The Topology of Preferential Attachment

The Asymptotics of the Expected Betti Numbers of Preferential Attachment Clique Complexes

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So, preferential attachment…

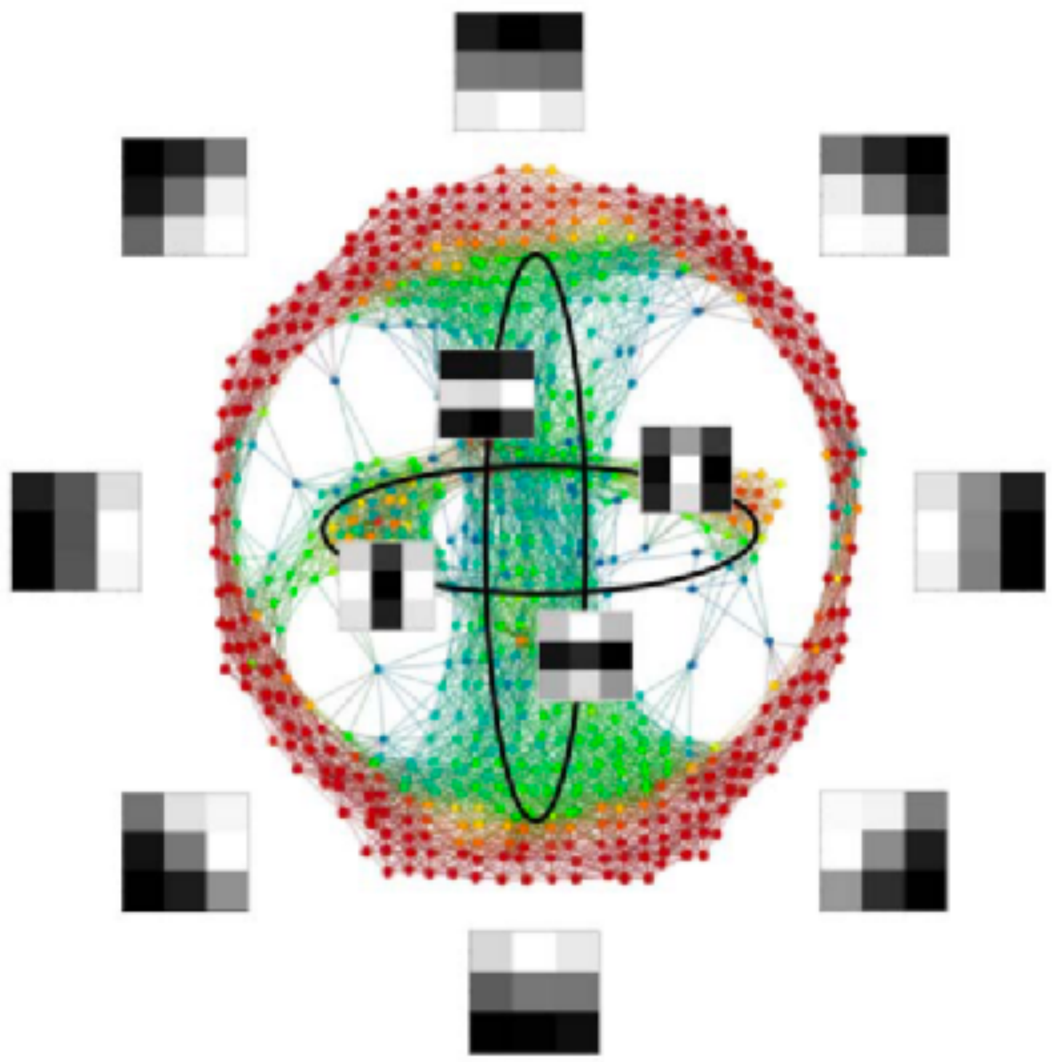
(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Just a bouquet of circles?

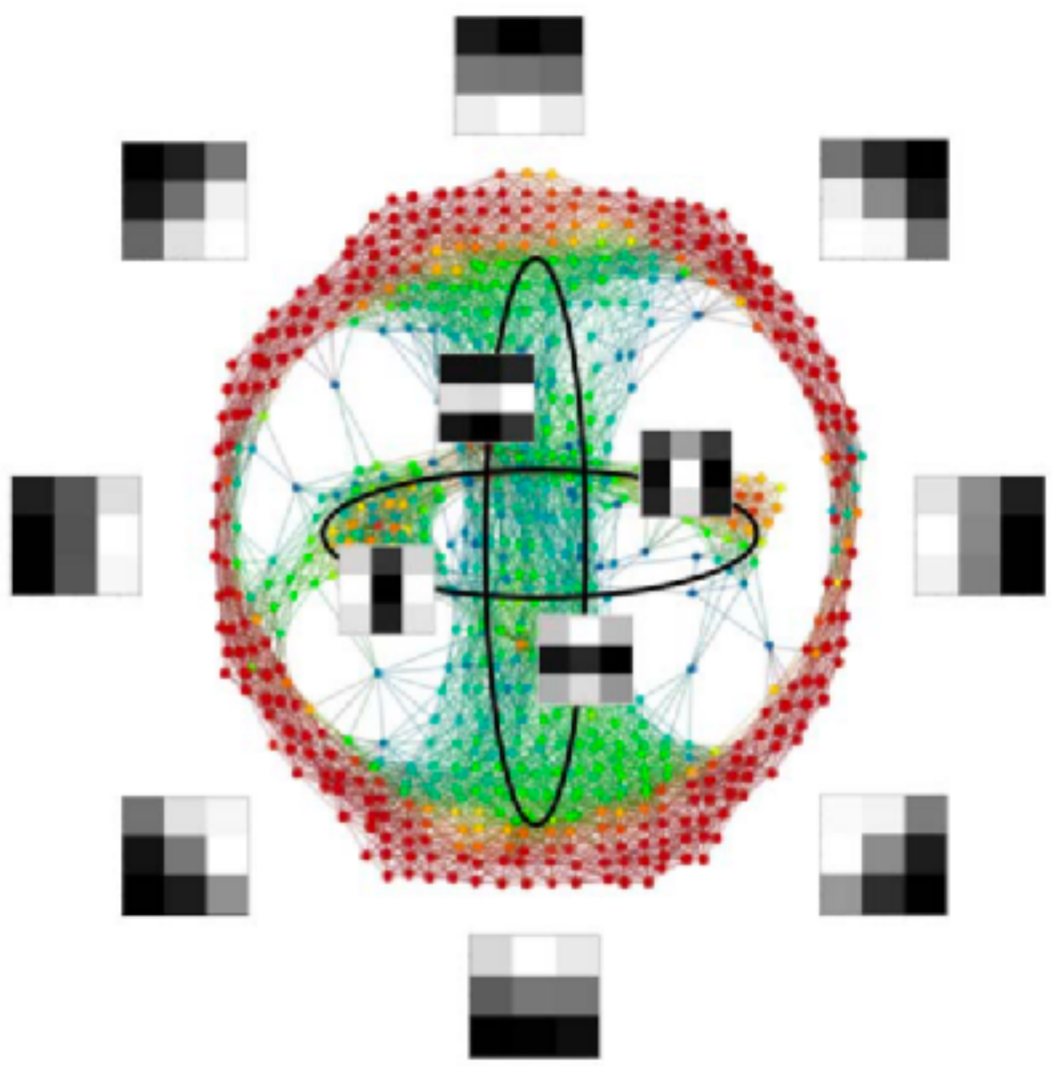
(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Agenda



topological data analysis

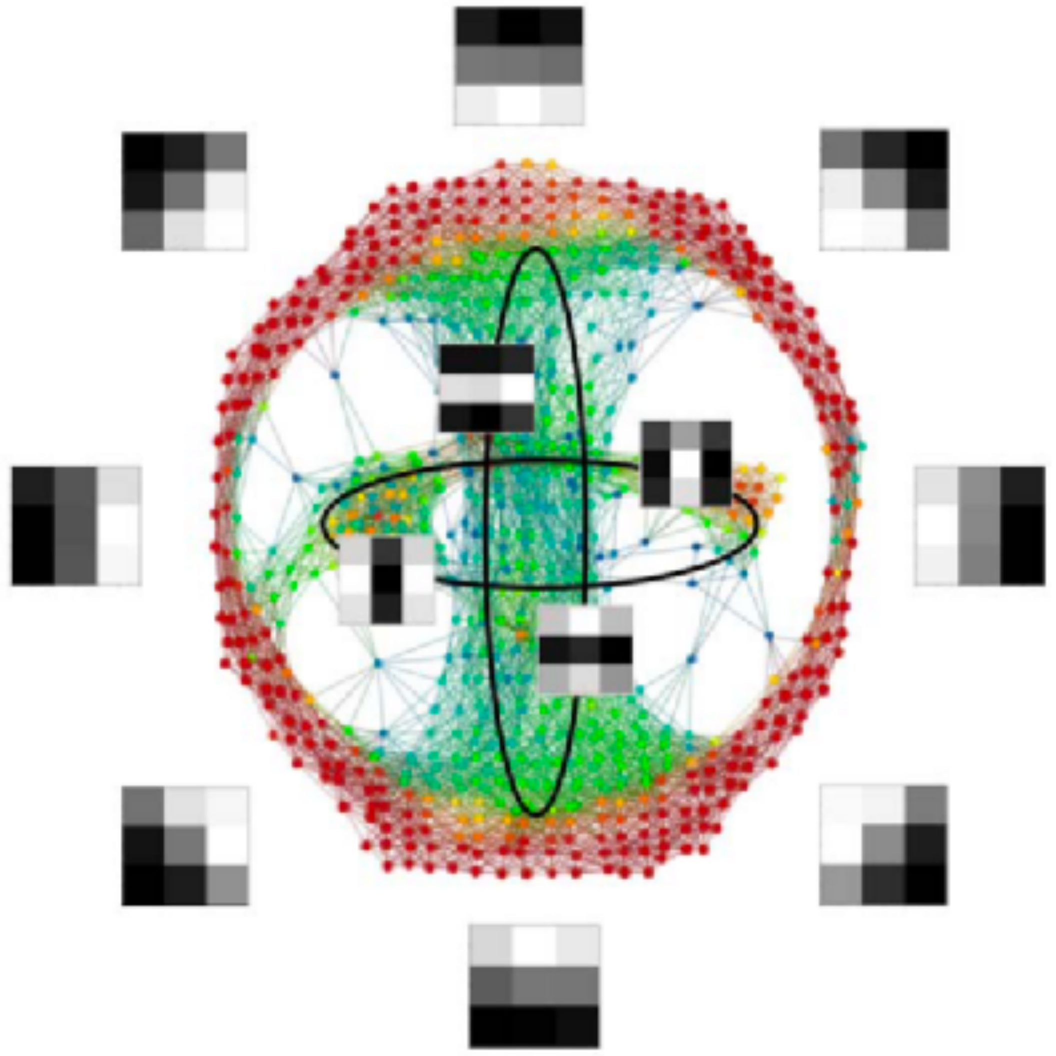
Agenda



topological data analysis

stochastic topology

Agenda



topological data analysis

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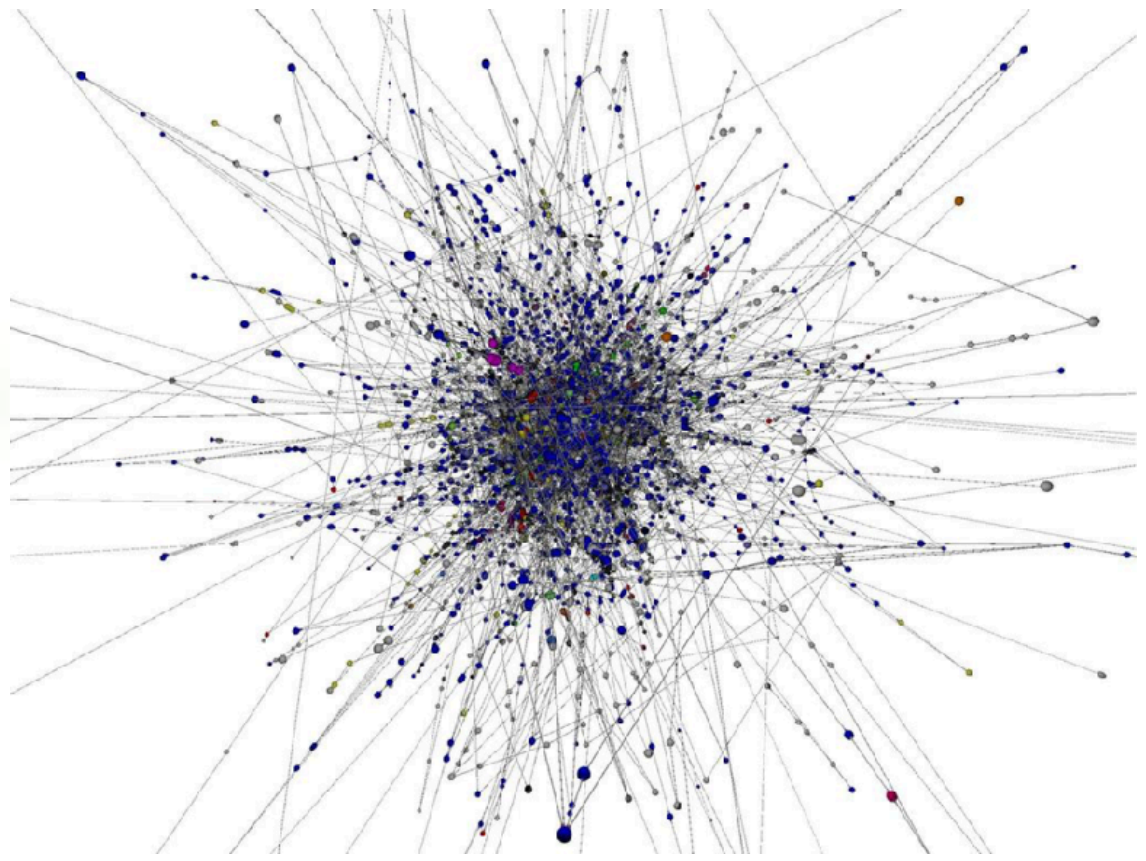
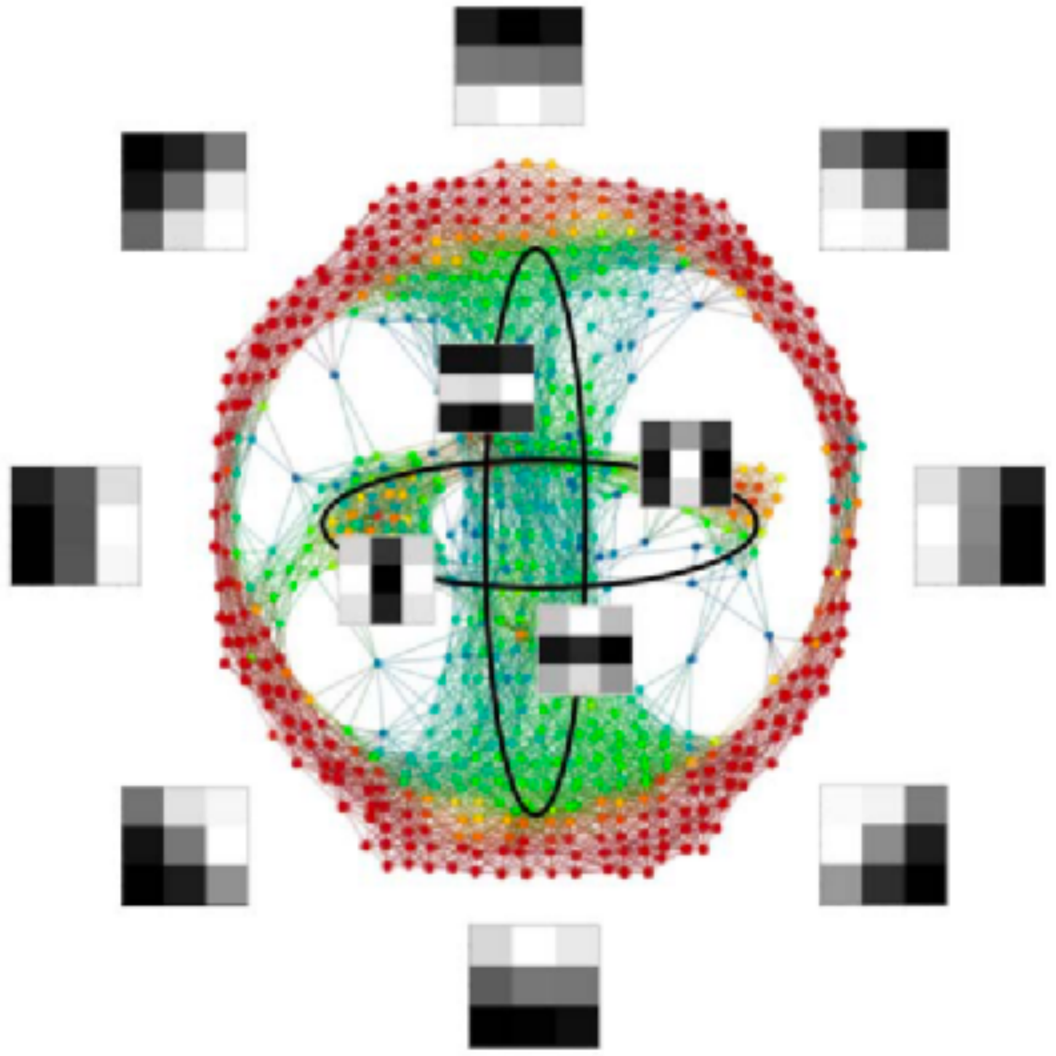


preferential attachment

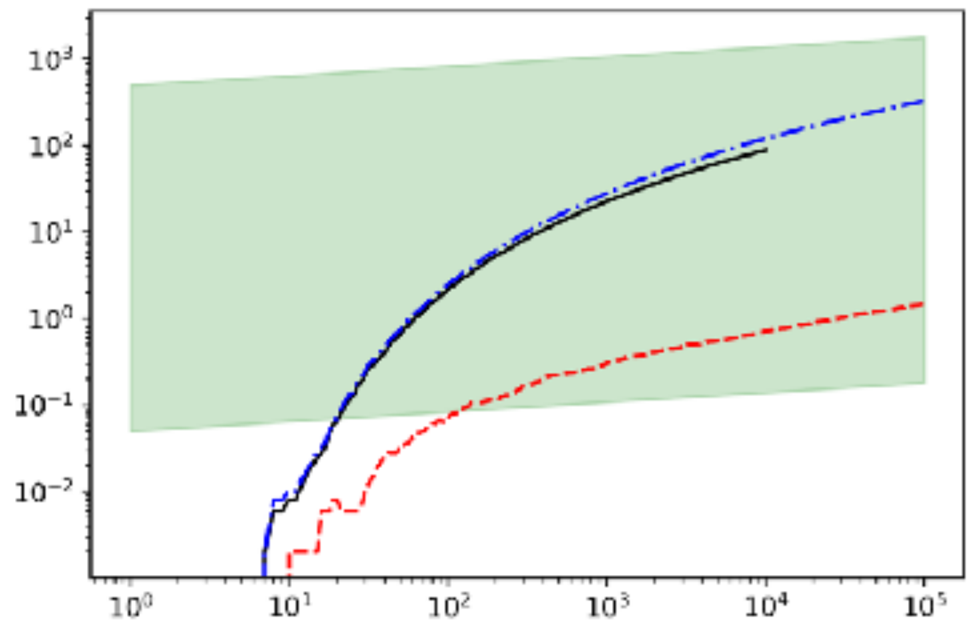
Agenda



topological data analysis

stochastic topology

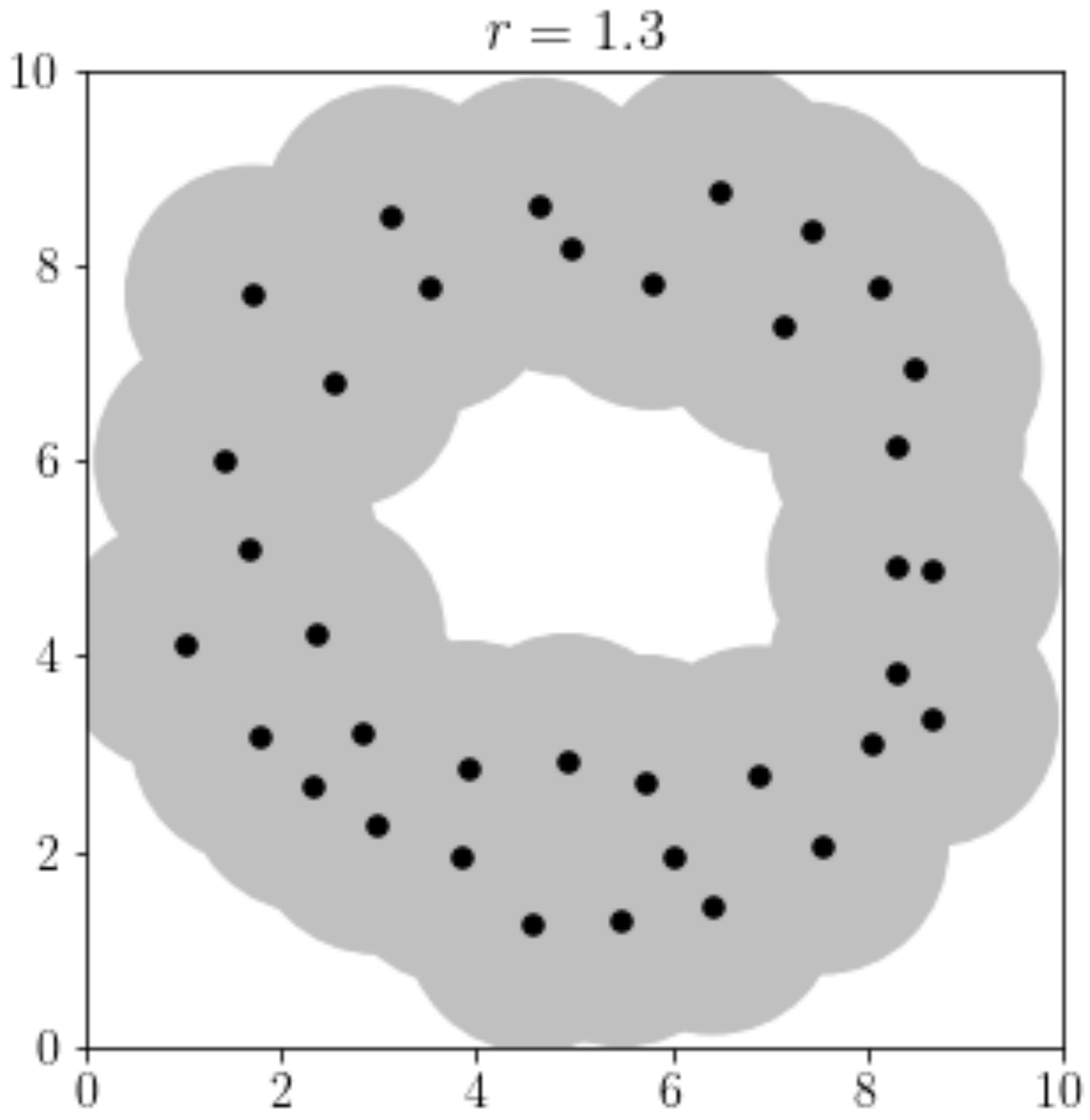
preferential attachment



our result

I. Topological Data Analysis

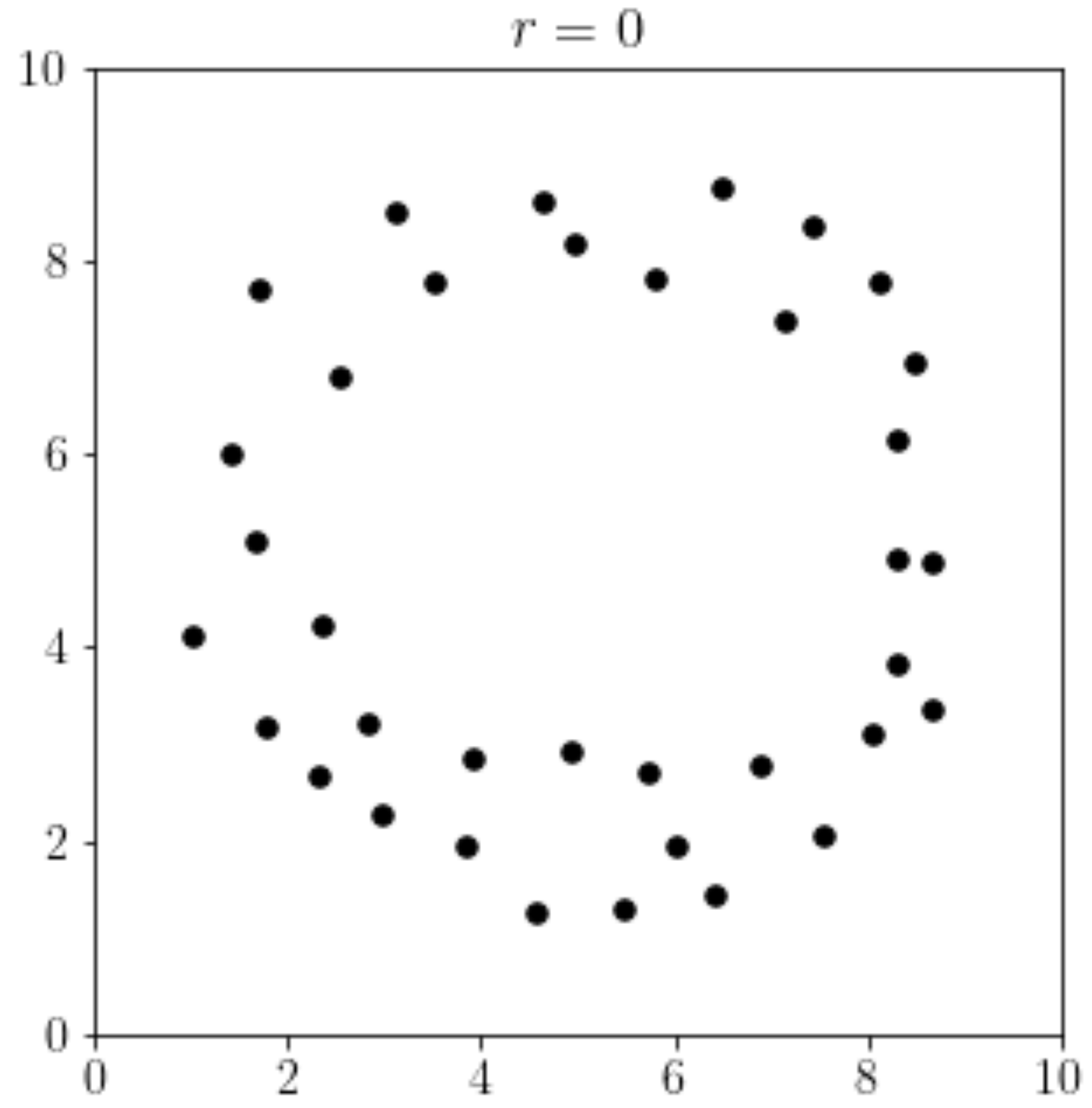
Two Approaches



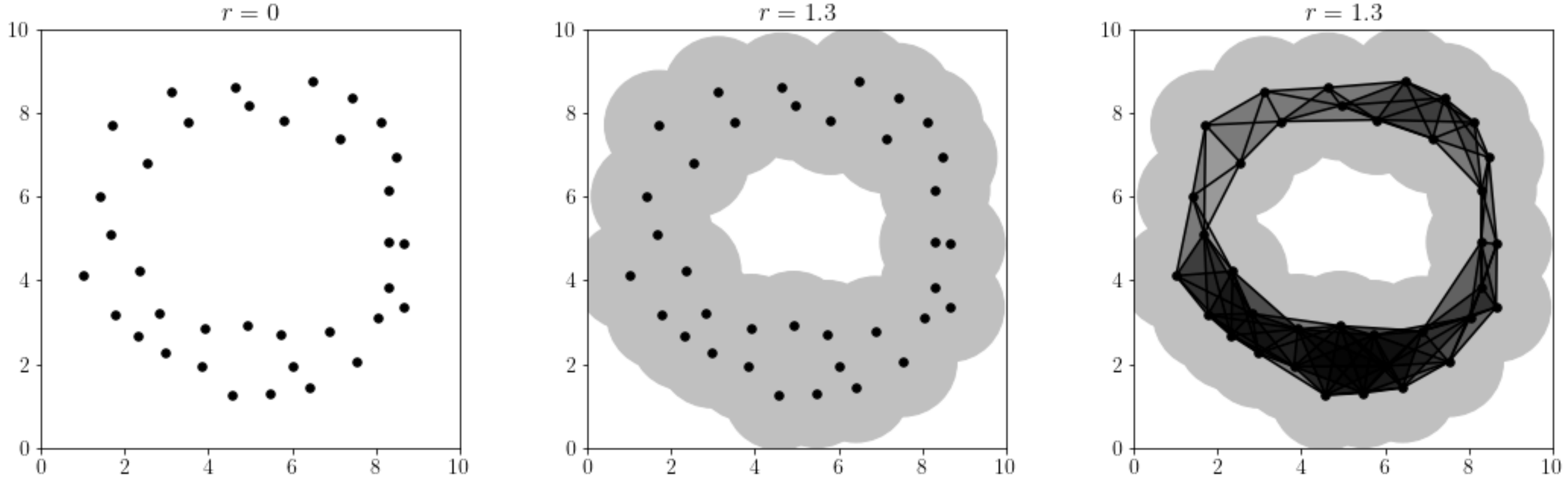
points

networks and complexes

Points



Points



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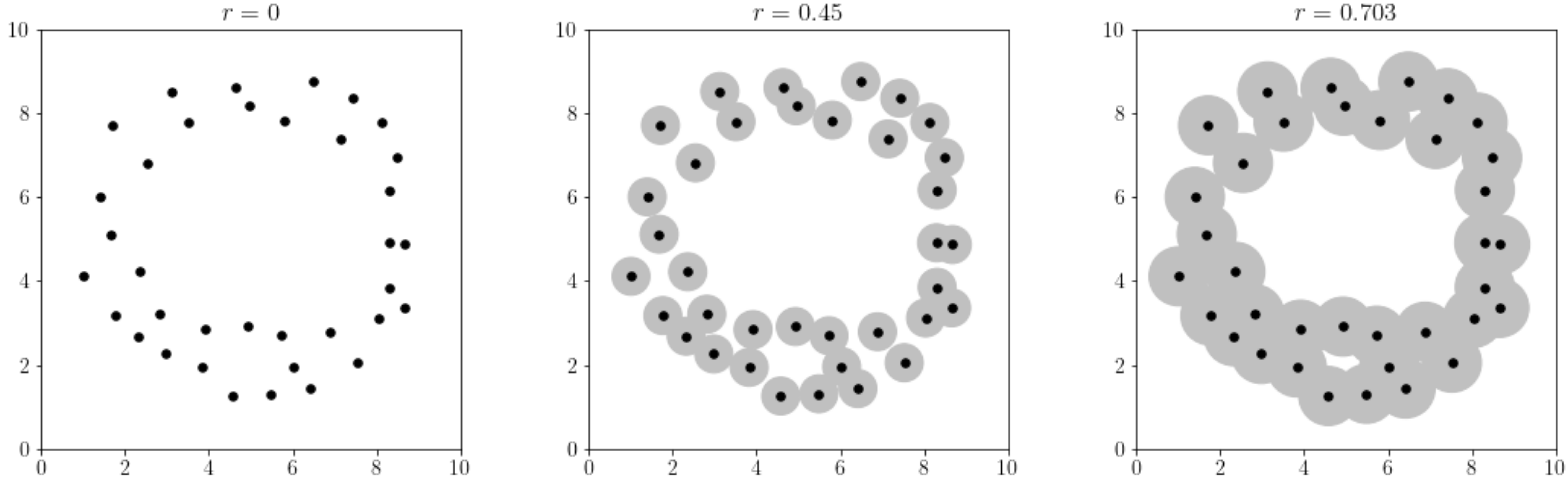


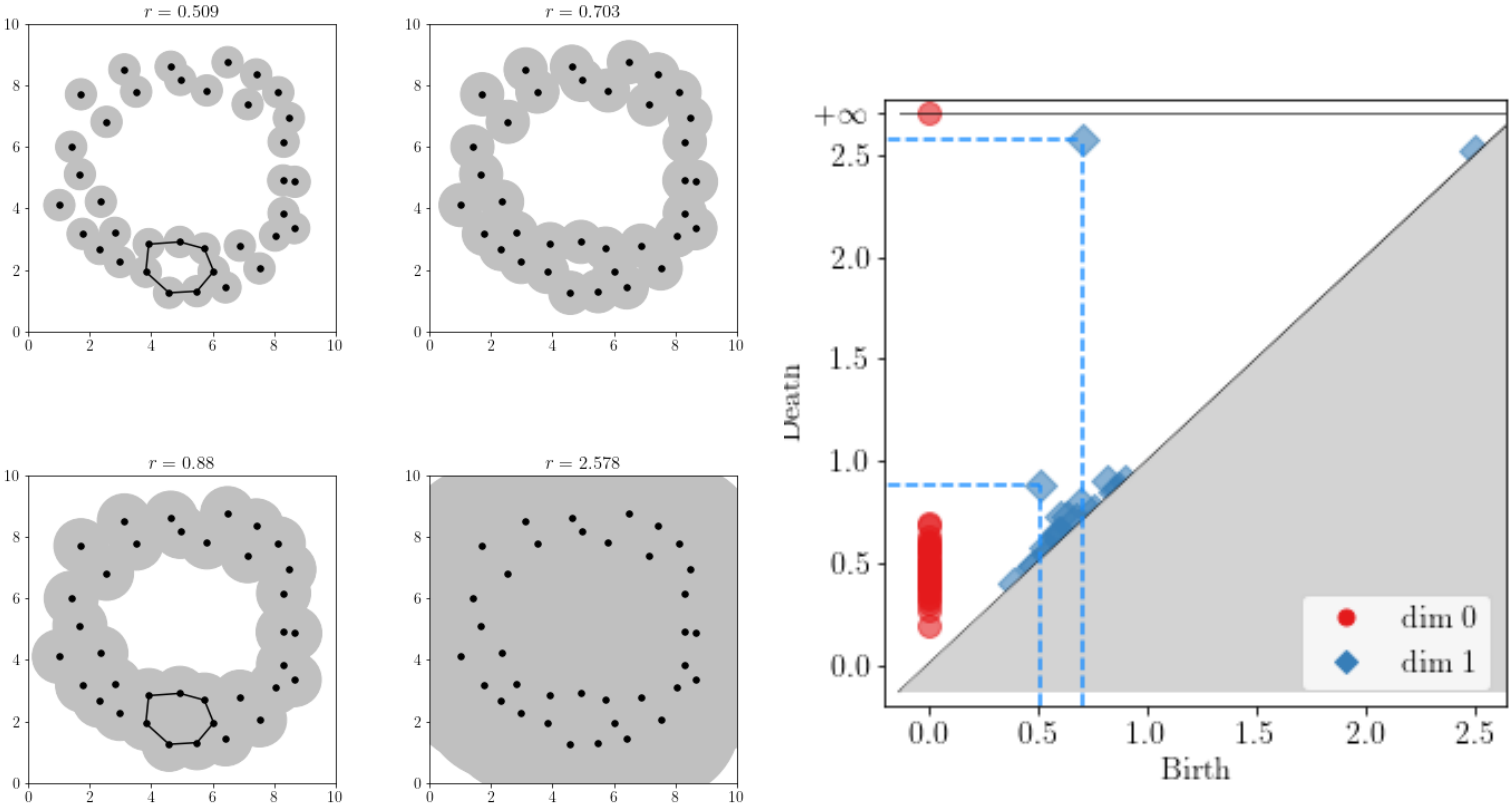
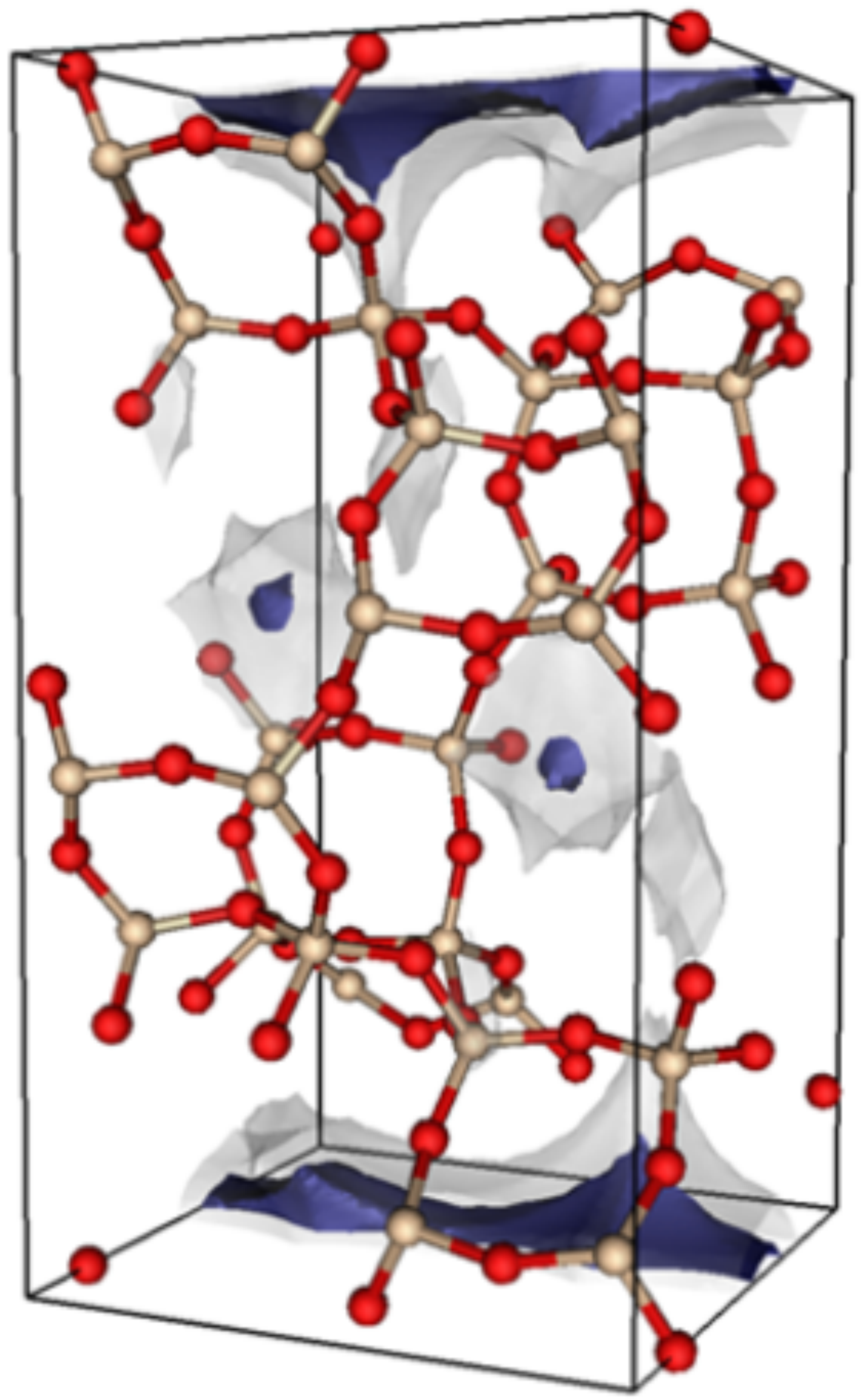


diagram credit: Andrey Yao

Zeolite crystals

[Krishnapriyan et al, 2020]



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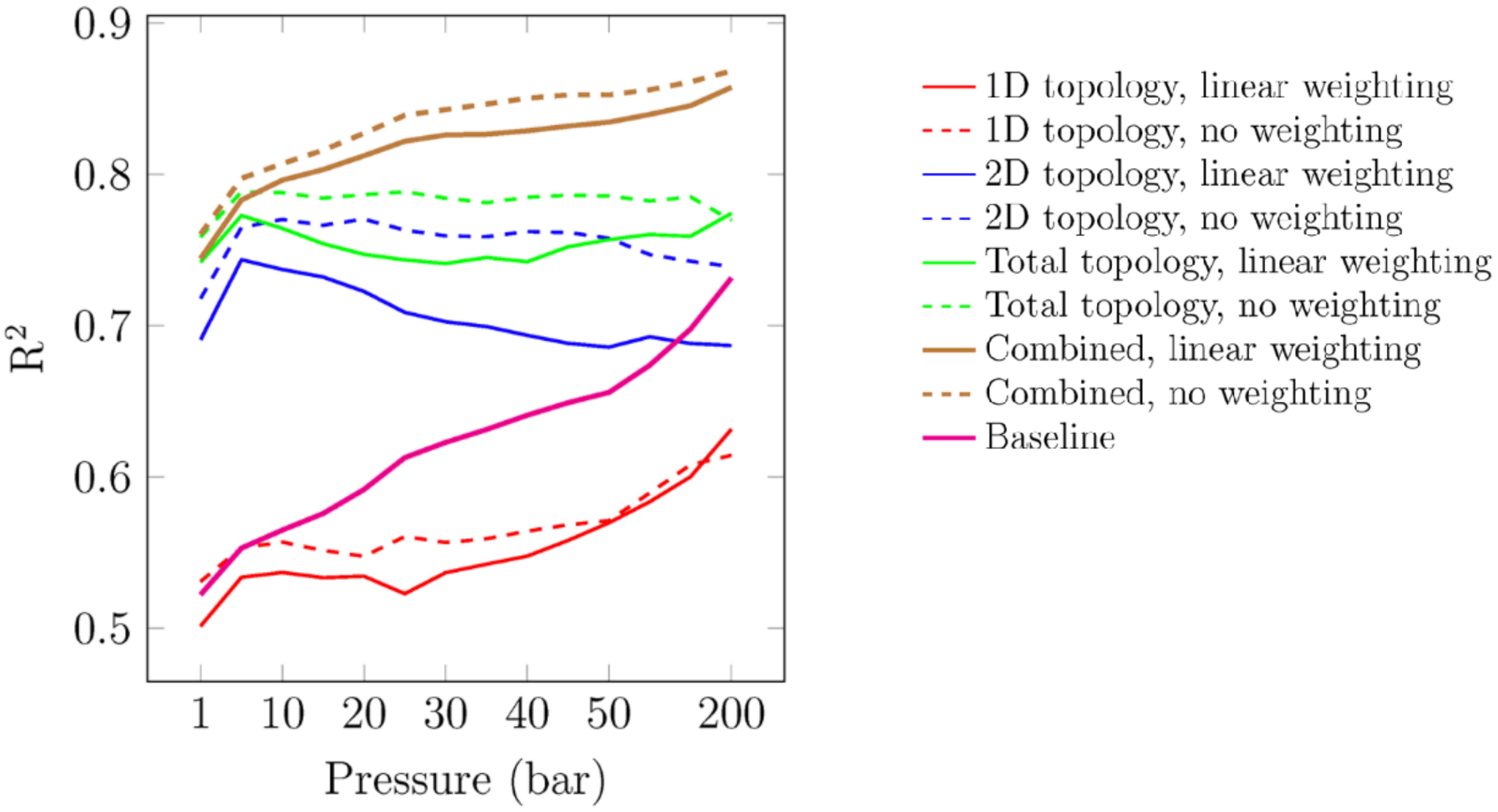

