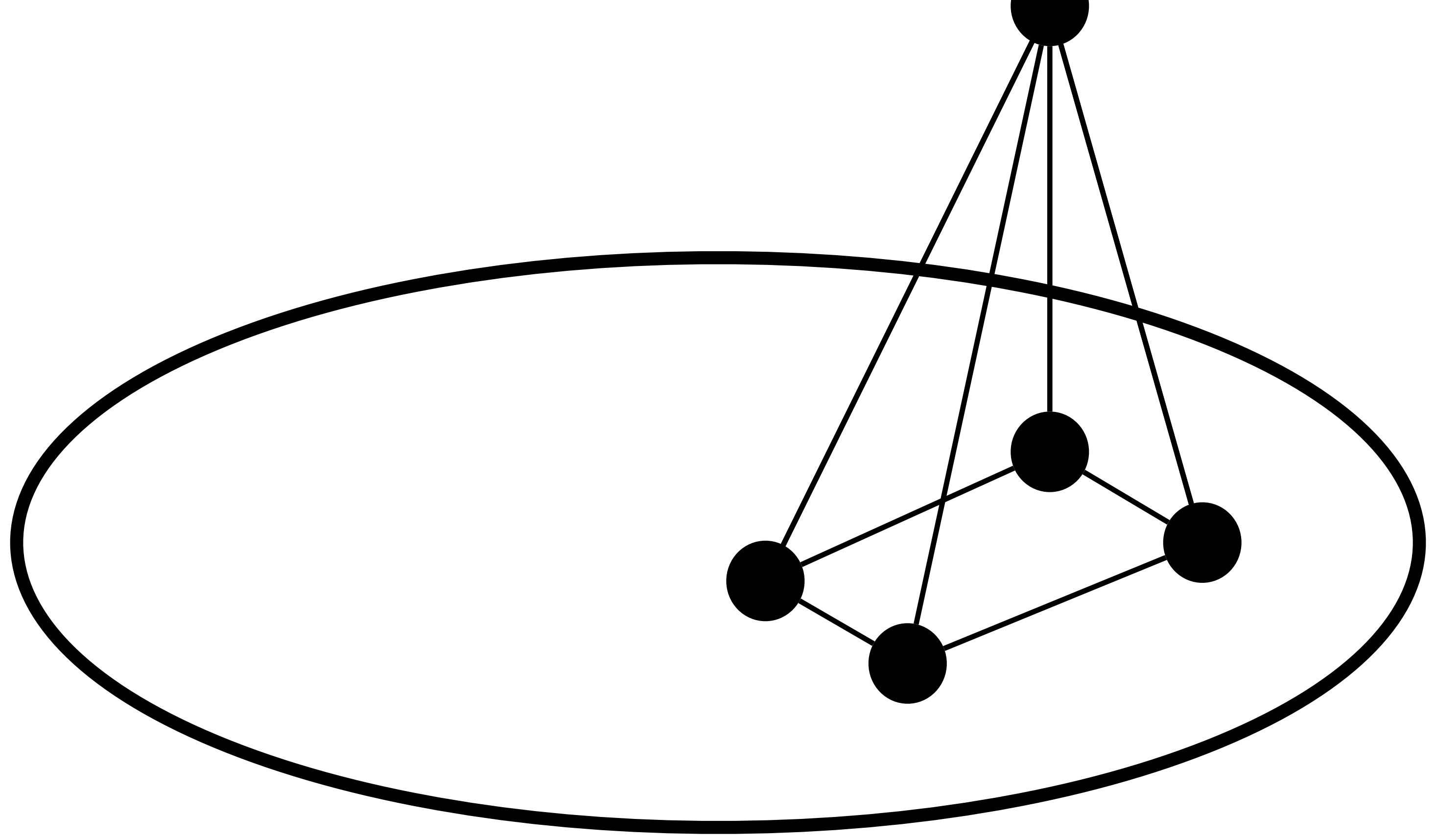

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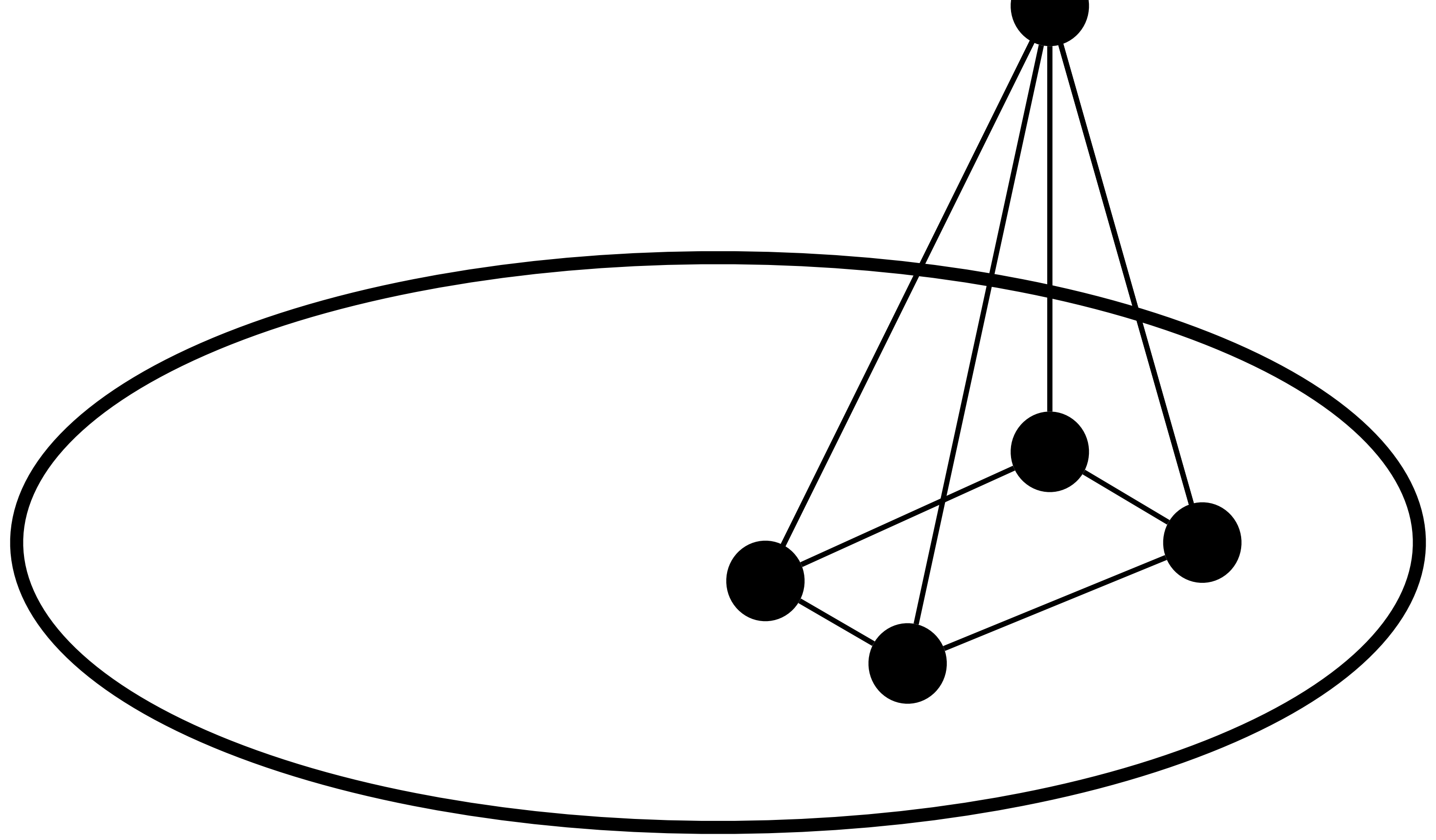

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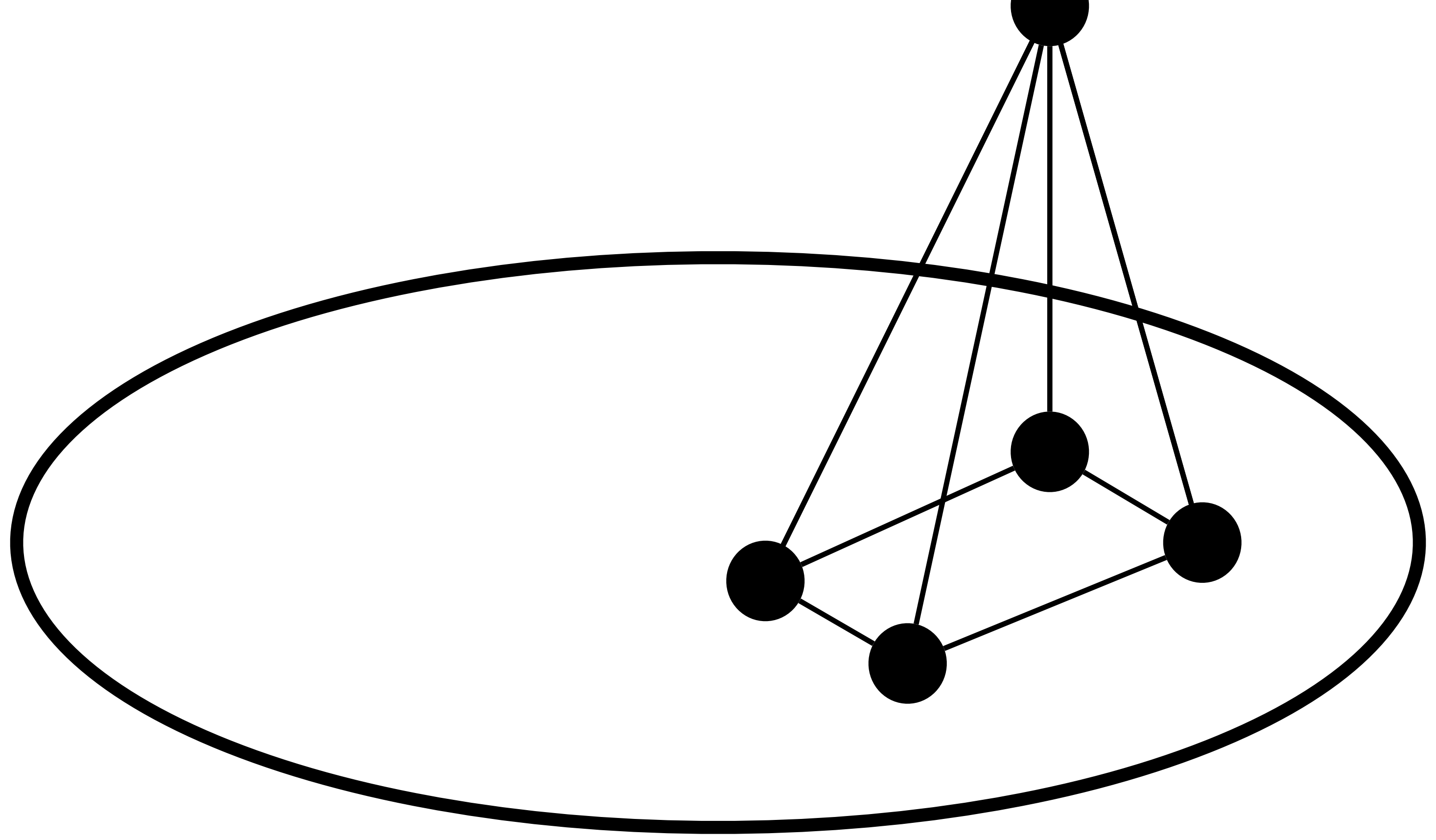

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