Zeolite crystals
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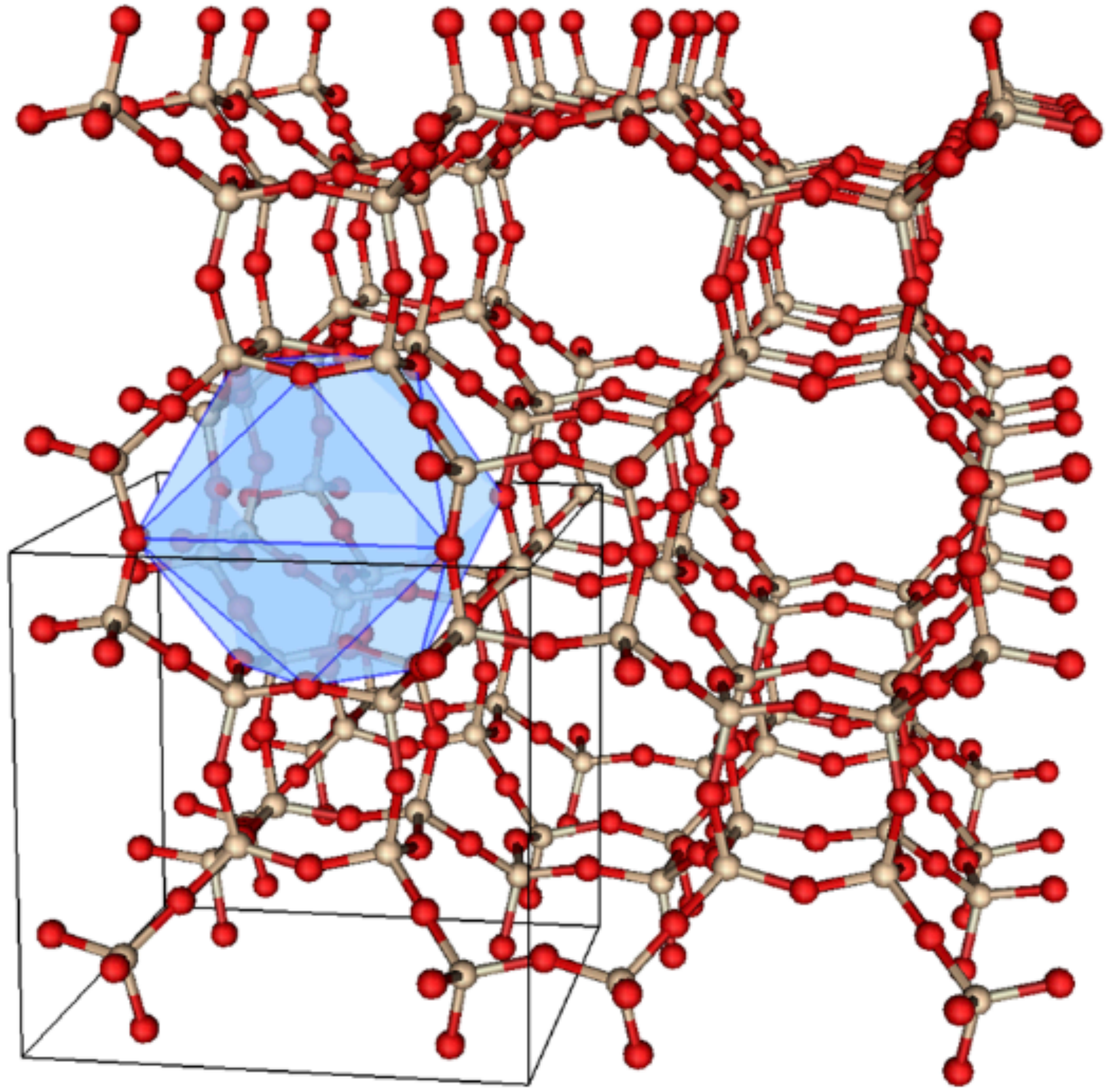
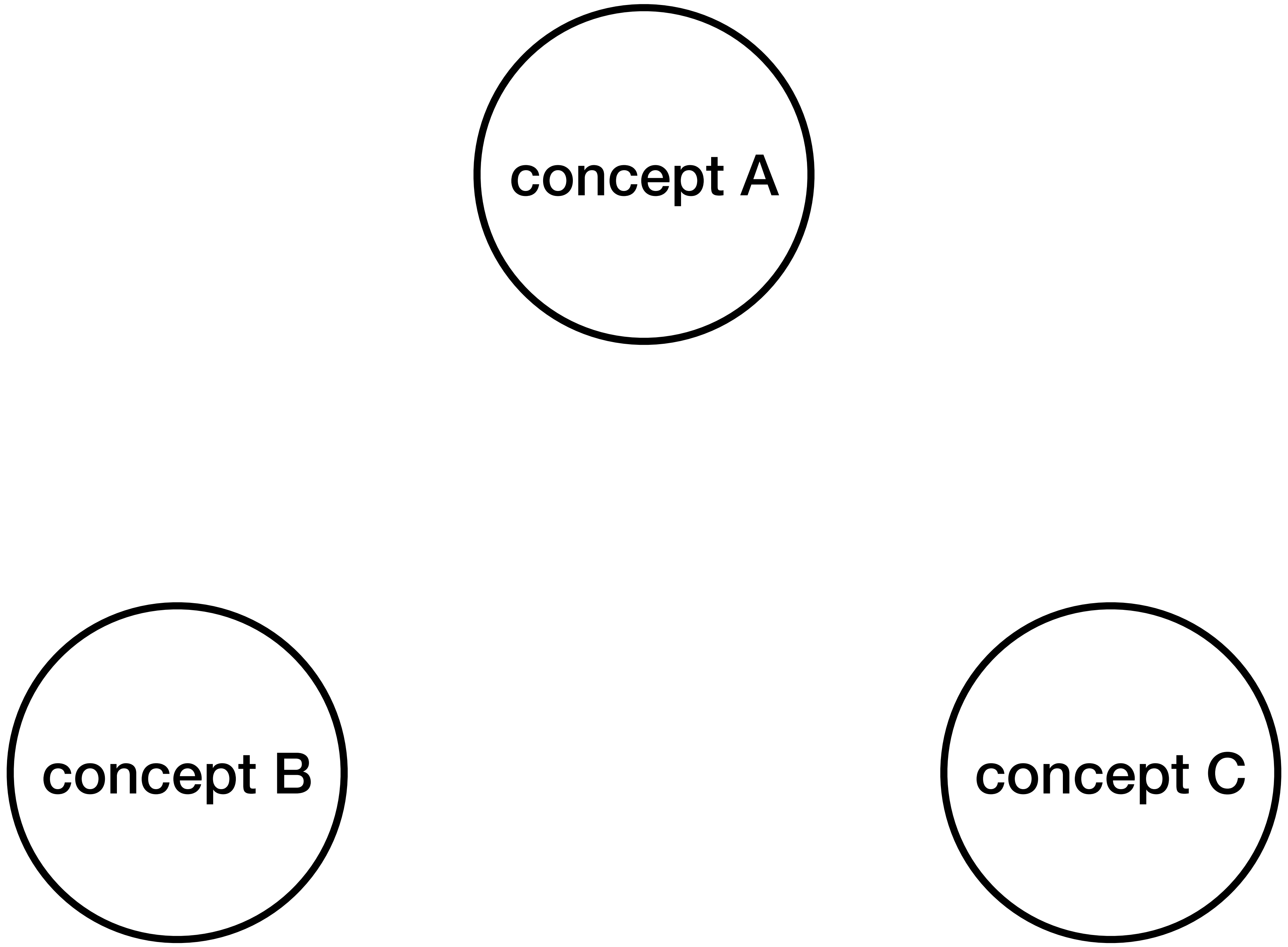
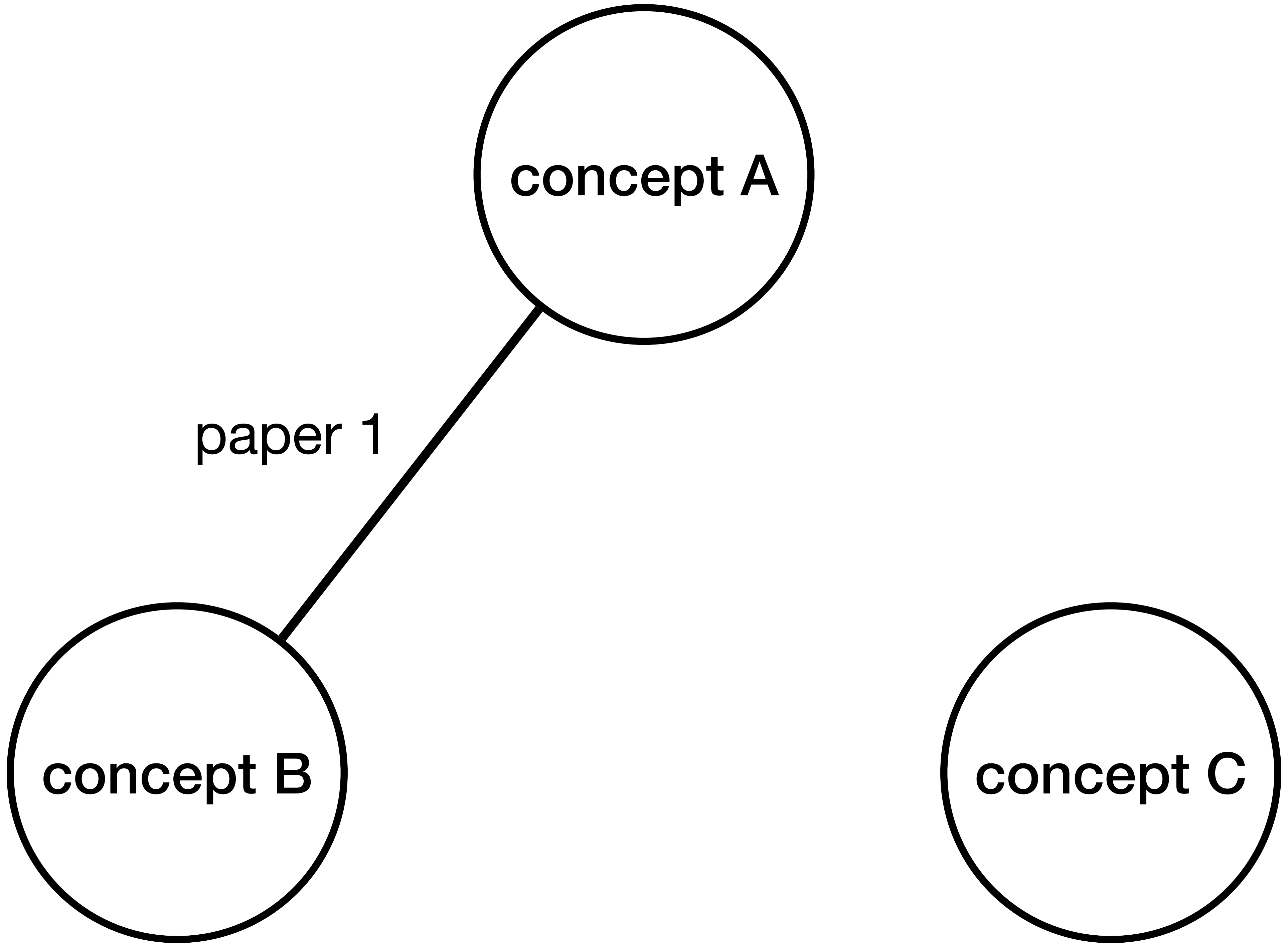


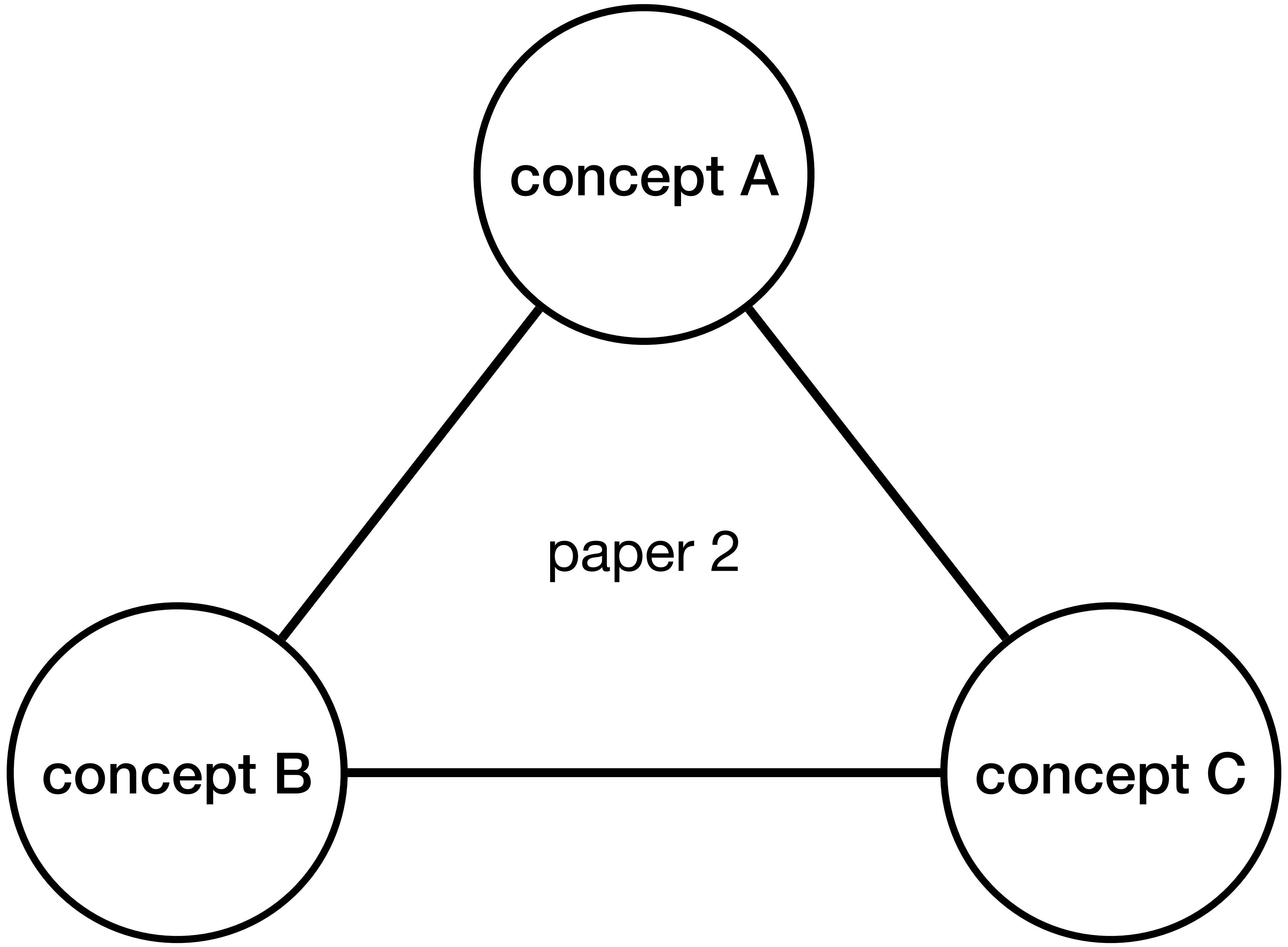
Networks and Complexes

Networks and Complexes

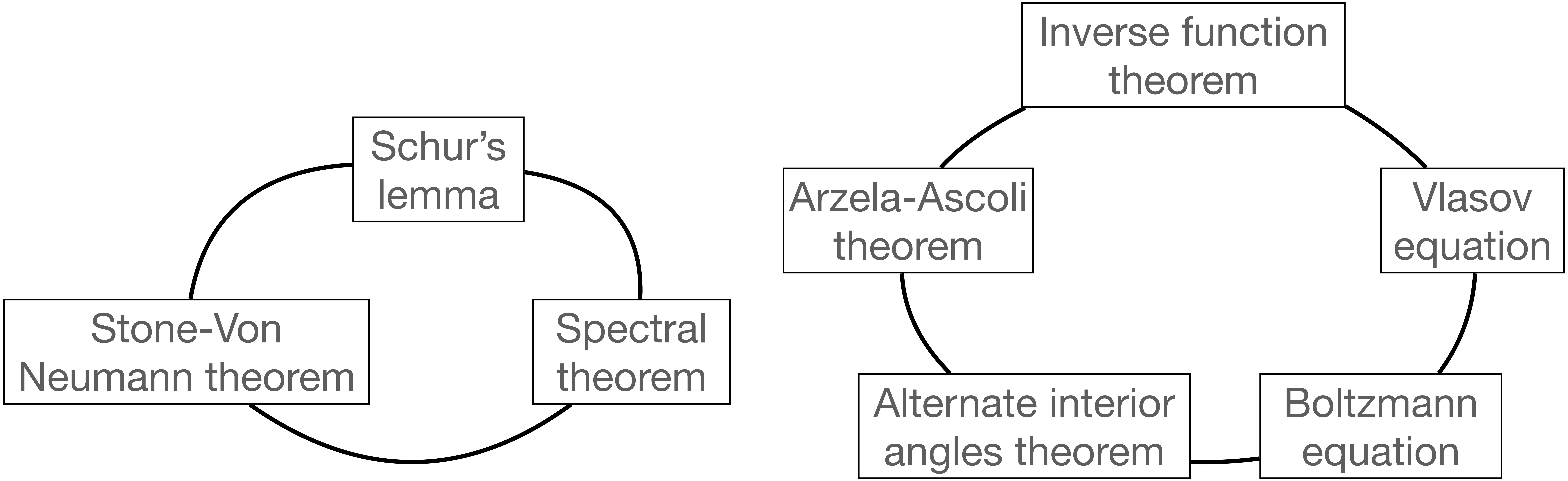
- Co-occurence complex in Math research paper [Salikov et al, 2018]


concept A
concept B
concept C


concept A
paper 1
concept B
concept C


concept A
paper 2
concept B
concept C

Gap in Understanding

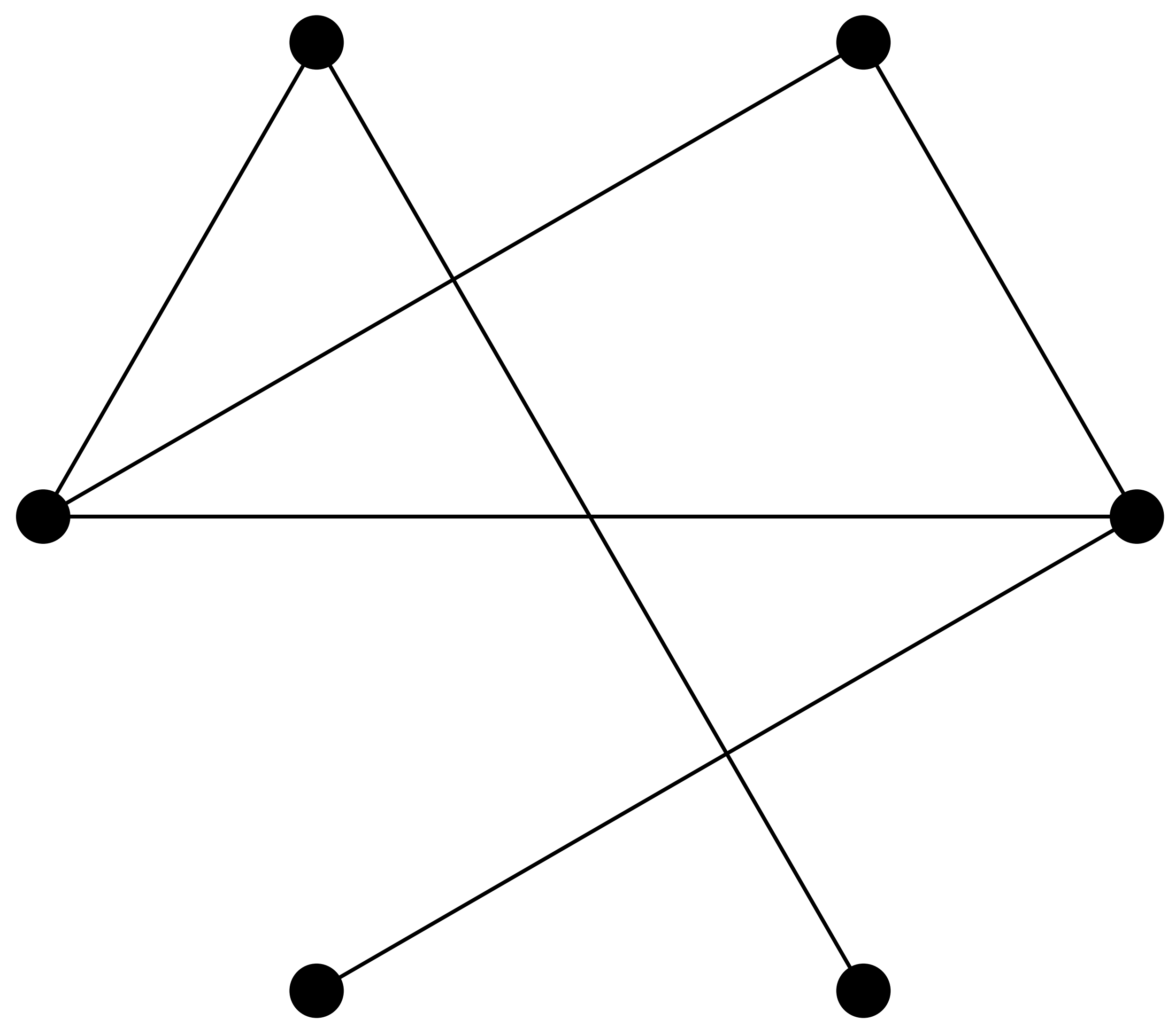


Benchmark of Comparison?

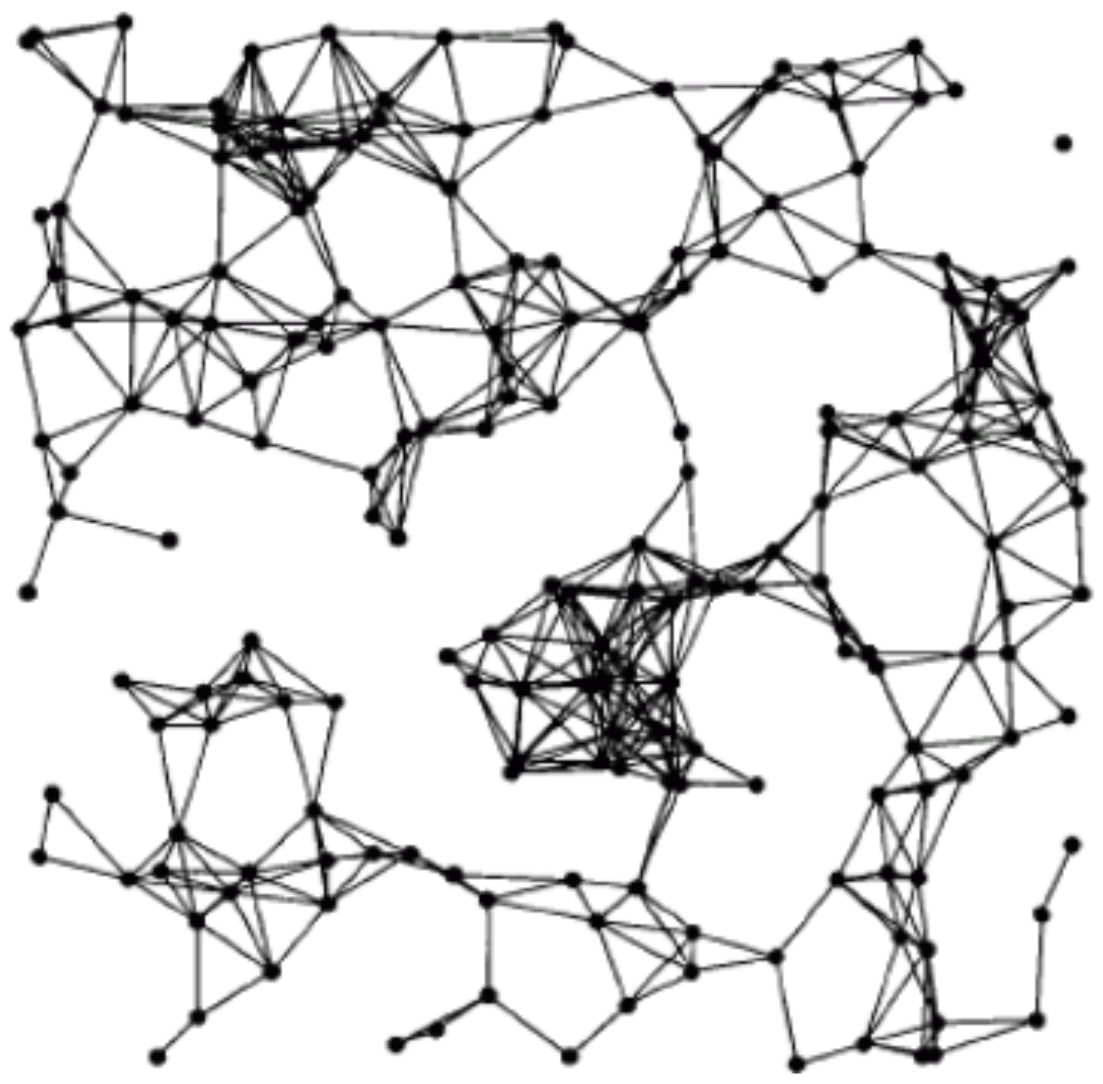
II. Stochastic Topology

Mug doesn’t play dice?

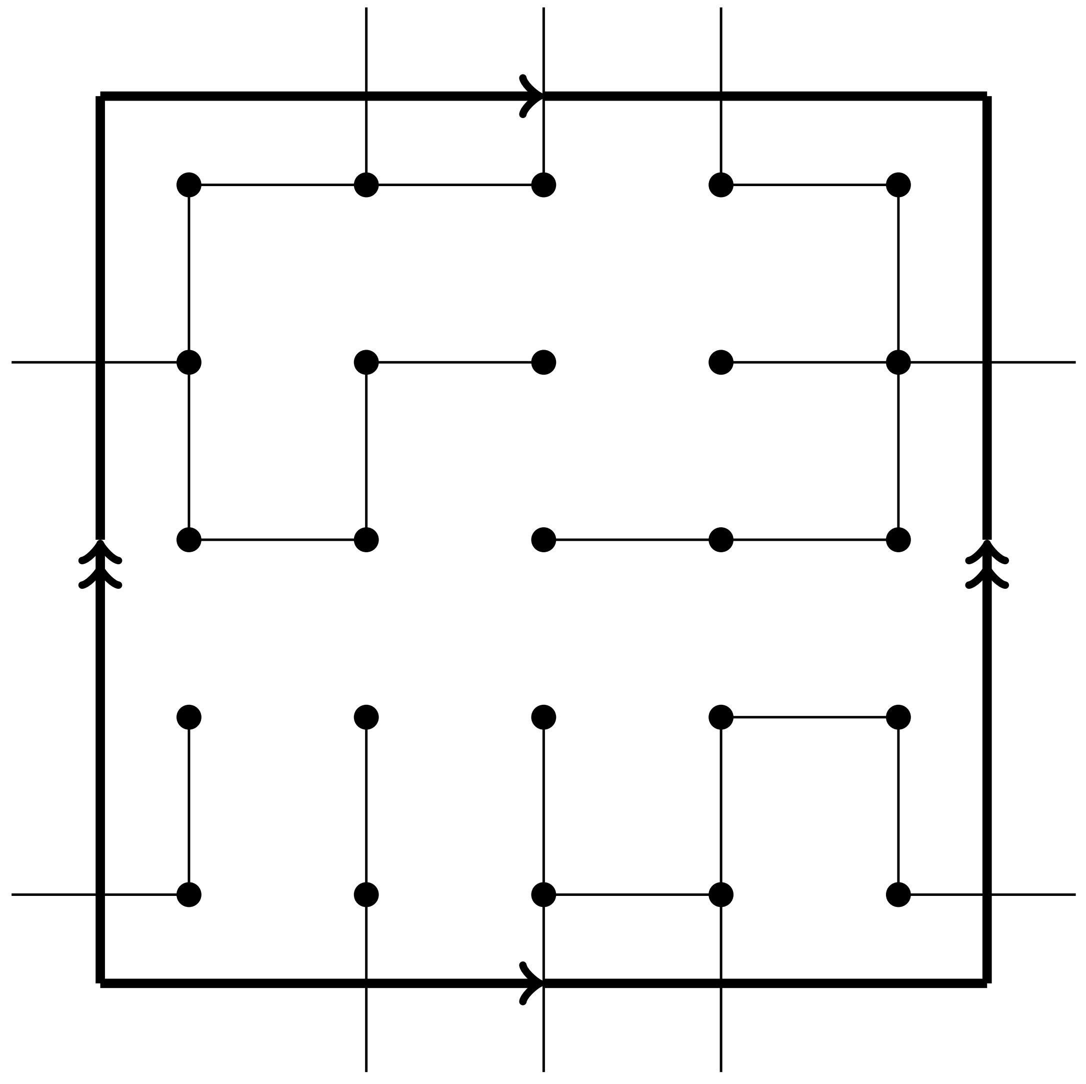
Tapas of Random Topology



Erdo-Renyi Complexes

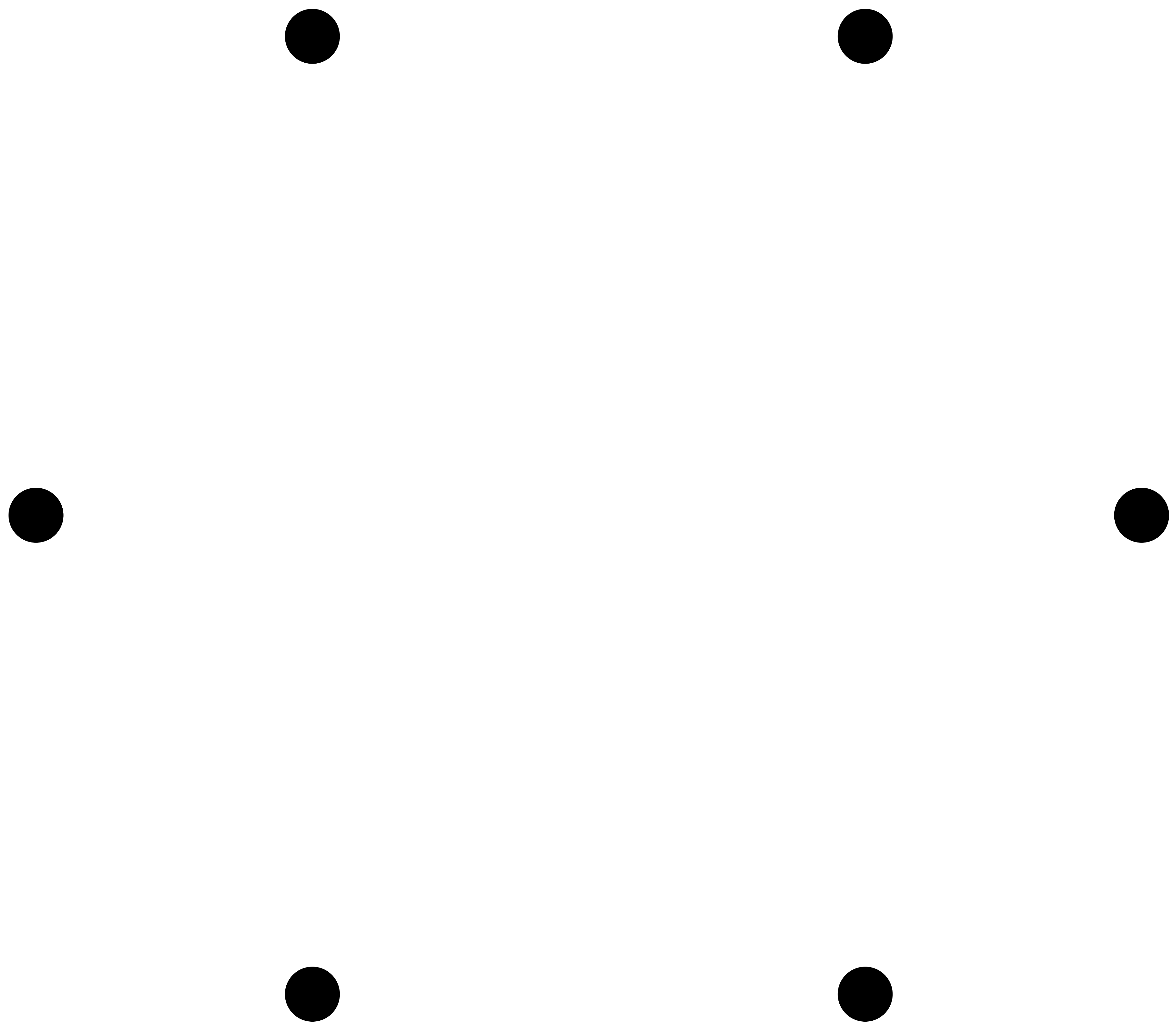


Geometric Complexes

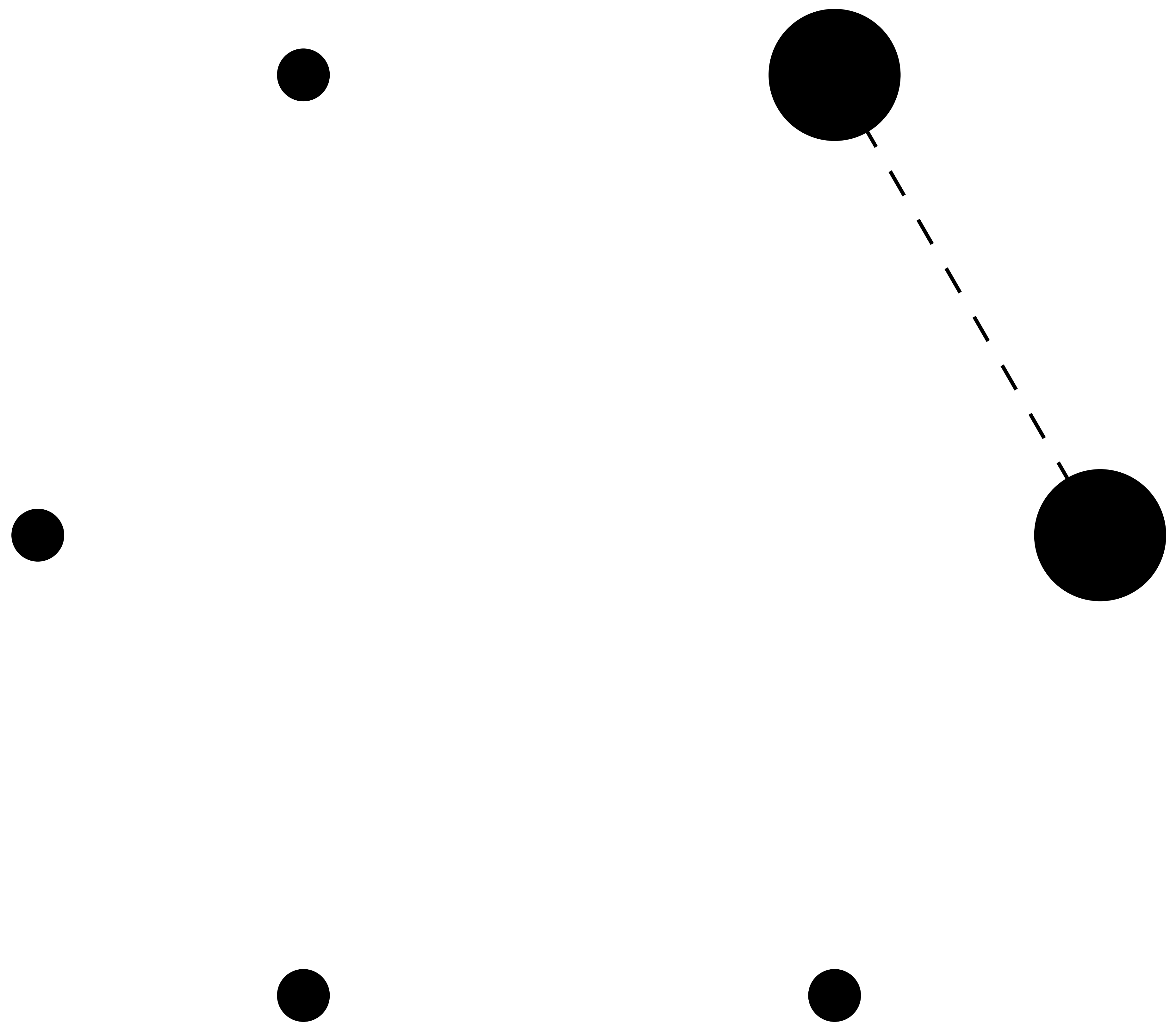


Topological Percolation

Erdos-Renyi graphs



Erdos-Renyi graphs



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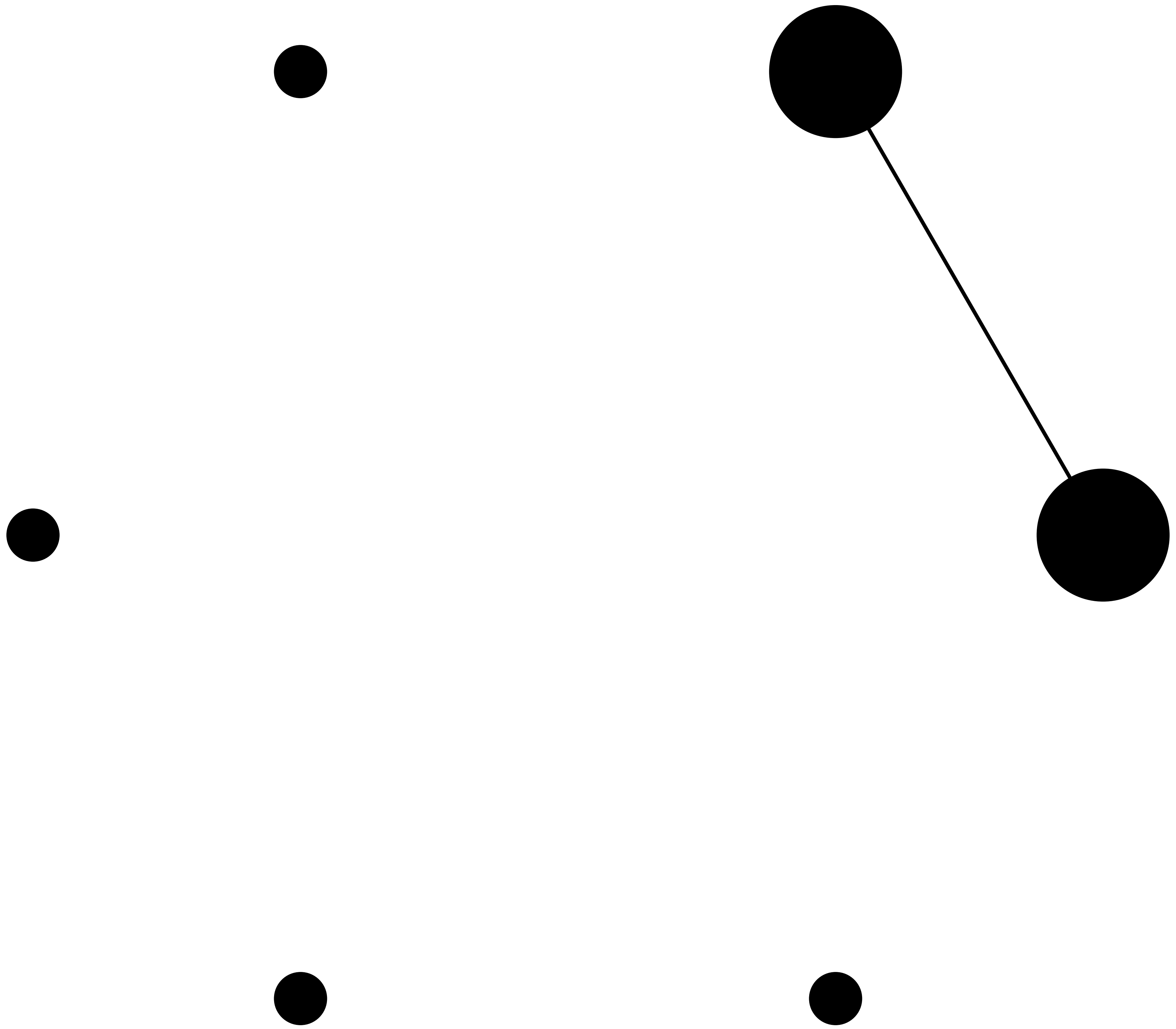

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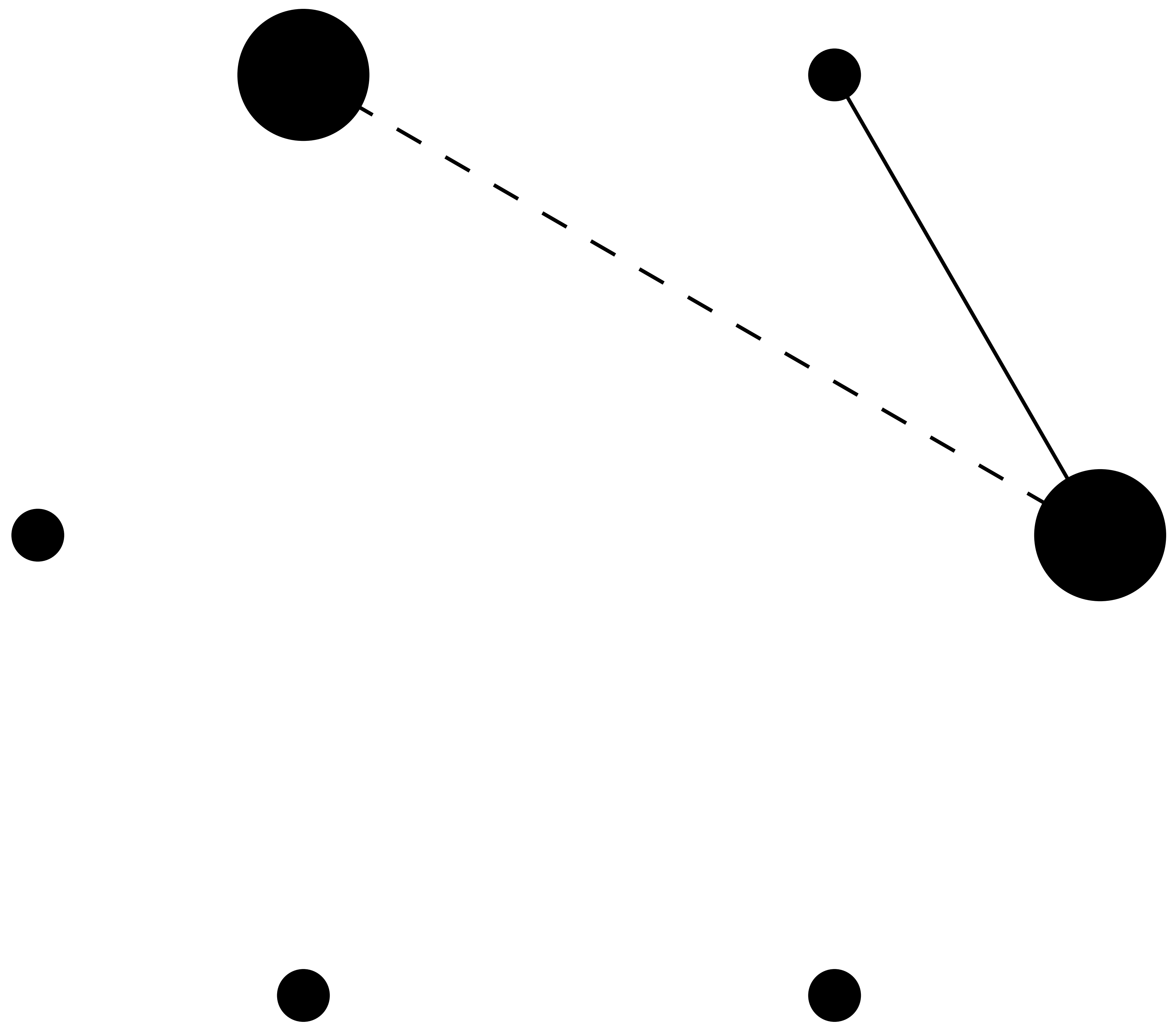

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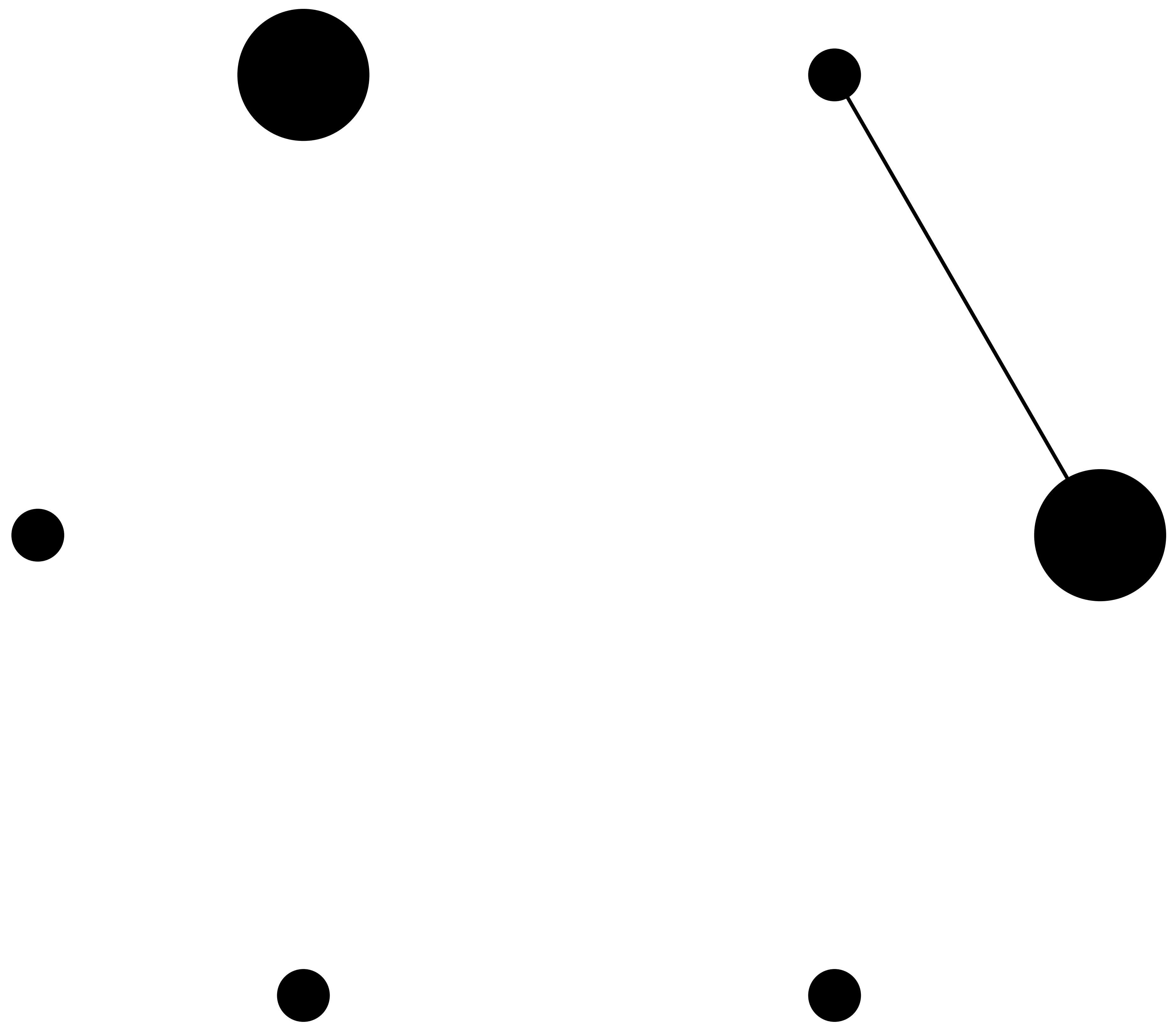