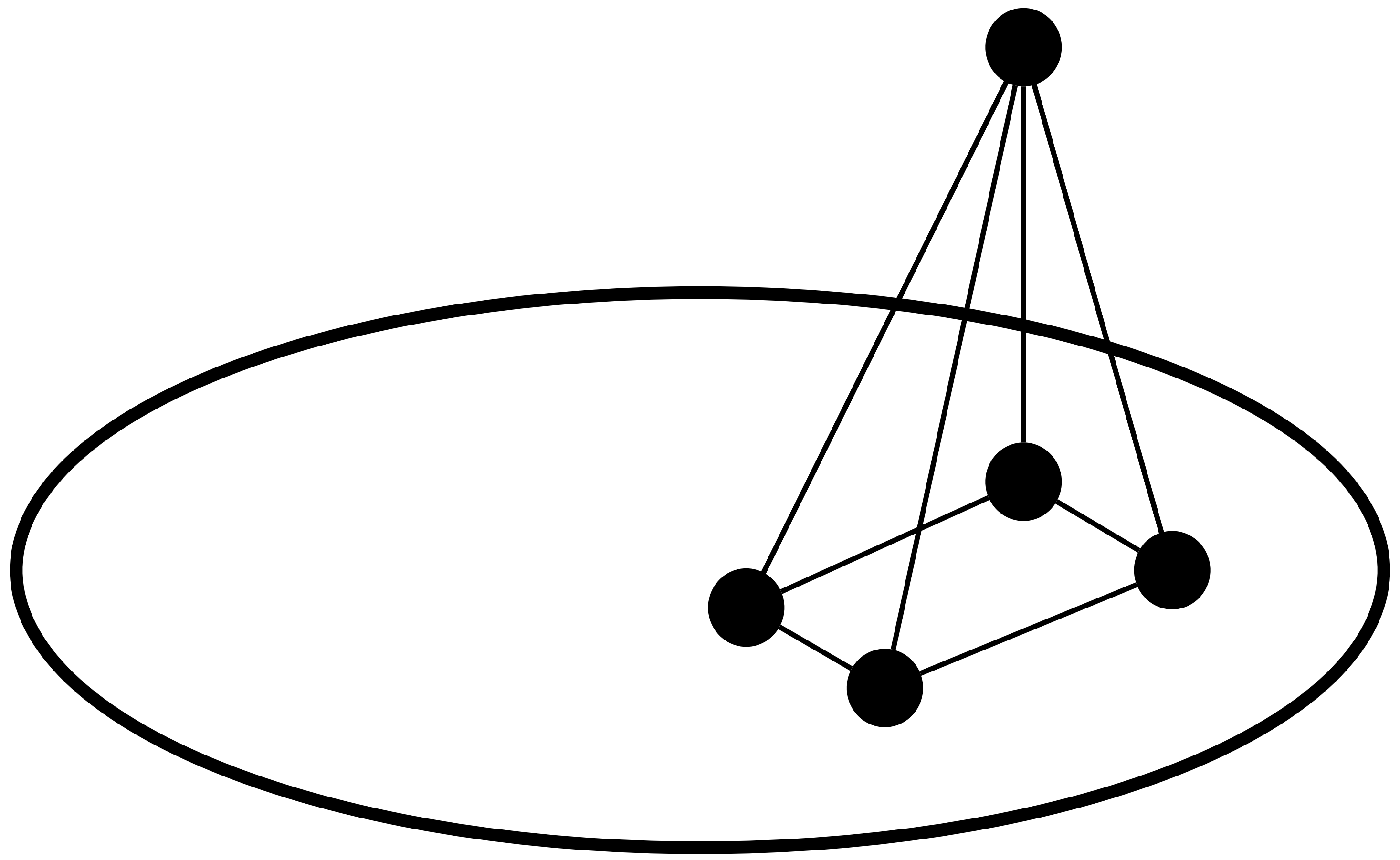
  - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$