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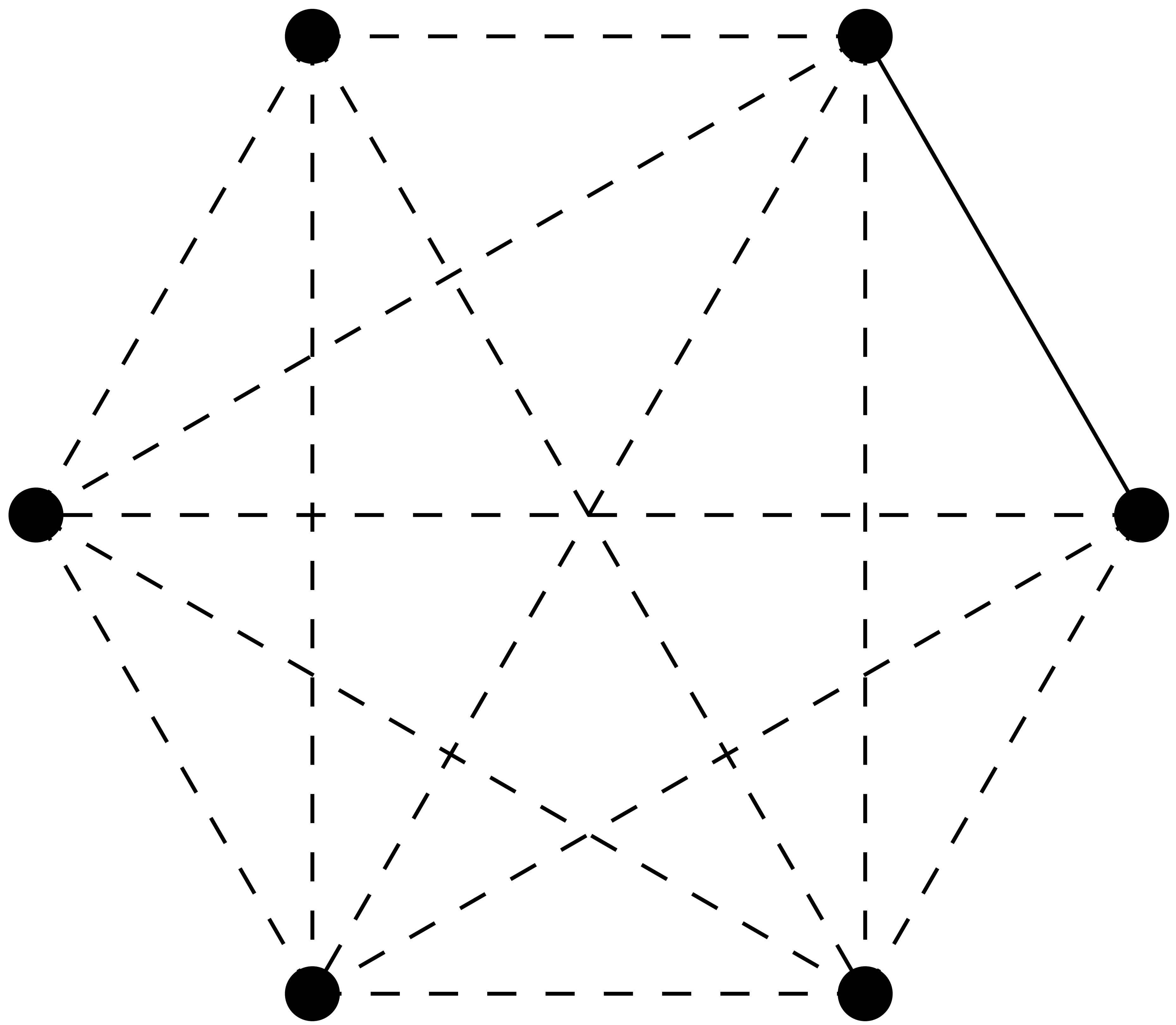

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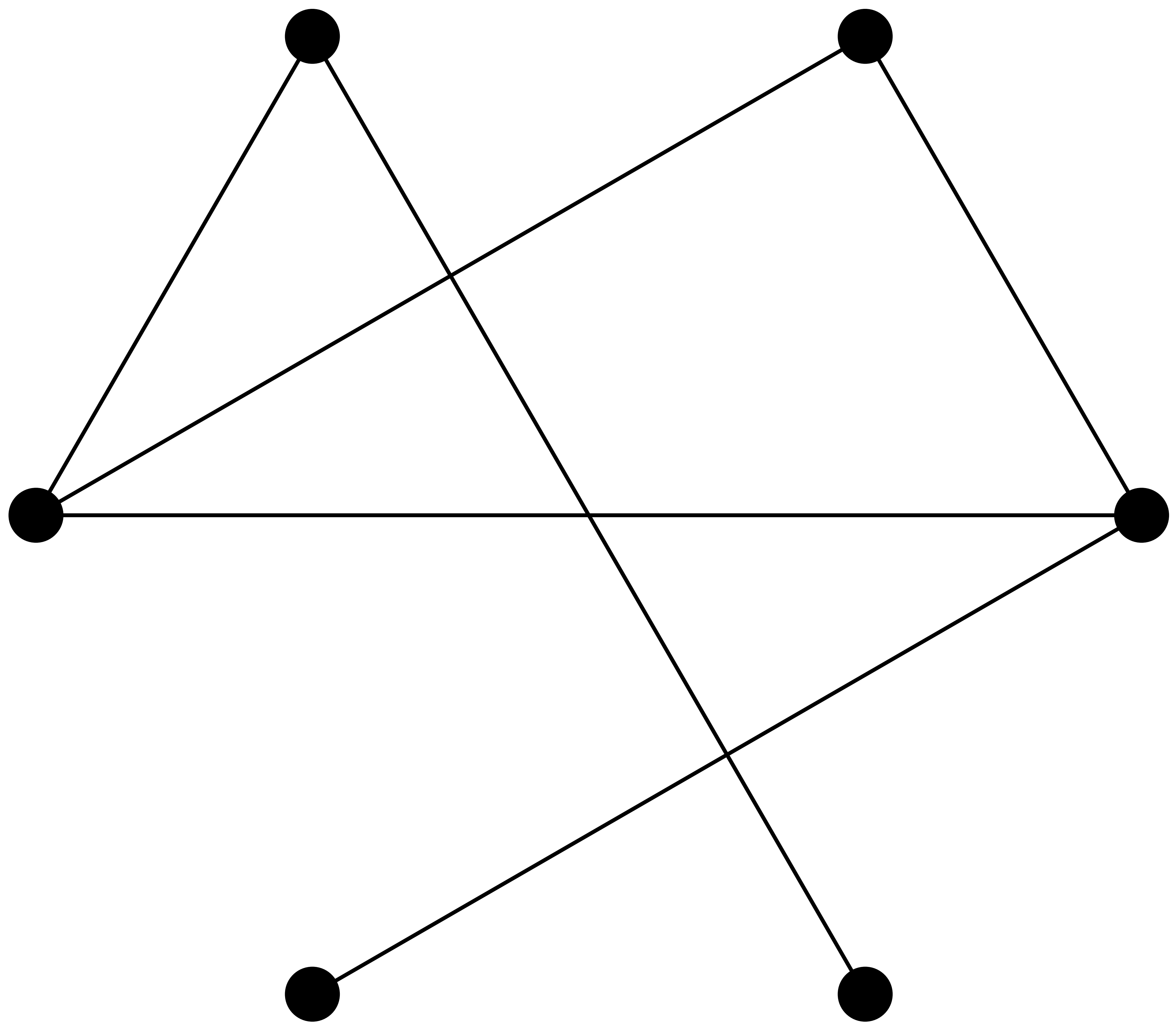

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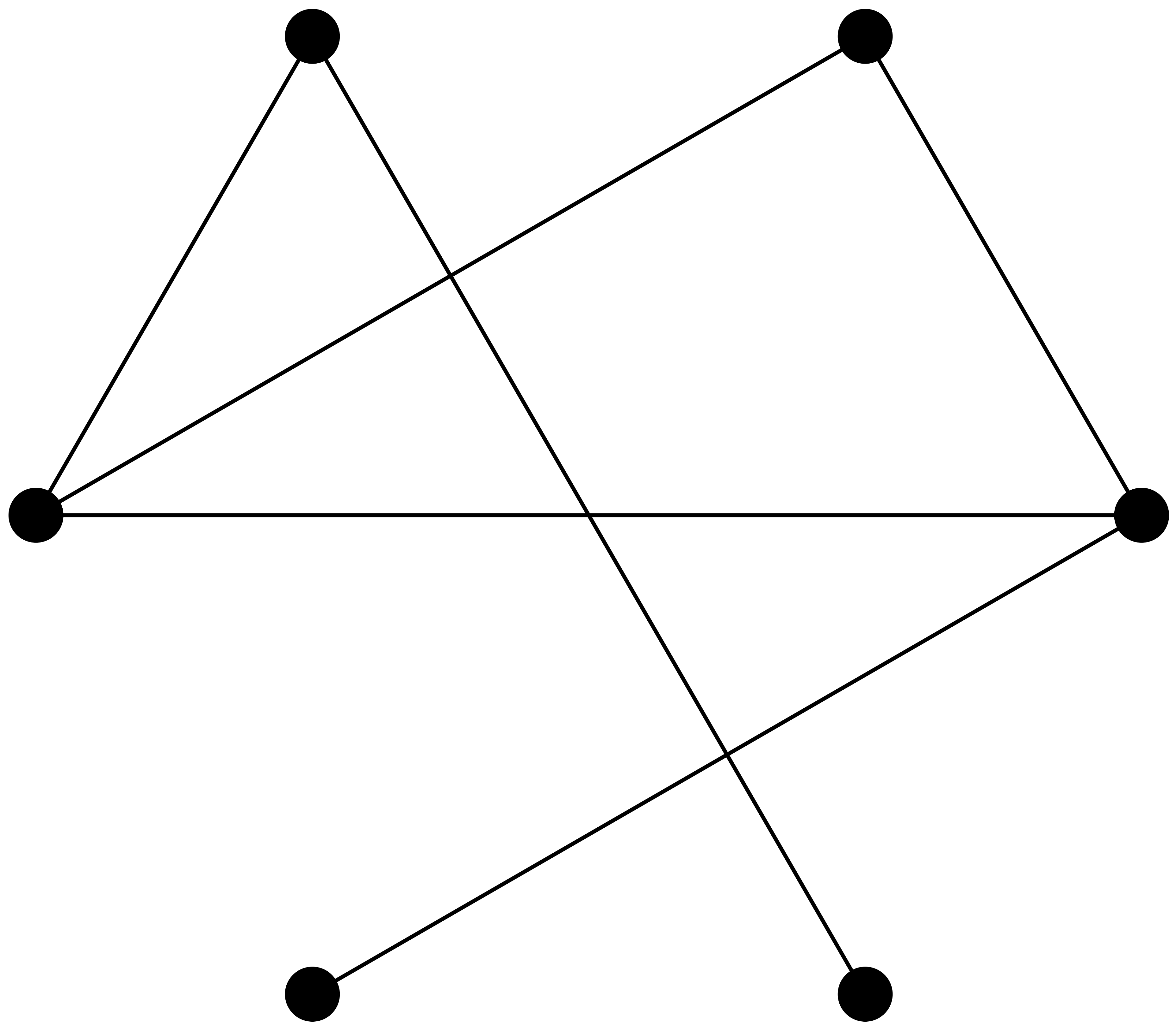
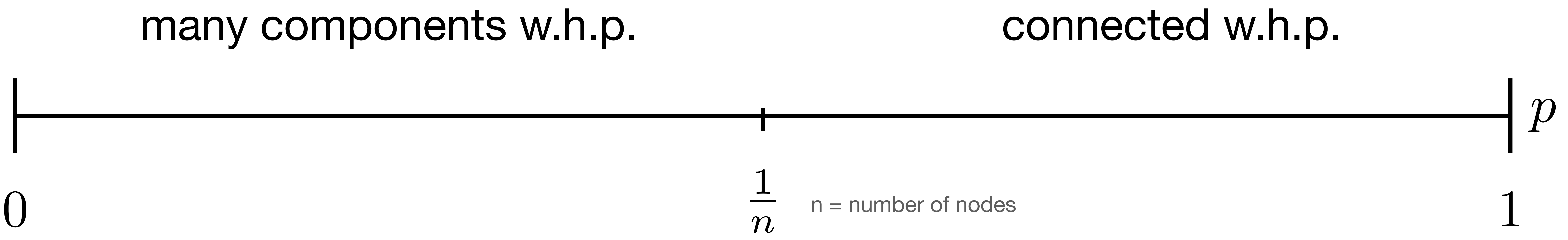
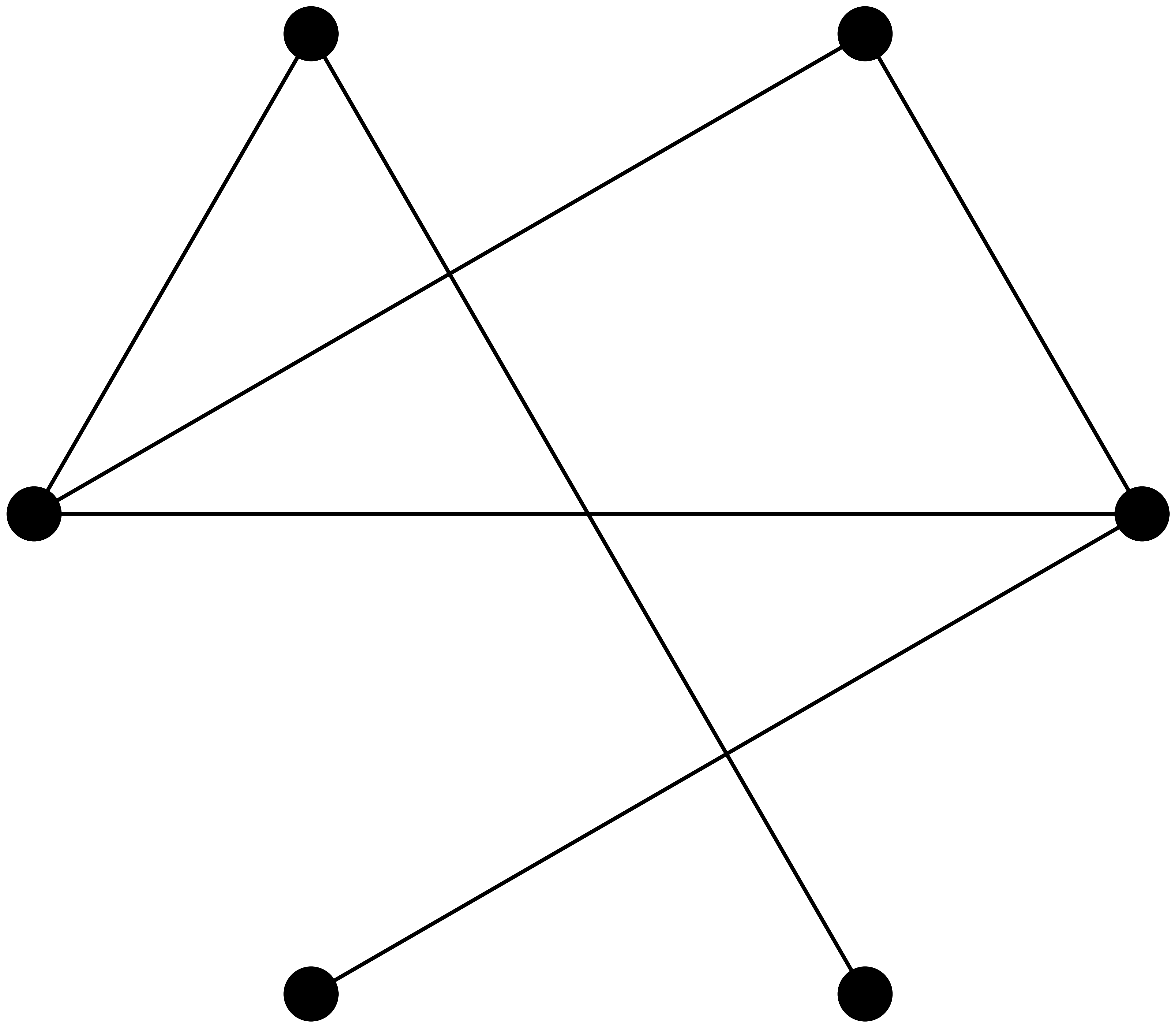


Phase Transition
[Erdos-Renyi 1960]

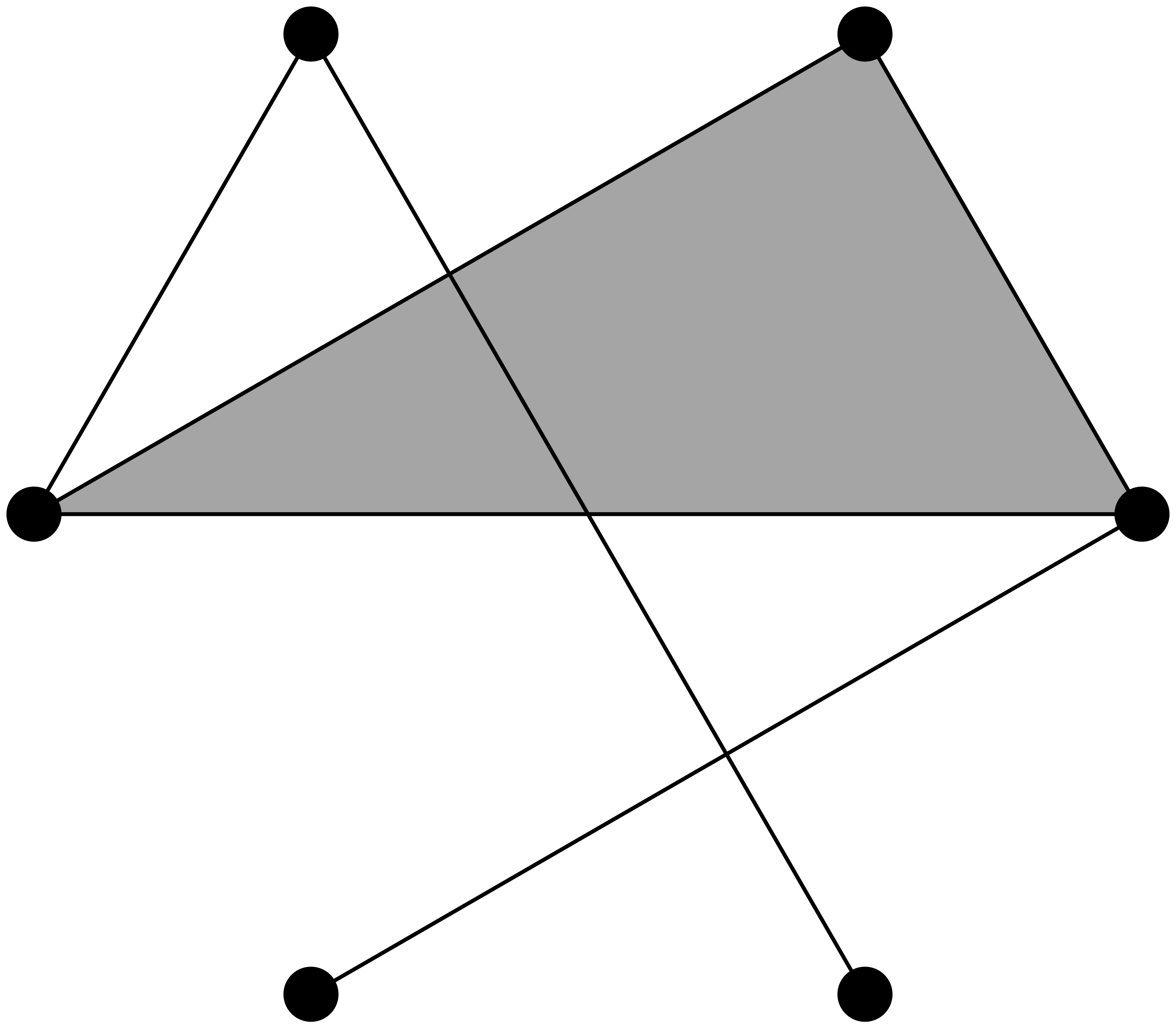


all log terms and constants forgone

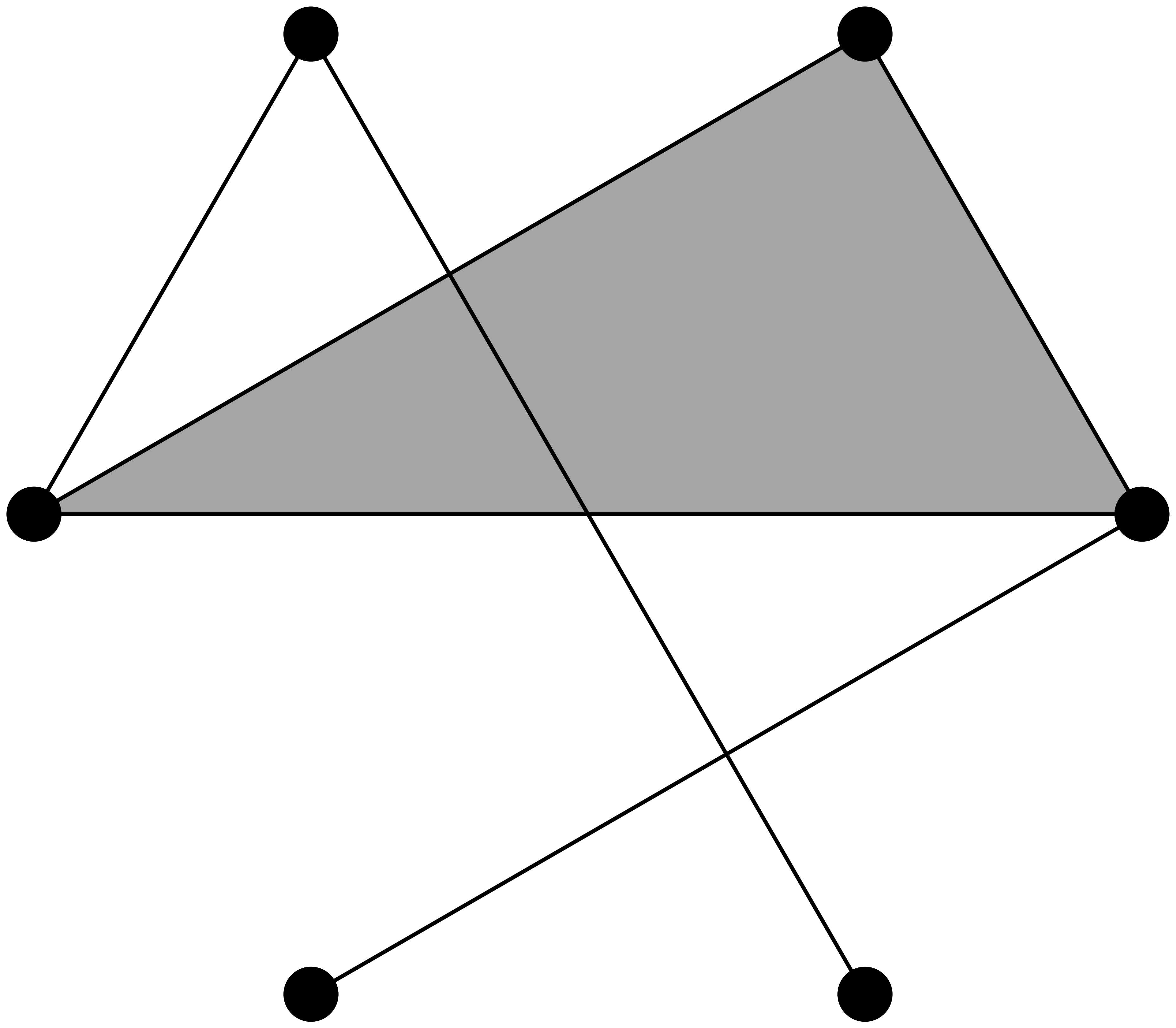
Erdos-Renyi Clique Complex

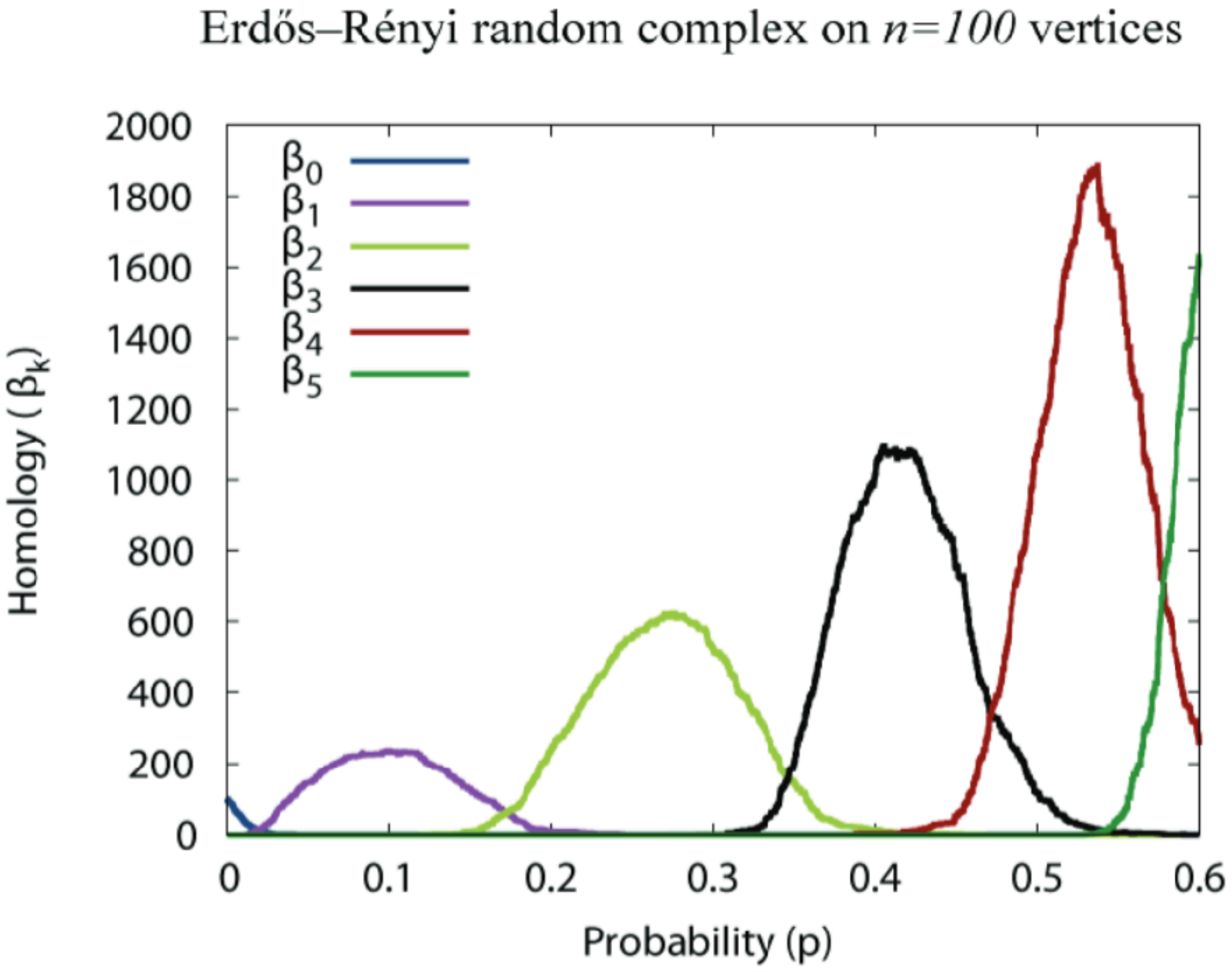


Erdos-Renyi Clique Complex



Betti Numbers

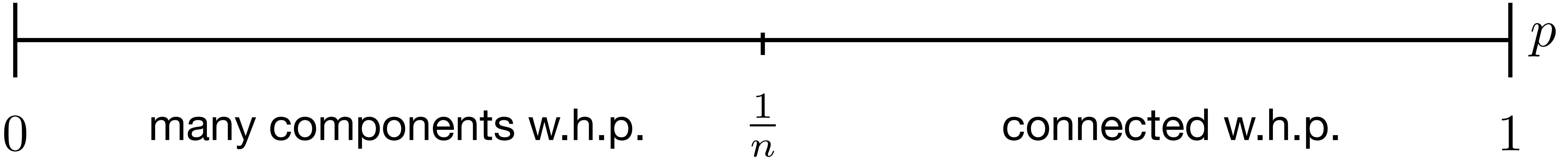




computation and plotting done by Zomorodian

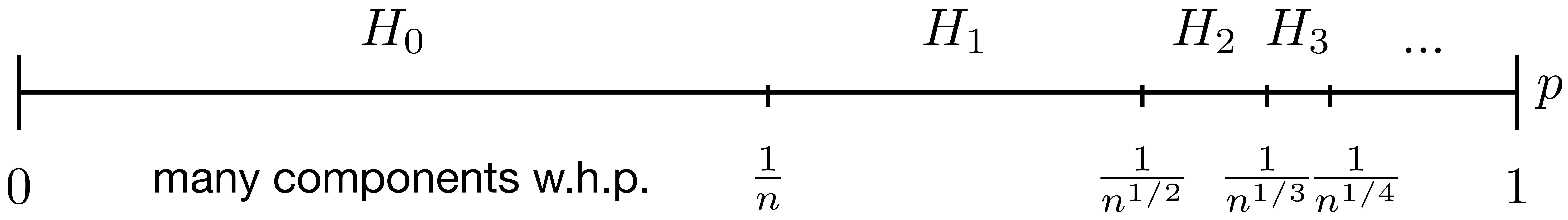
Phase Transition

[Erdos-Renyi 1960]



n = number of nodes
all log terms and constants forgone

Phase Transition
[Kahle 2009, 2014]



n = number of nodes
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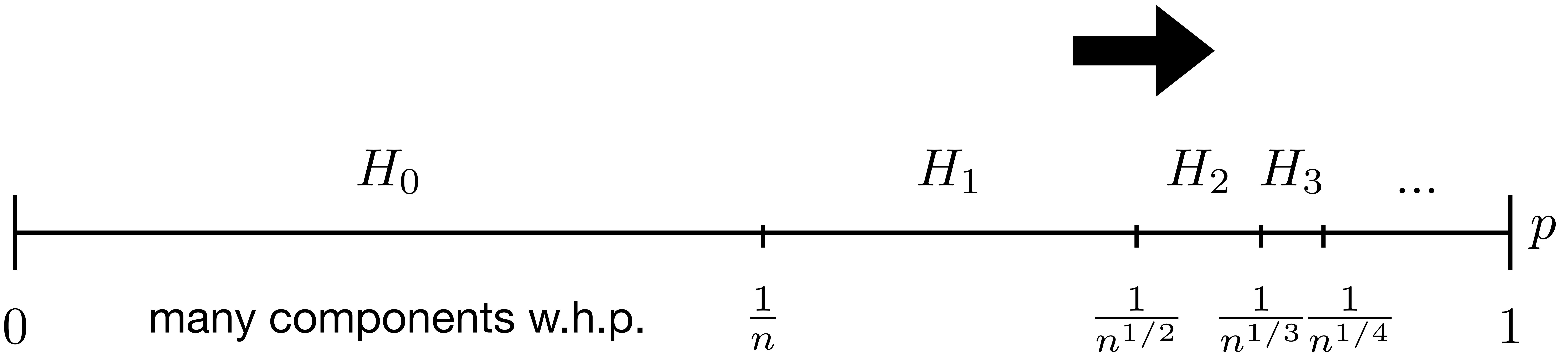
Phase Transition

[Kahle 2009, 2014]

Holes get filled.



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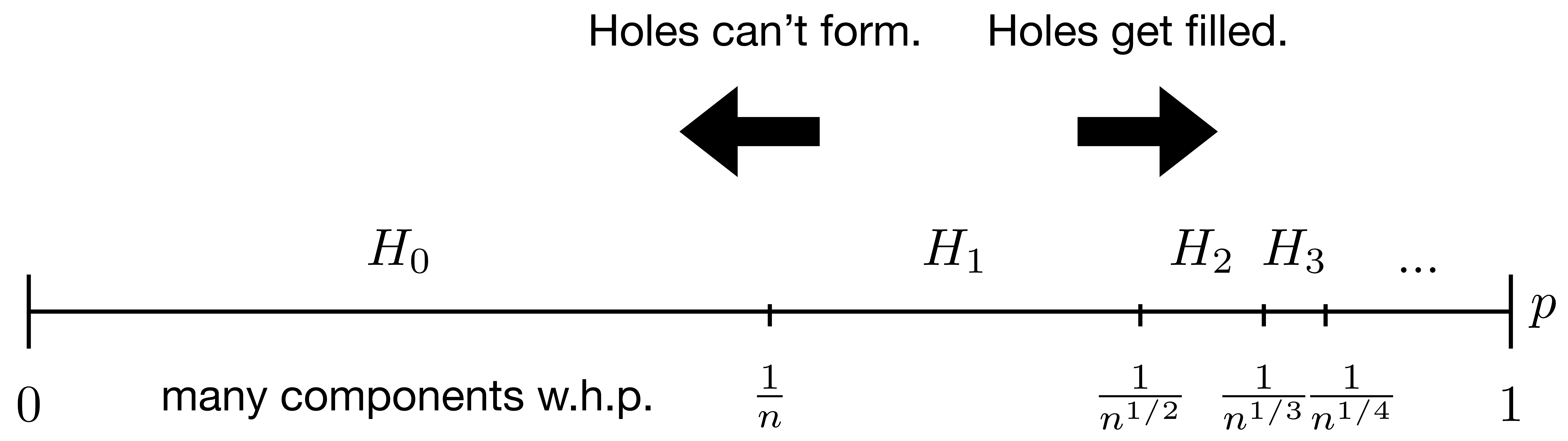

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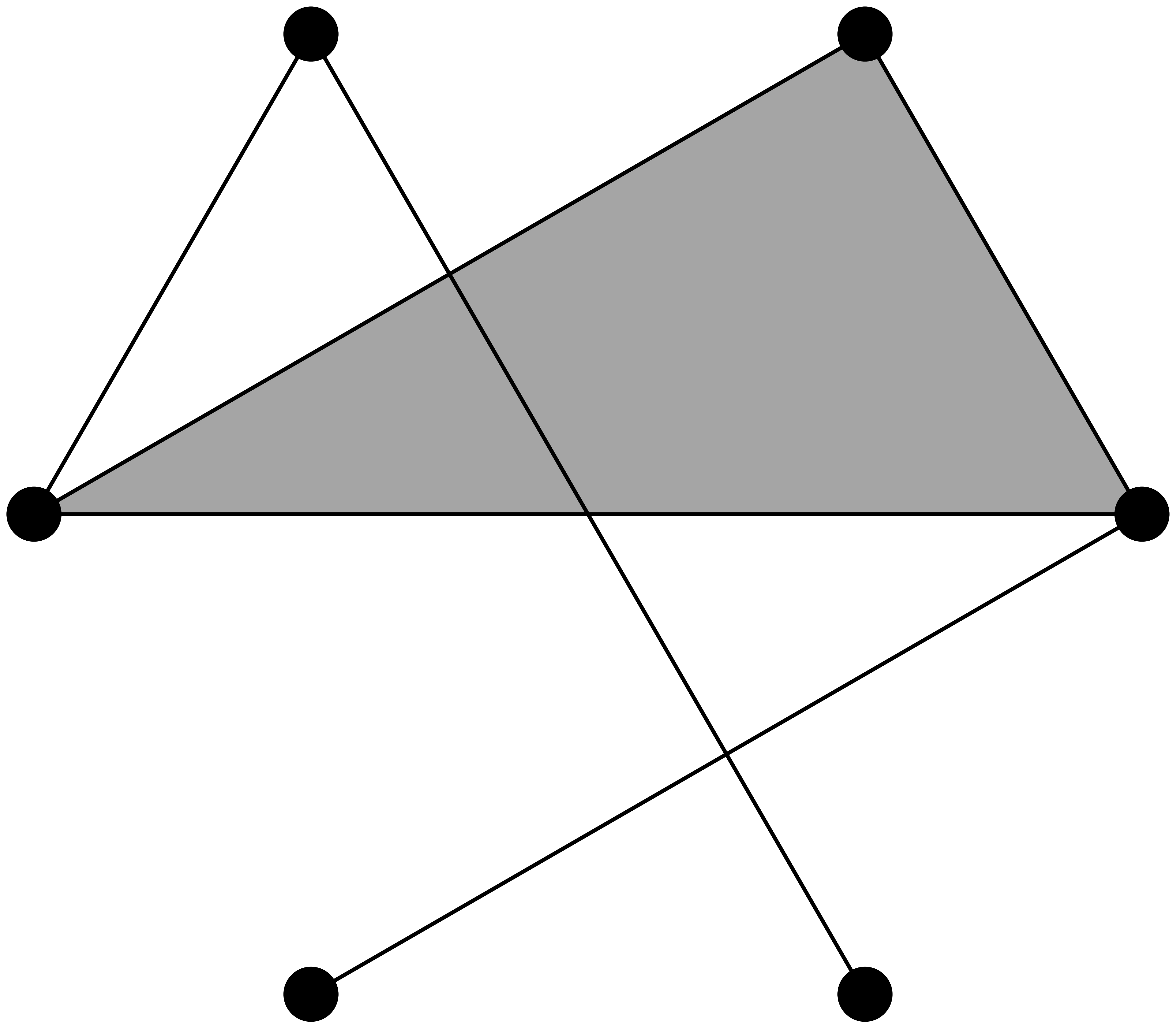
Erdos-Renyi Clique Complex



Geometric Complexes

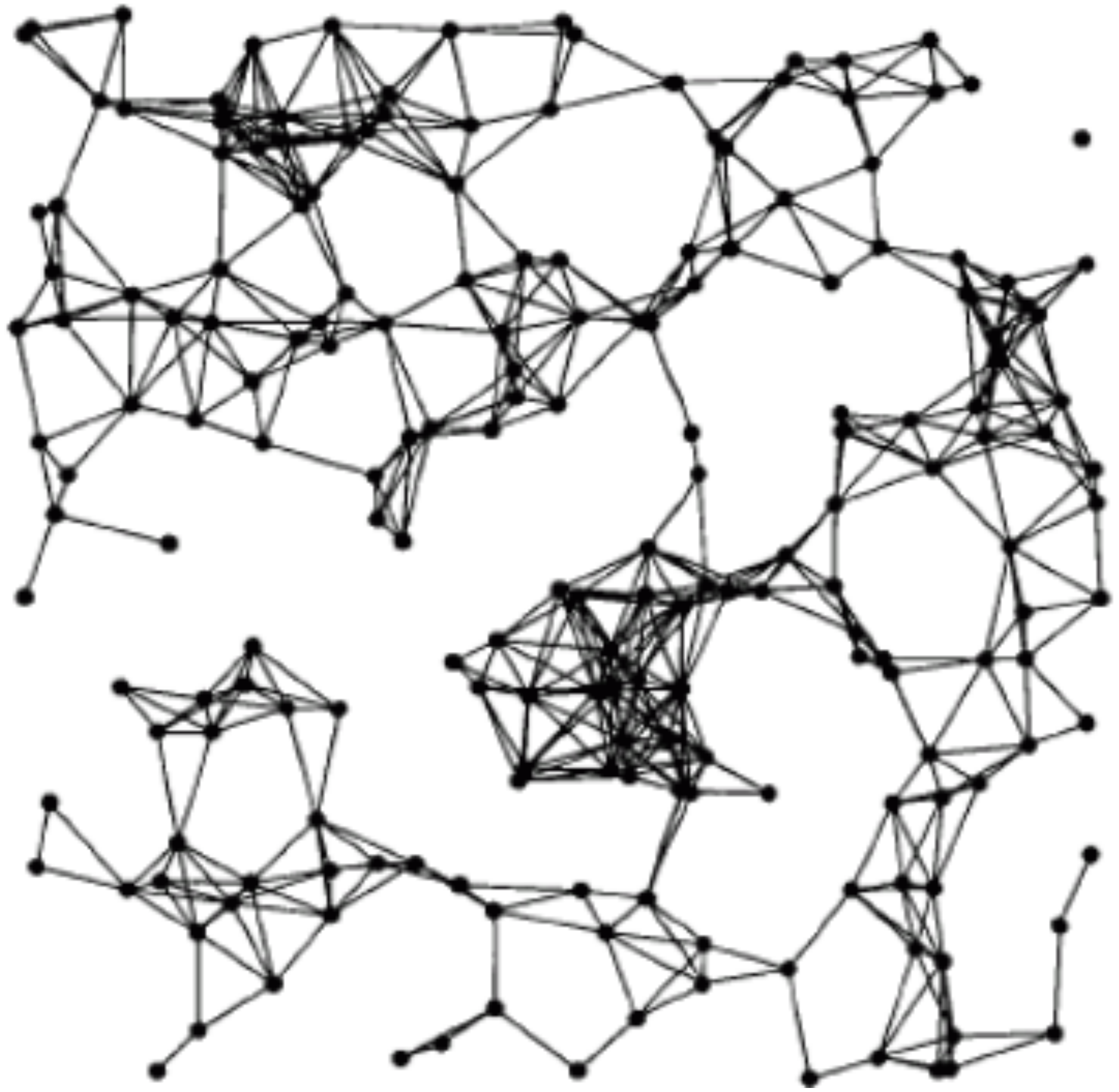
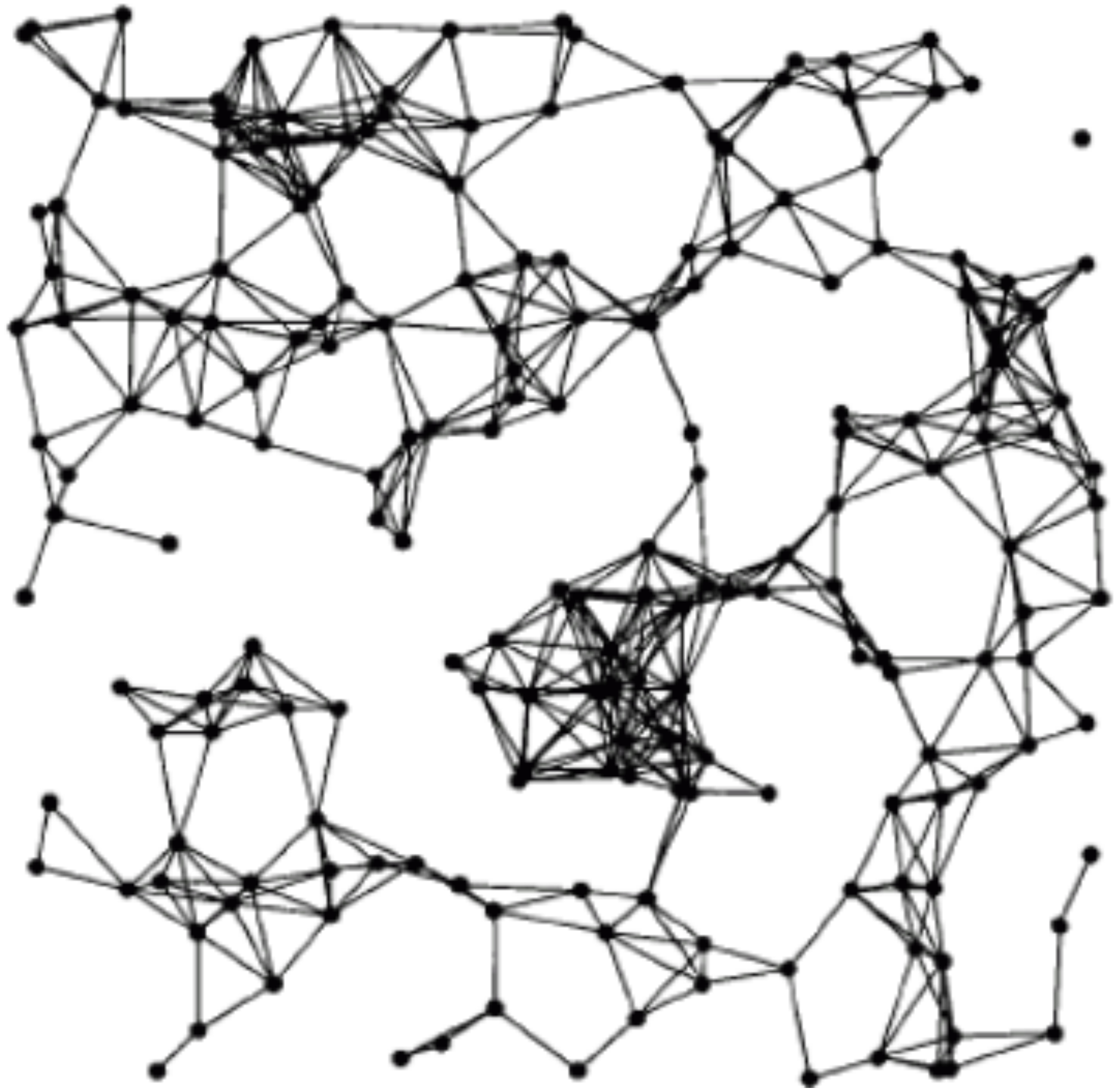


image credit: Penrose

Geometric Complexes

- Rips
- Cech



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Geometric Complexes

- Rips (clique)
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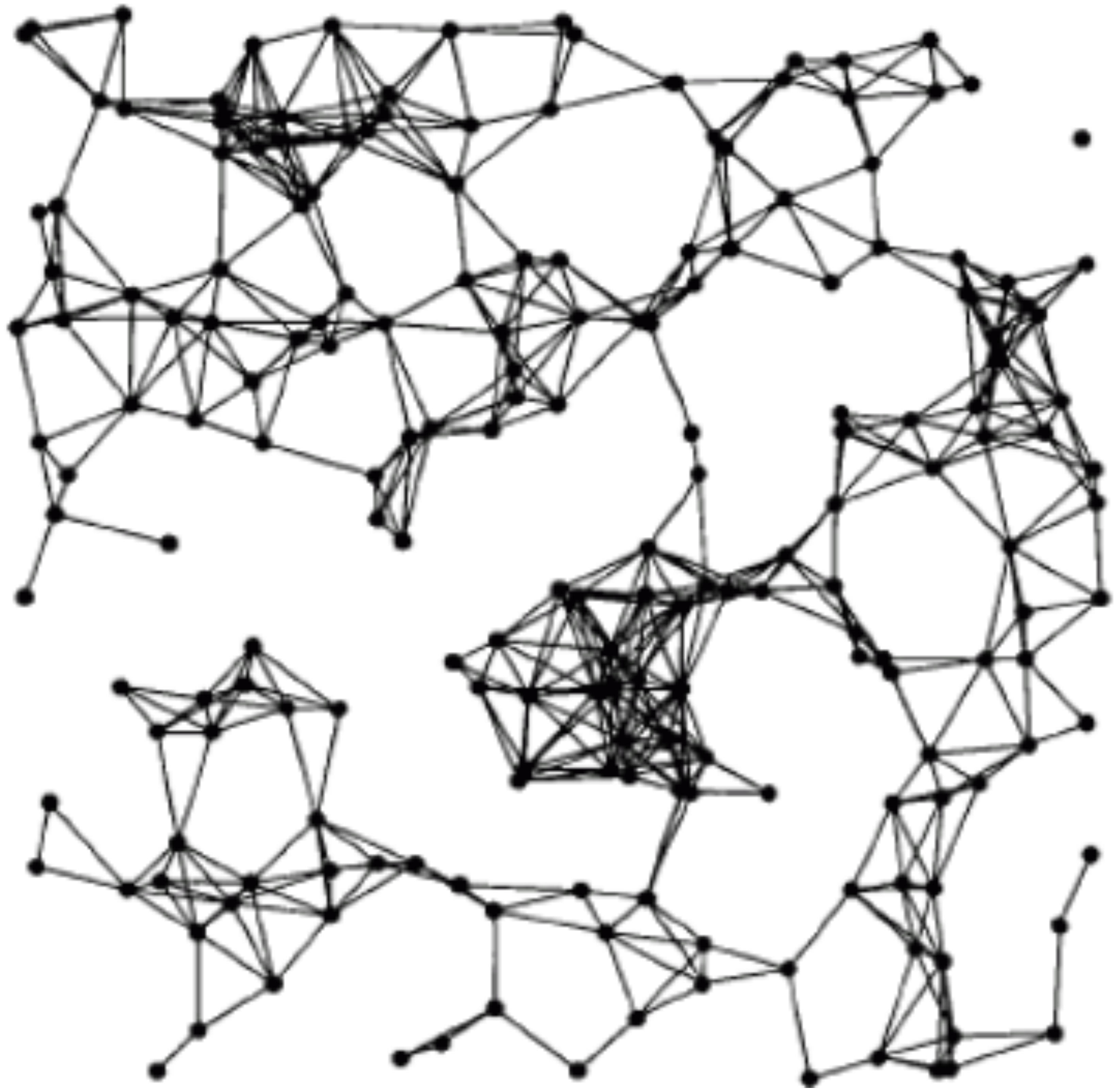

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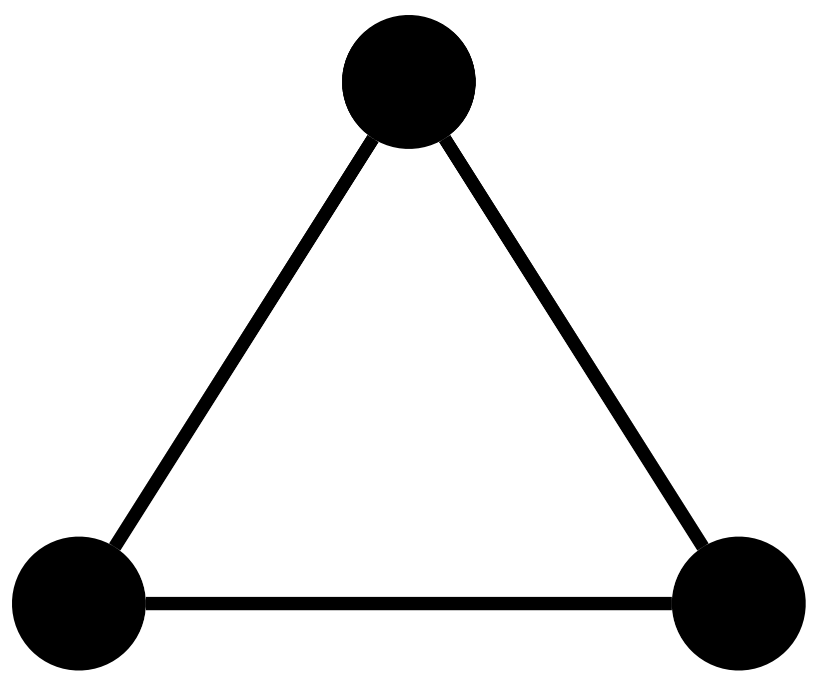

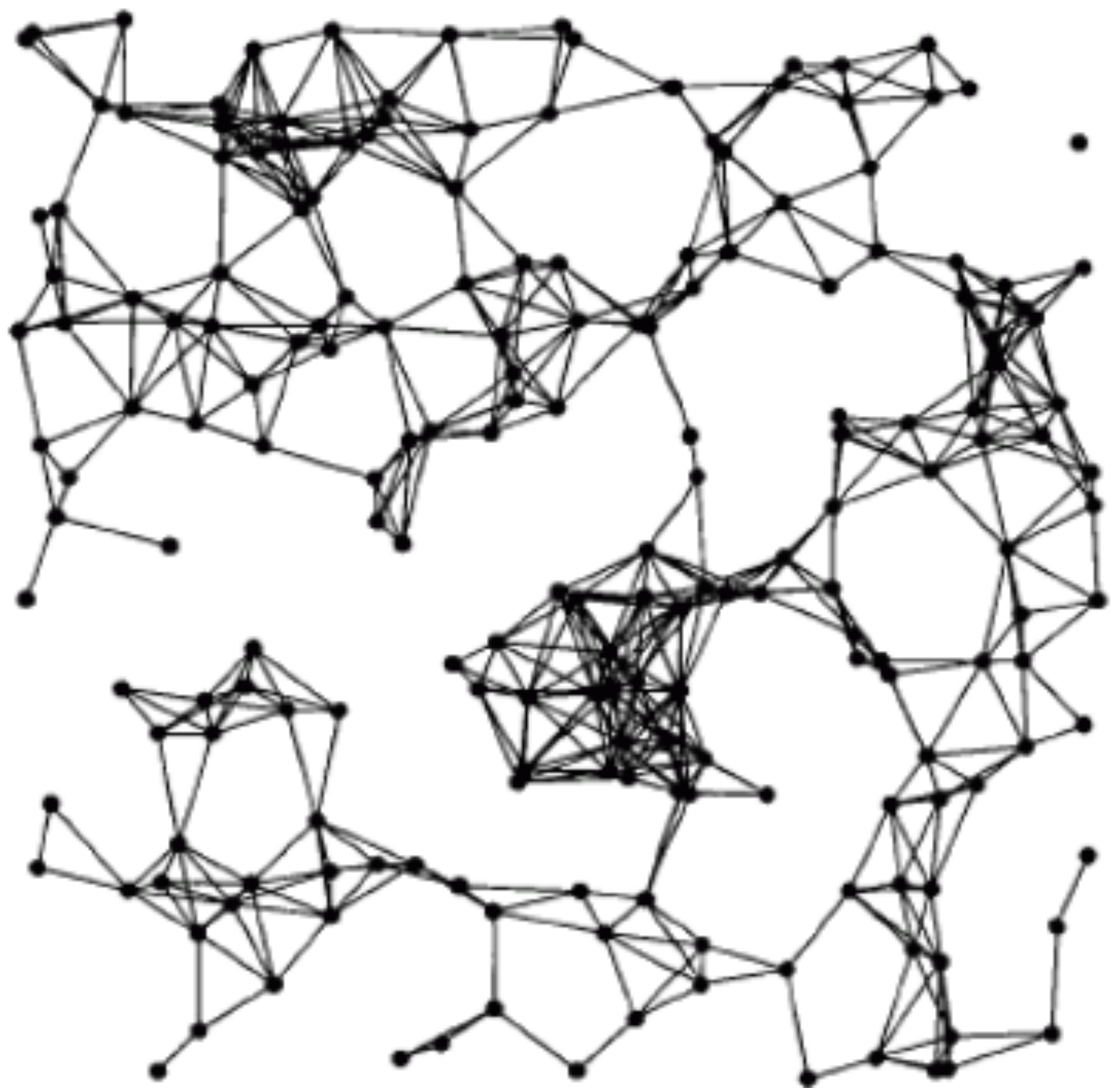

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Geometric Complexes

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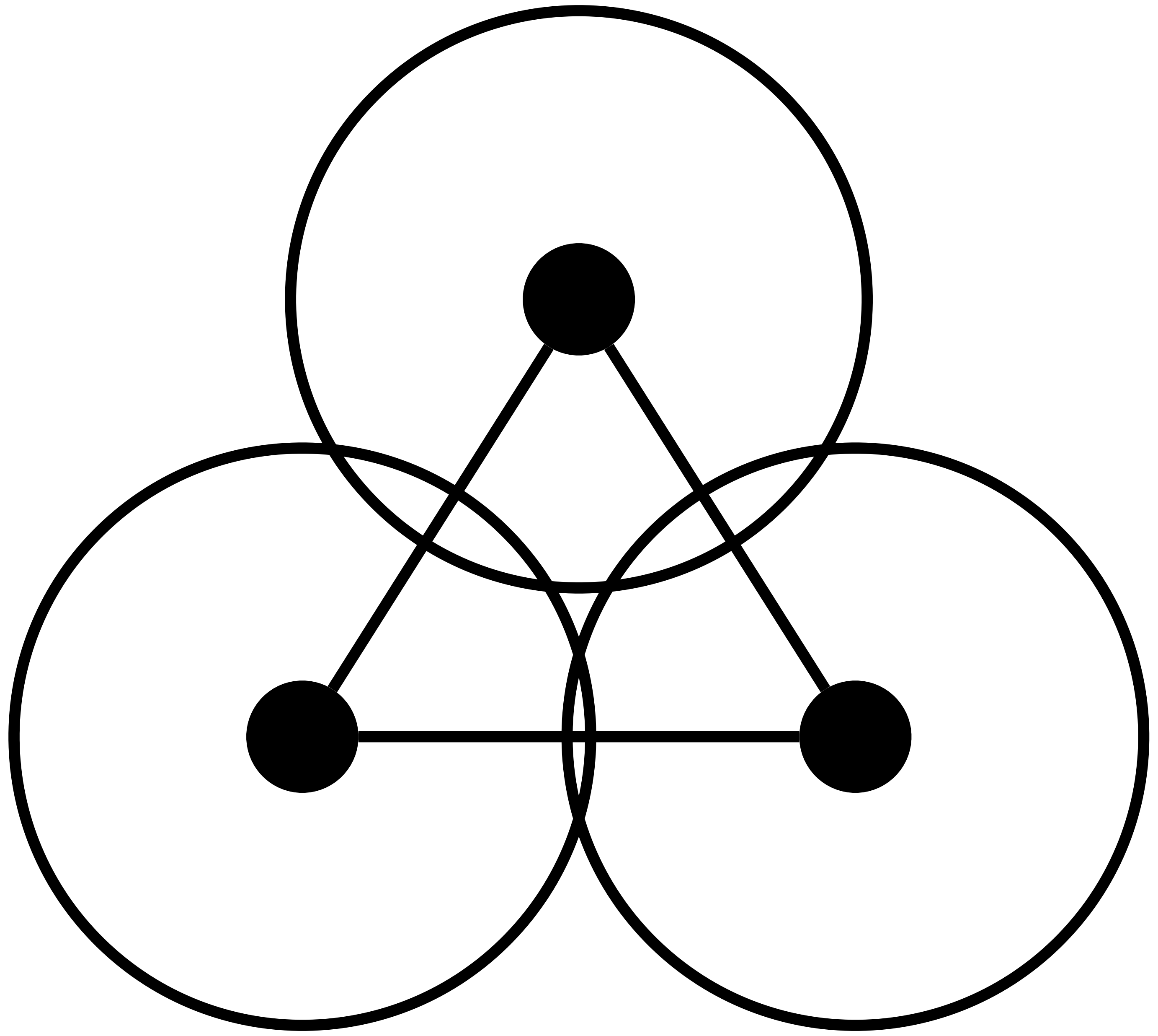

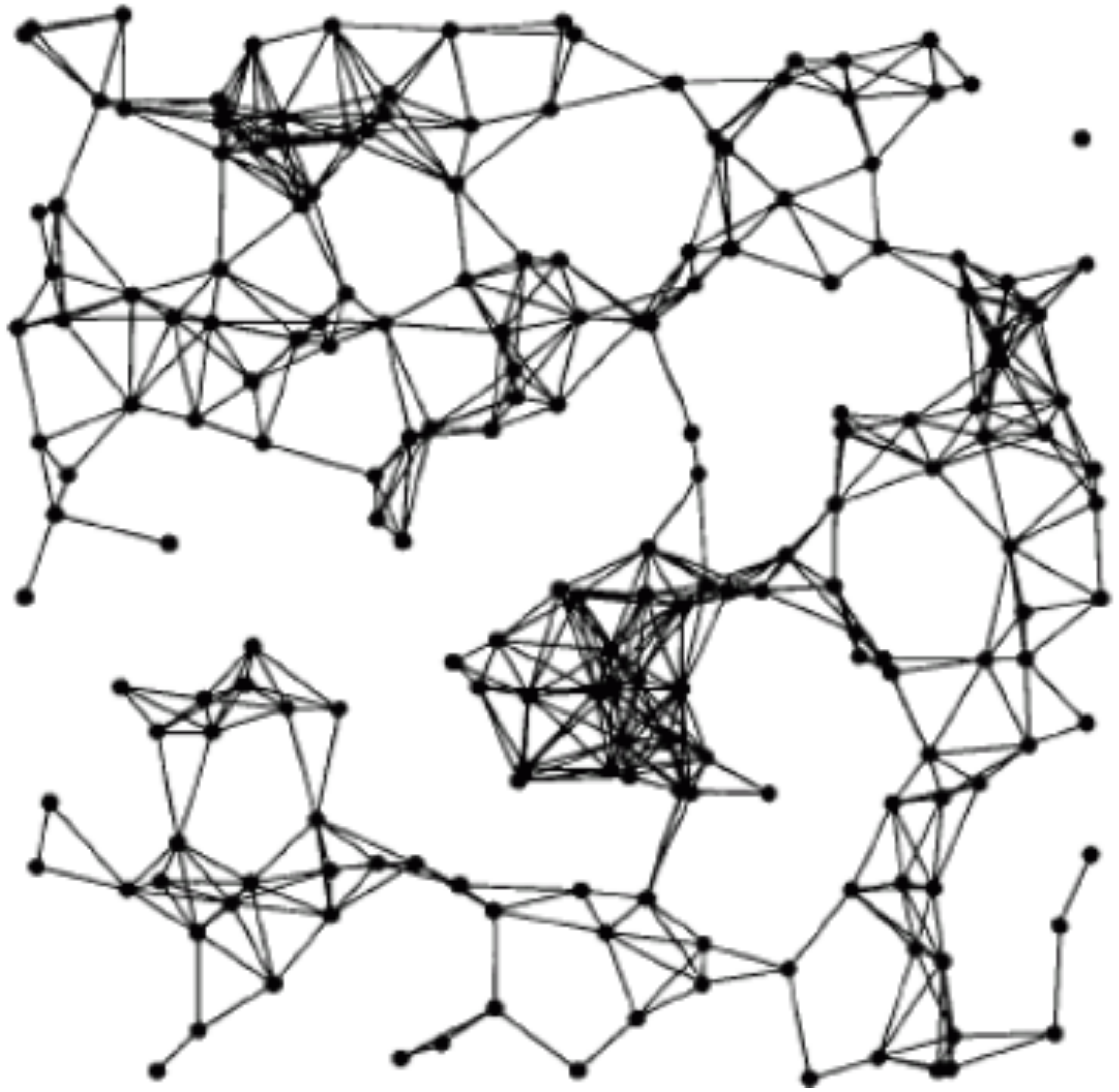

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Expected Betti numbers at dimension k

[Kahle 2011]

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- n, the number of points

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- n, the number of points
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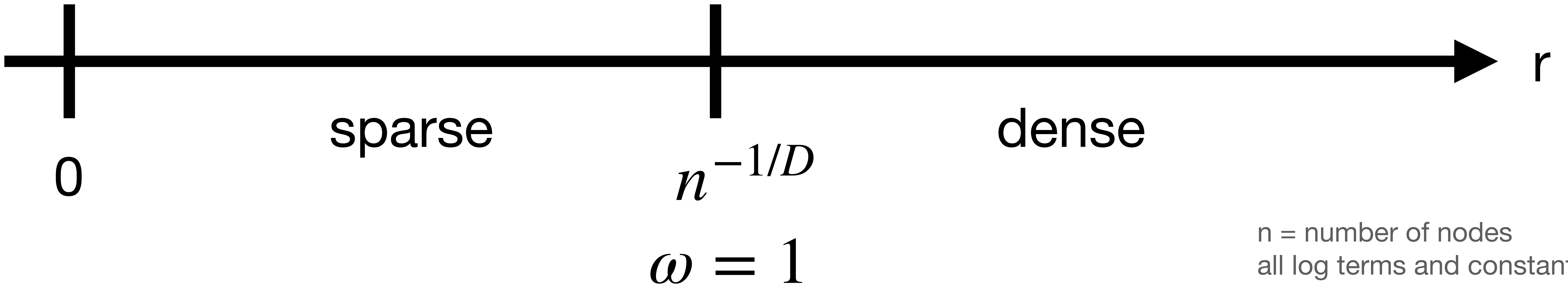

n = number of nodes
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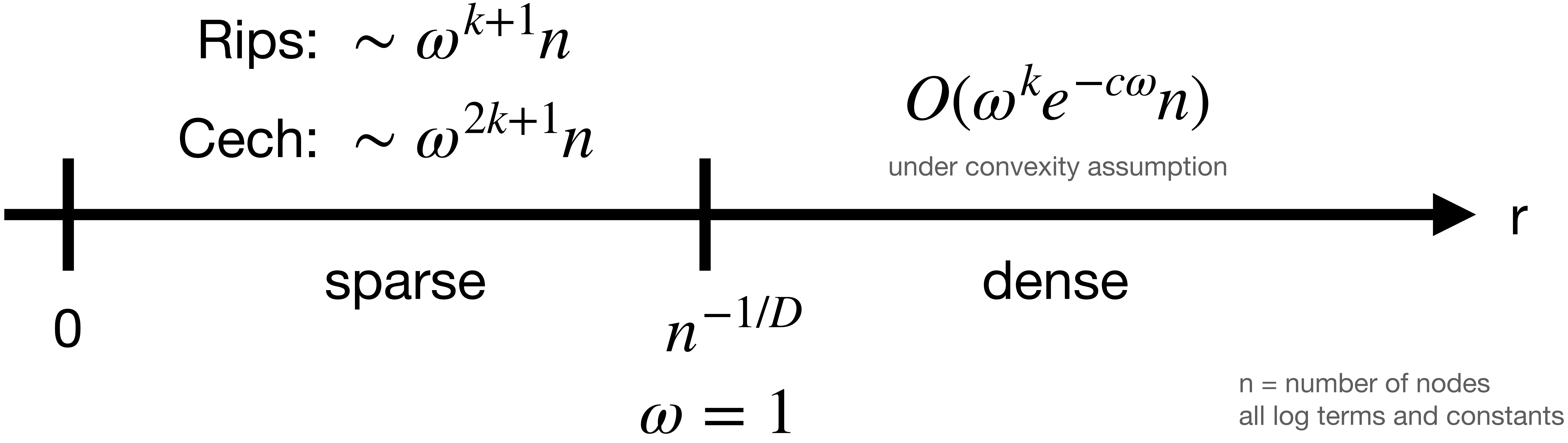

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Expected Betti numbers at dimension k
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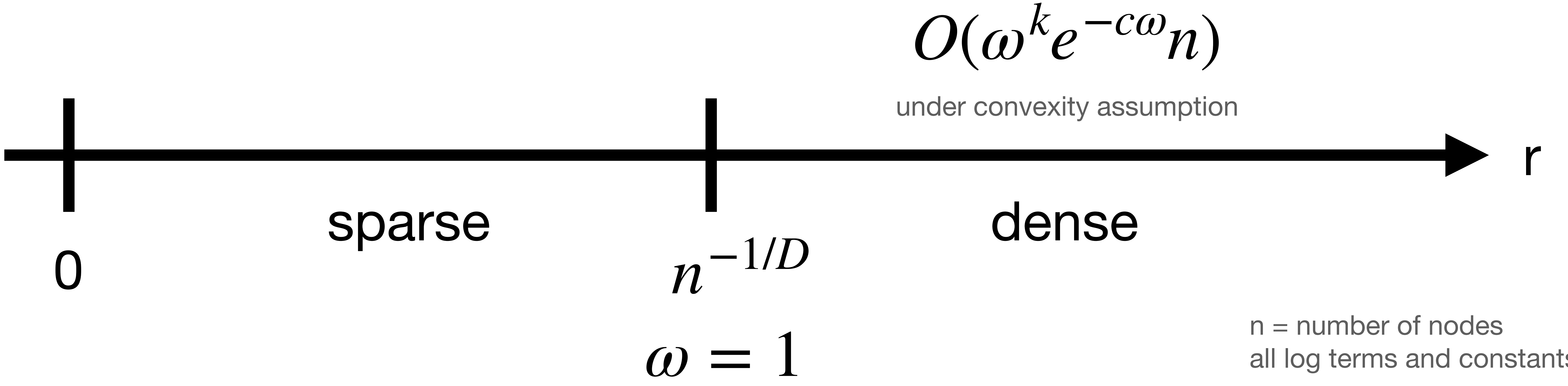
- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension
- $E\beta_k(\text{Cech}) \sim \omega^{2k+1}n$



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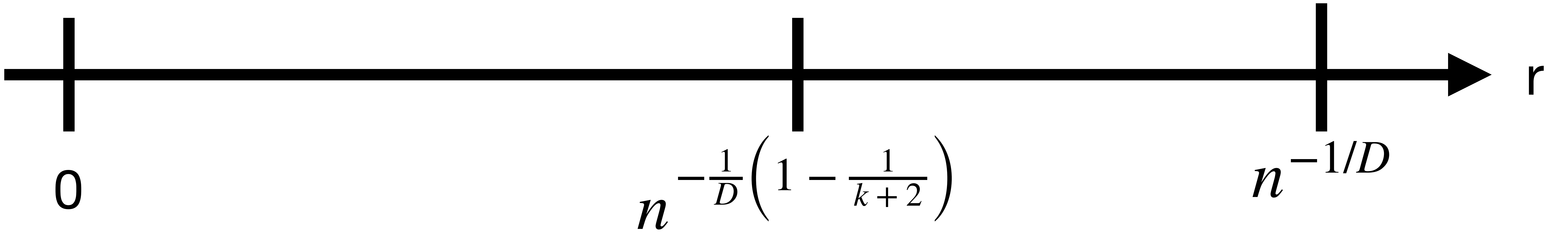

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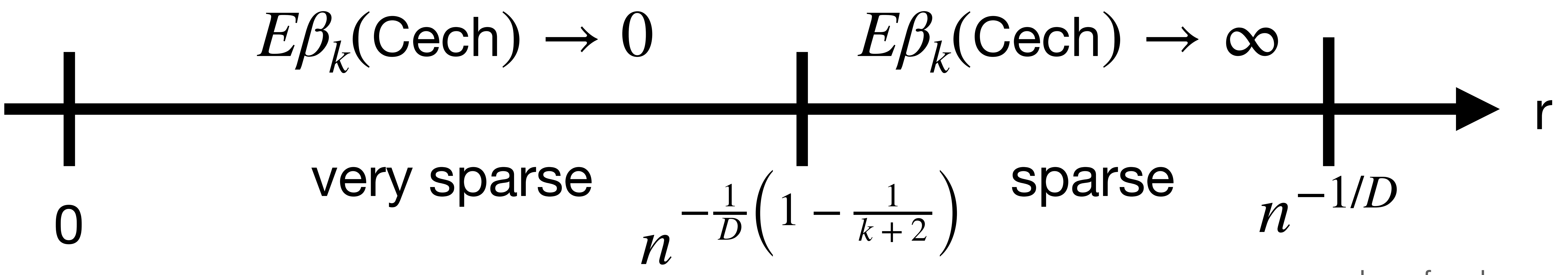

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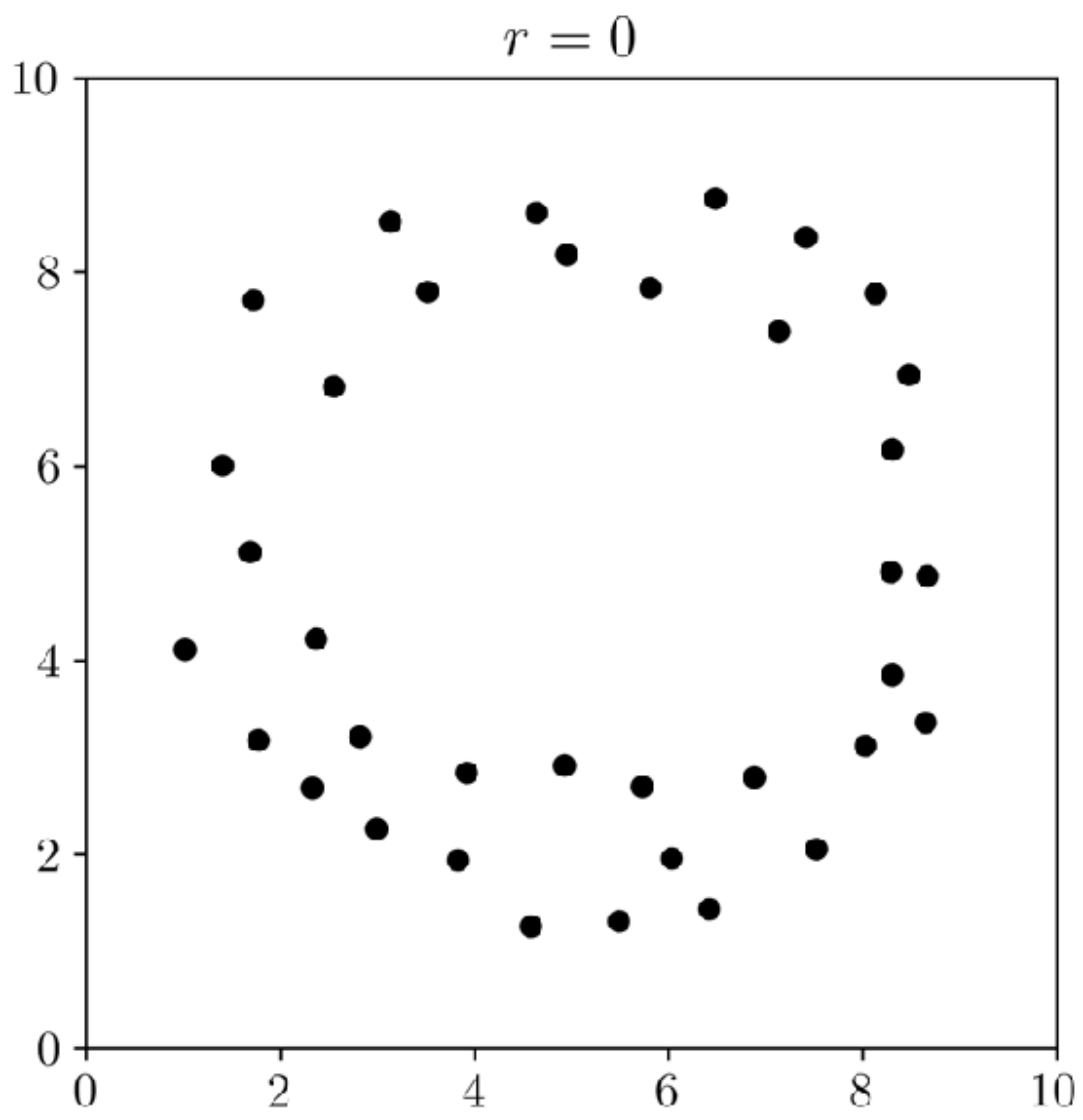
Maximally Persistent Cycles



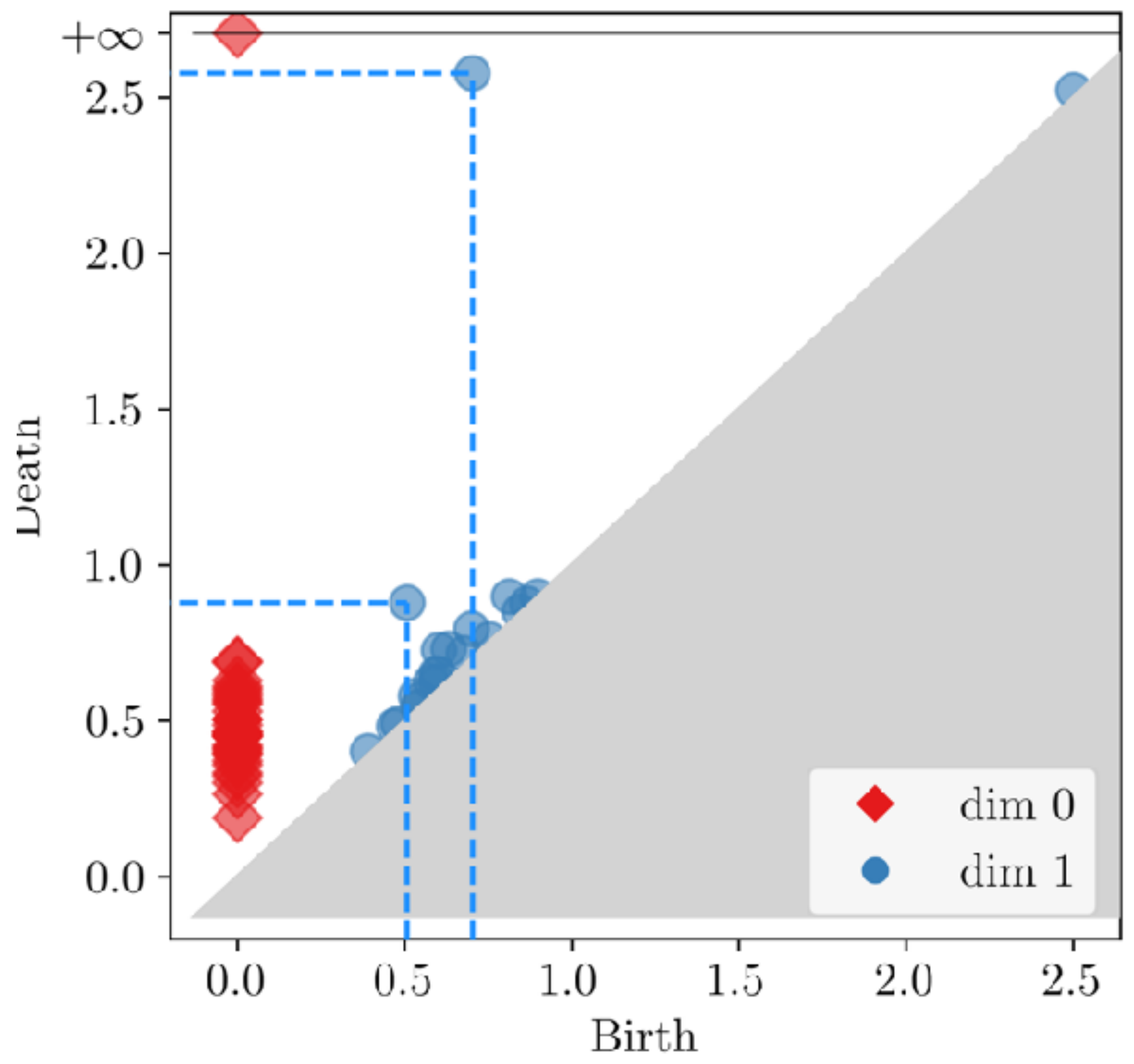


image credit: Andrey Yao

Maximally Persistent Cycles

n points in expectation

k-cycle

Maximally Persistent Cycles

[Bobrowski-Kahle-Skraba 2017]

n points in expectation

k-cycle

$$c\left(\frac{\log n}{\log\log n}\right)^{1/k} \leq \text{max persistence} \leq C\left(\frac{\log n}{\log\log n}\right)^{1/k}$$

a.a.s.

Geometric Complexes

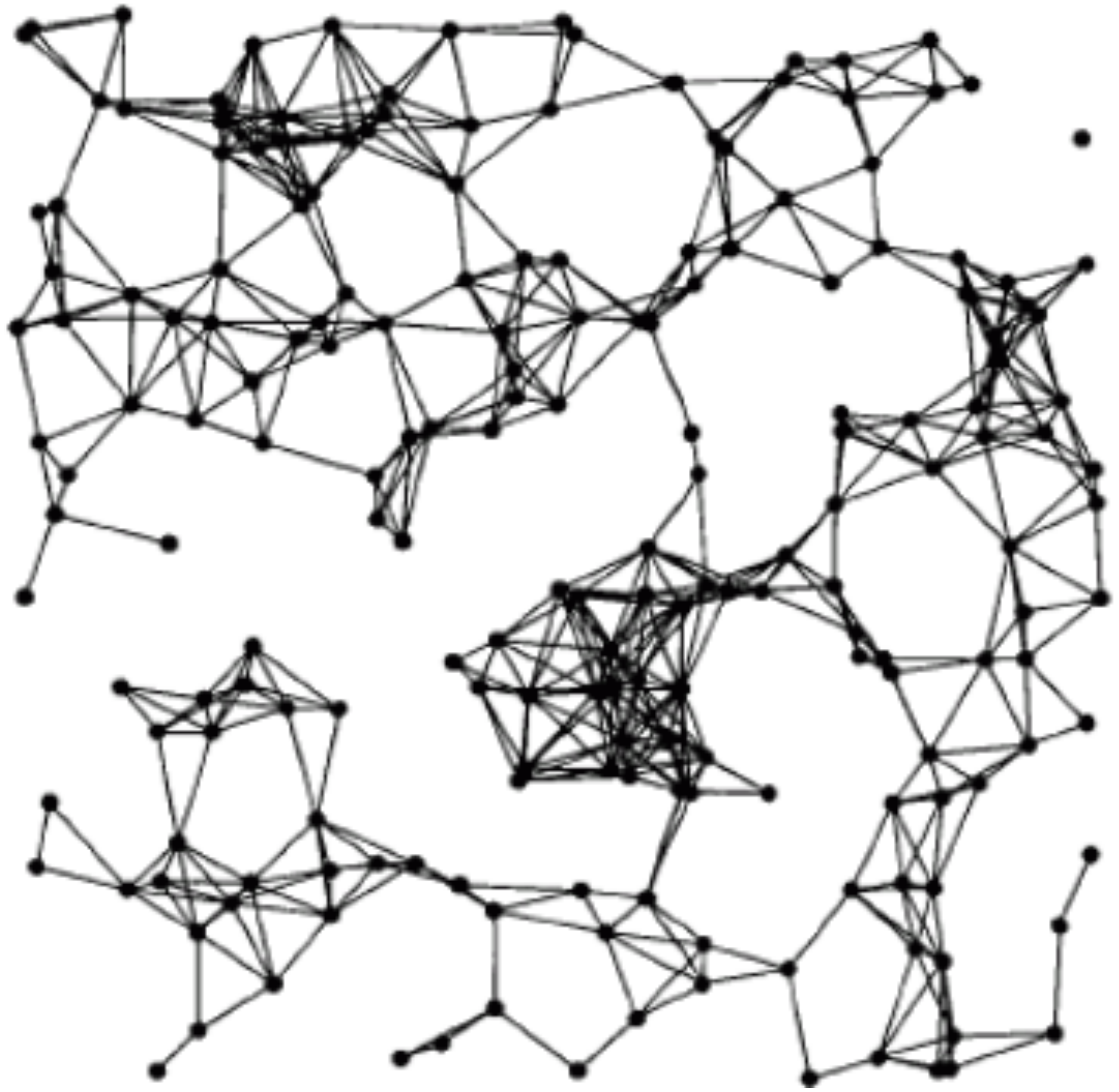
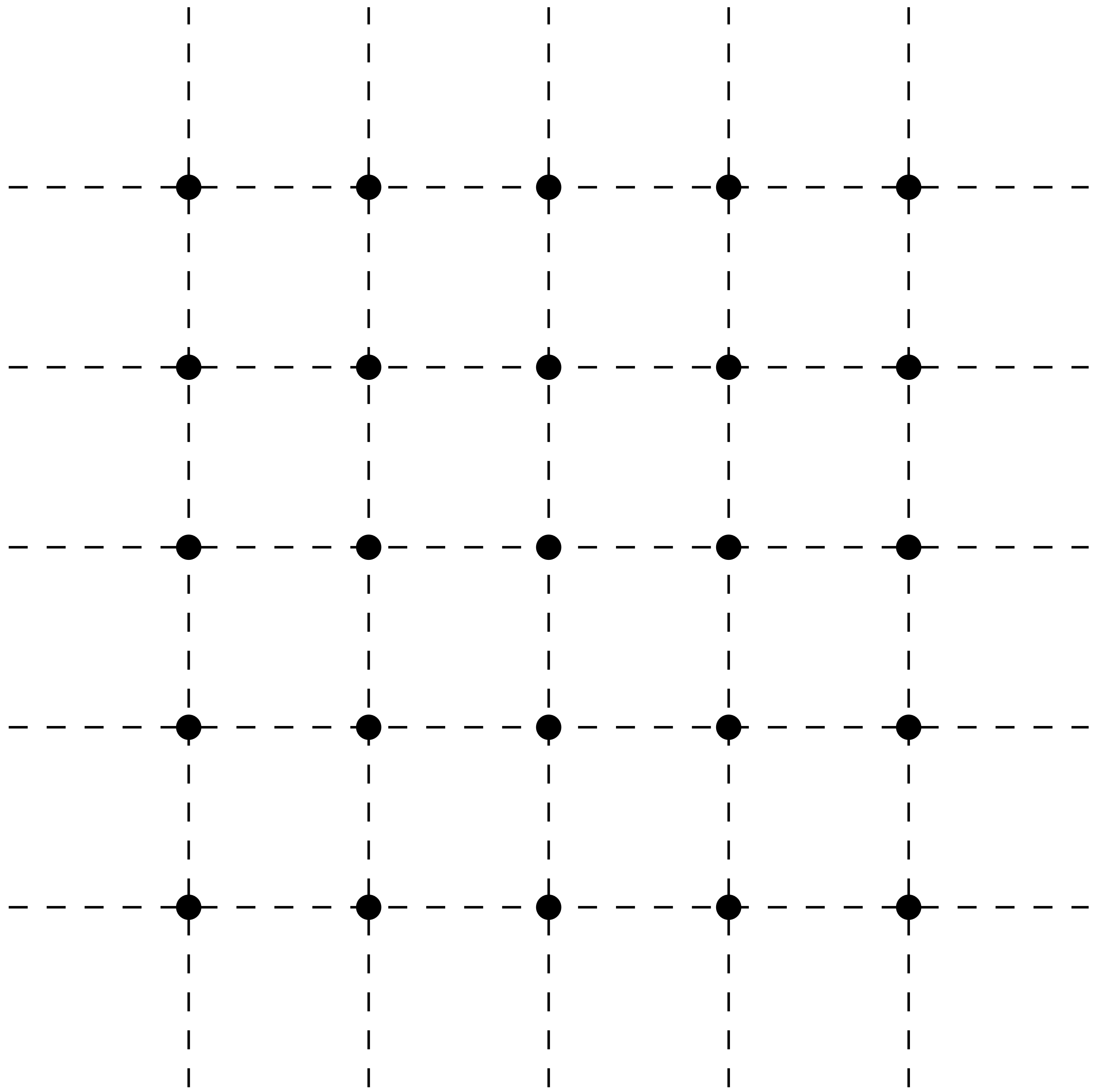


image credit: Penrose

Bernoulli Bond Percolation



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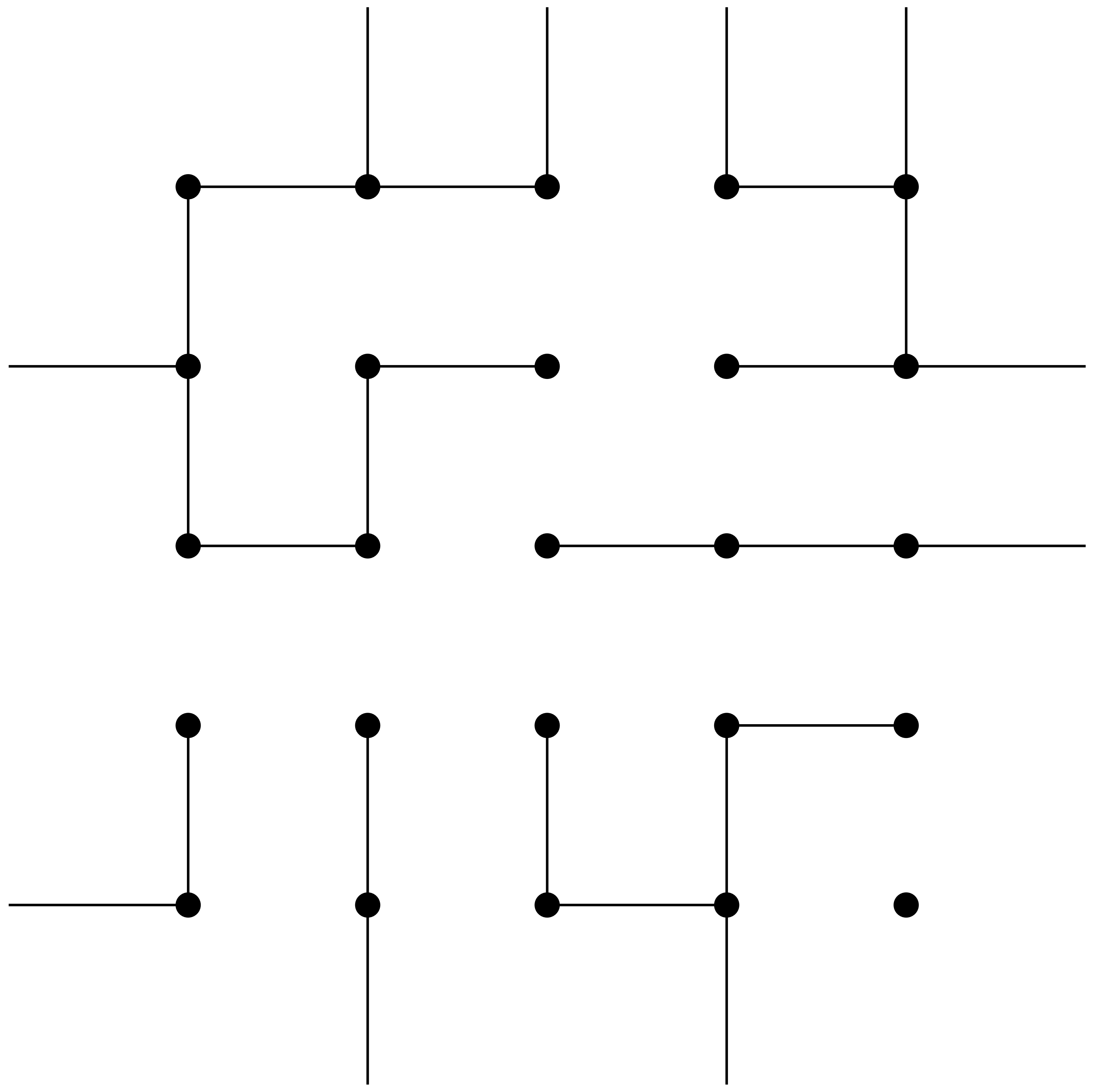
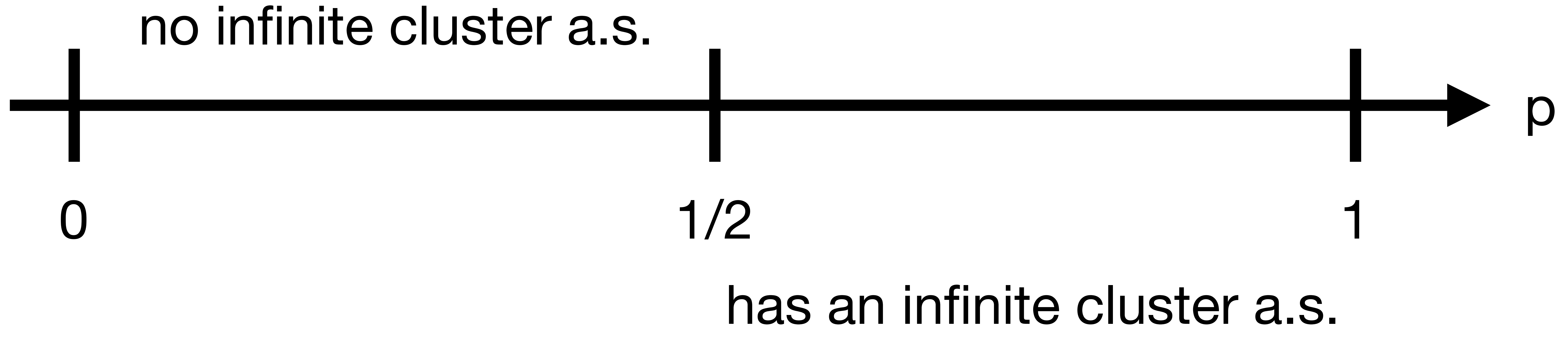