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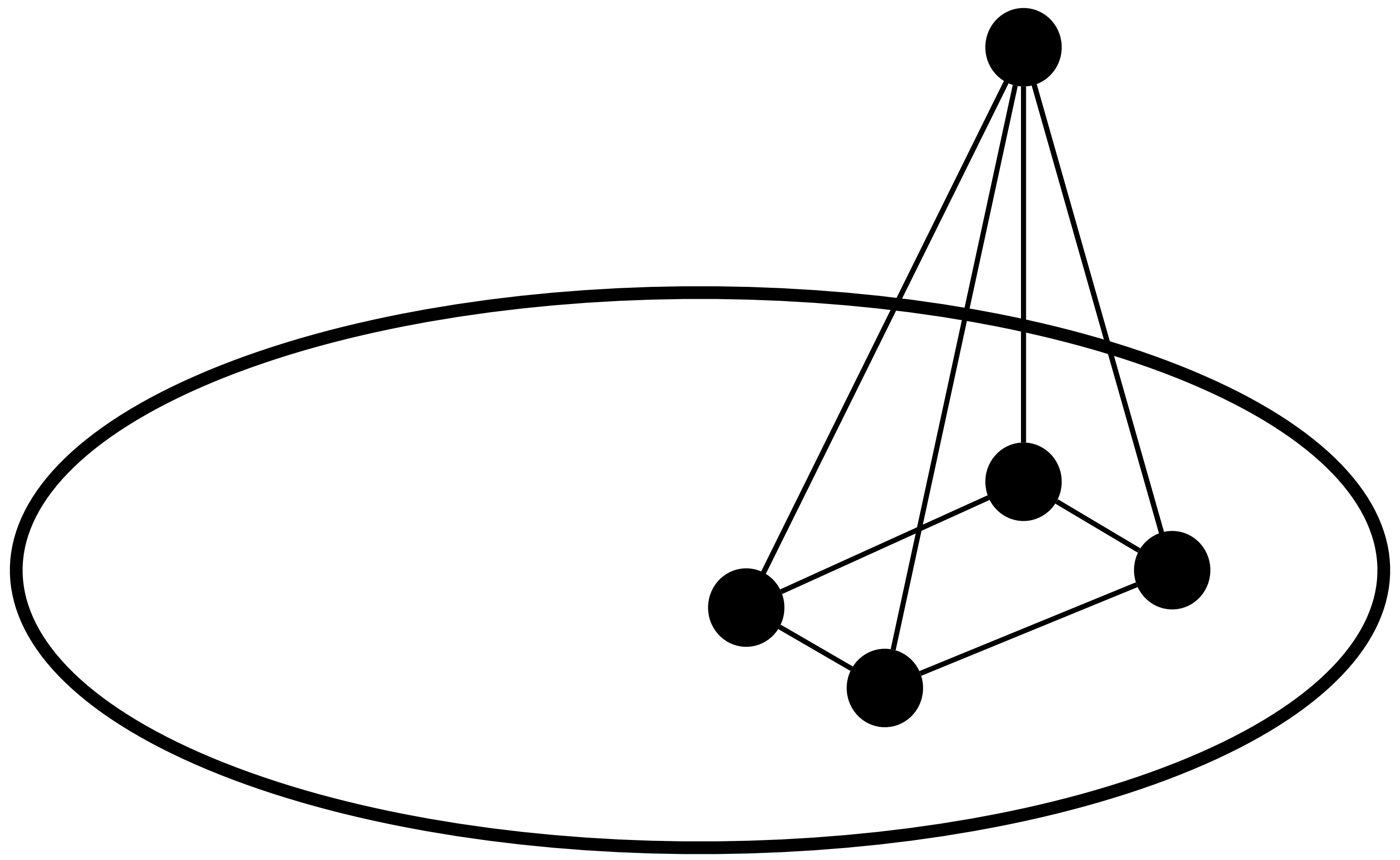
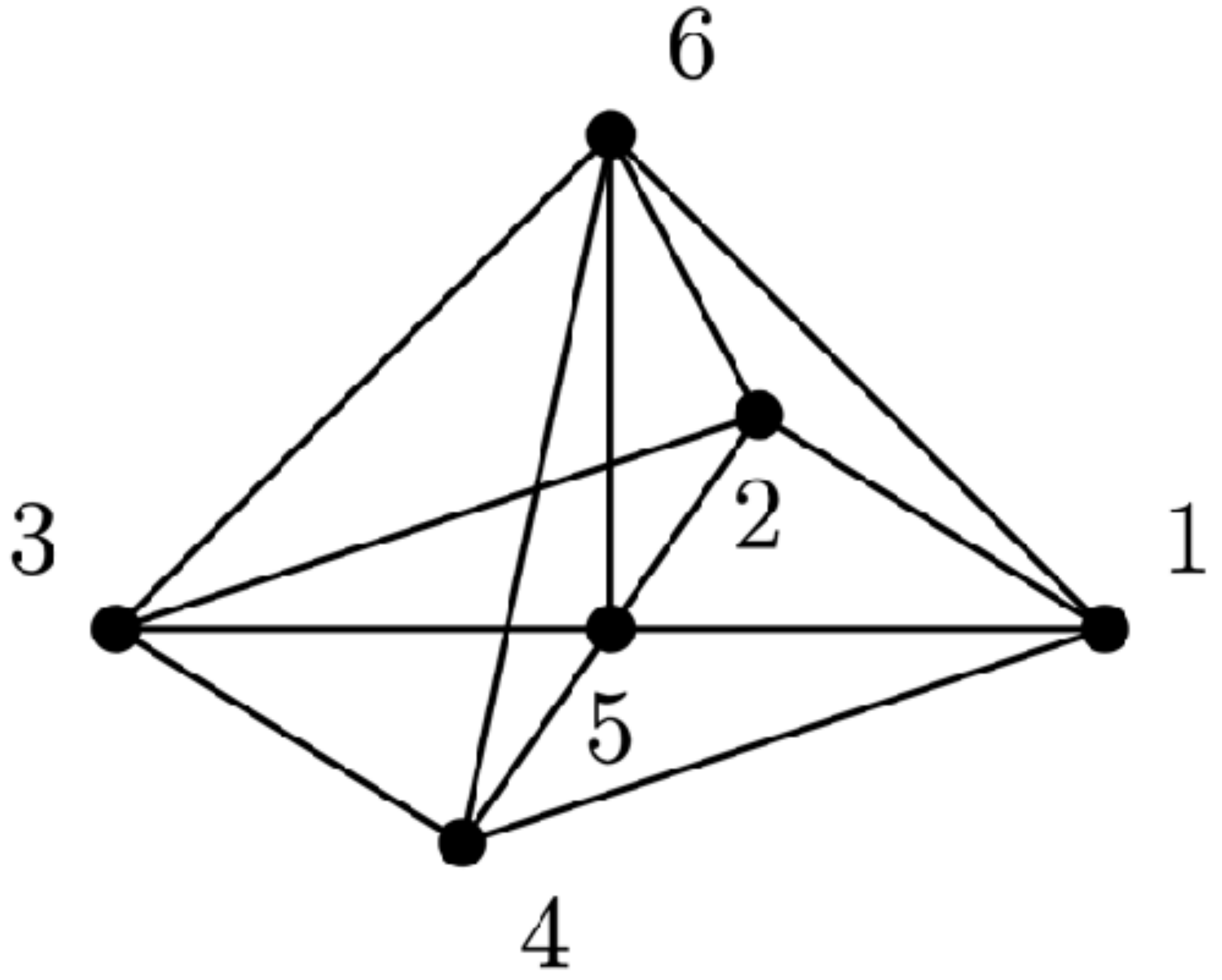