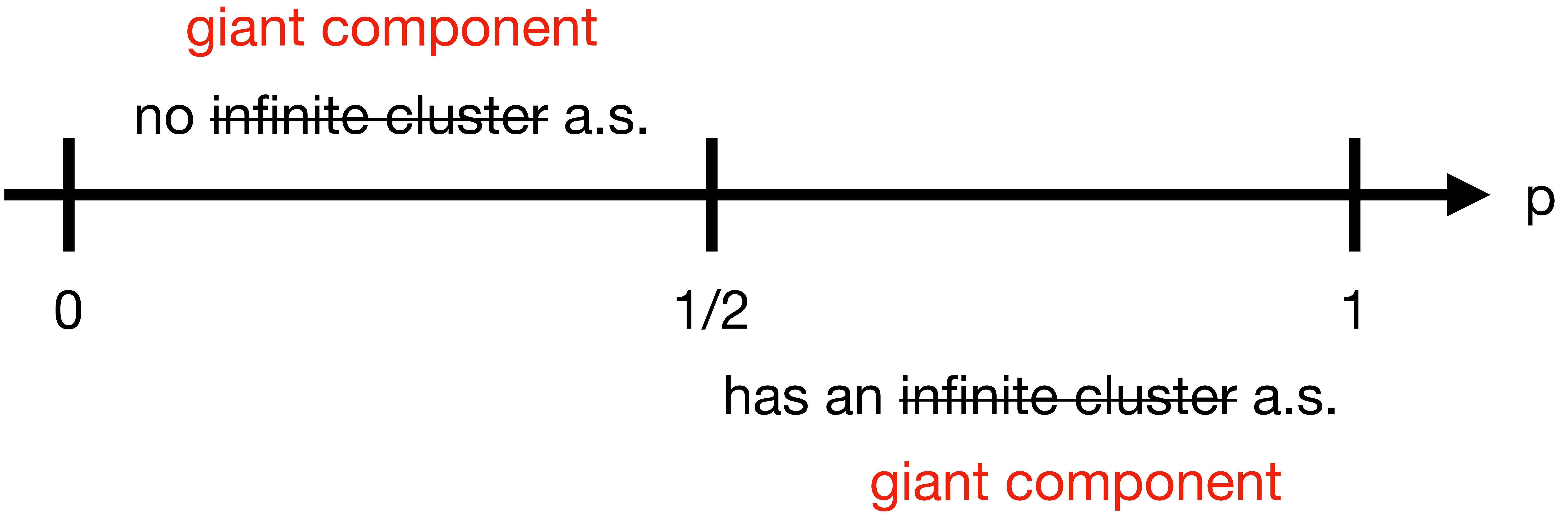
Phase Transition

[Harris 1960, Kesten 1980]



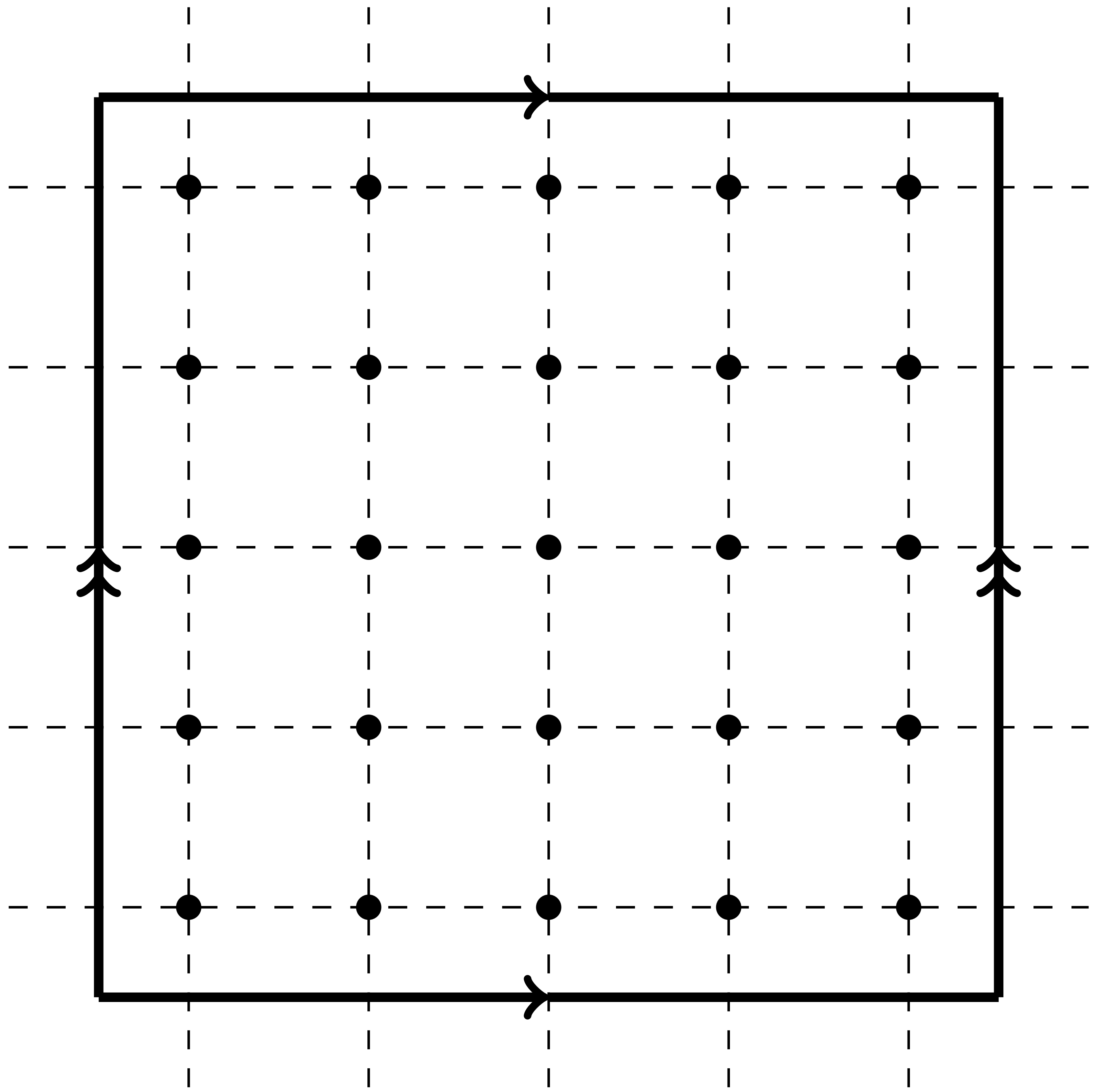
Phase Transition

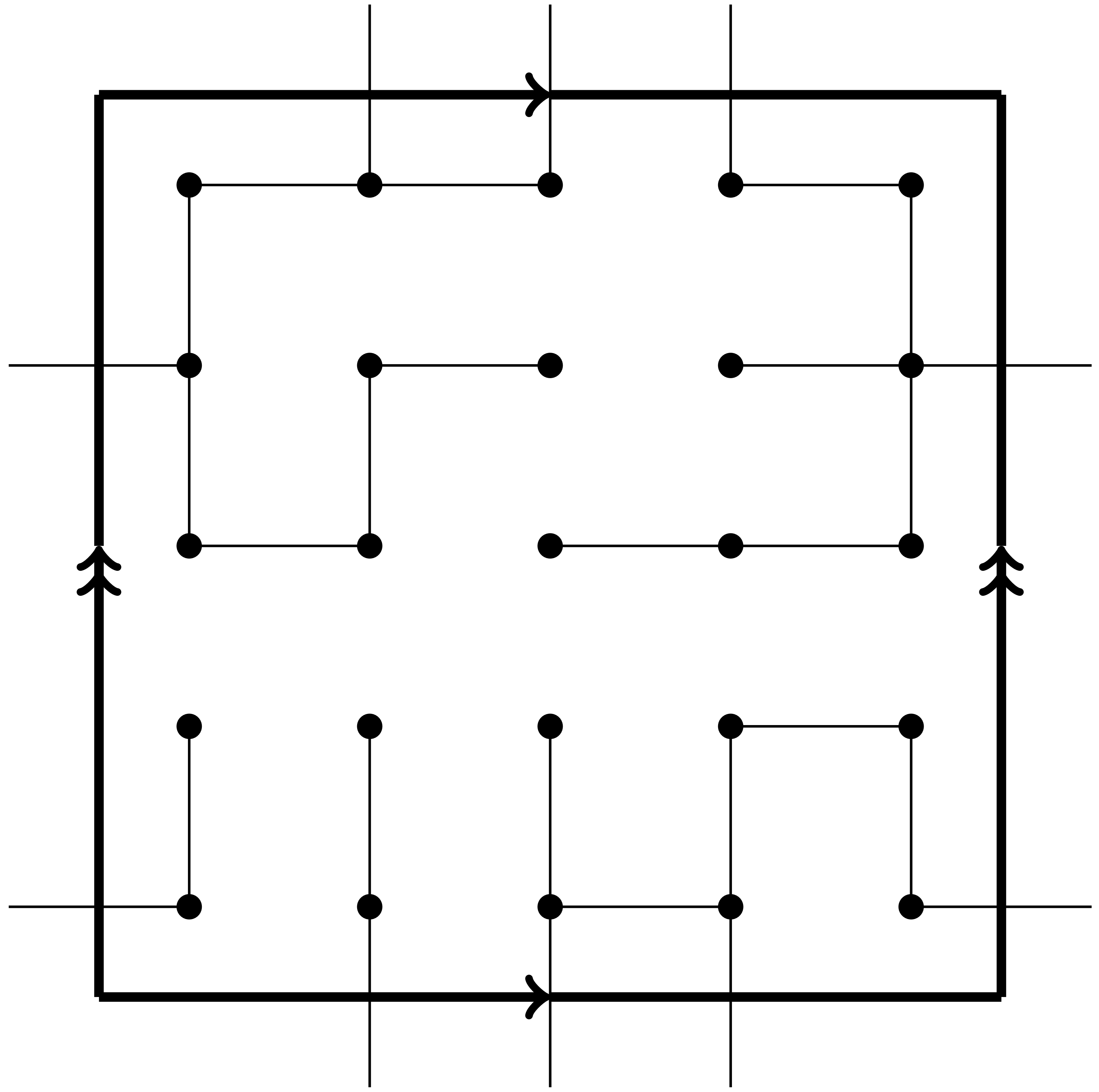
[Harris 1960, Kesten 1980]



Giant Cycles?

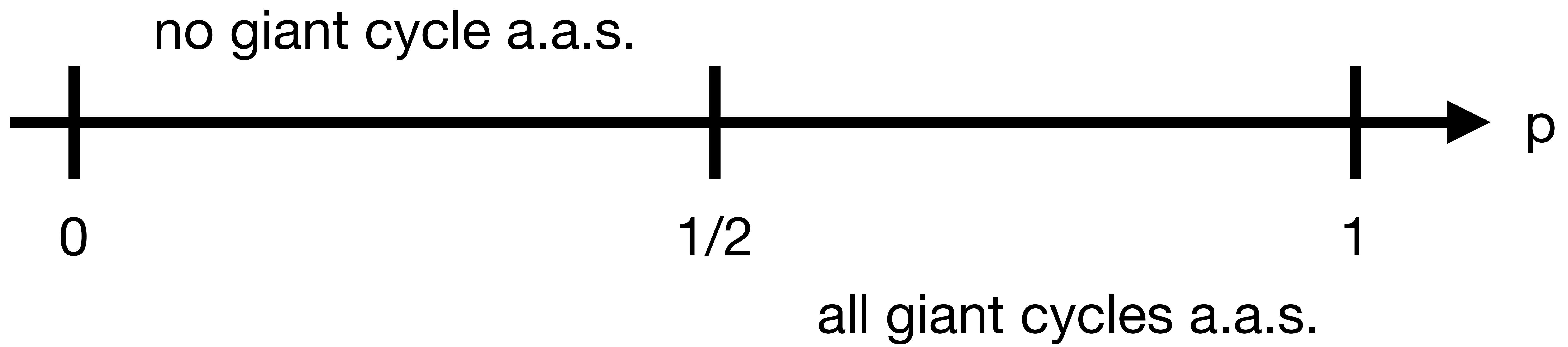
Bernoulli Bond Percolation



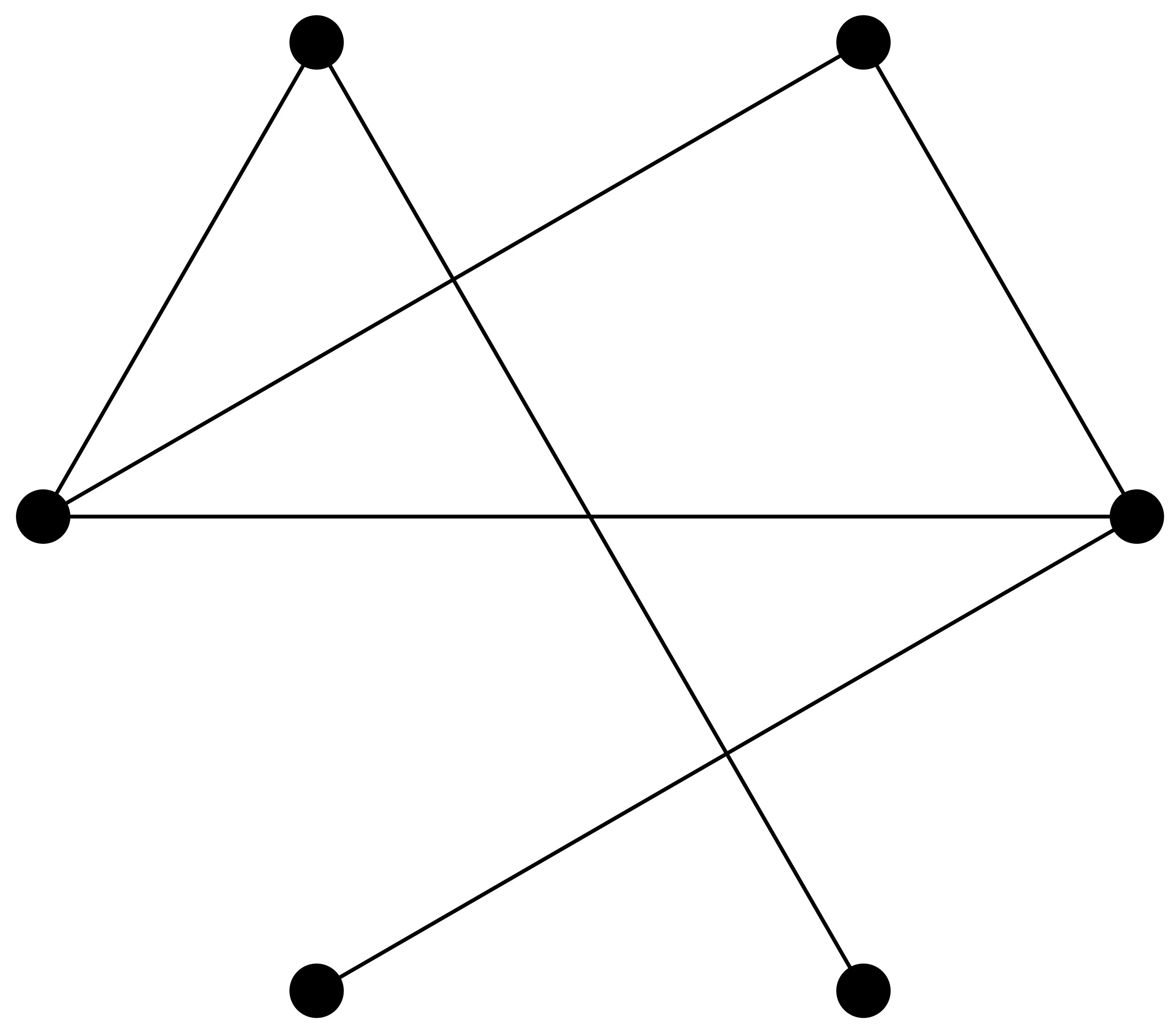


Phase Transition

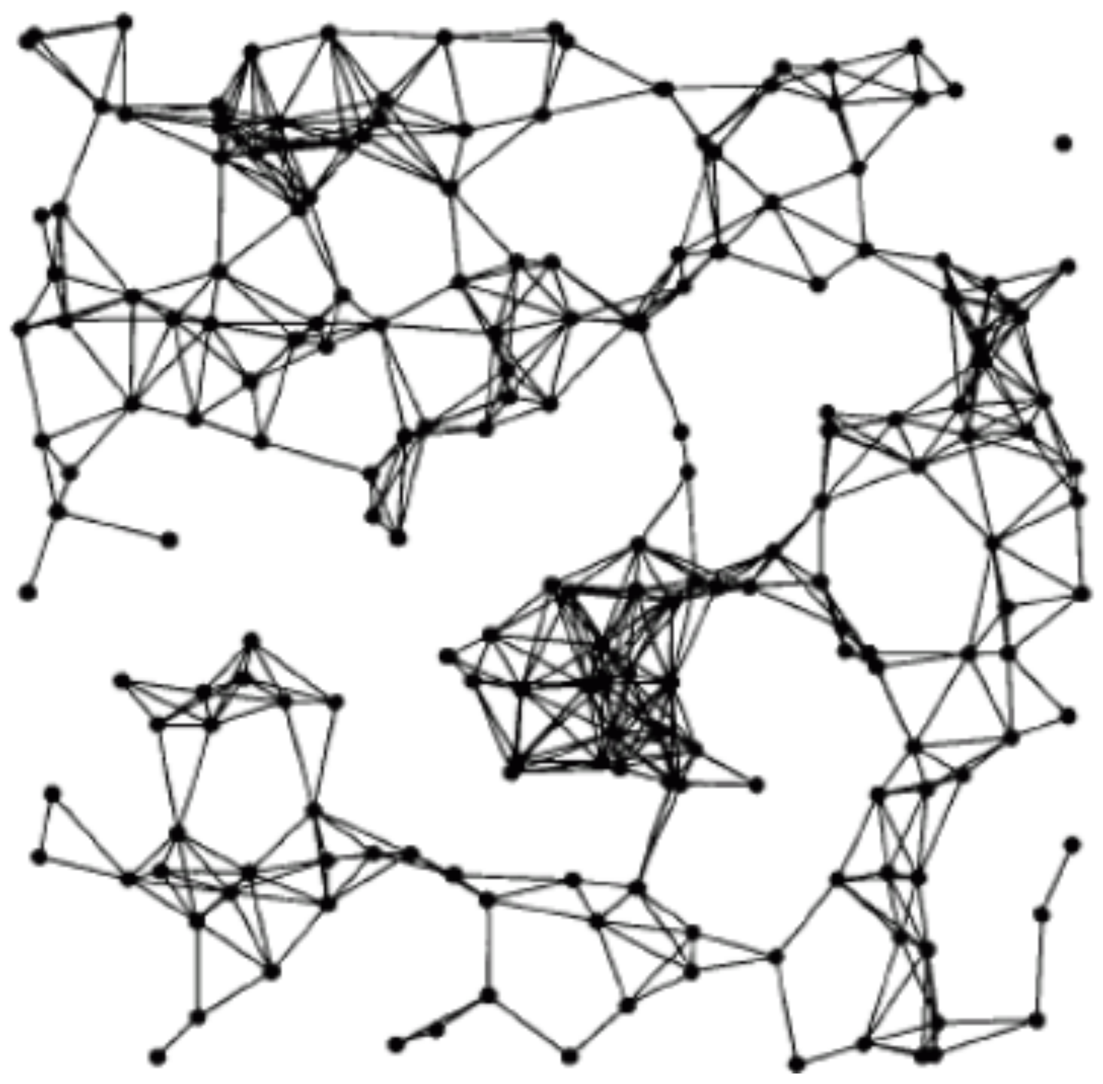
[Duncan-Kahle-Schweinhart, 2021]



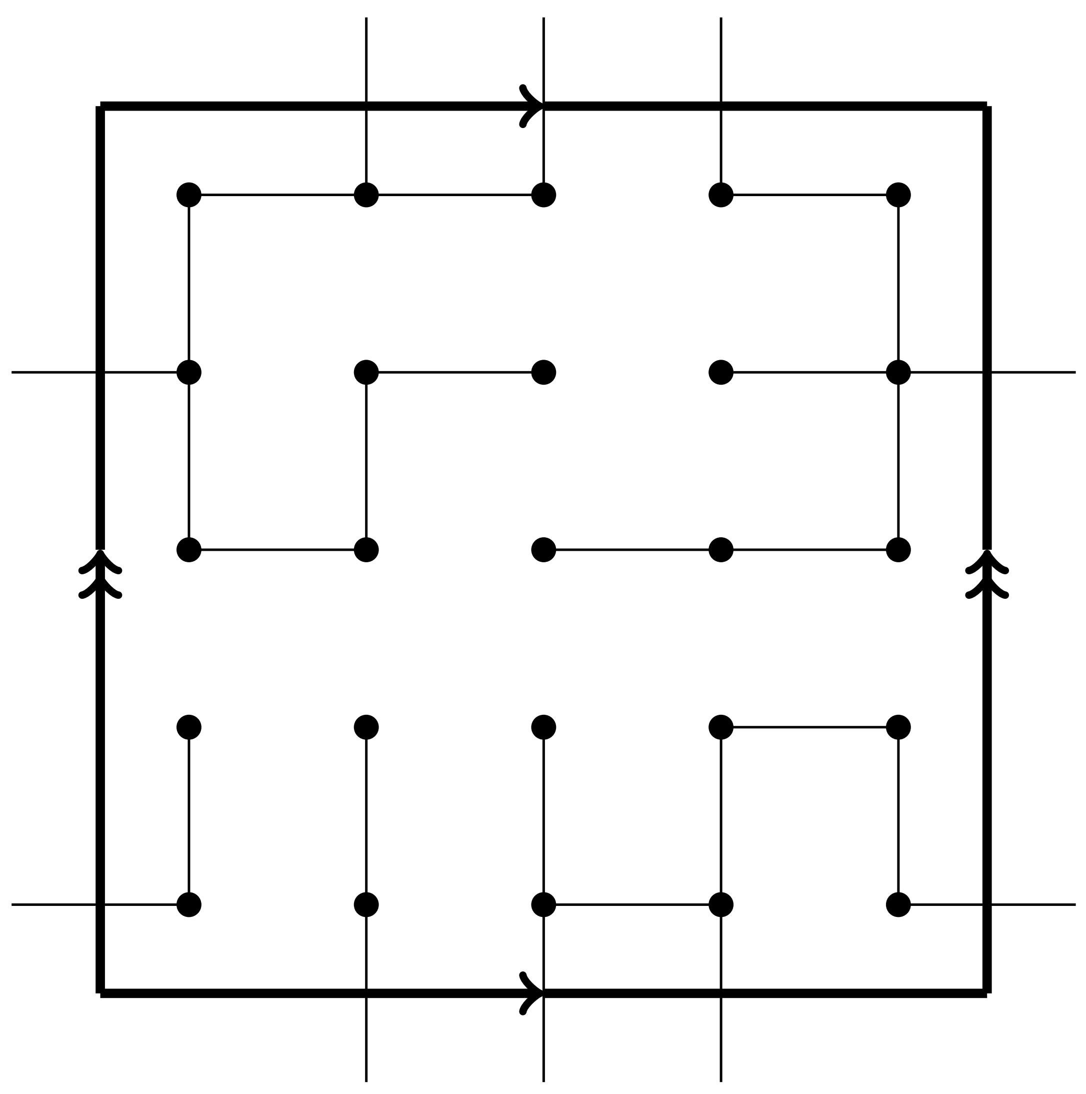
Tapas at Random Topology



Erdo-Renyi Complexes



Geometric Complexes



Topological Percolation

III. Preferential Attachment

A Non-Homogeneous Model

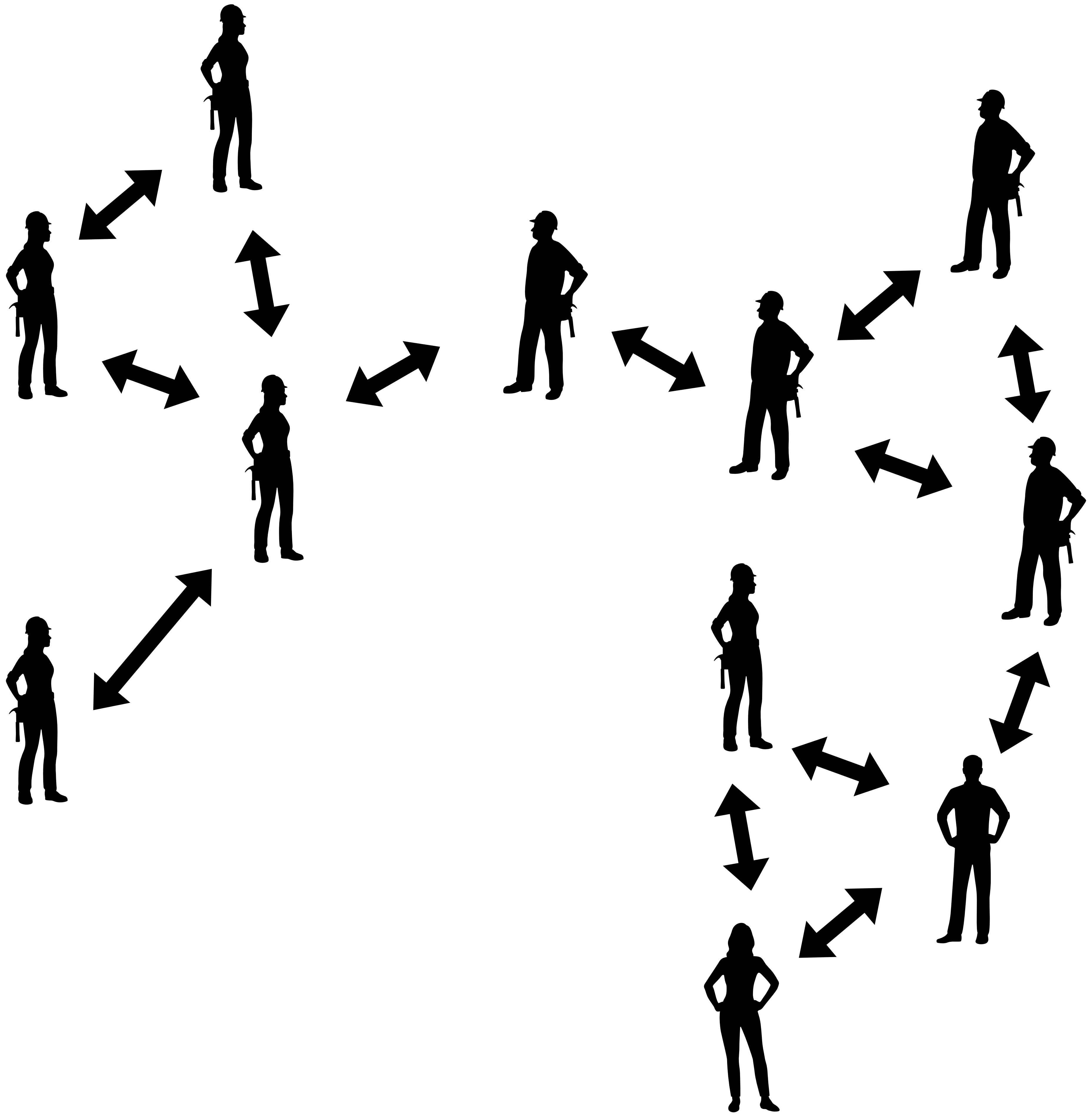
Preferential Attachment

[Albert and Barabasi 1999]

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Preferential Attachment

[Albert and Barabasi 1999]



Preferential Attachment

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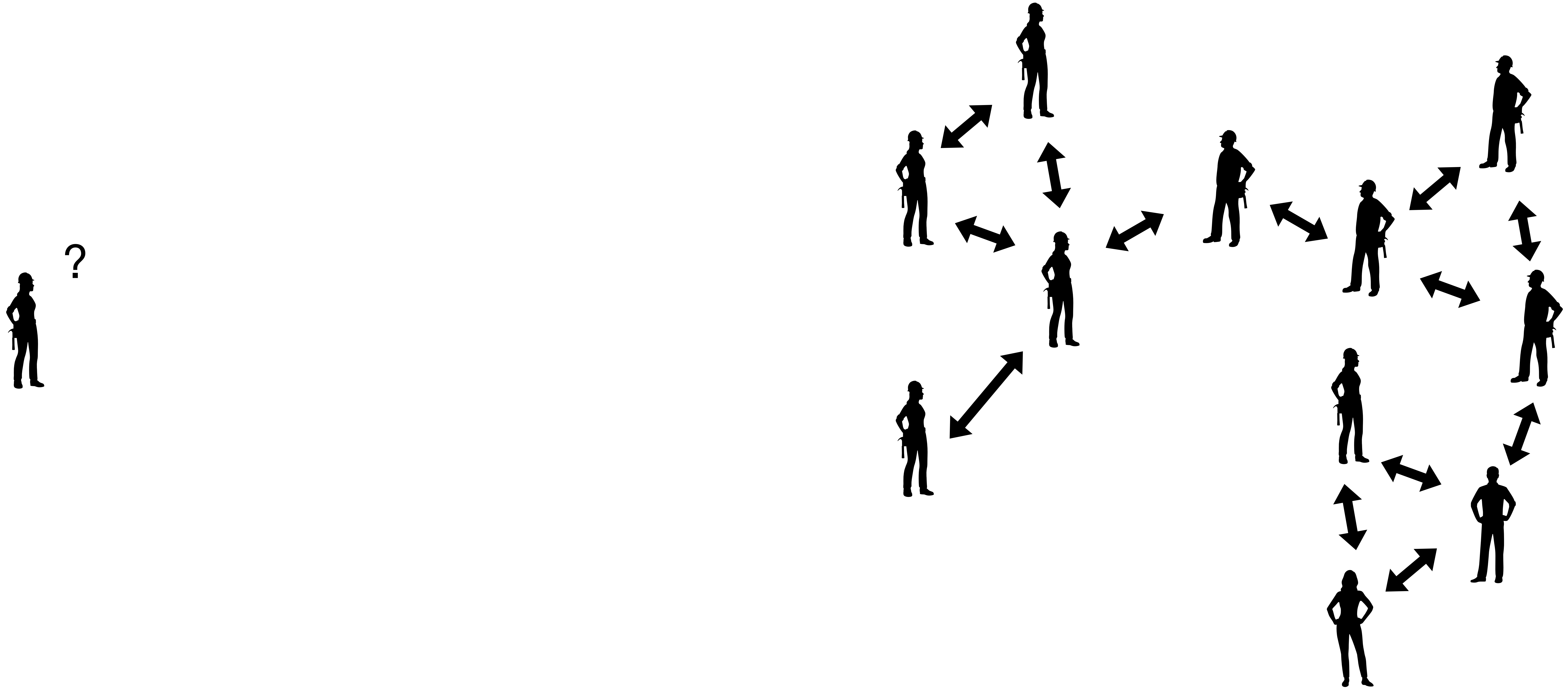

Preferential Attachment

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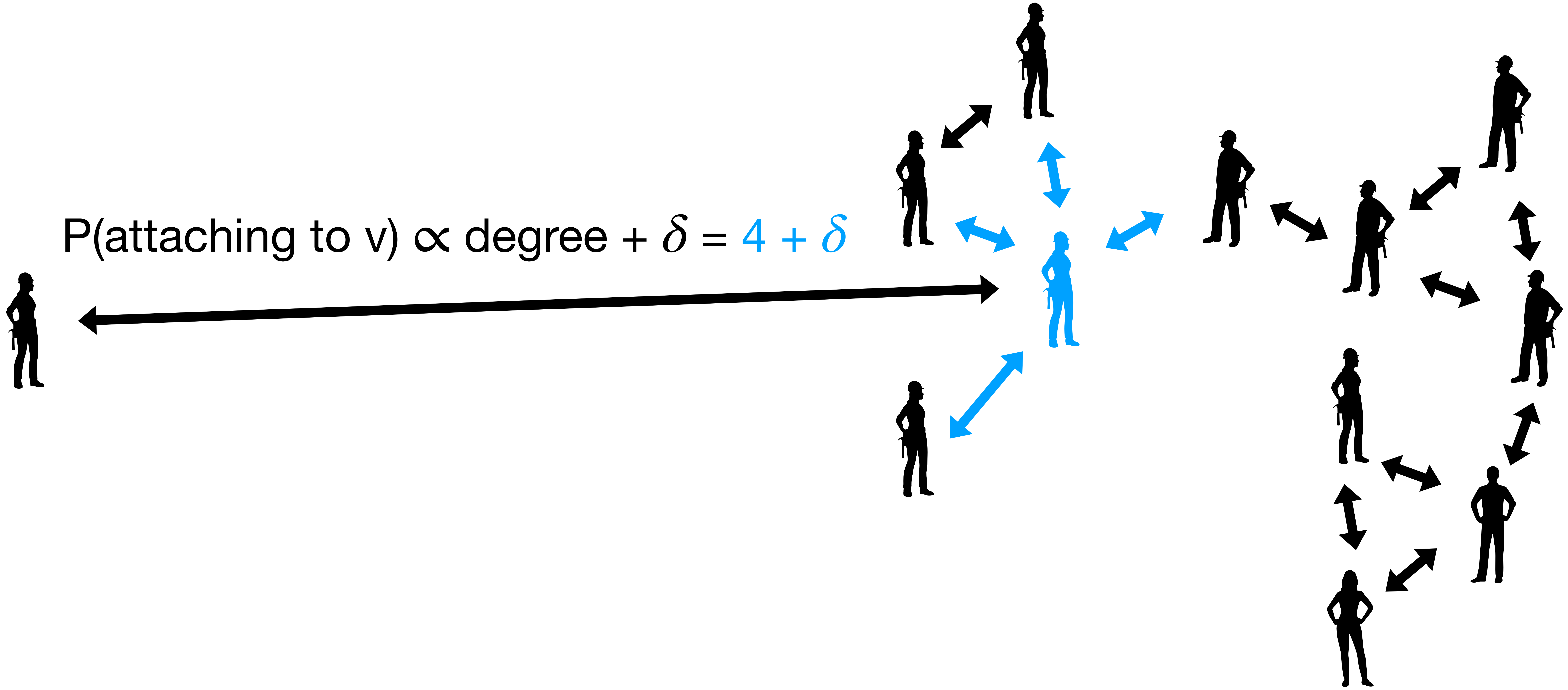
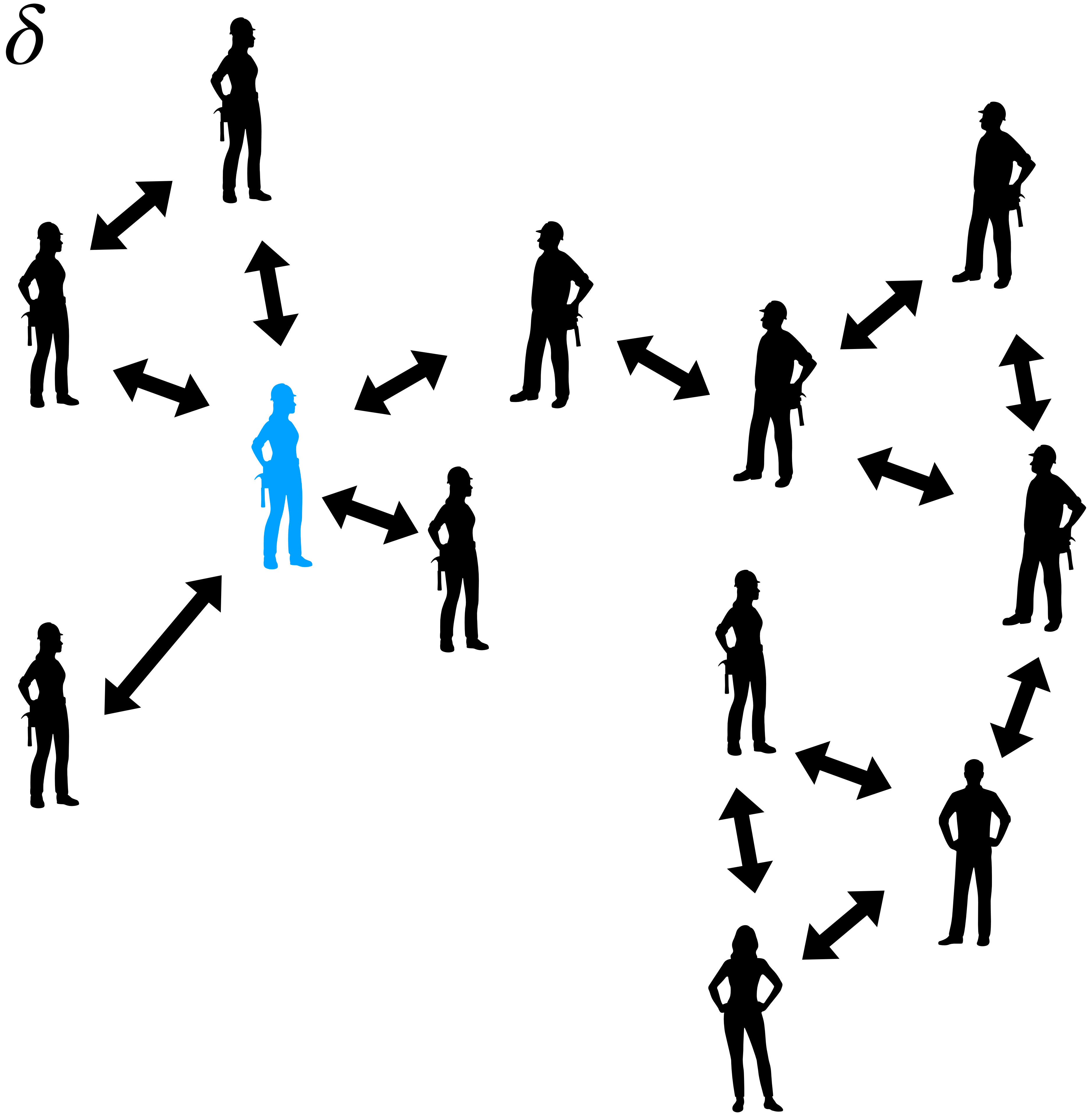


Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) $\propto$ degree + a tuning parameter δ



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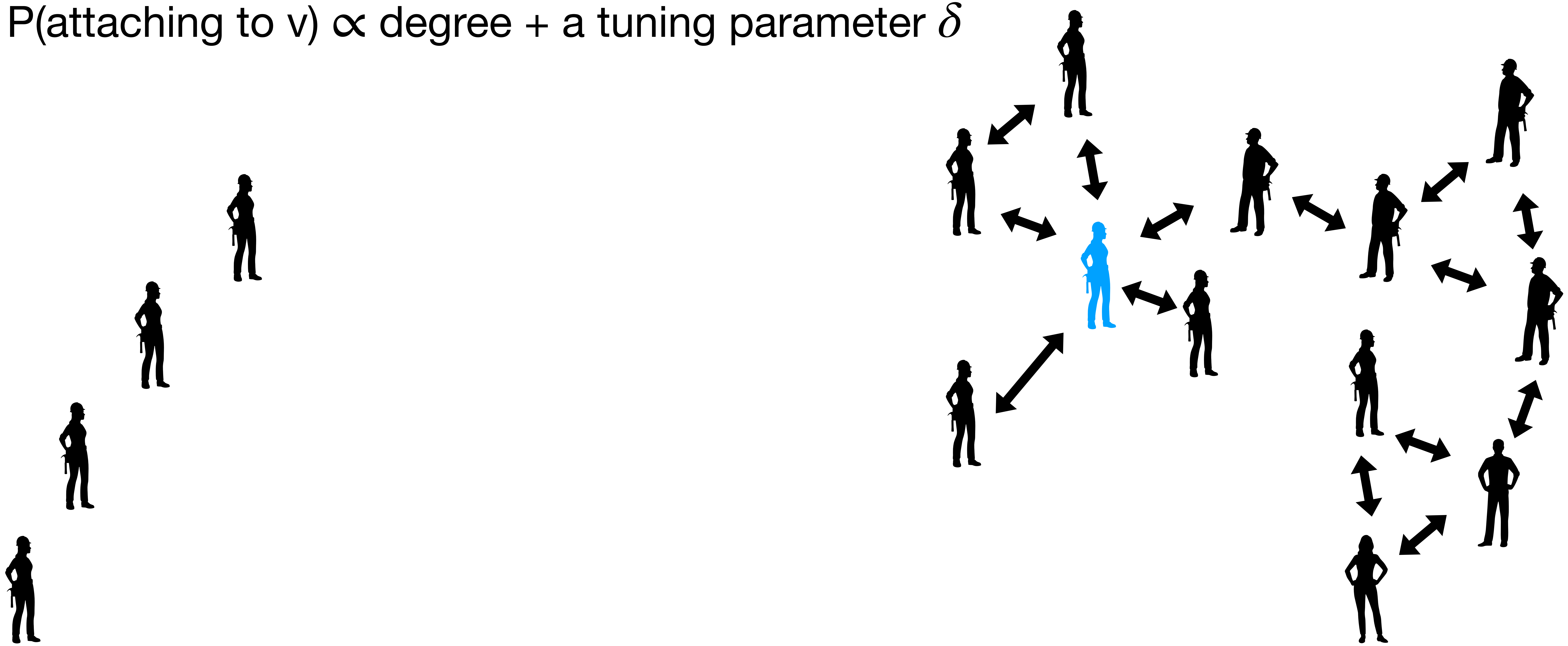

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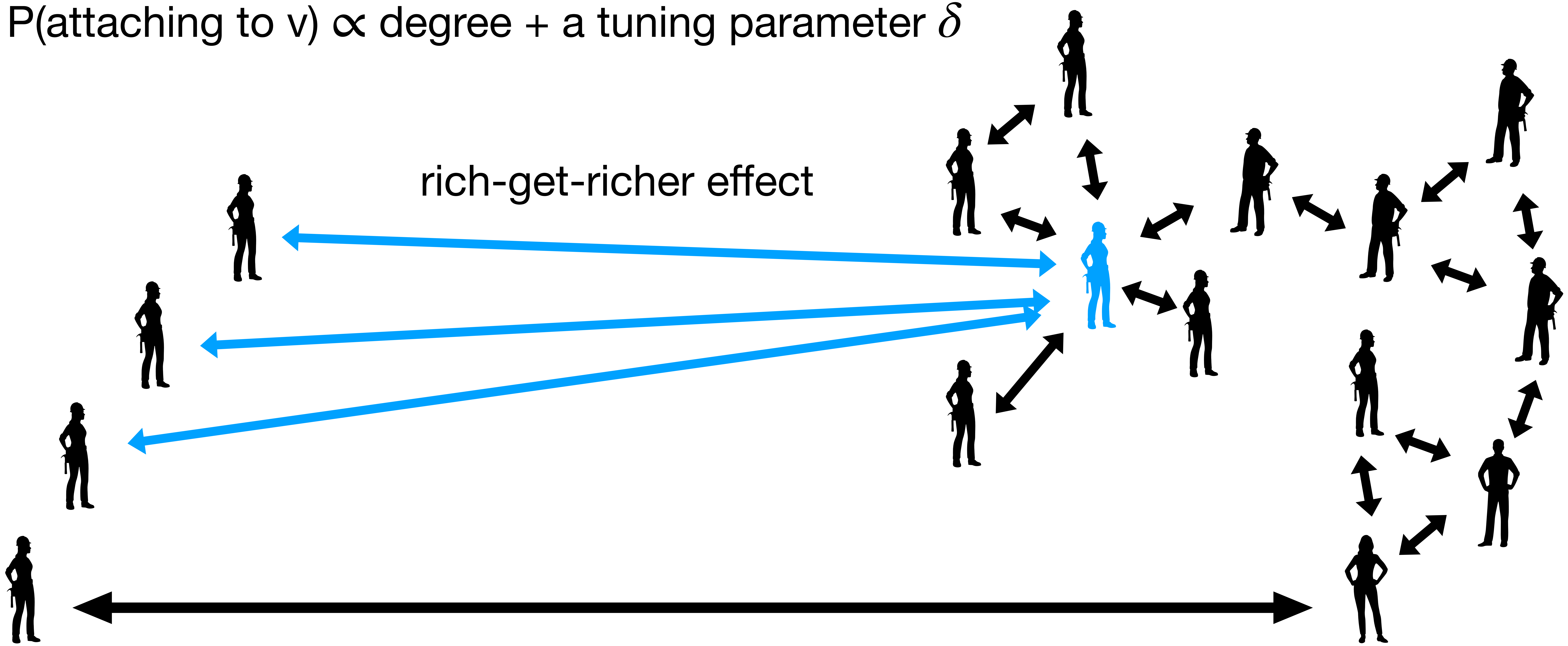

What do we know?

What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]

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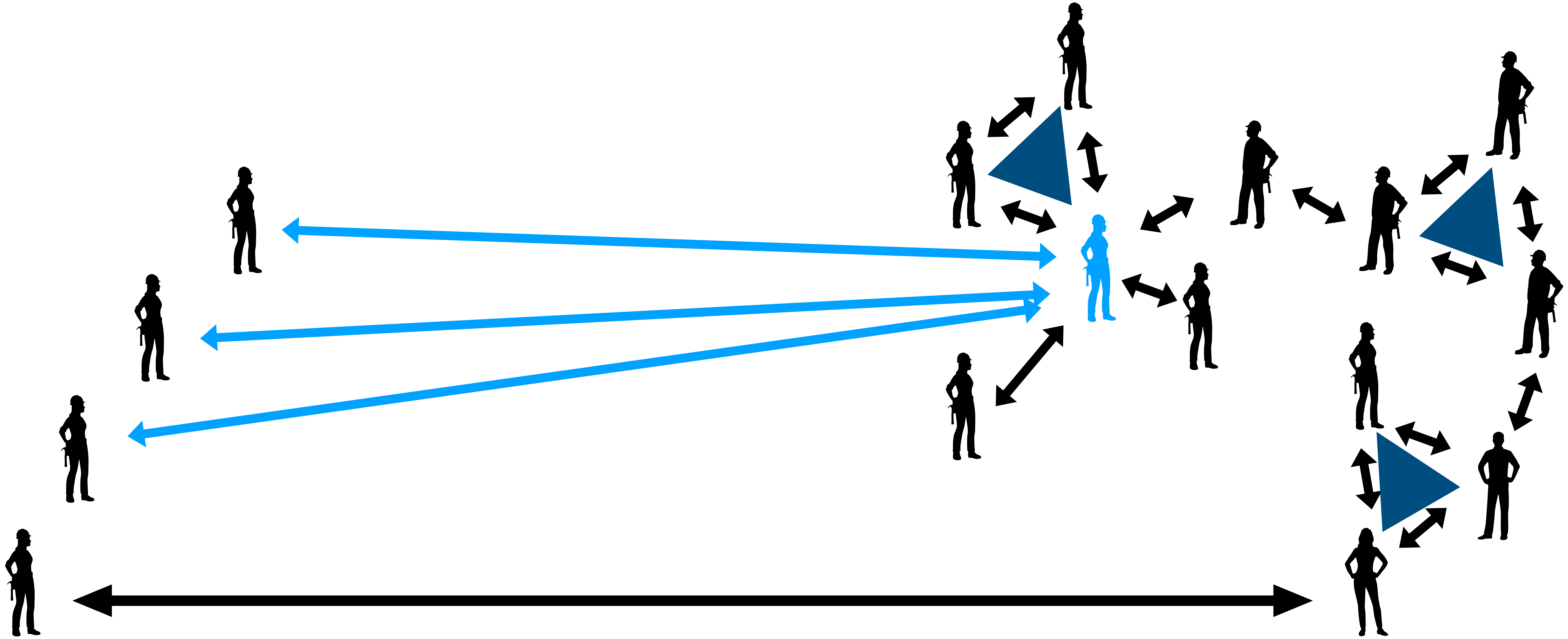
- subgraph counts [Garavaglia and Steghuis 2019]

What do we know?

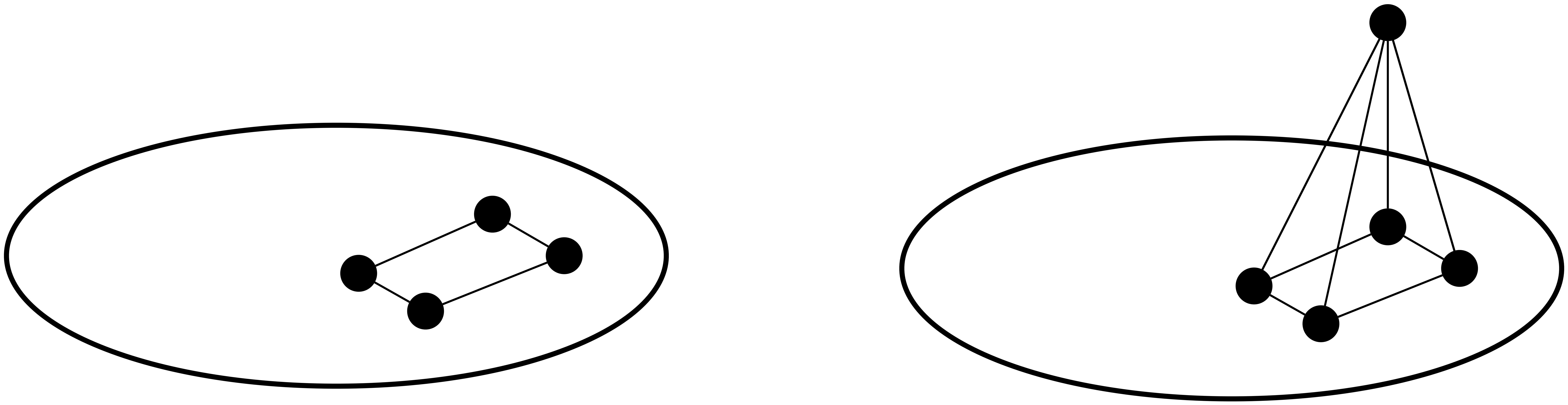
- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]
- and more…

Clique Complex

aka Flag Complex

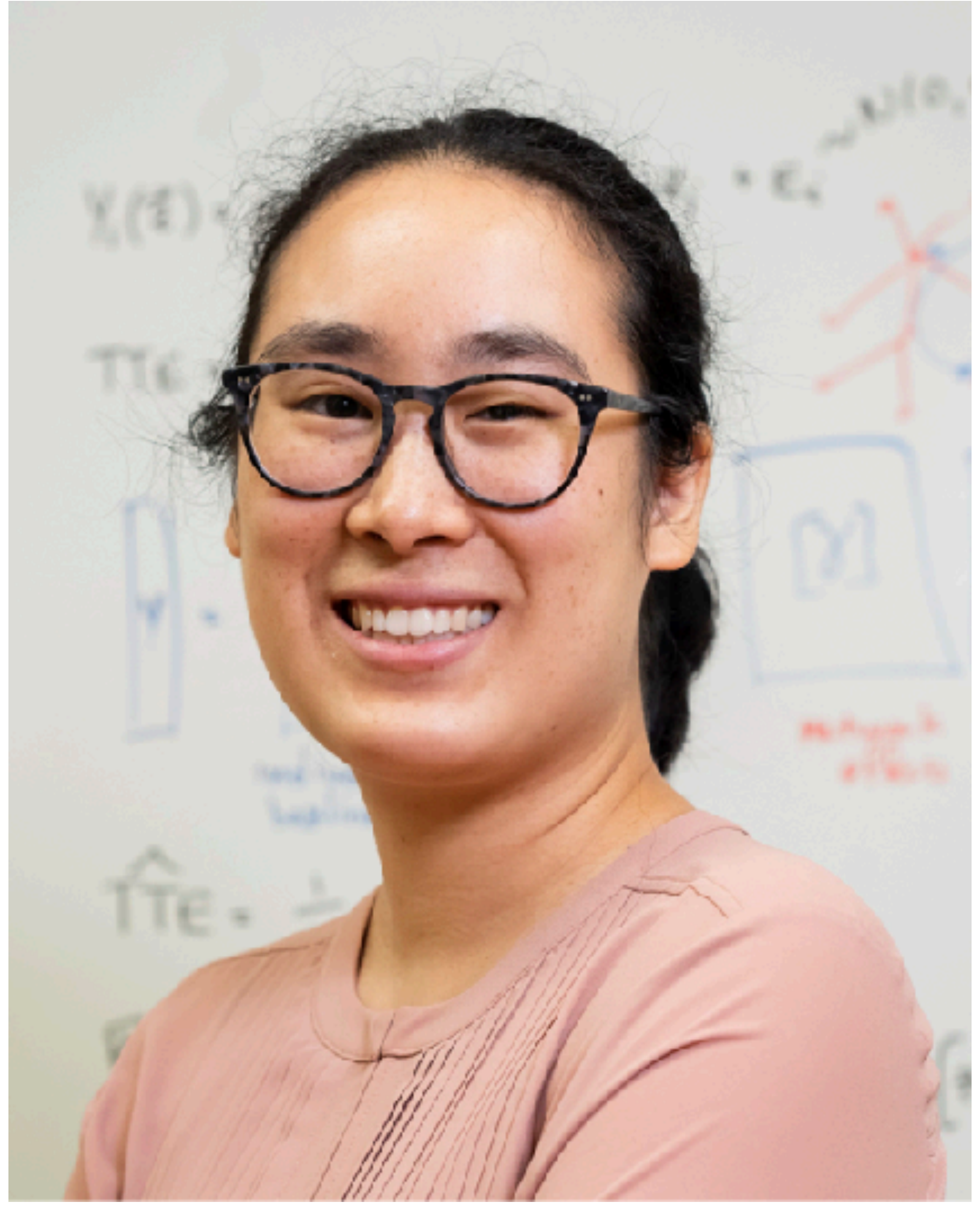


Clique Complex = Mapping Cone



III Topology of Preferential Attachment

My Lovely Collaborators



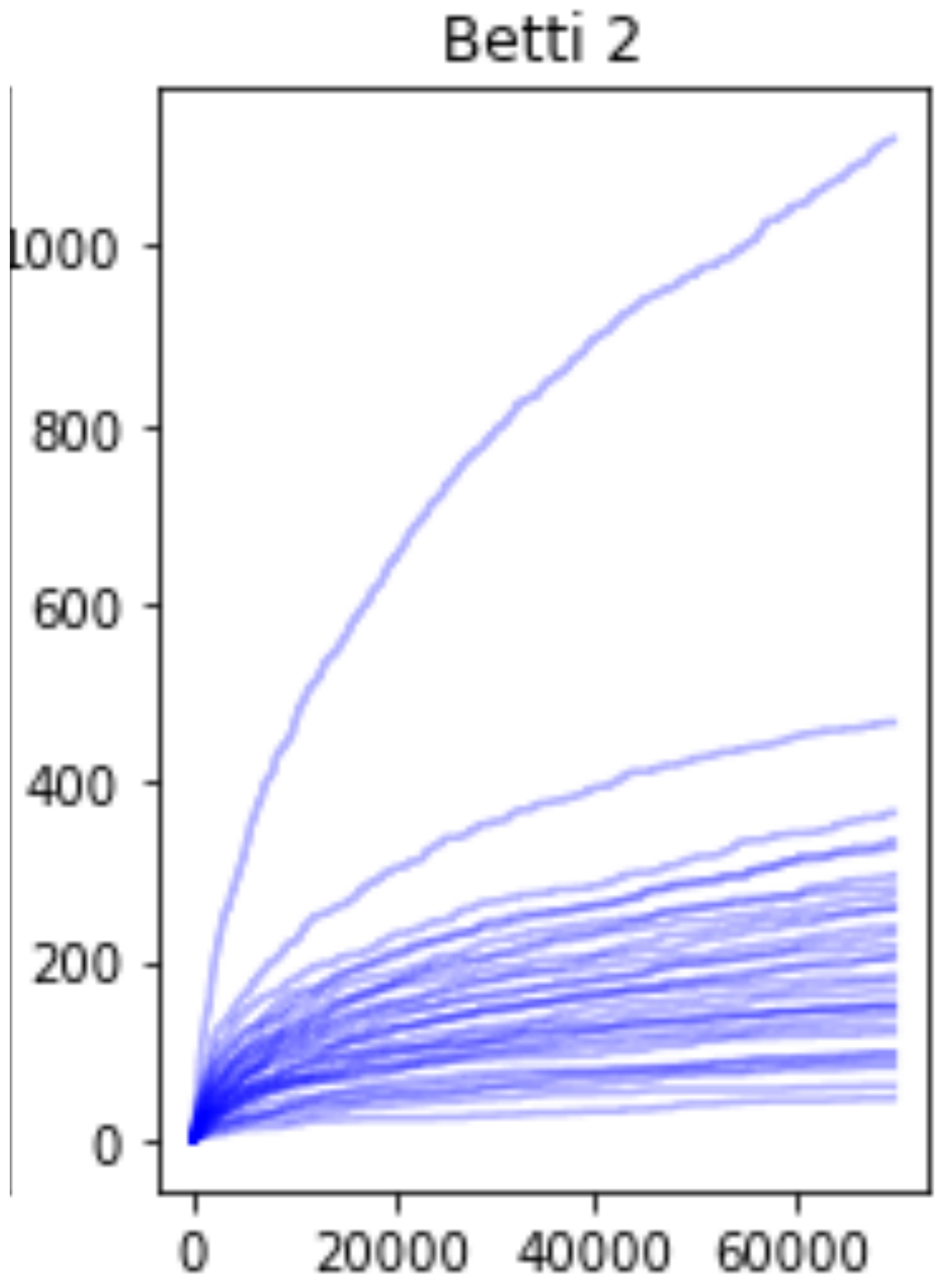
Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

Expected Betti Number $E[\beta_q]$

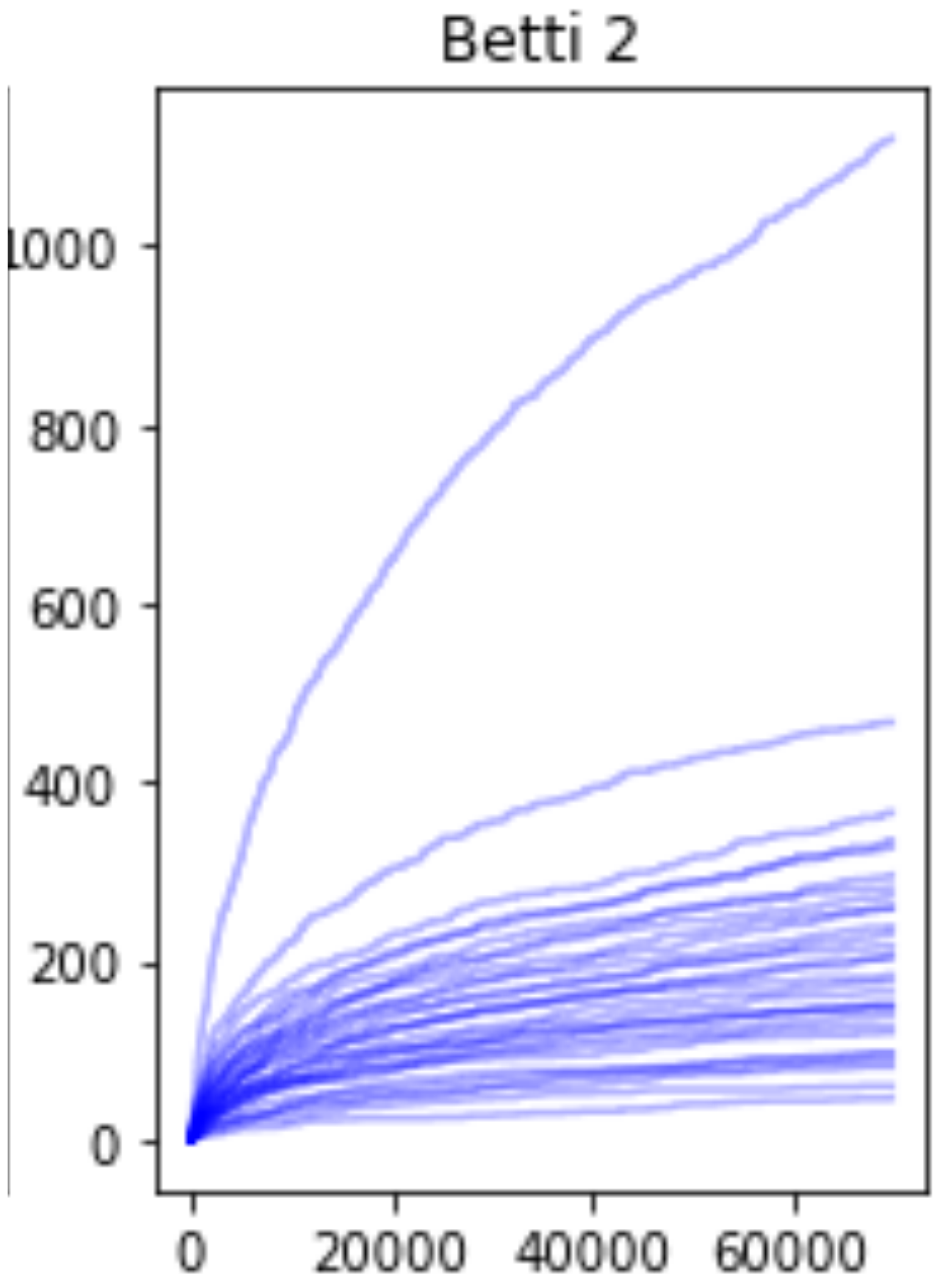
Expected Betti Number $E[\beta_q]$



Different curves, different random seeds.
All curves have the same model parameters.

Expected Betti Number $E[\beta_q]$

- increasing trend



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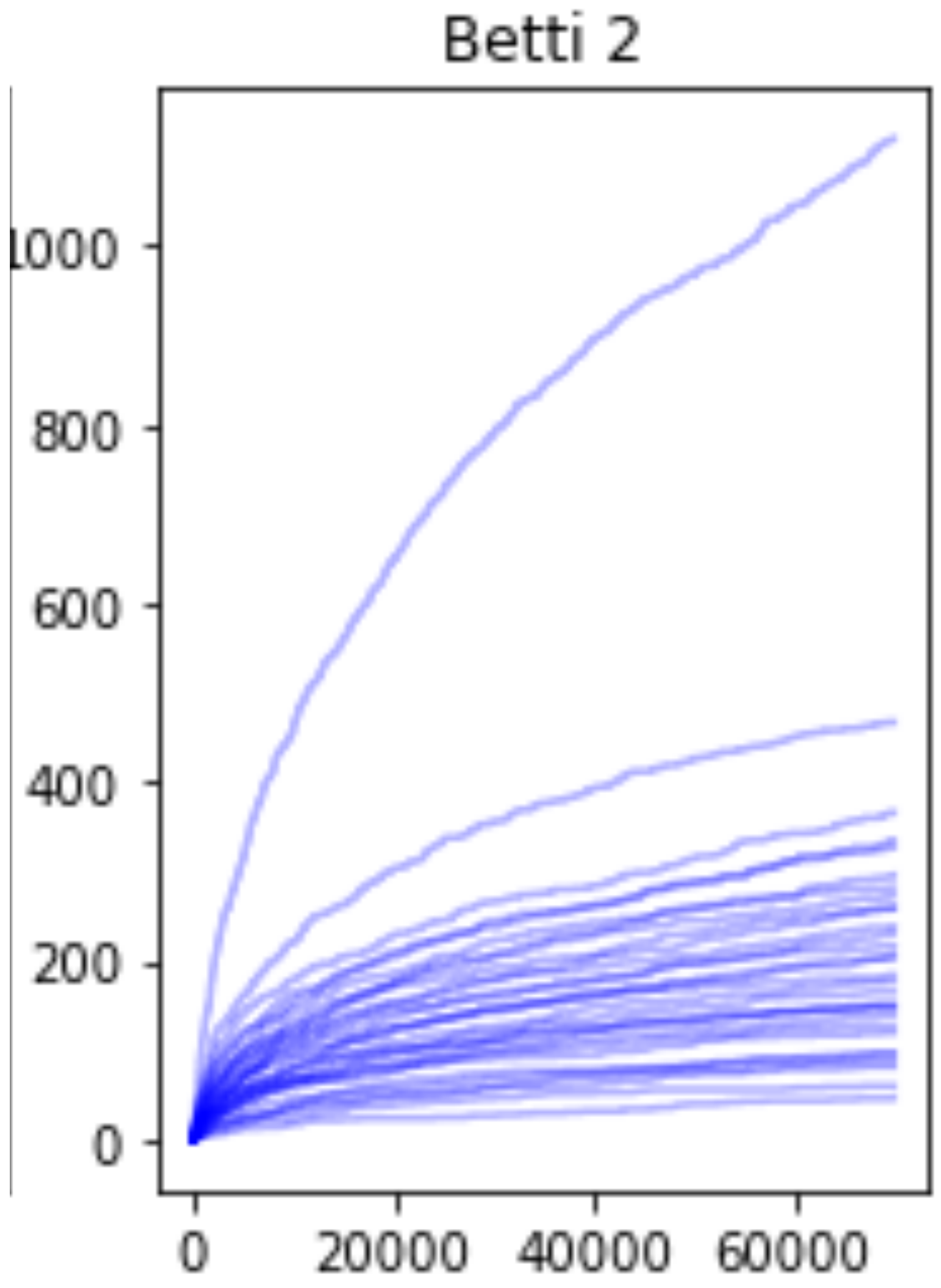
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth



Different curves, different random seeds.
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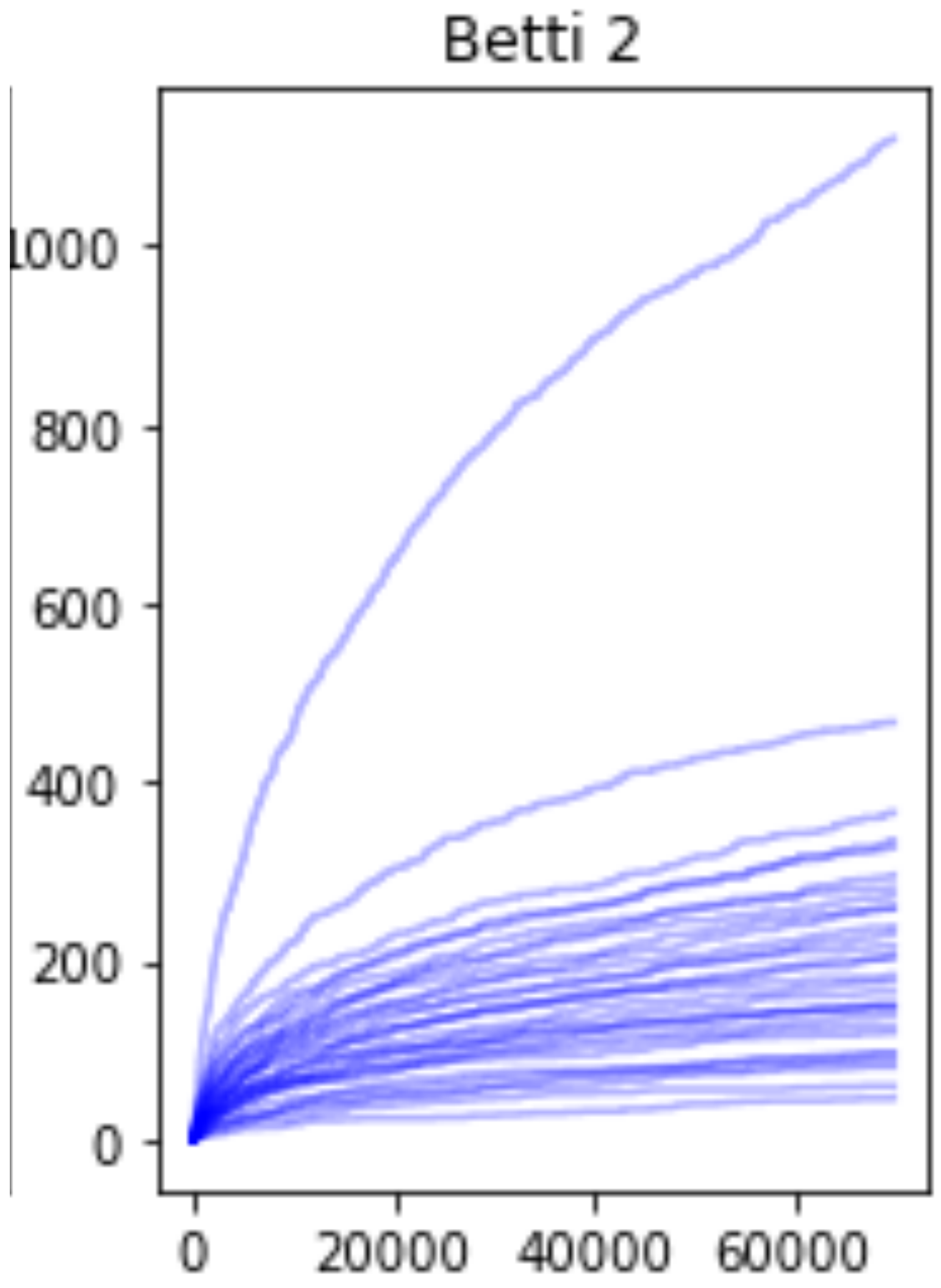
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth
- outlier



Different curves, different random seeds.
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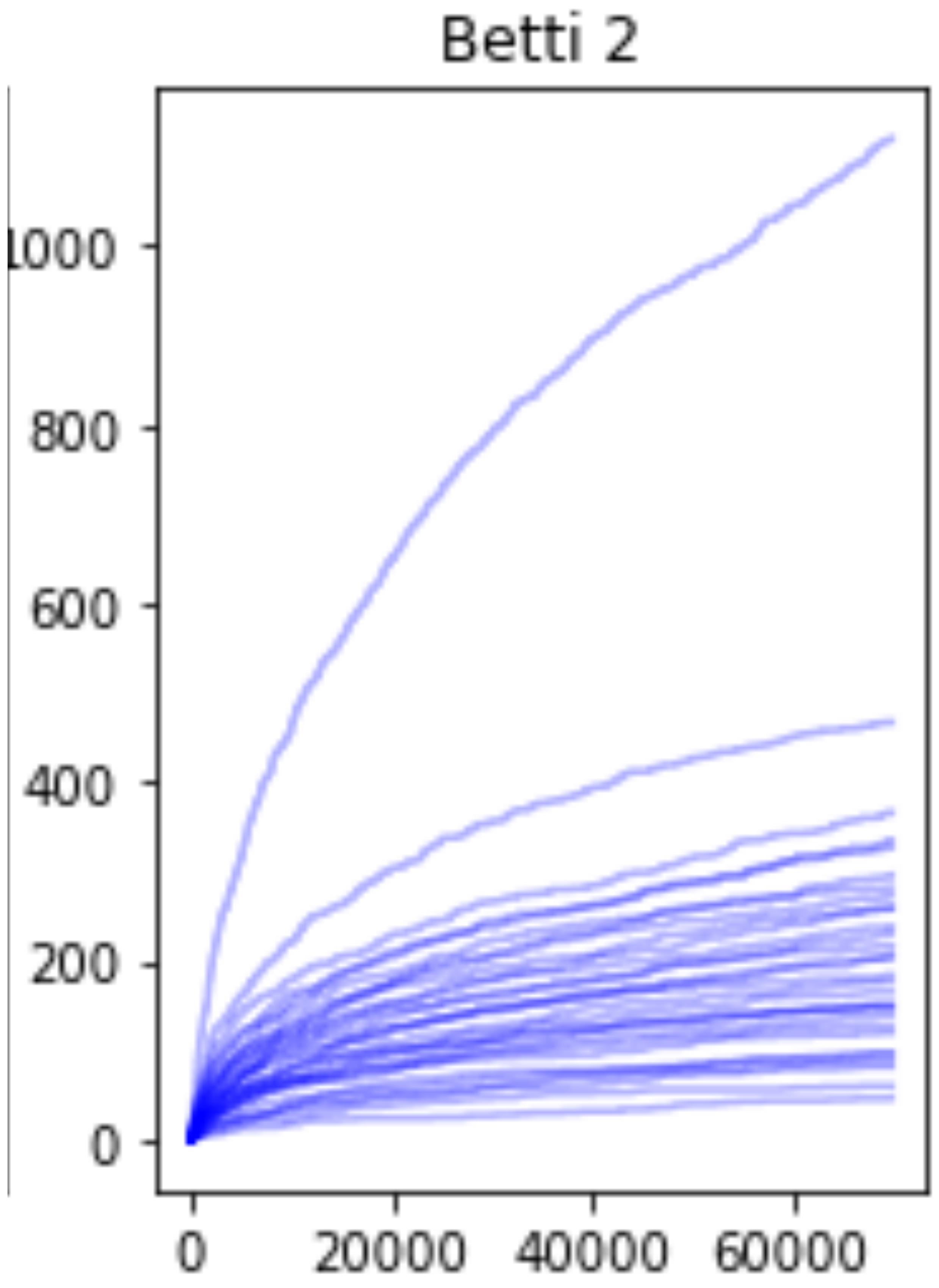
Expected Betti Number $E[\beta_q]$

- $c(\text{num of nodes}^{1-4x}) \leq E[\beta_2] \leq C(\text{num of nodes}^{1-4x})$
  under mild assumptions
  - $x \in (0,1/2)$ depends on the preferential attachment strength.



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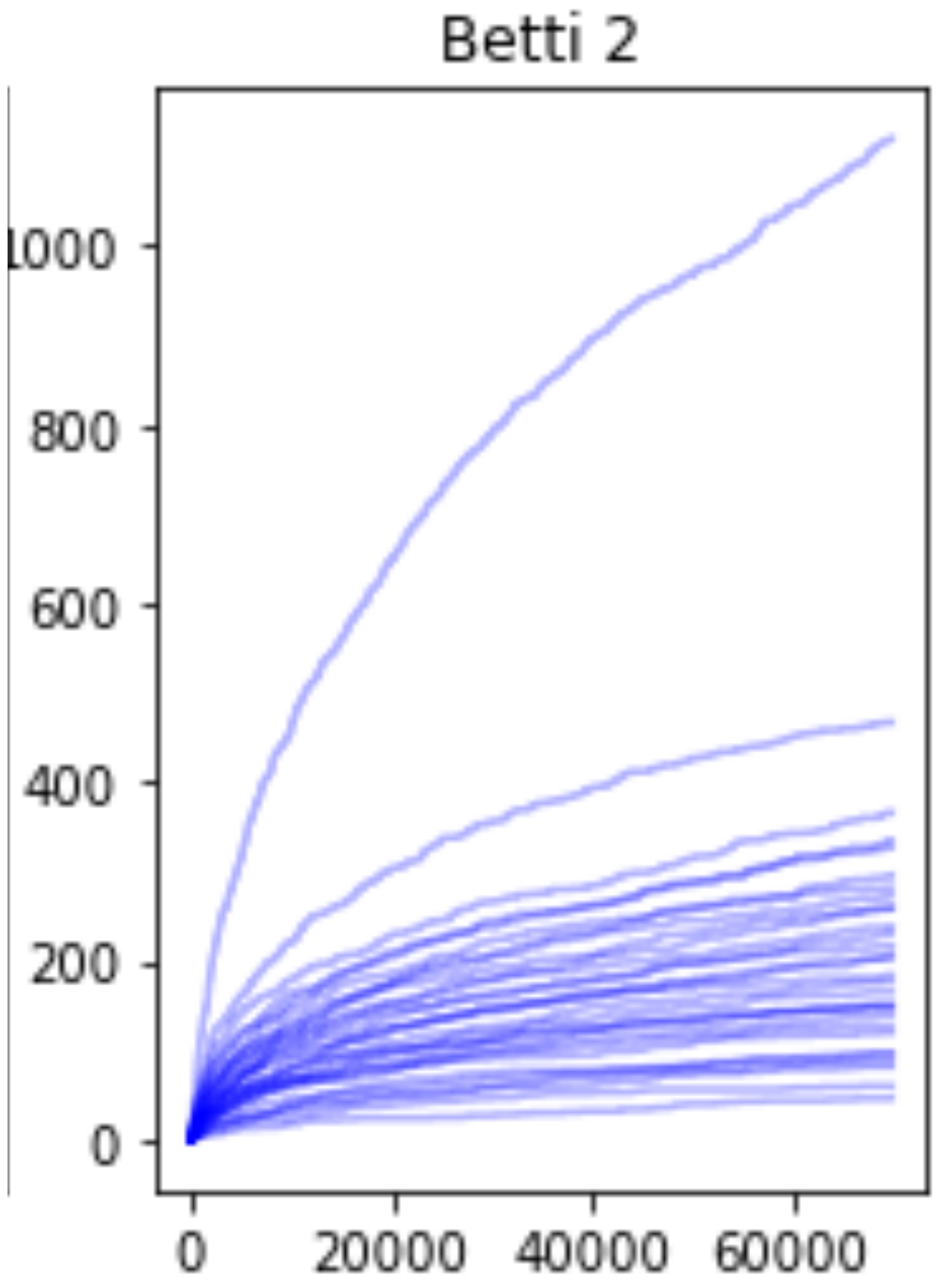
  - If $1 - 4x < 0$, then $E[\beta_2] \leq C$.



Expected Betti Number $E[\beta_q]$

- c(num of nodes^{1-4x}) $\leq E[\beta_2] \leq C$(num of nodes^{1-4x})
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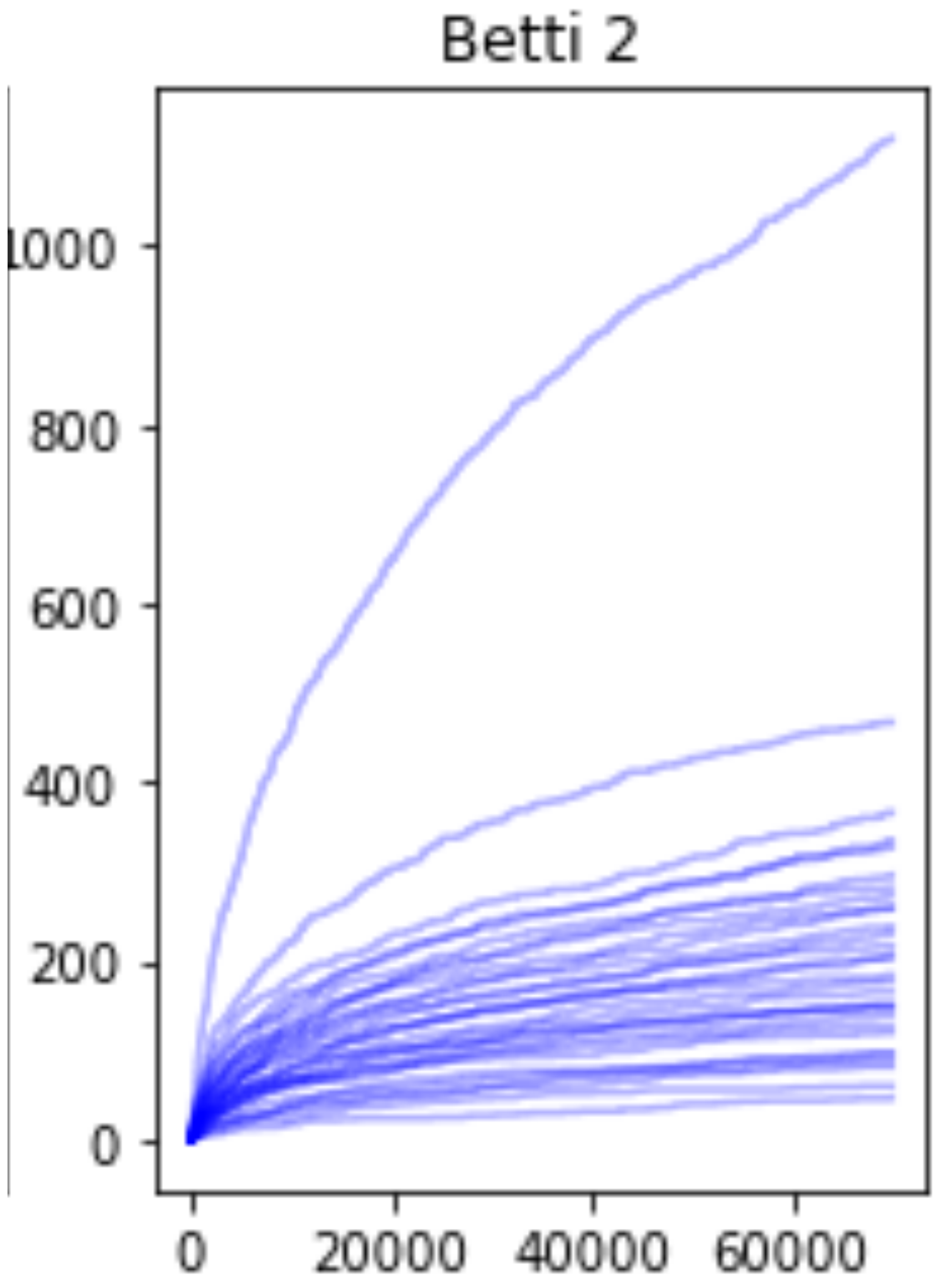
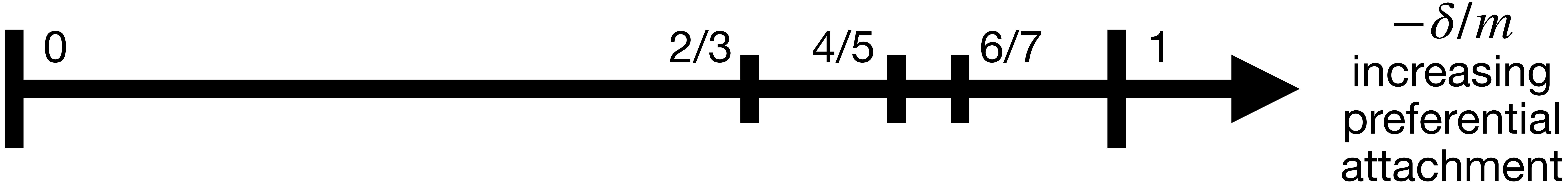
- c(num of nodes^{1-2qx}) $\leq E[\beta_q] \leq C$(num of nodes^{1-2qx}) for $q \geq 2$.



Phase transition

Recall
P(attaching to v) $\propto$ degree + δ
m = number of edges per new node



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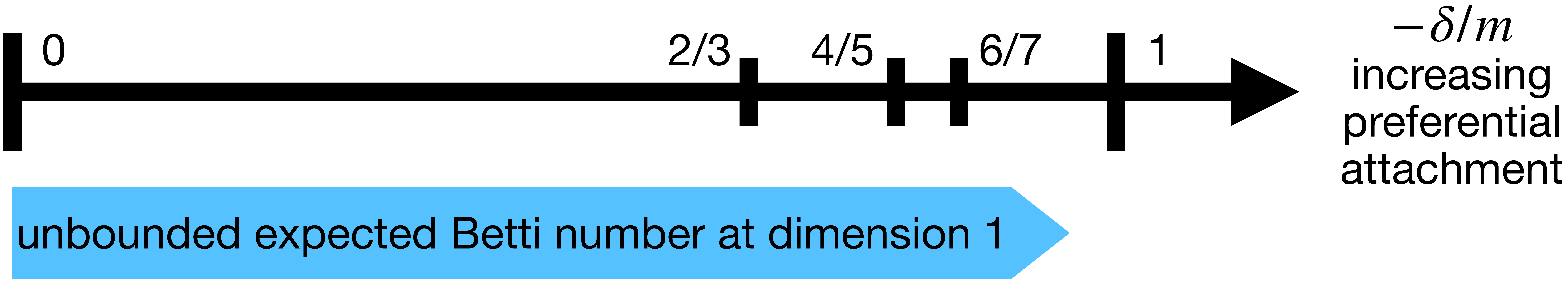

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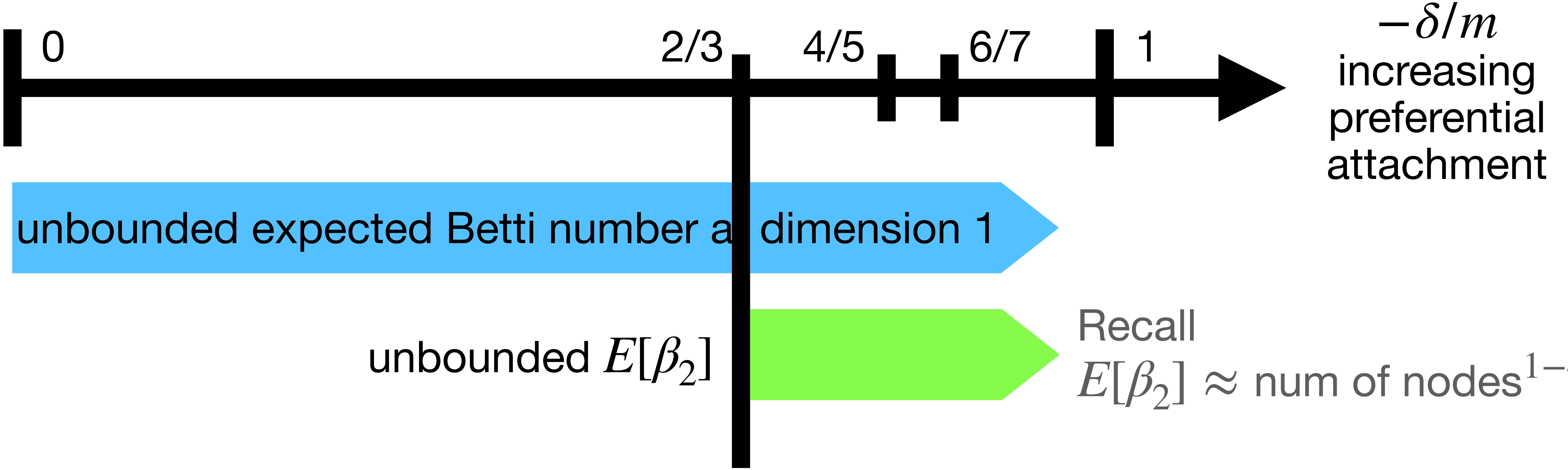

Recall

$E[\beta_2] \approx$ num of nodes^{1-4x}

Phase transition

Recall

P(attaching to v) $\propto$ degree + δ

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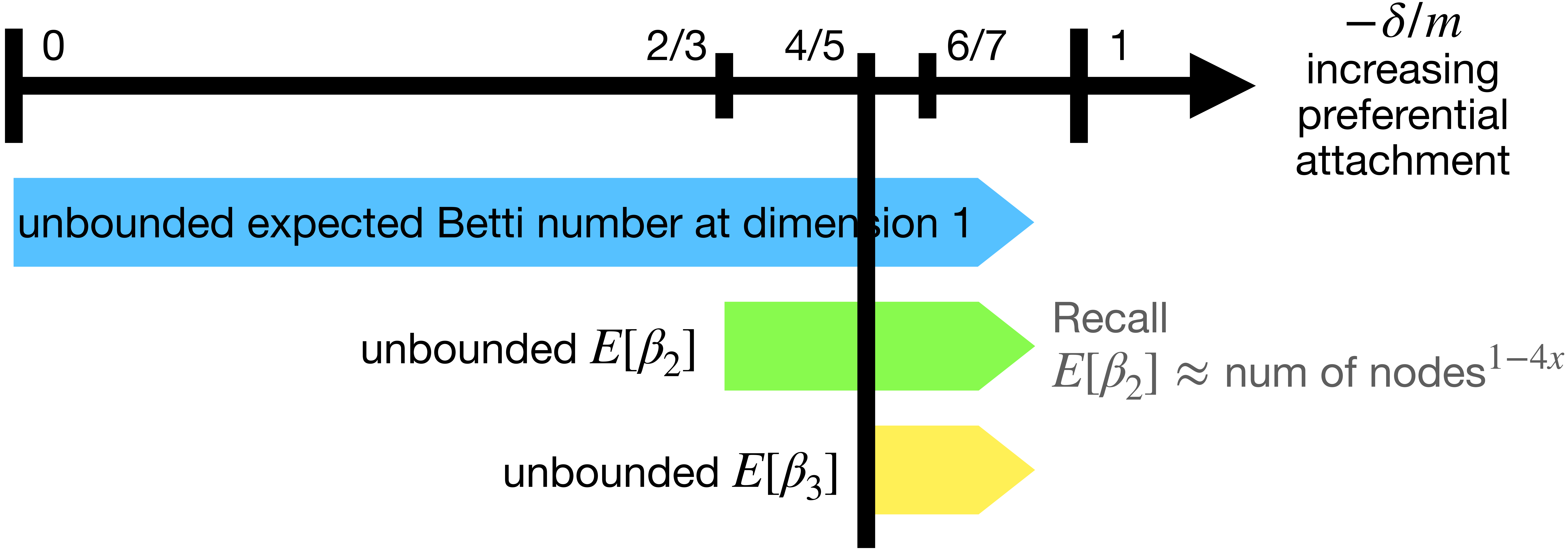

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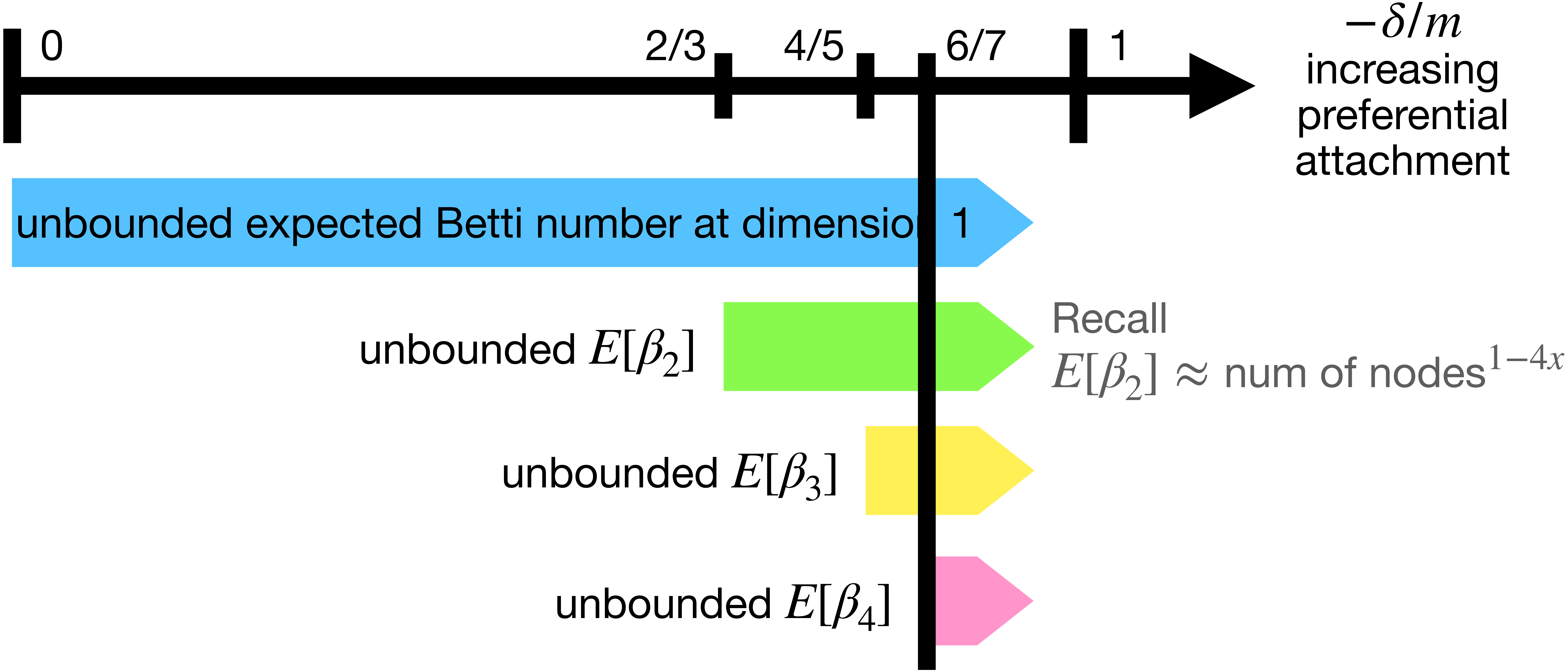