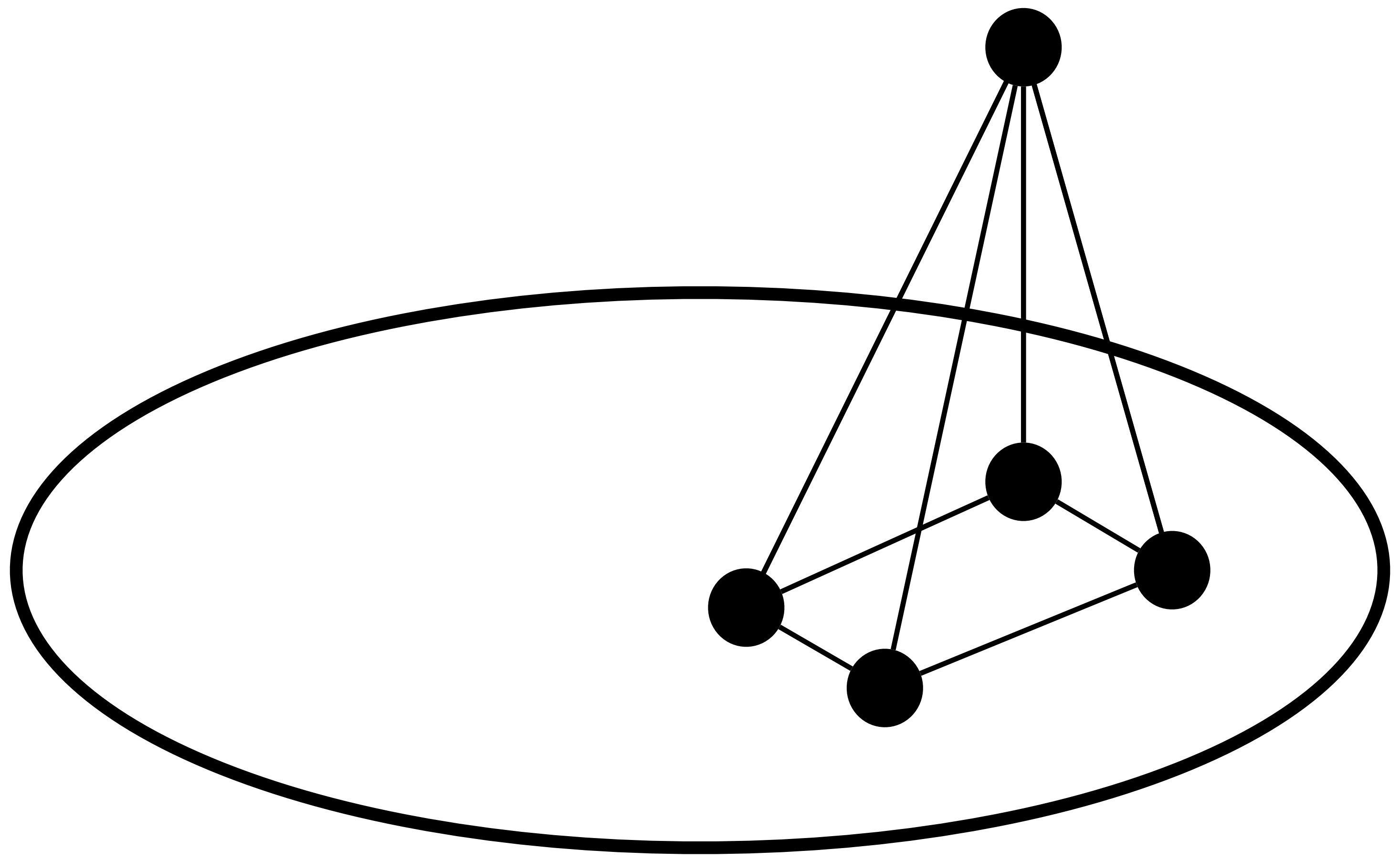
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra
  - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$

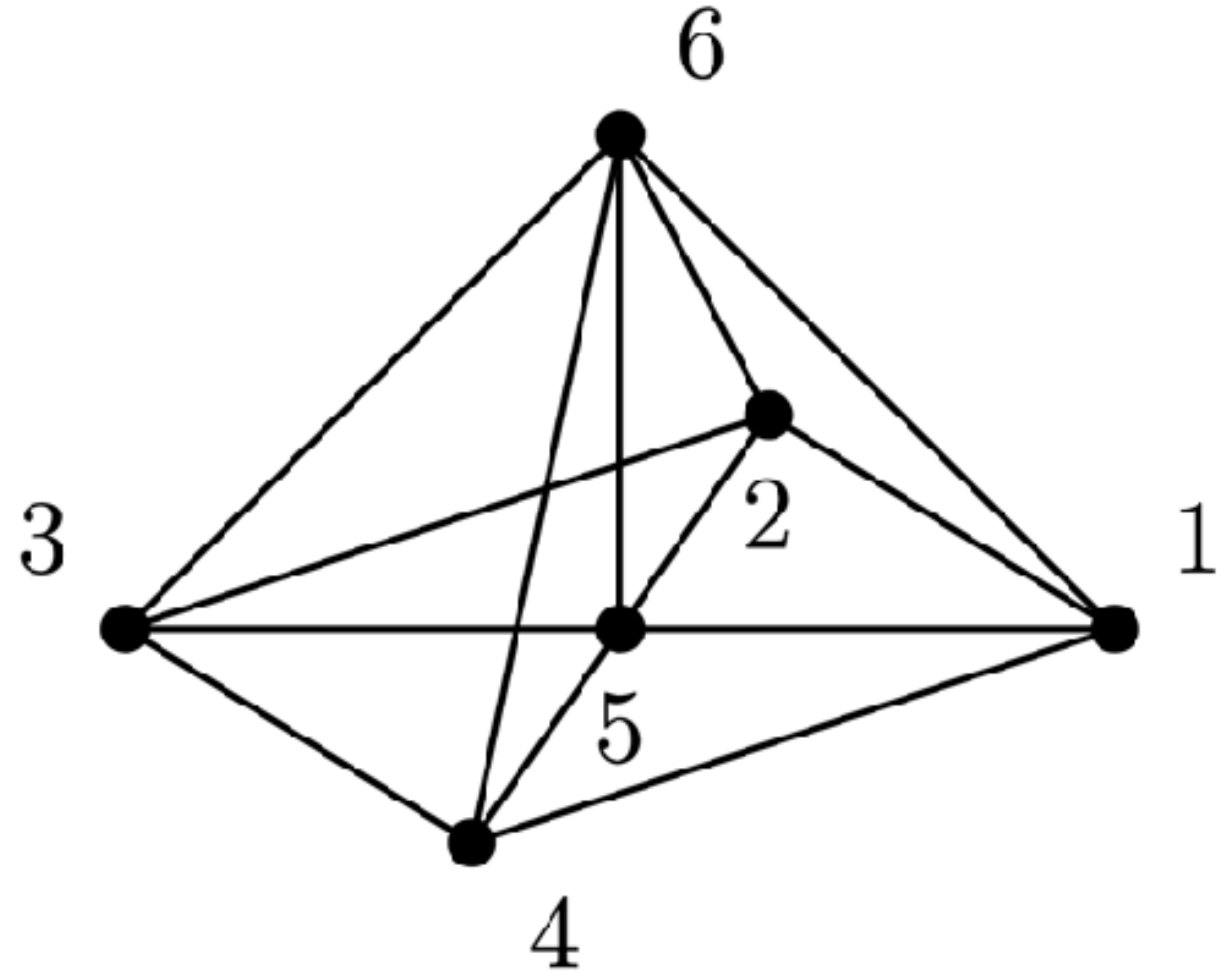


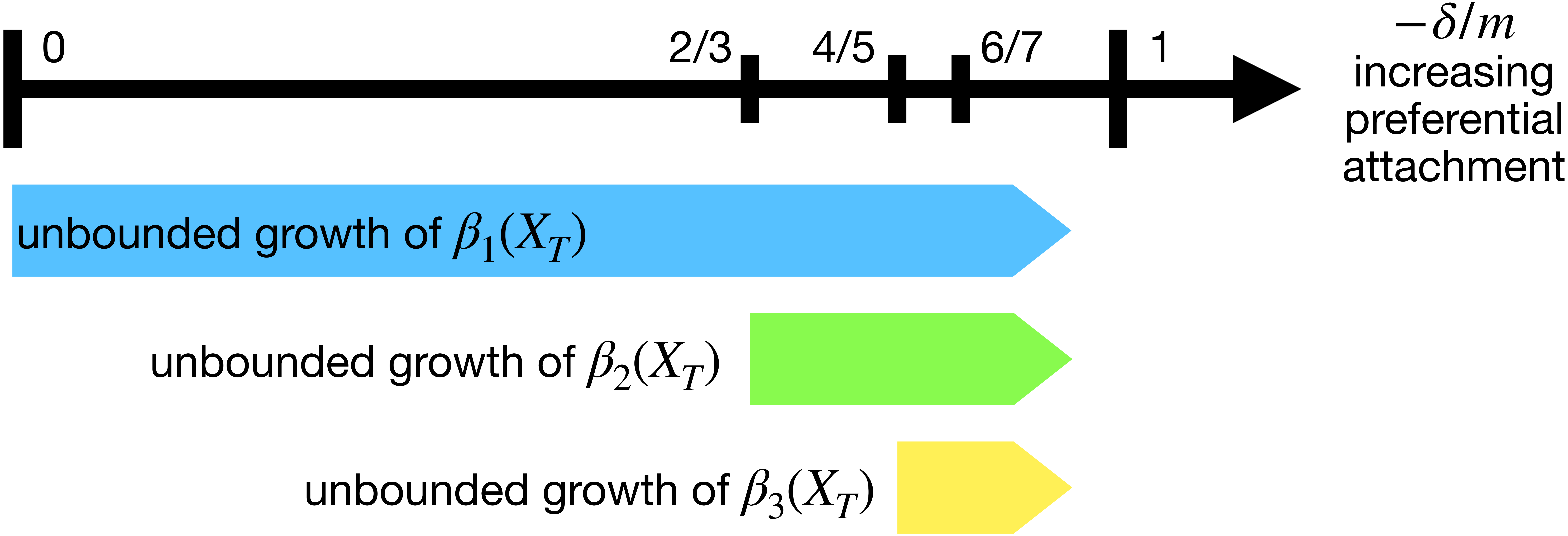


Subtleties

- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra
  - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs






0
2/3
4/5
6/7
1
$-\delta/m$
increasing
preferential
attachment
unbounded growth of $\beta_1(X_T)$
unbounded growth of $\beta_2(X_T)$
unbounded growth of $\beta_3(X_T)$