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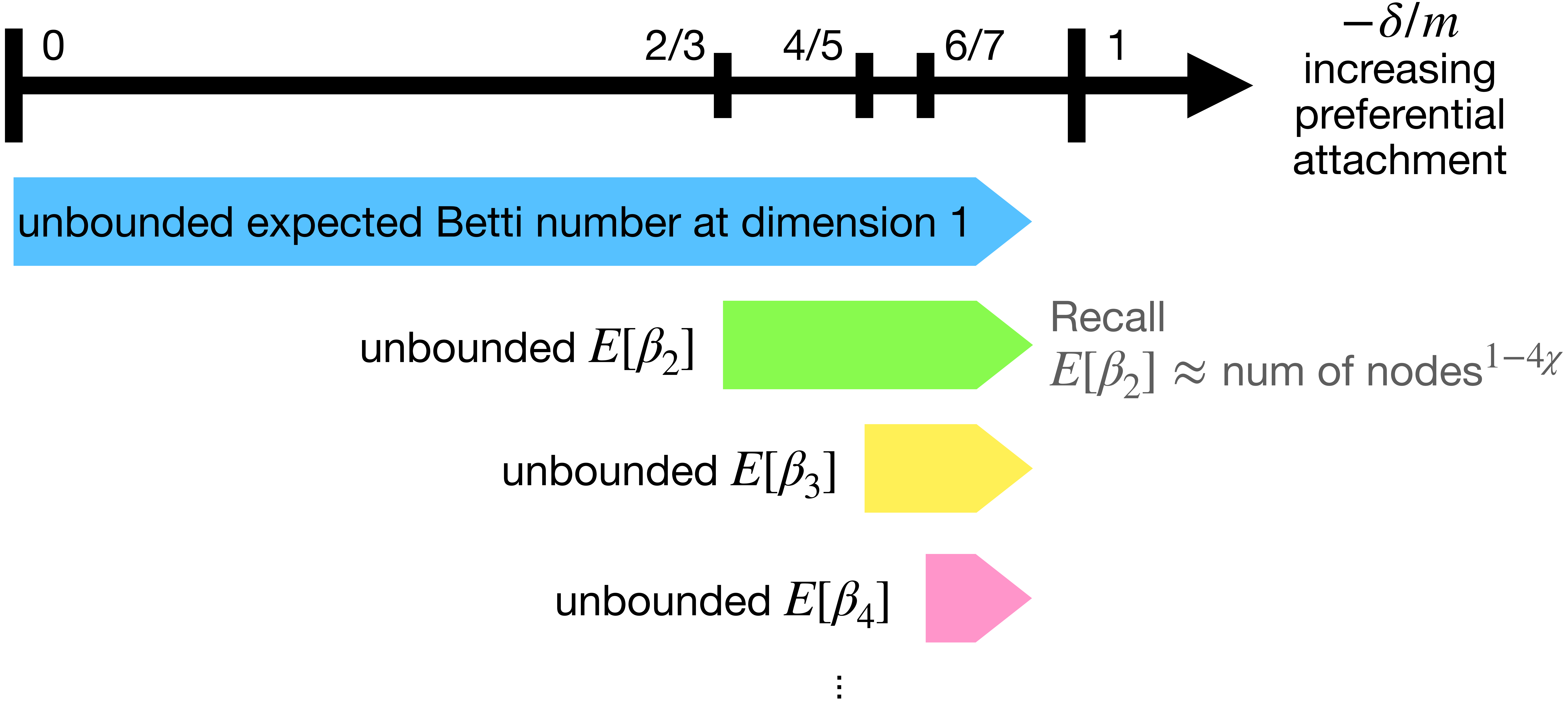
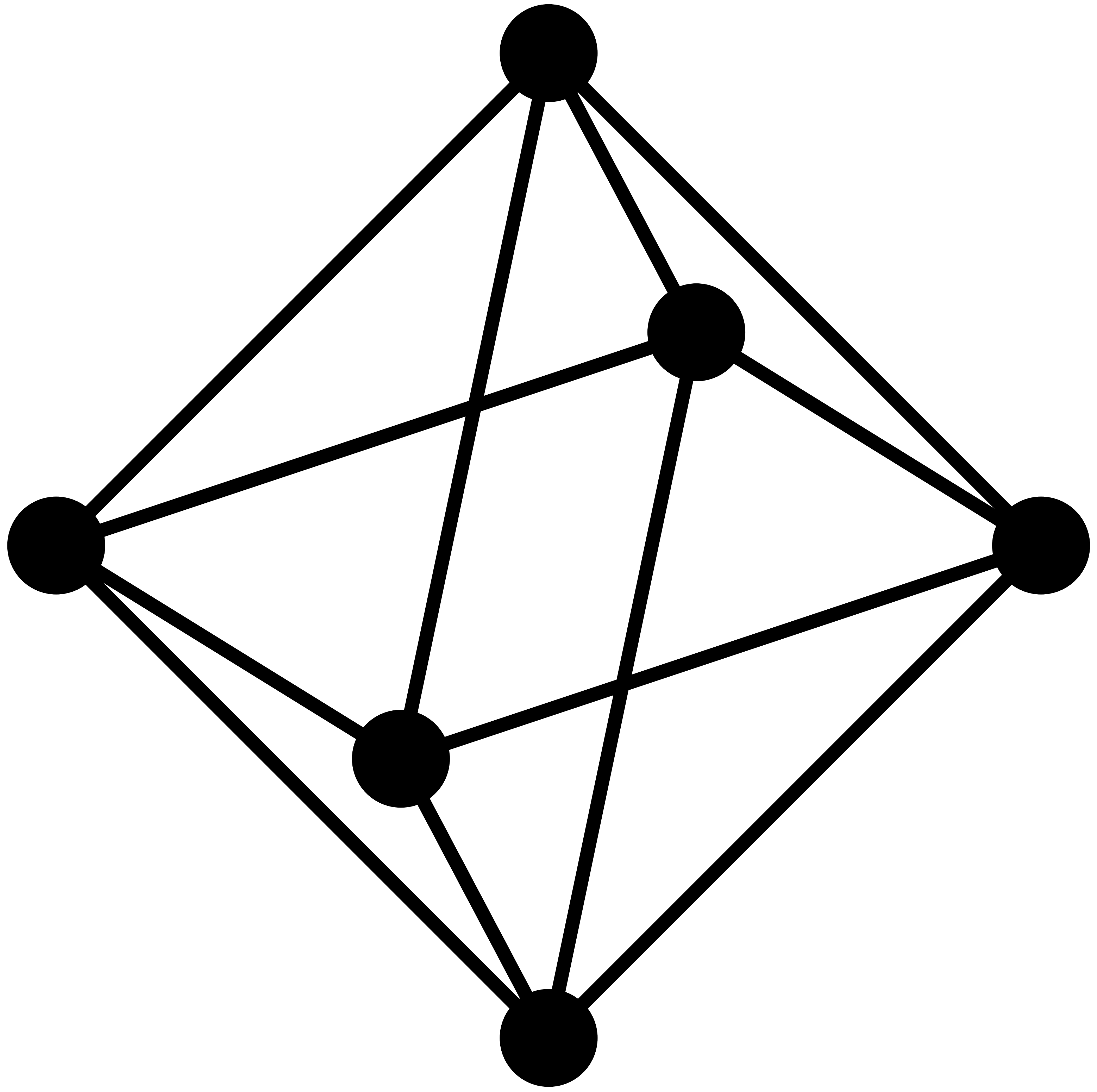


Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

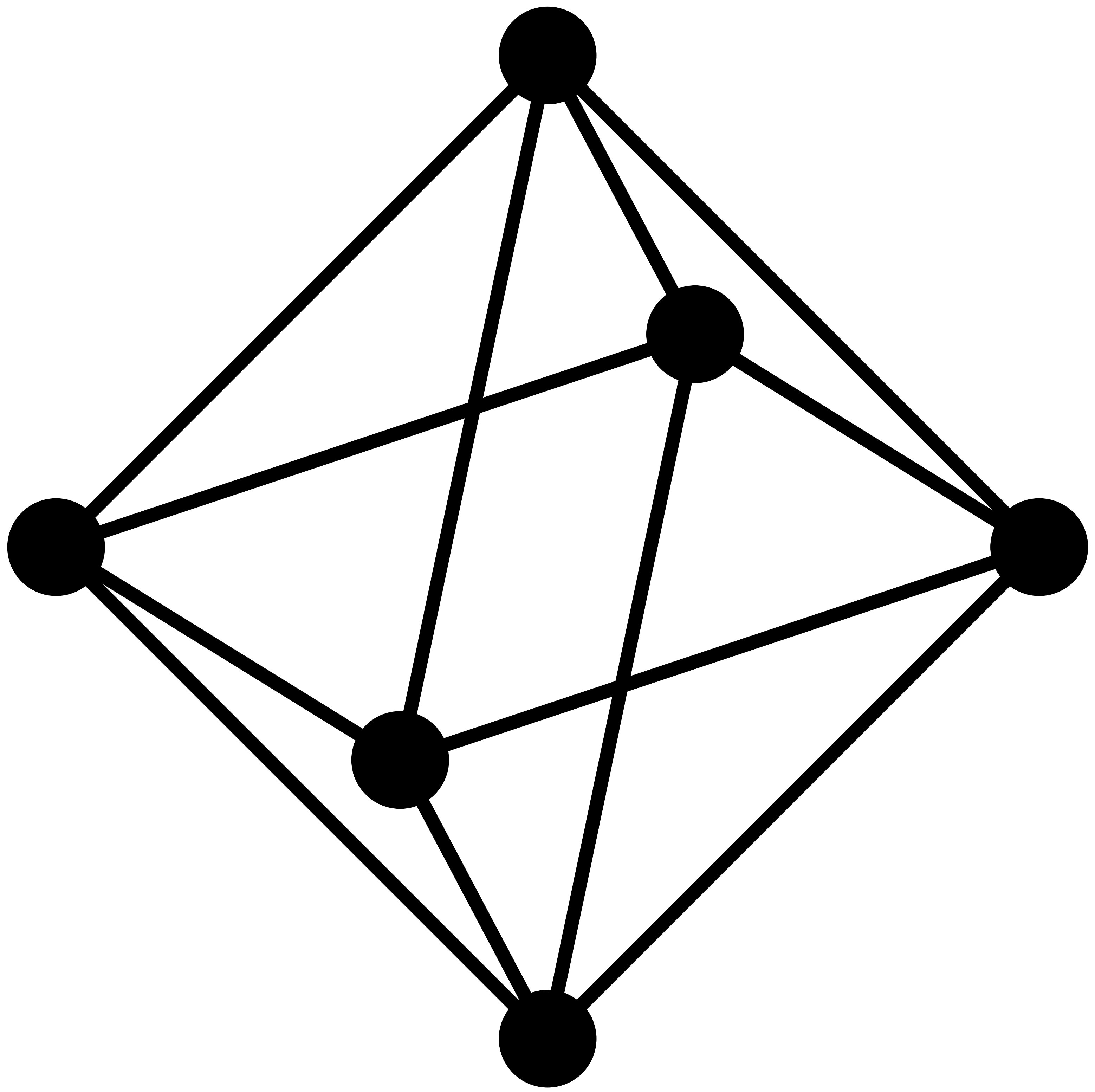
Proof?

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}

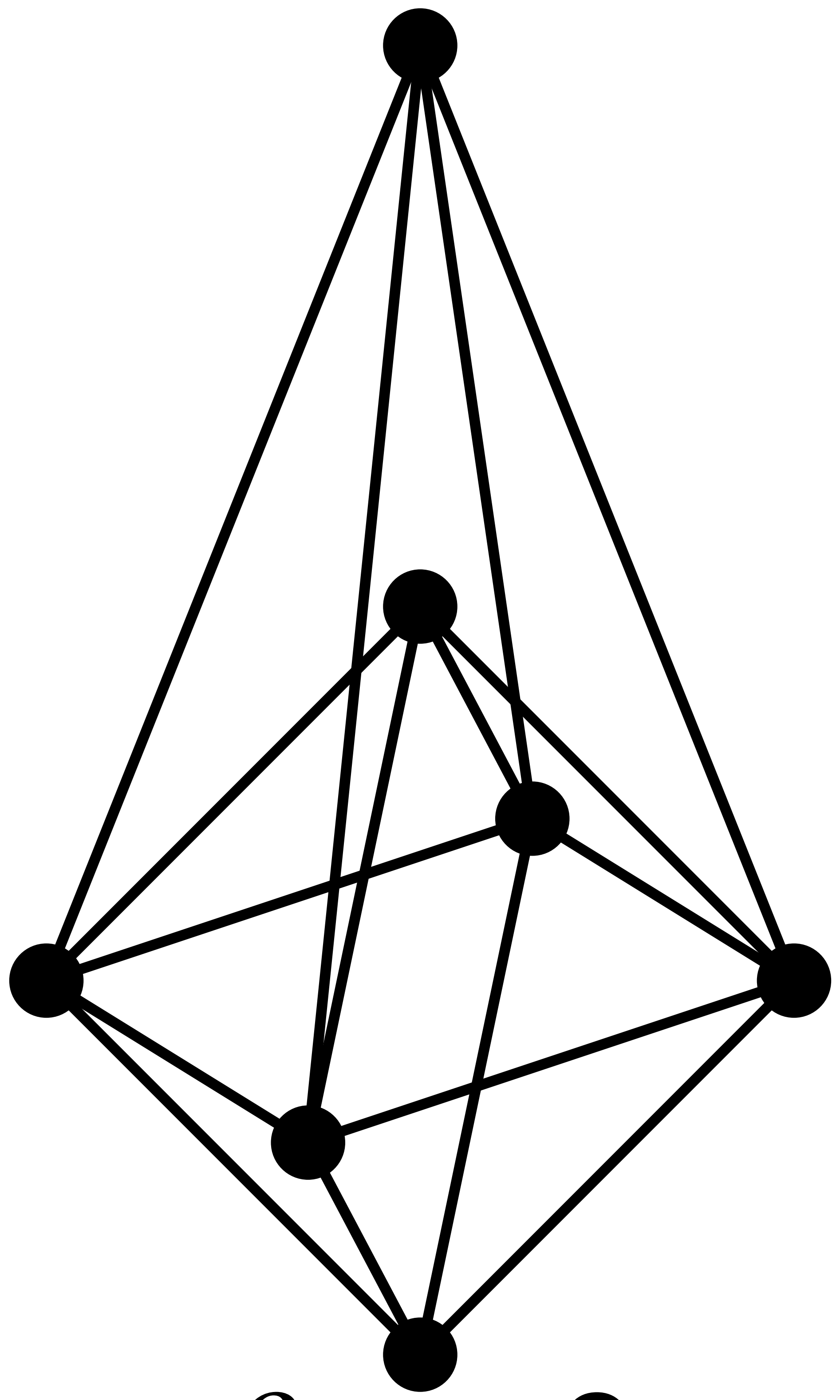


$\beta_2 = 1$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}

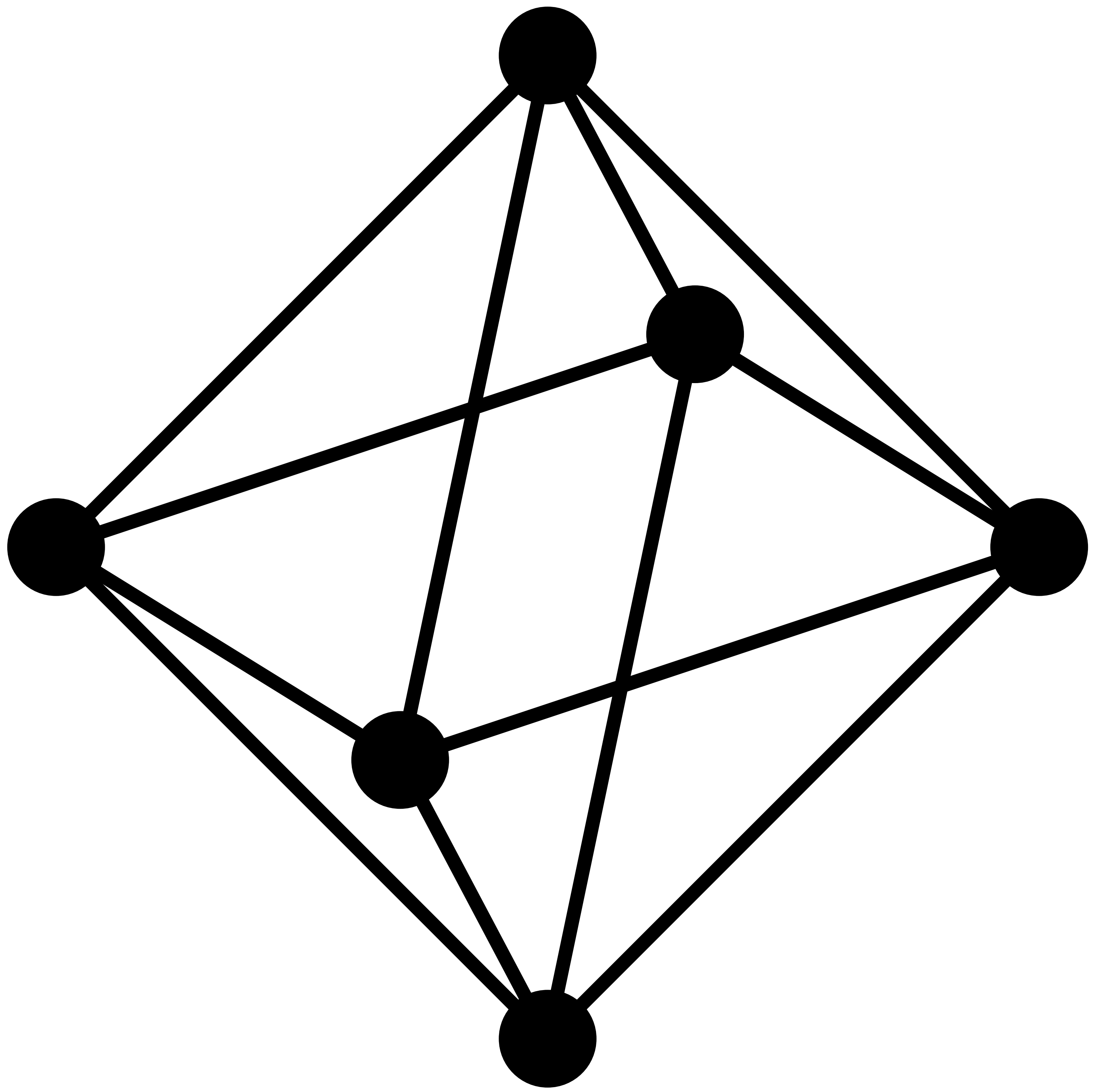


$\beta_2 = 1$

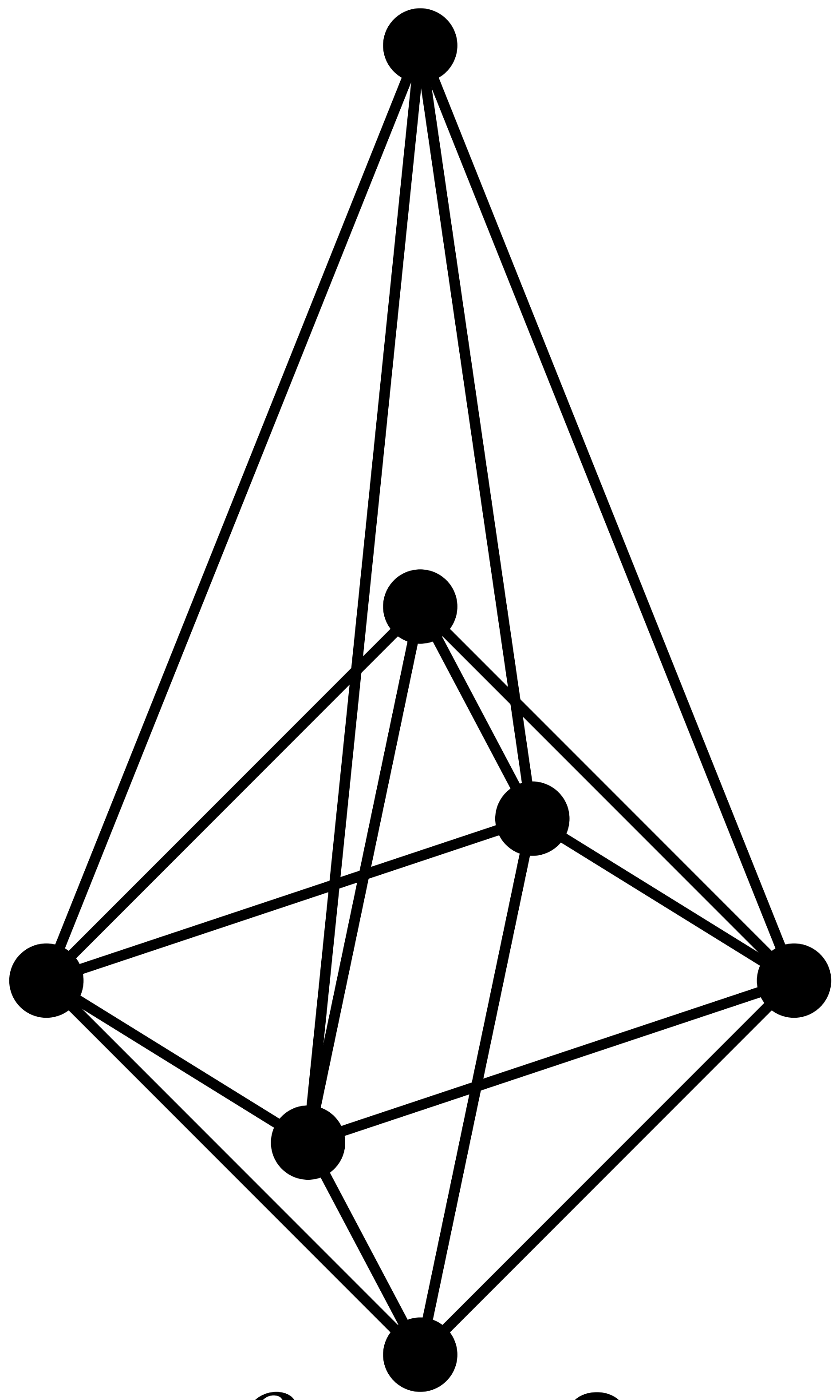


$\beta_2 = 2$

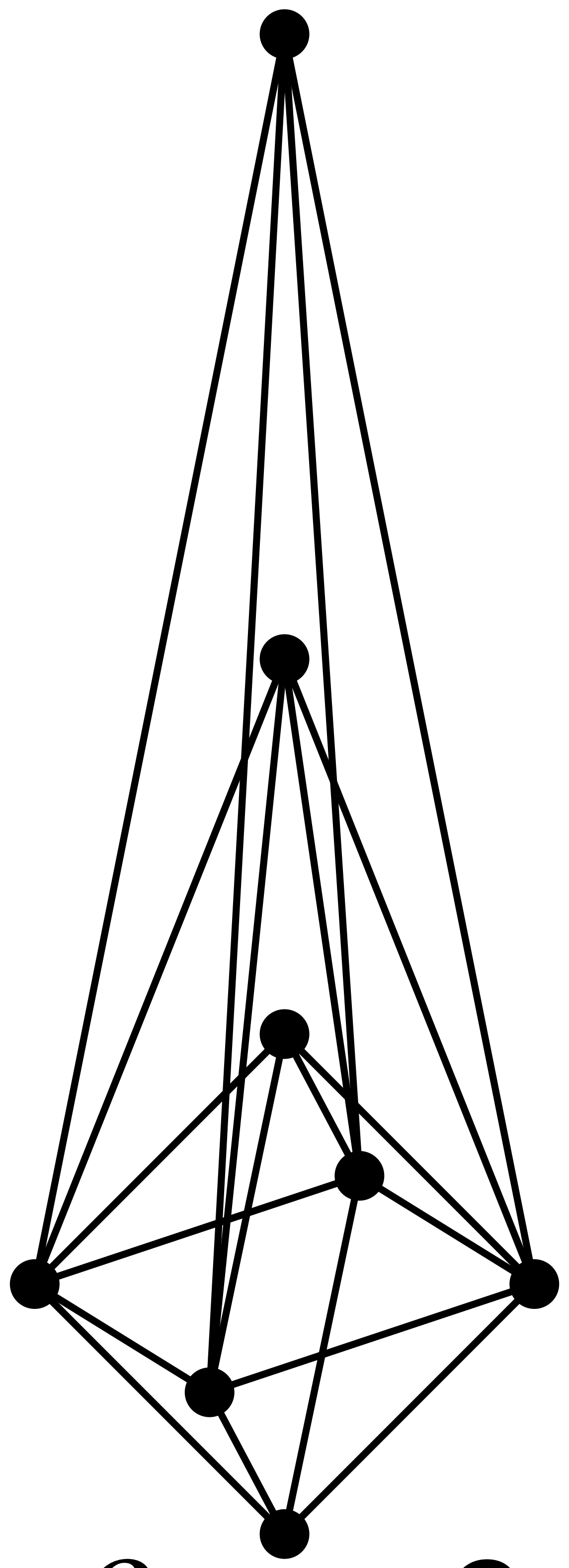
Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



$\beta_2 = 1$



$\beta_2 = 2$



$\beta_2 = 3$

Subtleties

- Need homological algebra to relate Betti numbers with counts

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- Need homological algebra to relate Betti numbers with counts
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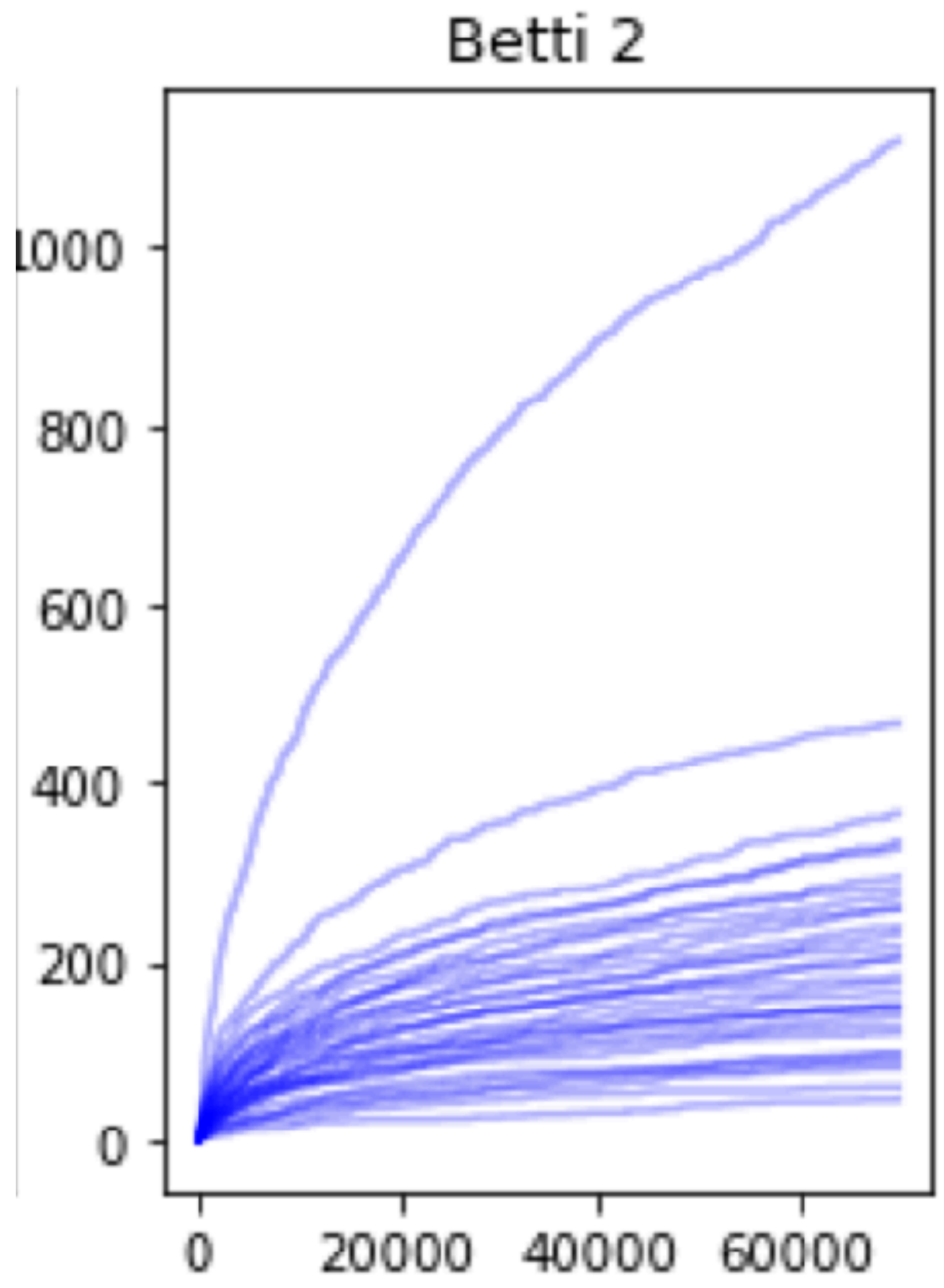
- Generalize minimal cycle results in the language of homological algebra
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs

Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

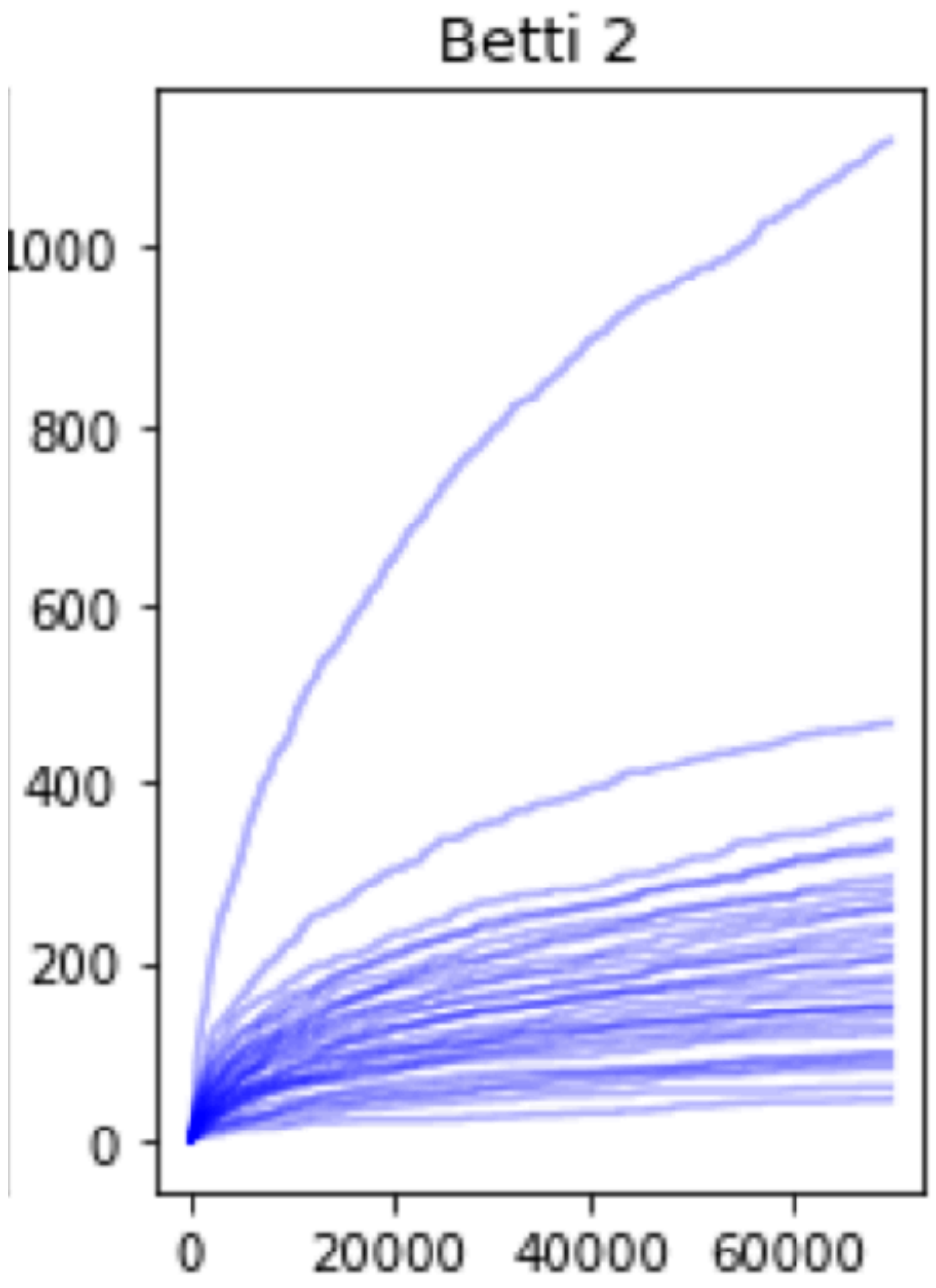
In practice???

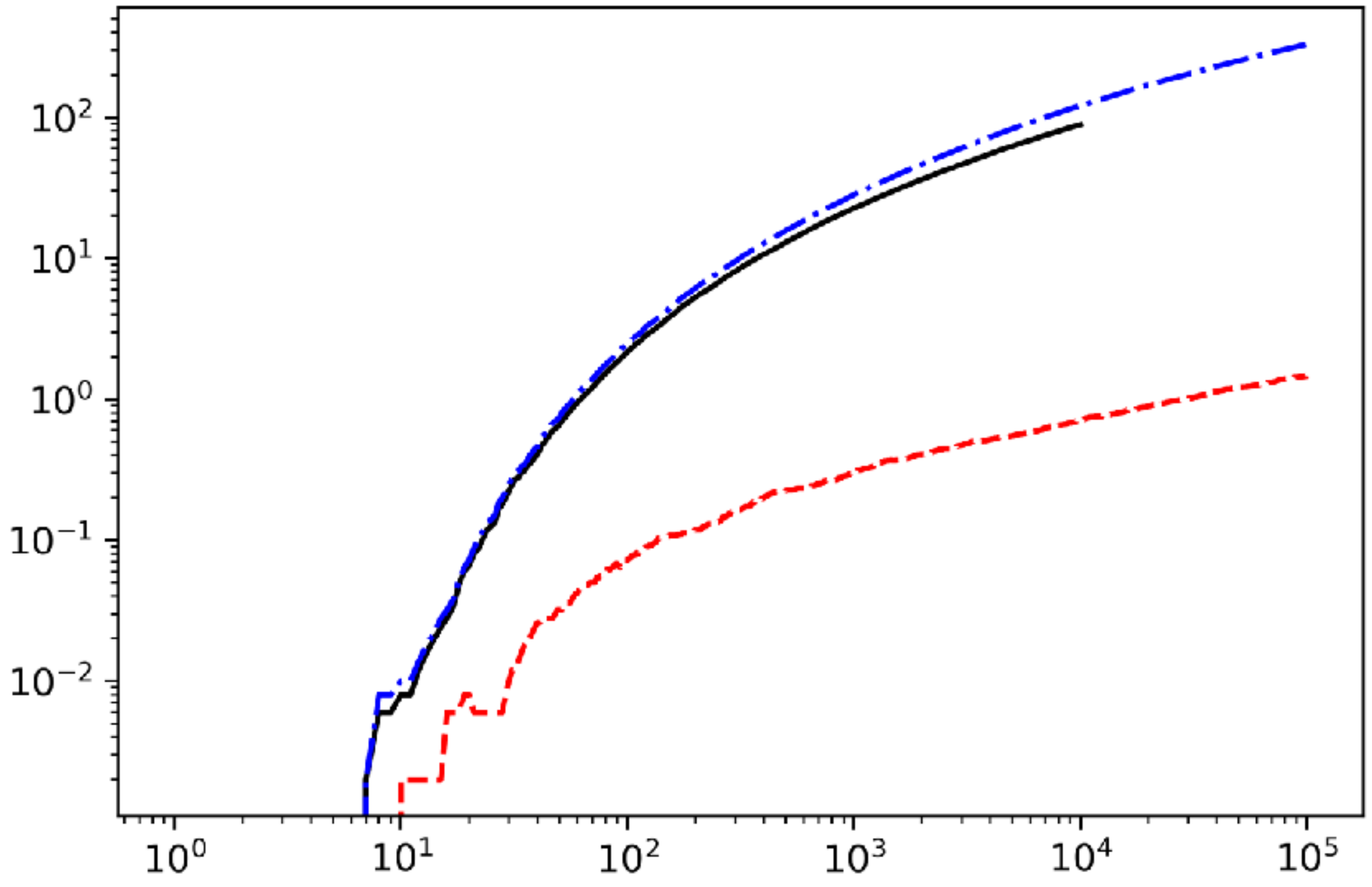
$E[\beta_2] \approx$ **num of nodes**$^{1-4x}$



$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

$$\log E[\beta_2] \approx (1-4x)\log(\textbf{num of nodes})$$





$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

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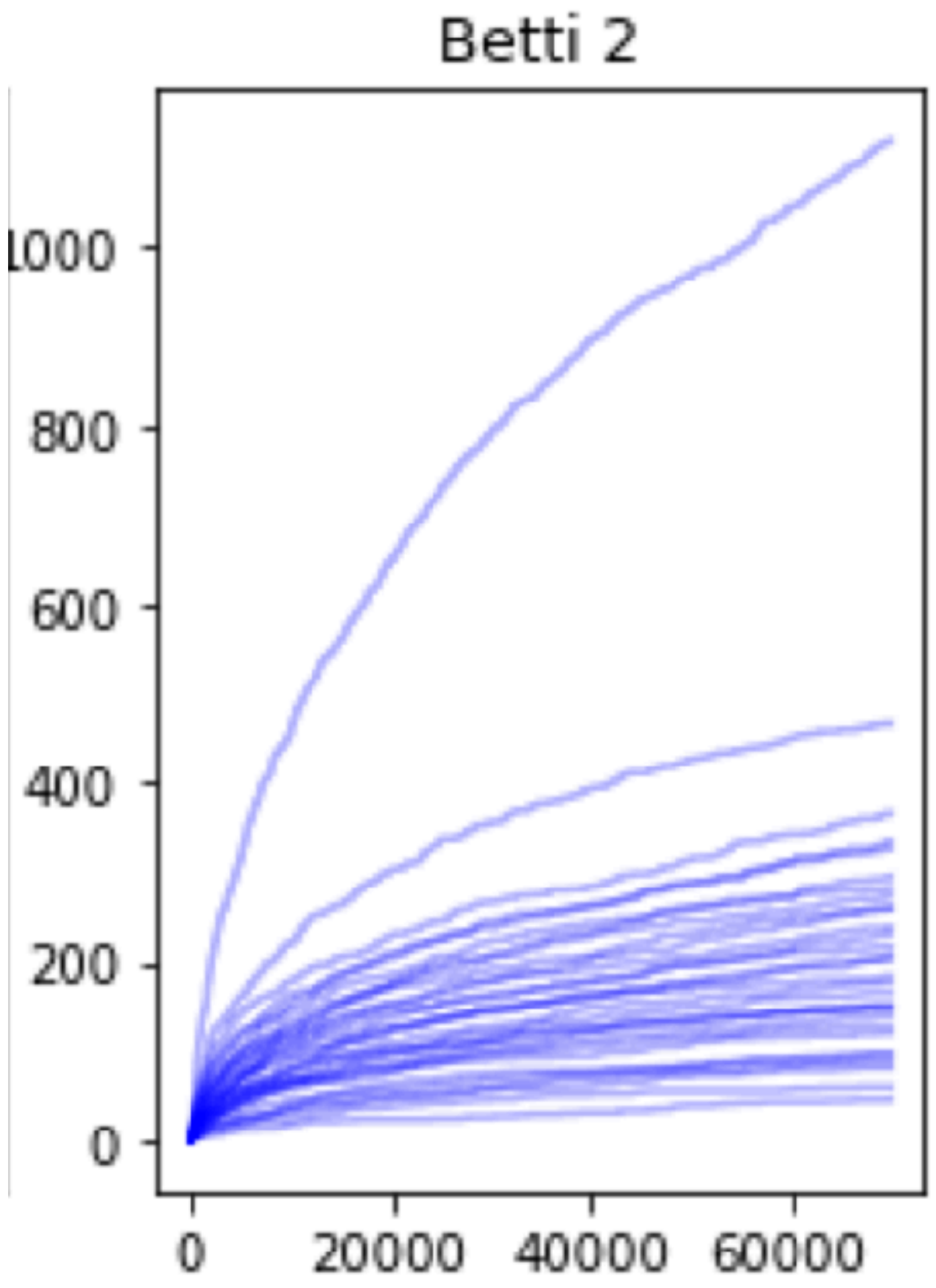

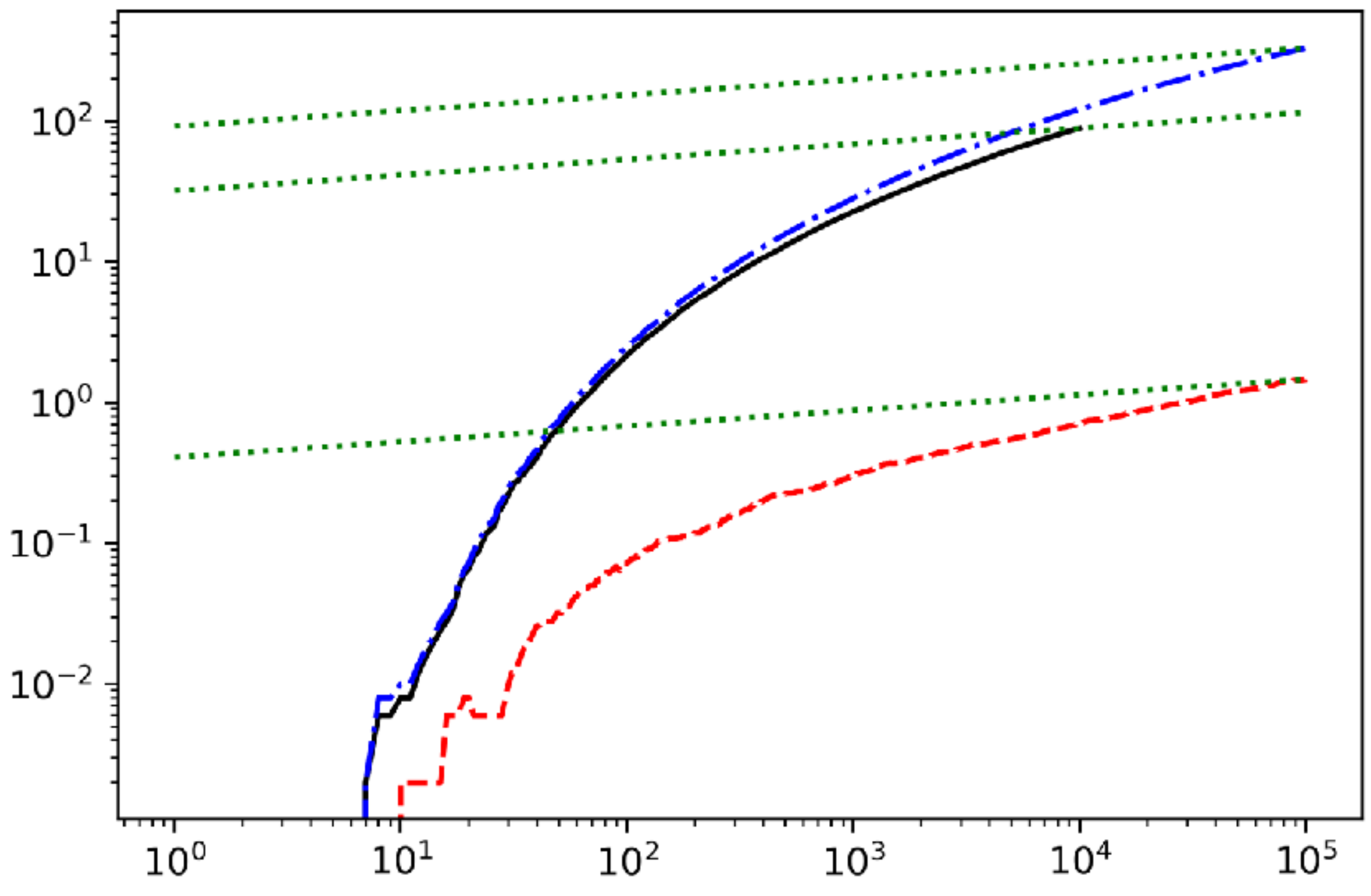

V. What lies ahead

order of magnitude of expected Betti numbers

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

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homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

simplicial preferential attachment?

parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

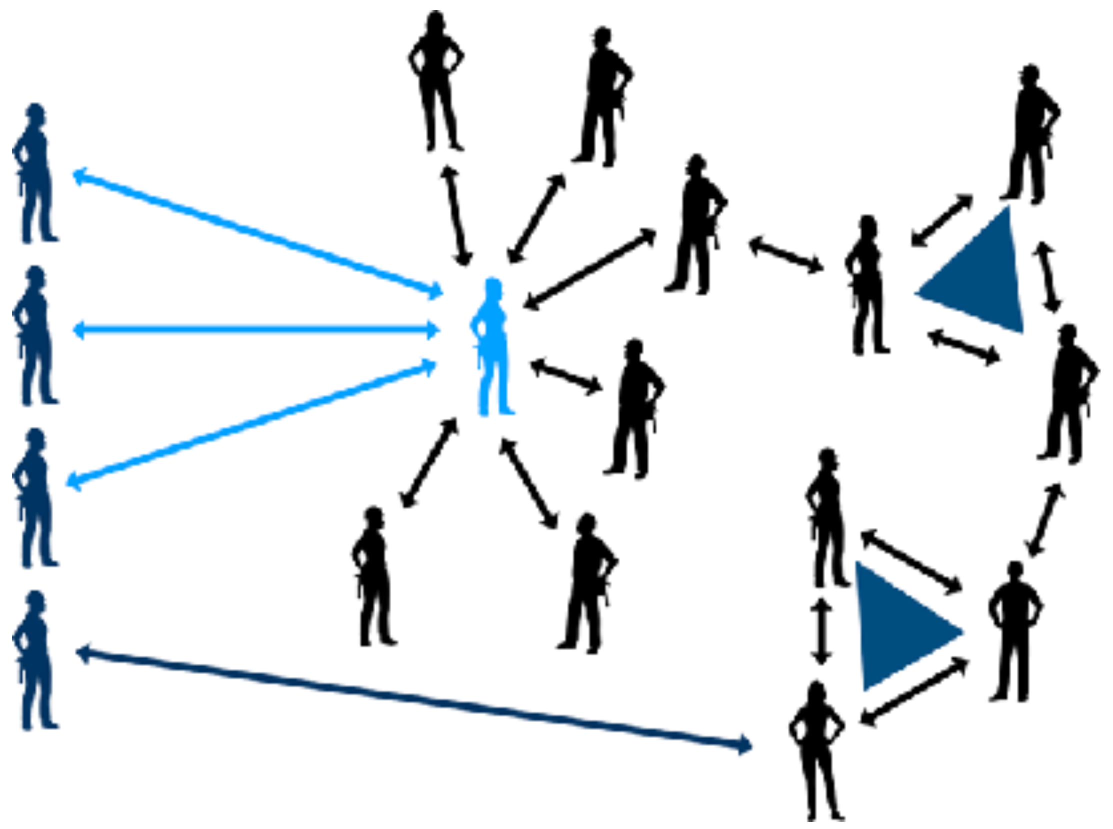
simplicial preferential attachment?

other non-homogeneous complexes?

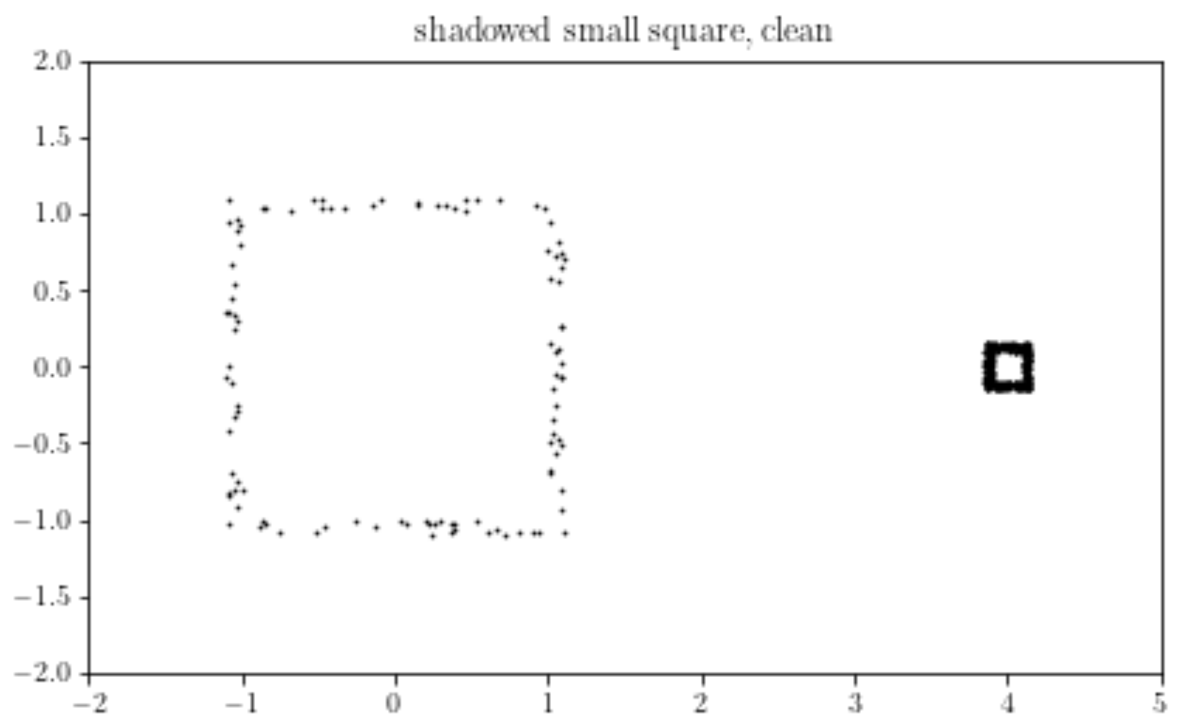
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

Chunyin Siu cs2323@cornell.edu
Cornell University



arxiv paper

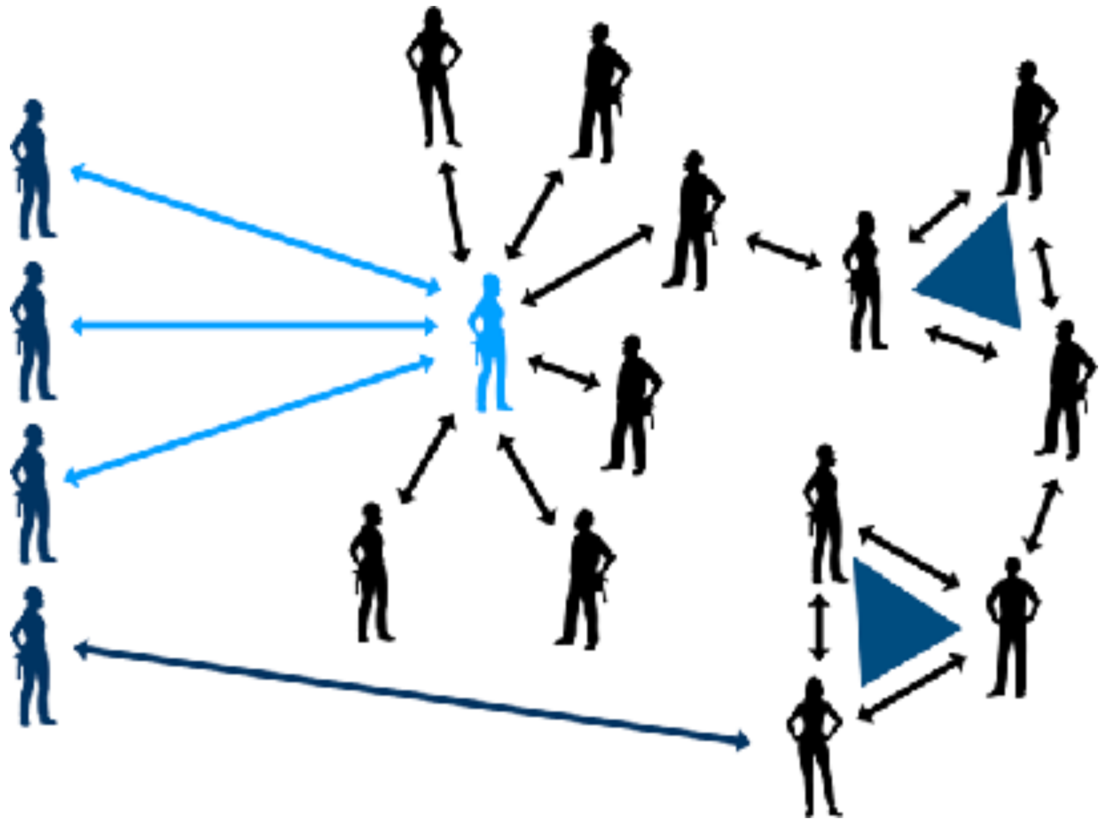


my video about small holes

Thank you!

Chunyin Siu
Cornell University

c-siu.github.io
cs2323@cornell.edu



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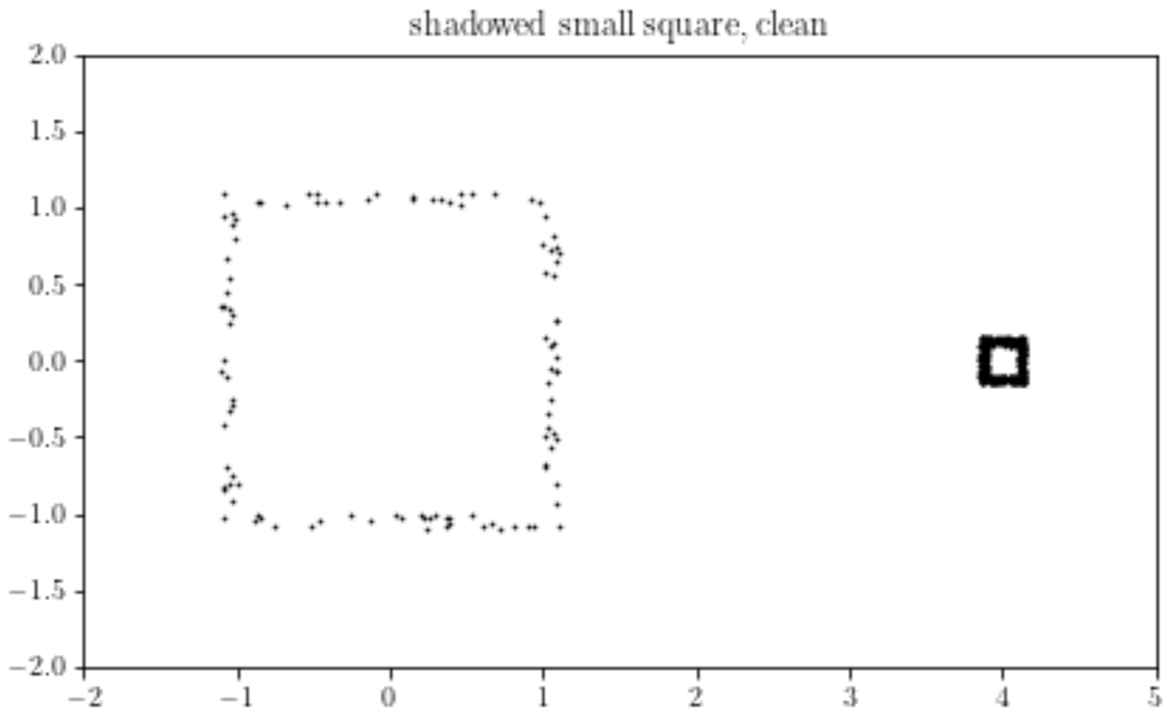

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Phase transition

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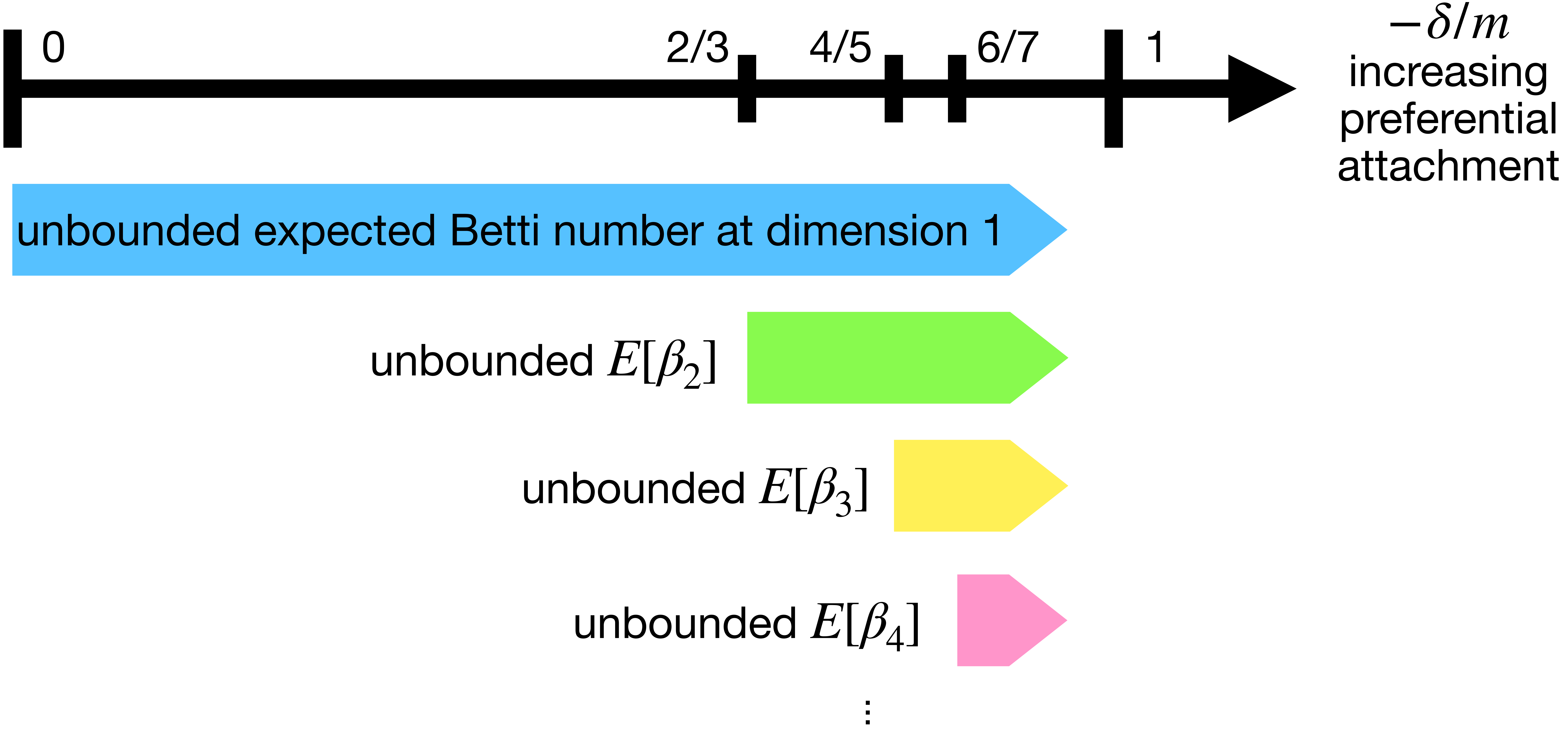

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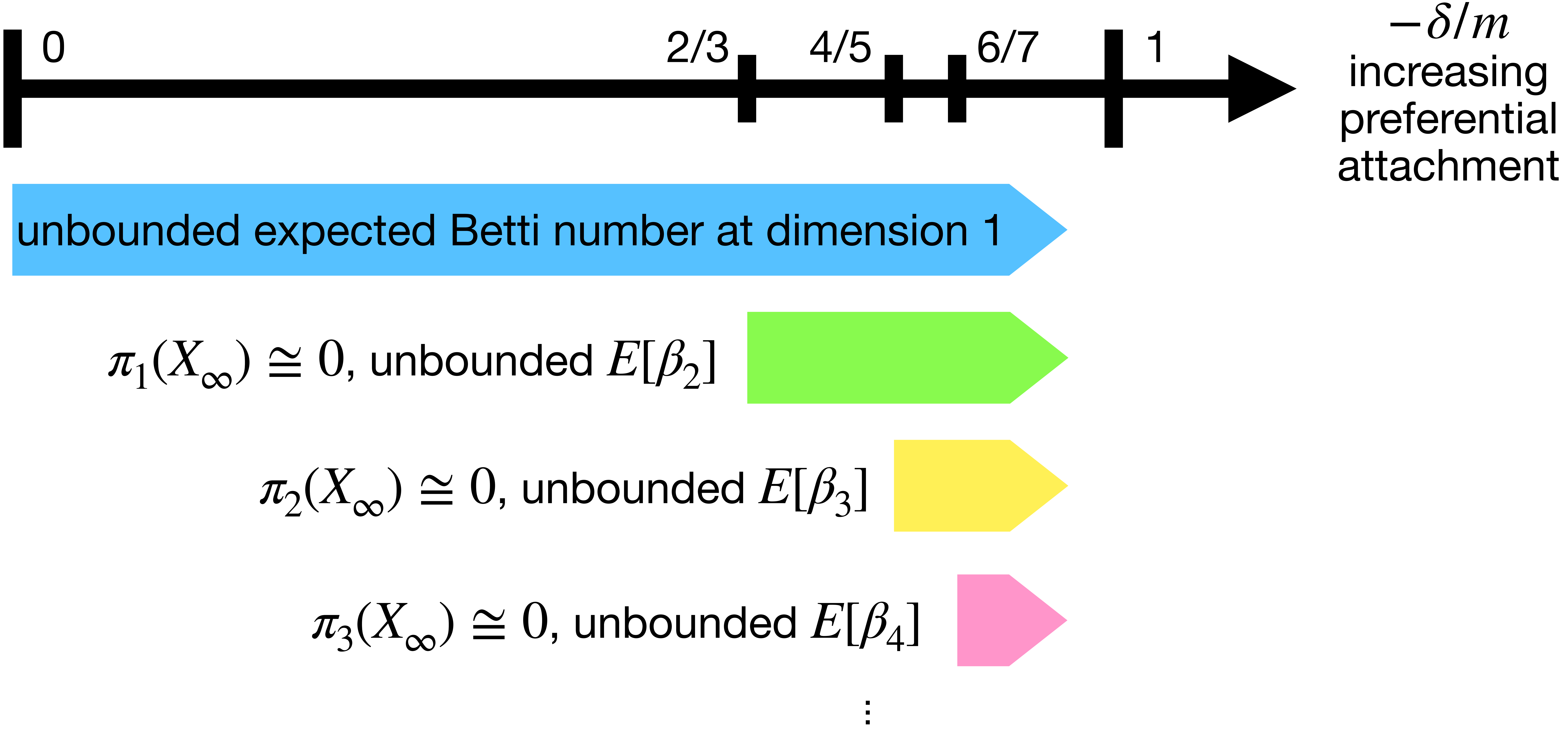

Subtleties

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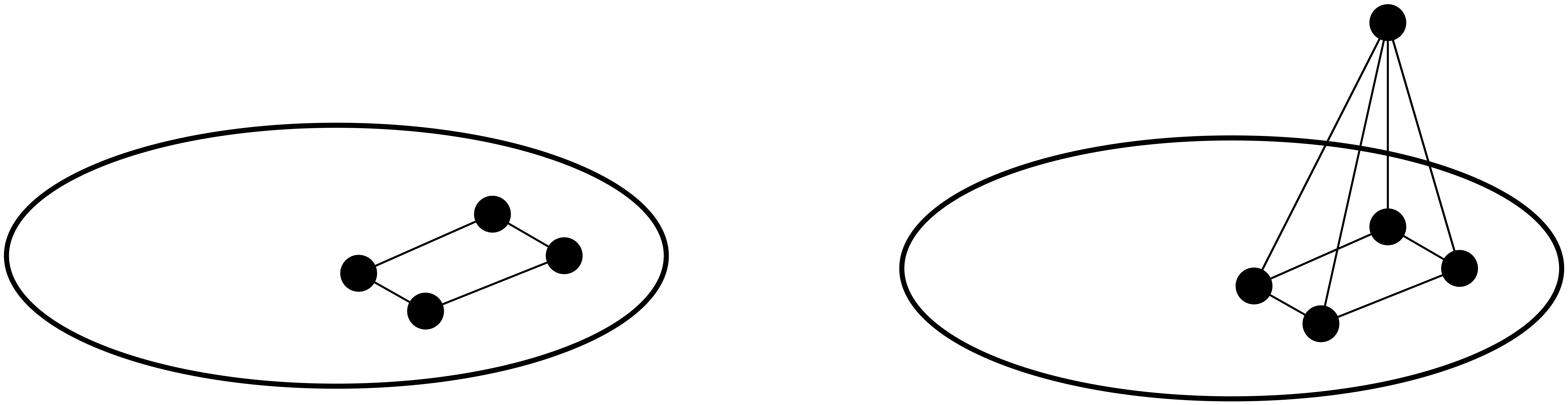

Subtleties

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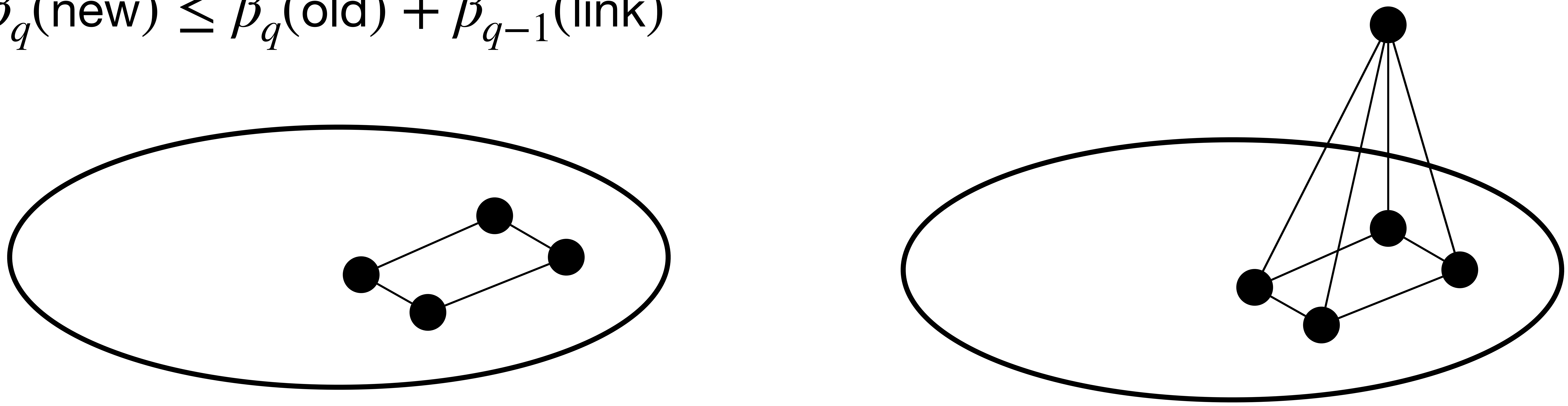
  - $\beta_q(\text{new}) \leq \beta_q(\text{old}) + \beta_{q-1}(\text{link})$



Subtleties

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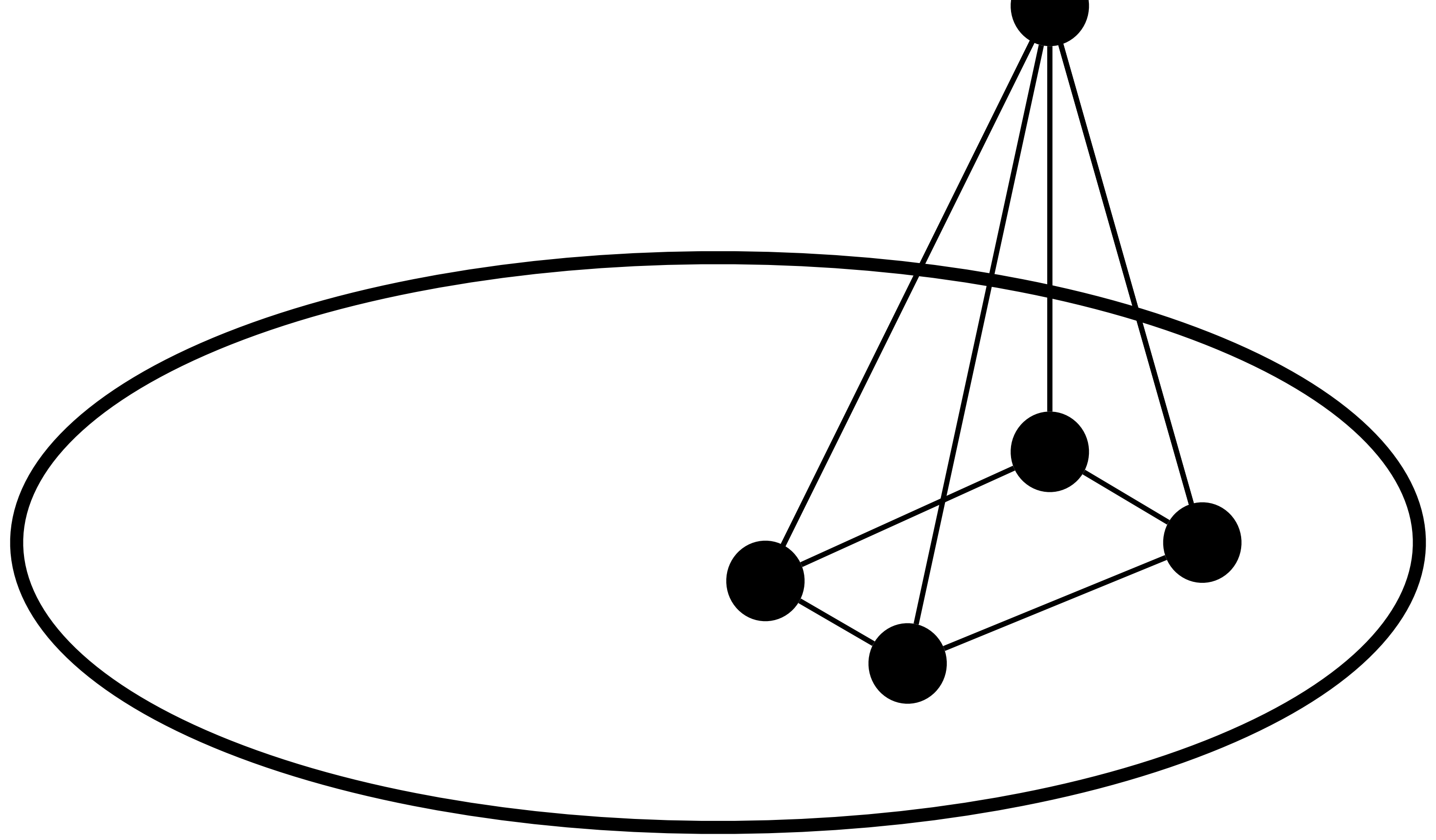
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$



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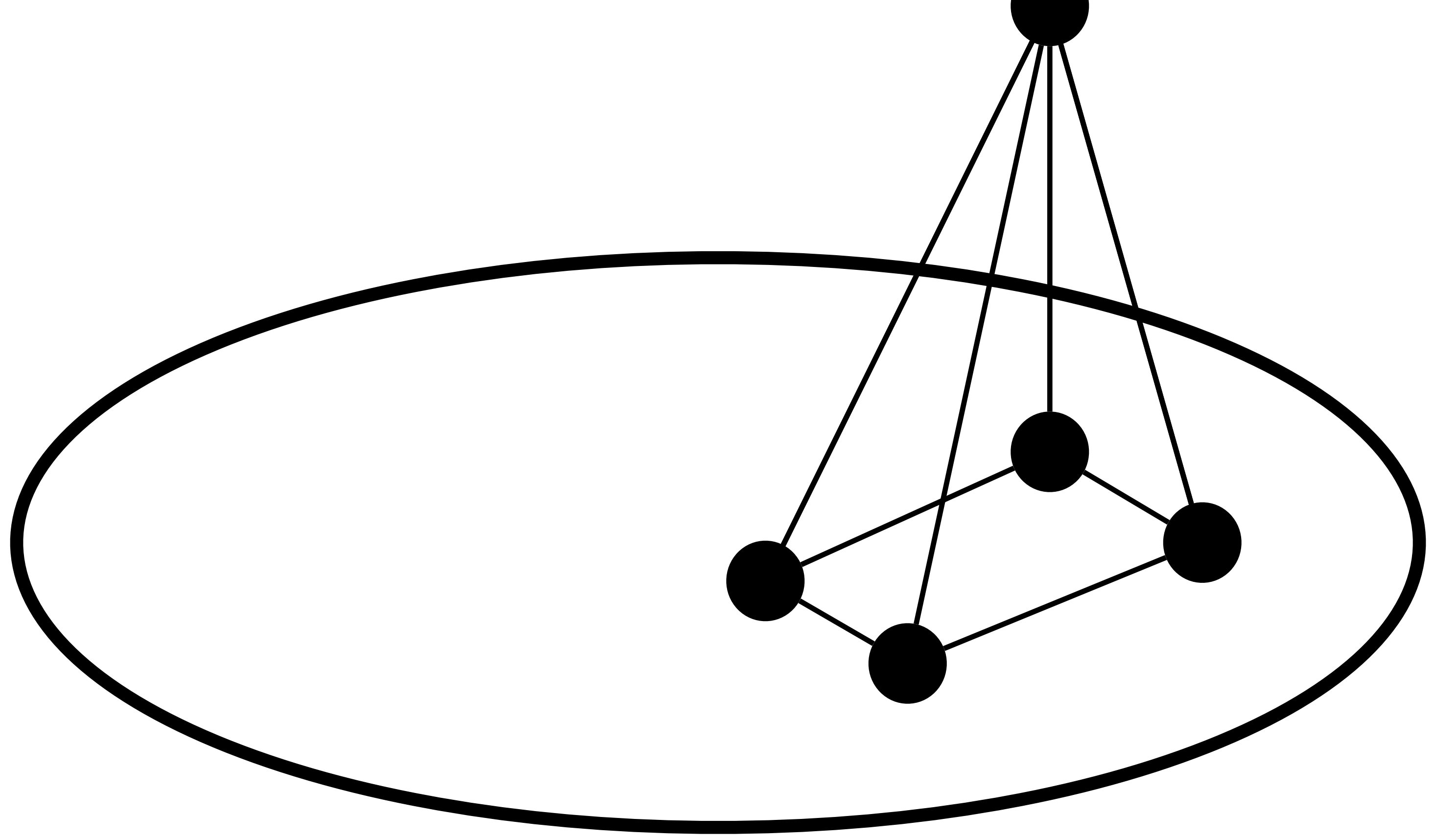

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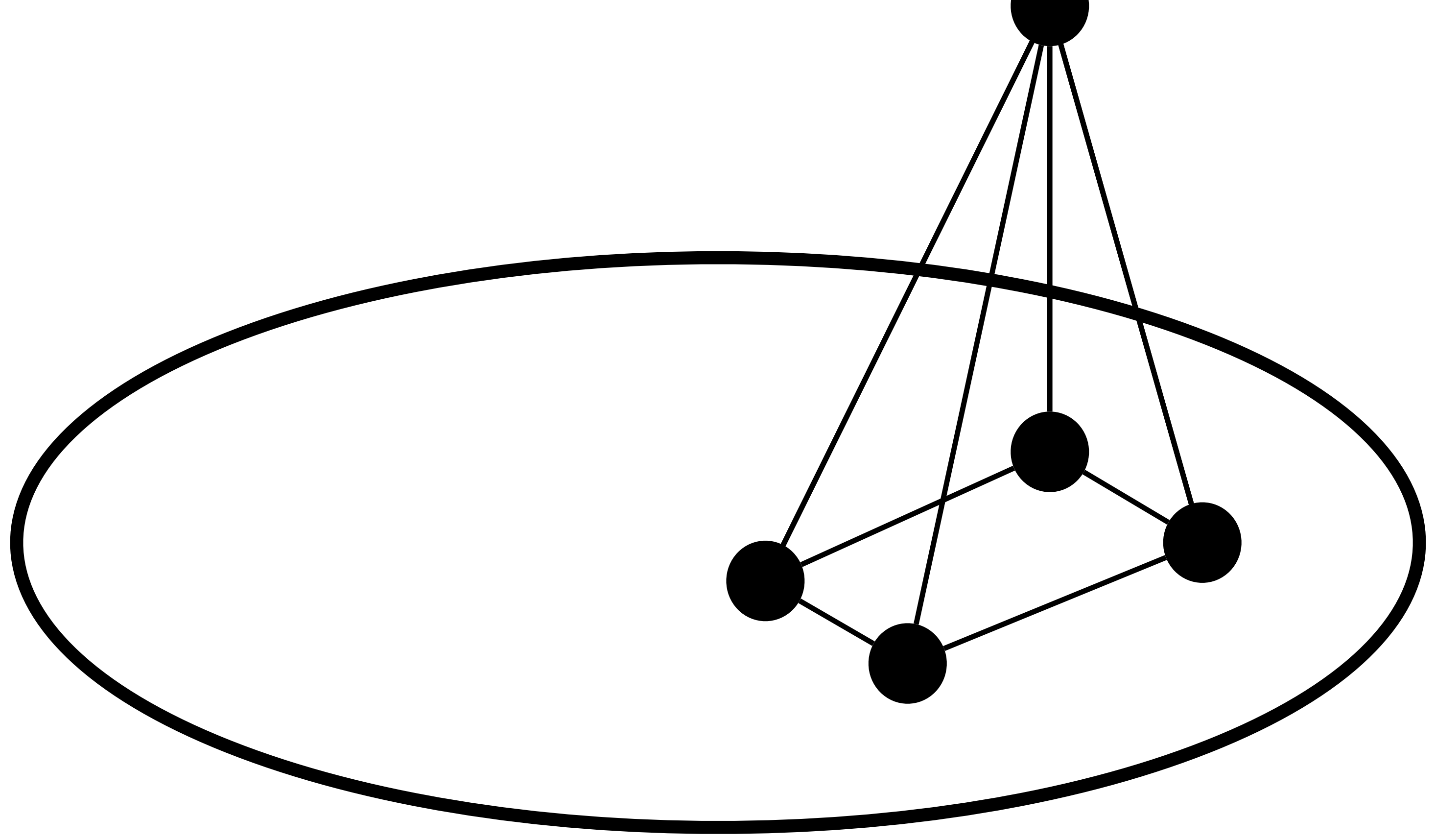

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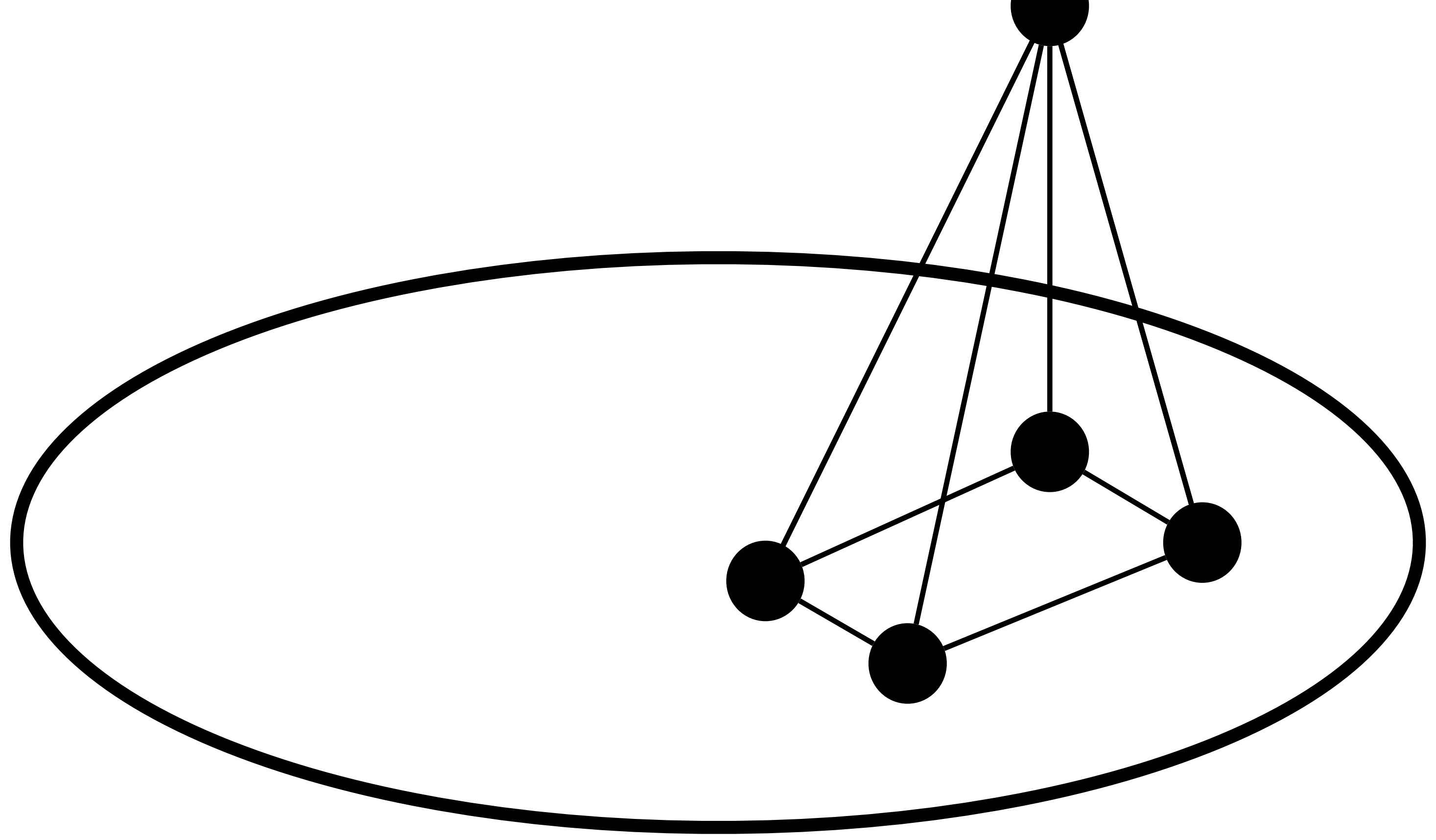

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  - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$

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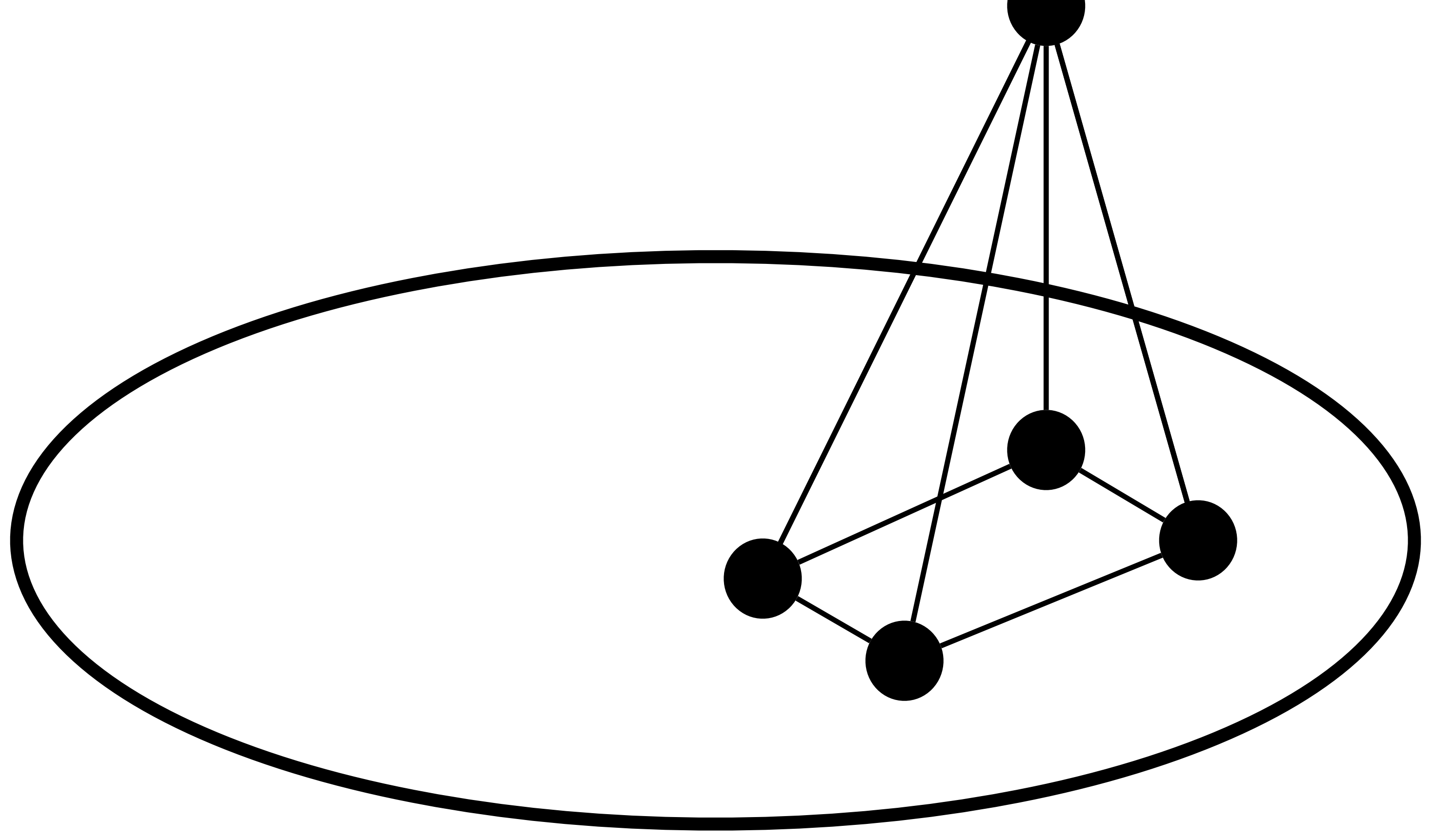

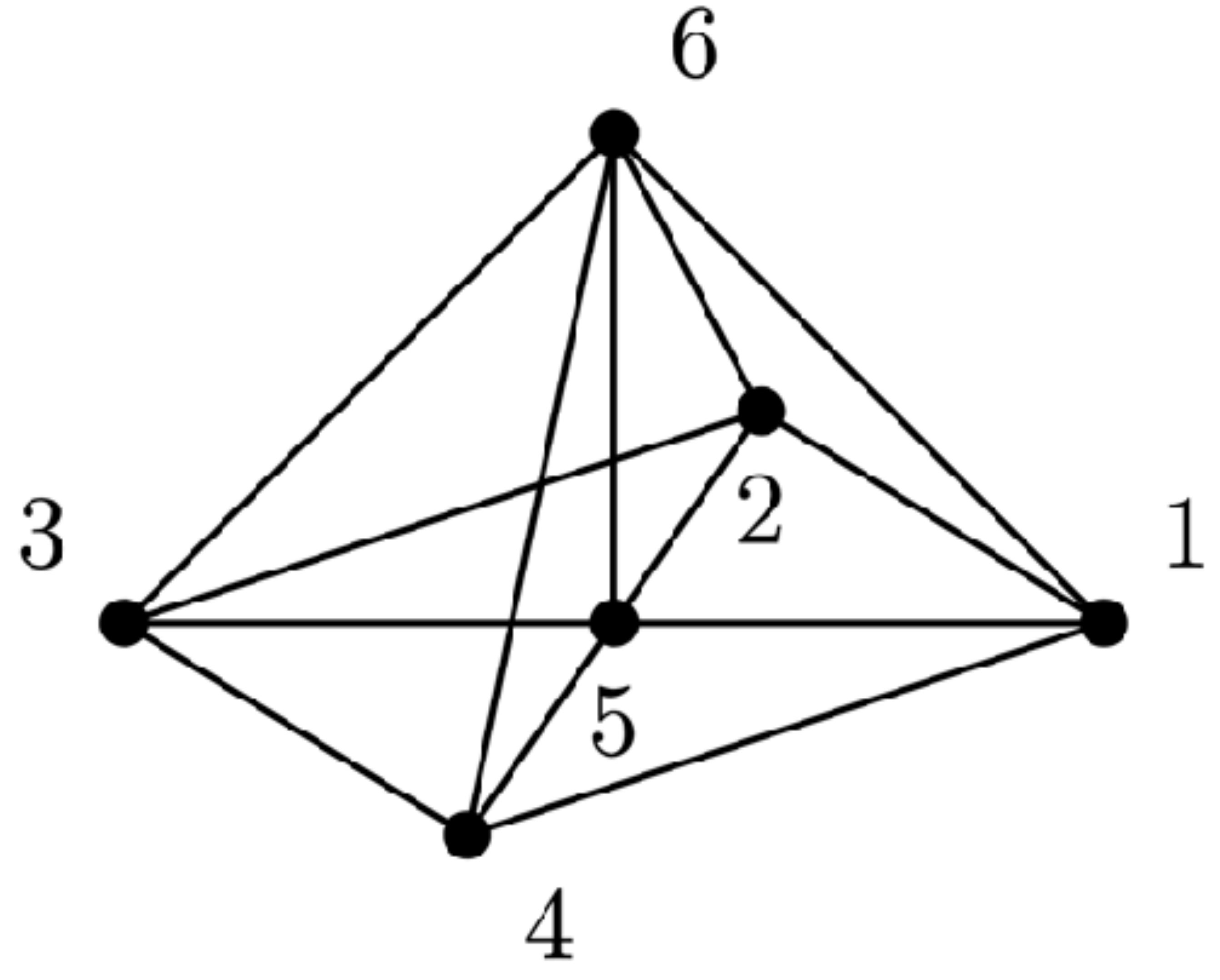

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- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs