The Topology of Preferential Attachment

The Asymptotics of the Expected Betti Numbers of Preferential Attachment Clique Complexes

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So, preferential attachment…

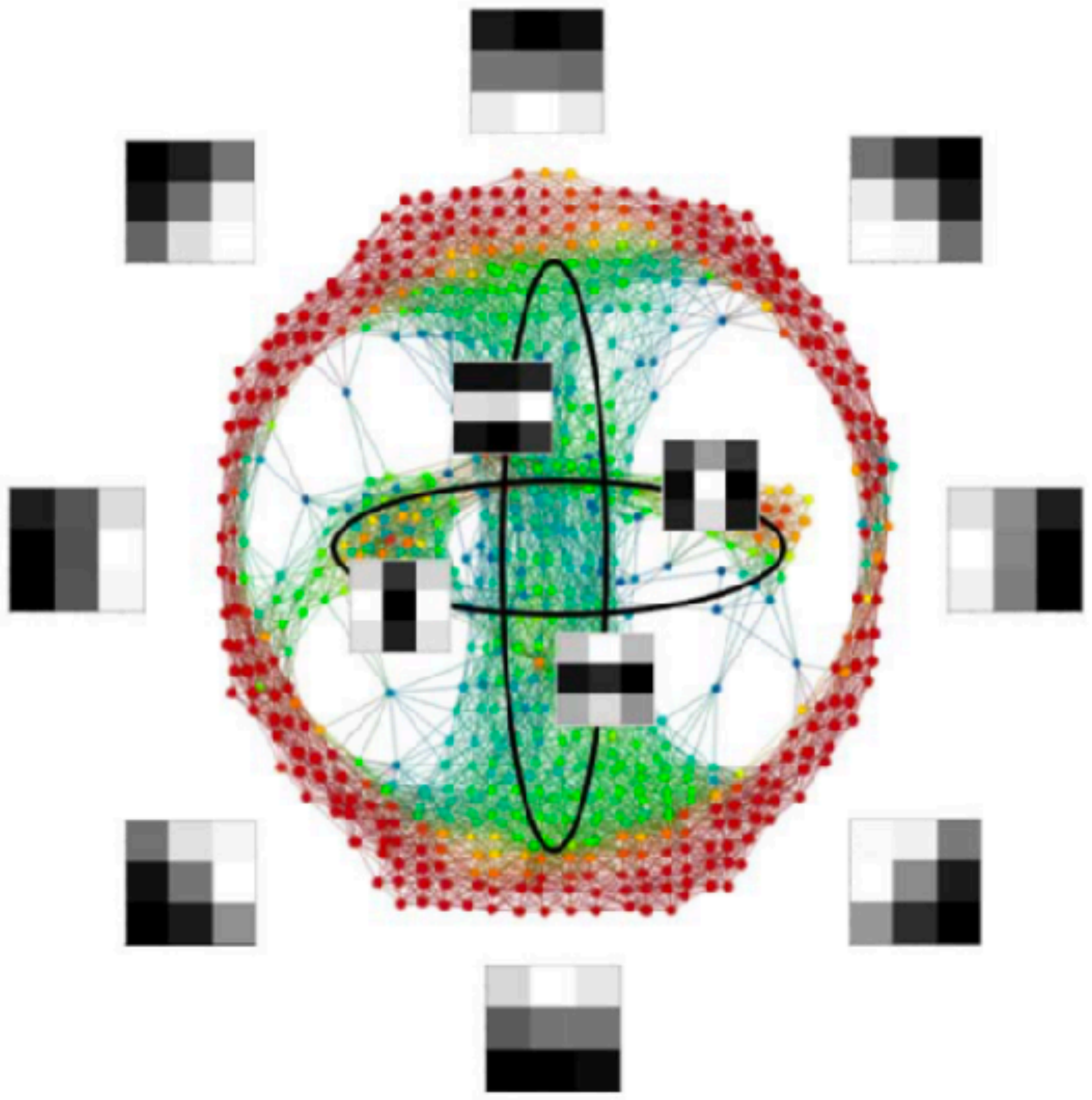
(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Just a bouquet of circles?

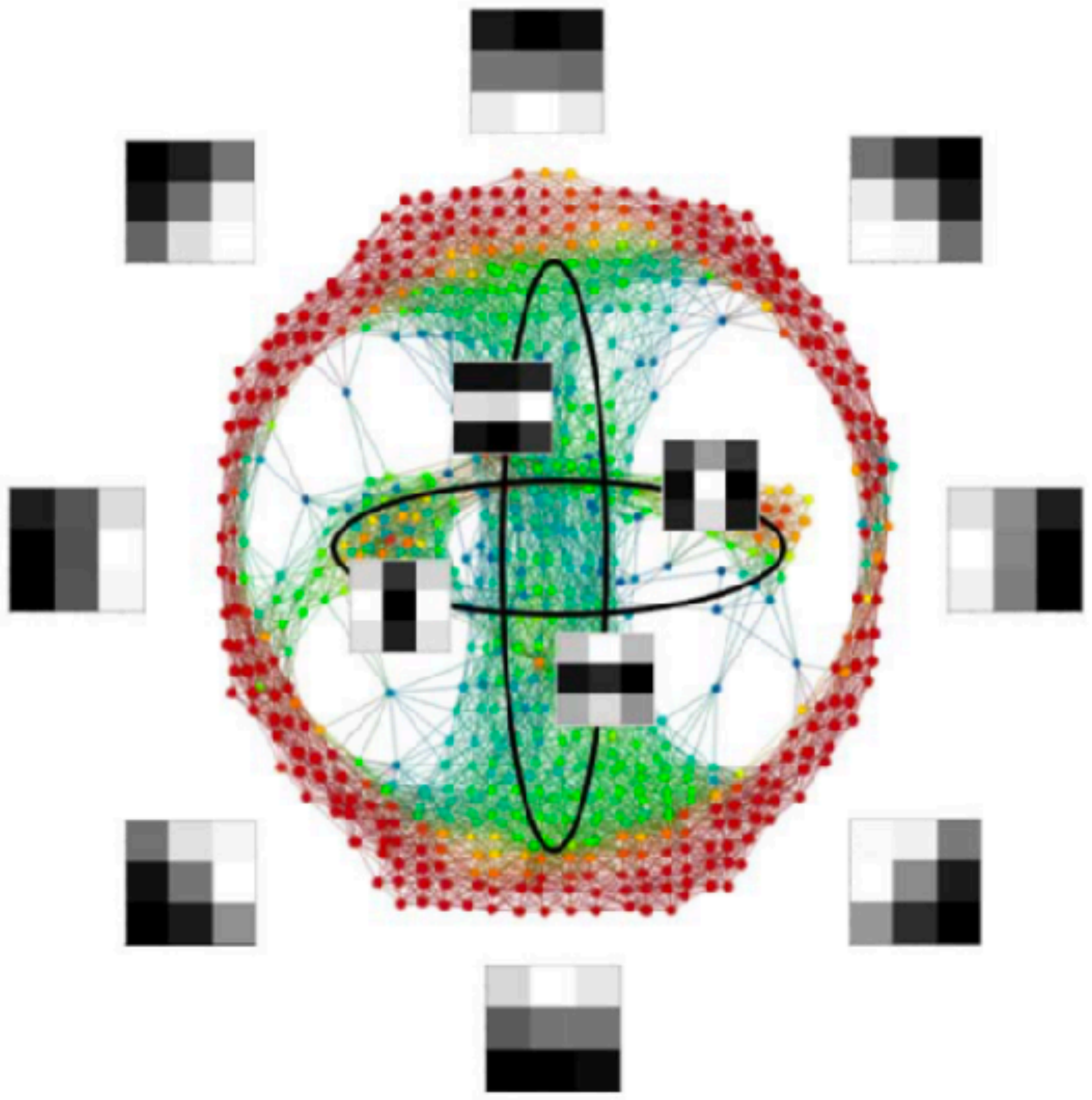
(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Agenda



Topological Data Analysis

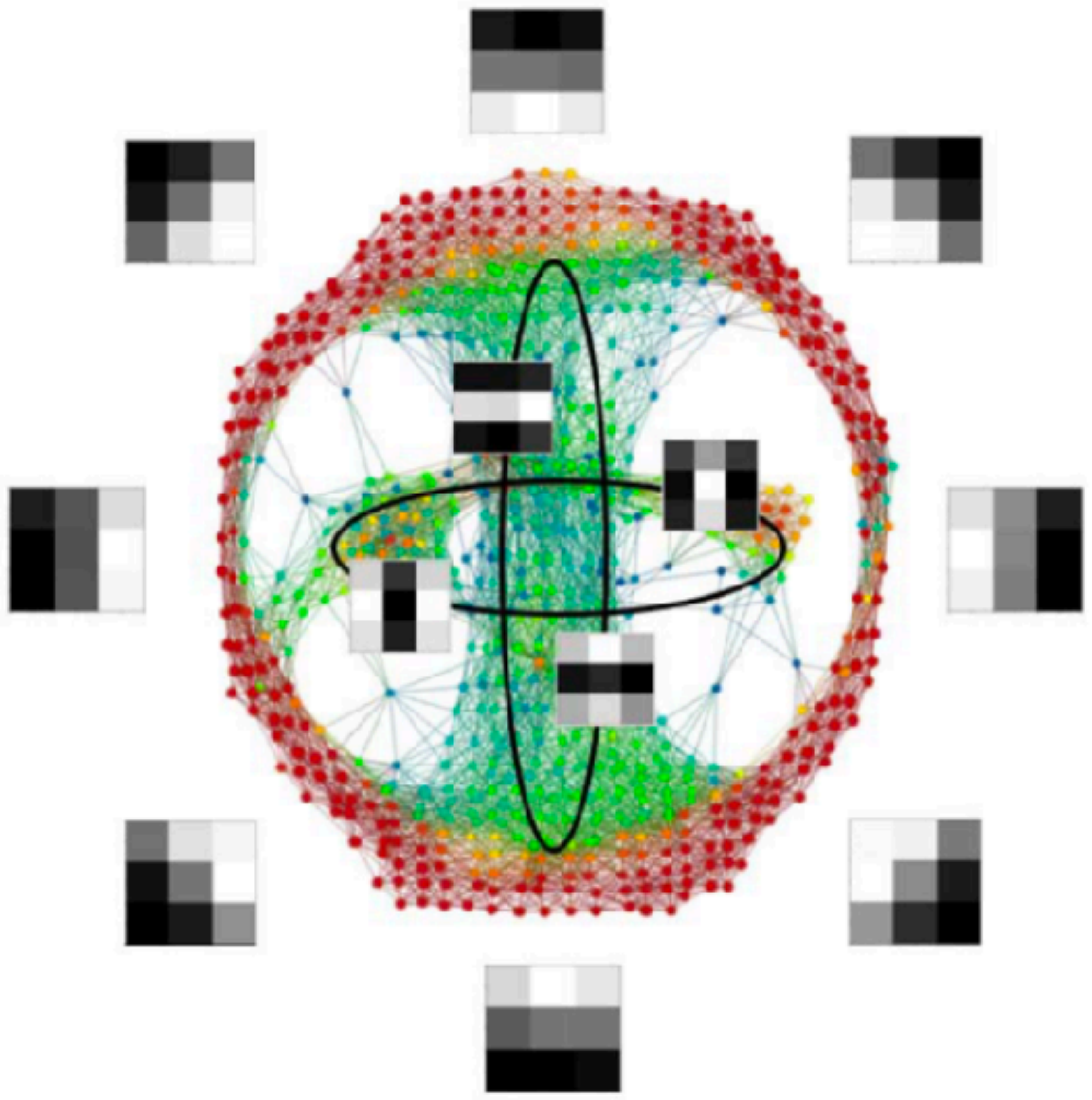
Agenda



random topology

preferential attachment

Agenda



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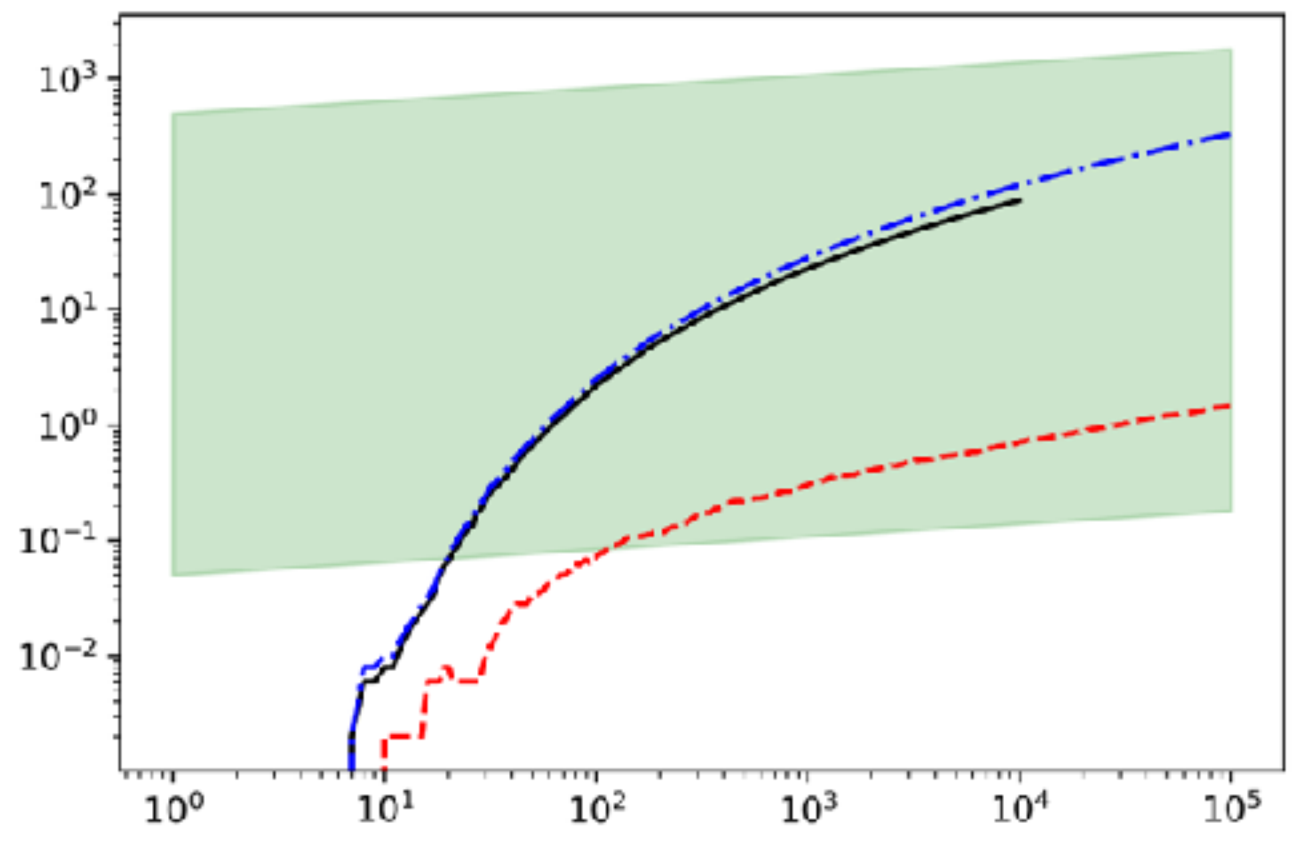
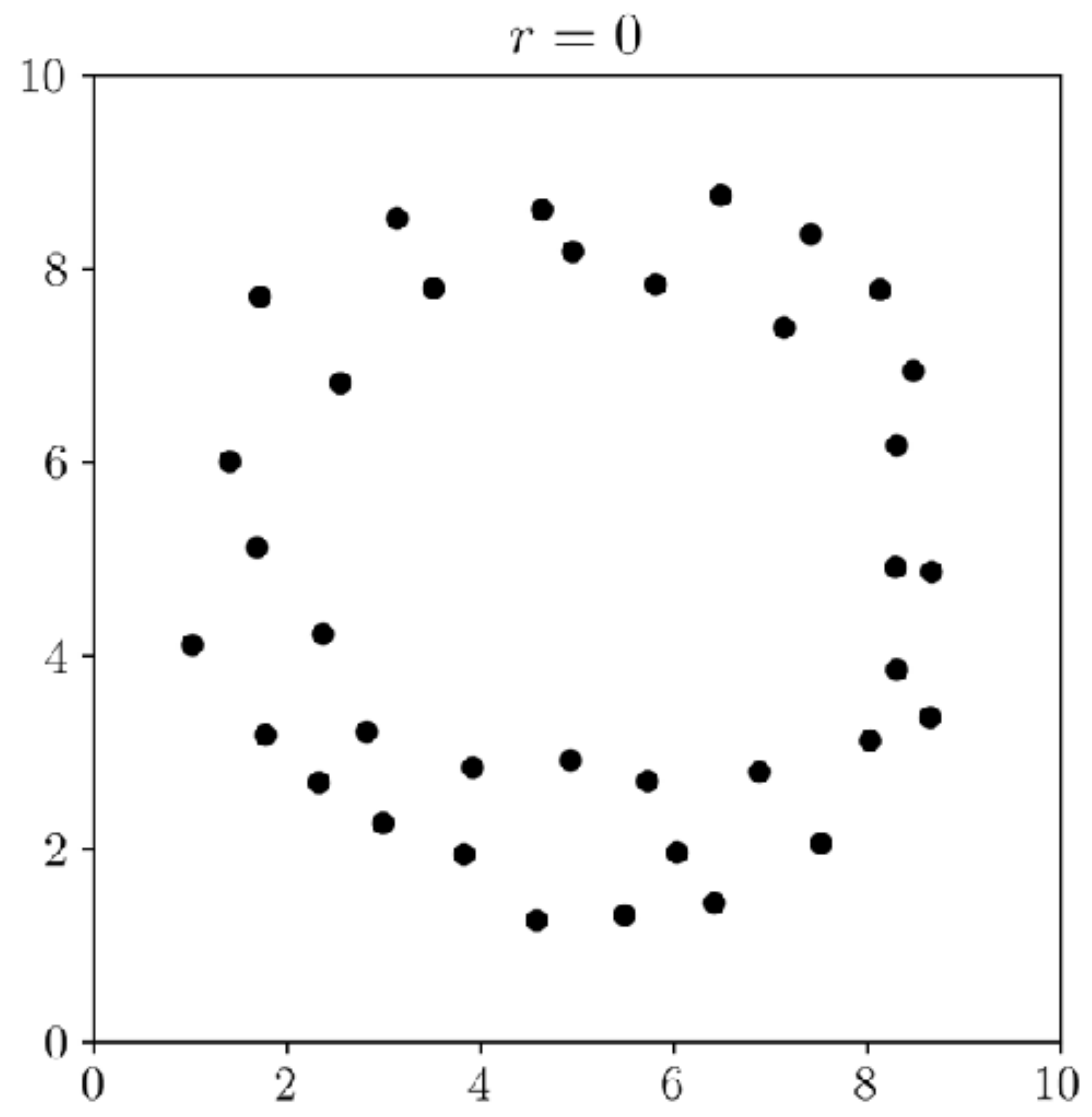
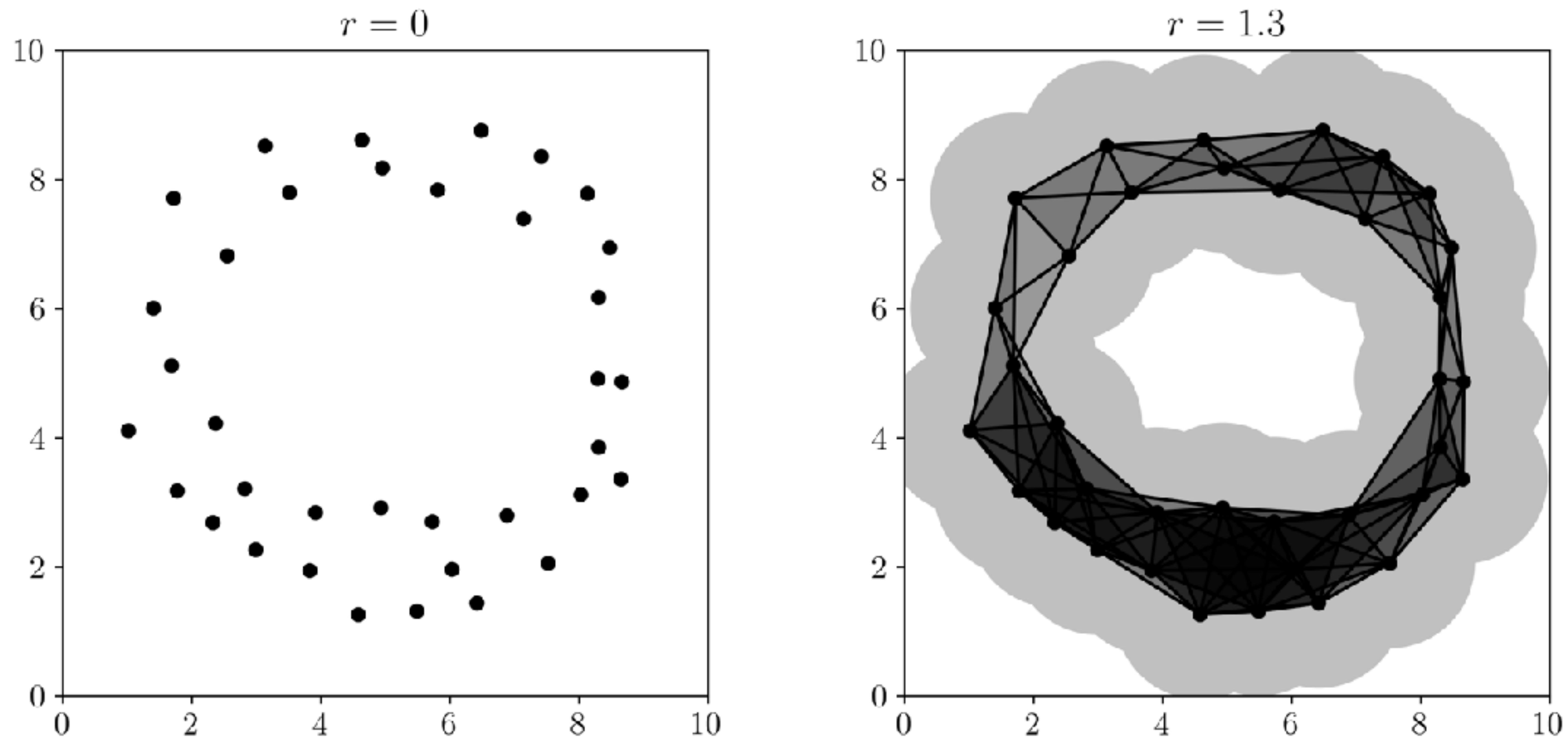


our result

I. Topological Data Analysis

Points



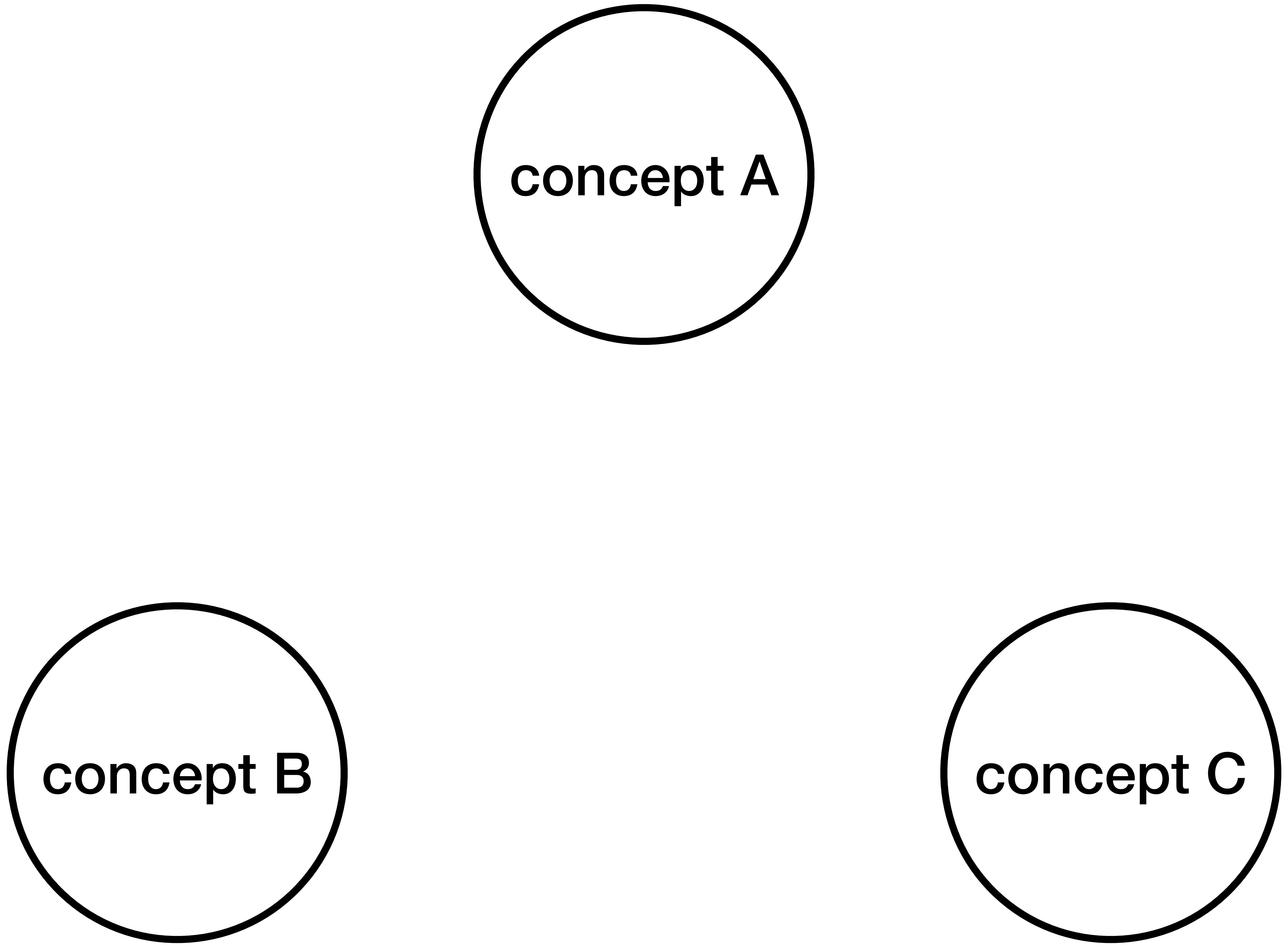


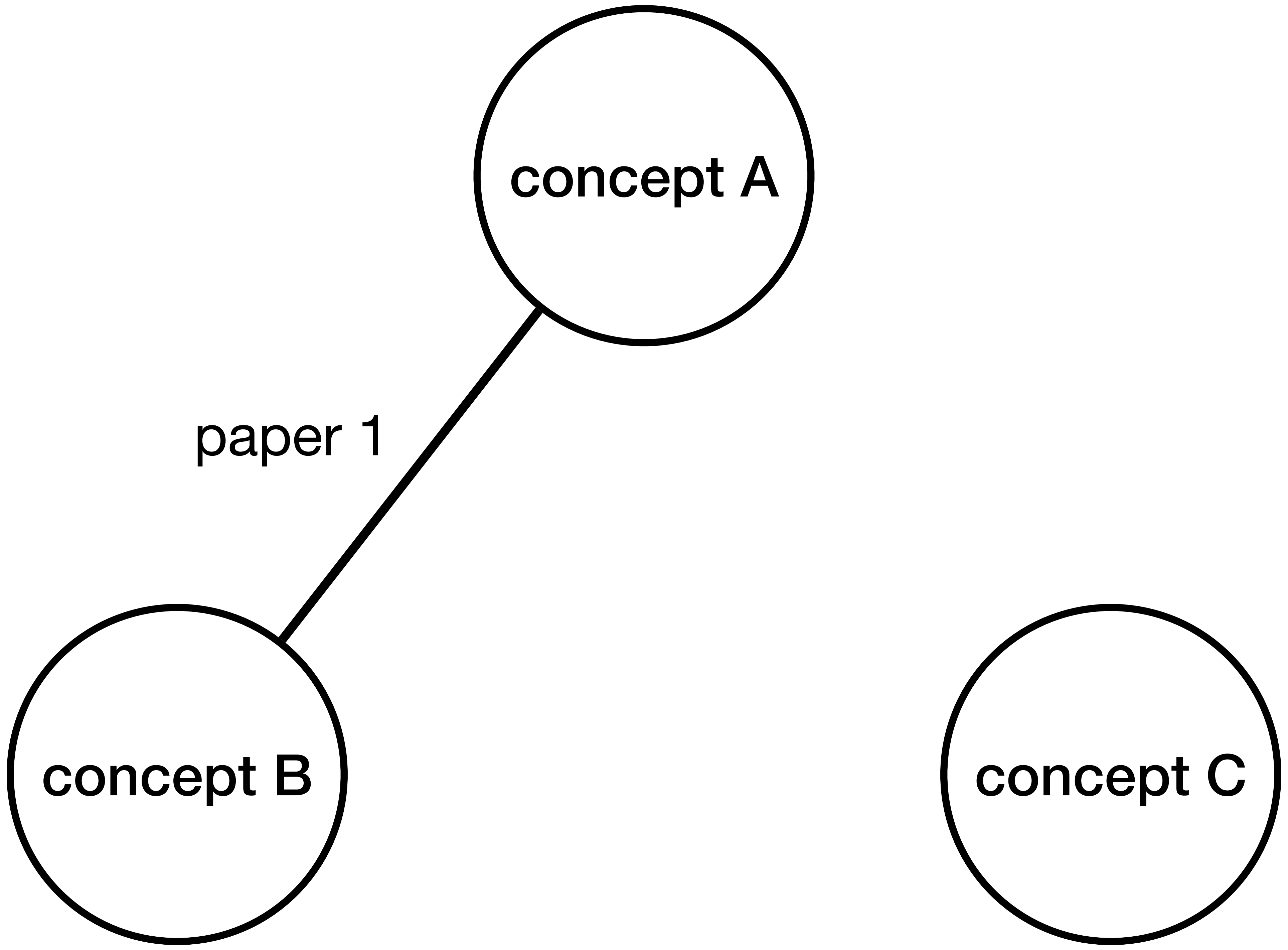
plots generated by Andrey Yao

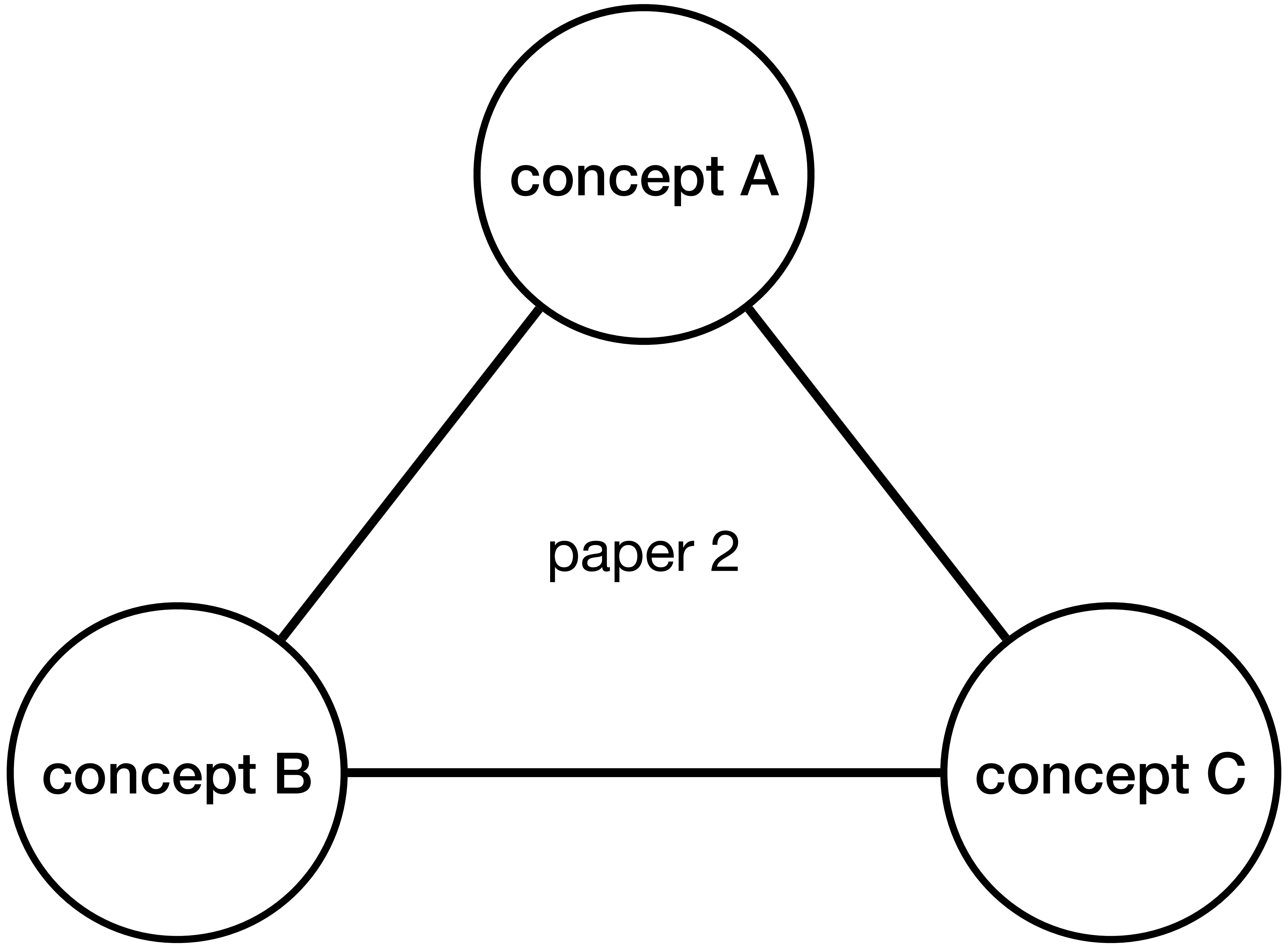
Networks and Complexes

Networks and Complexes

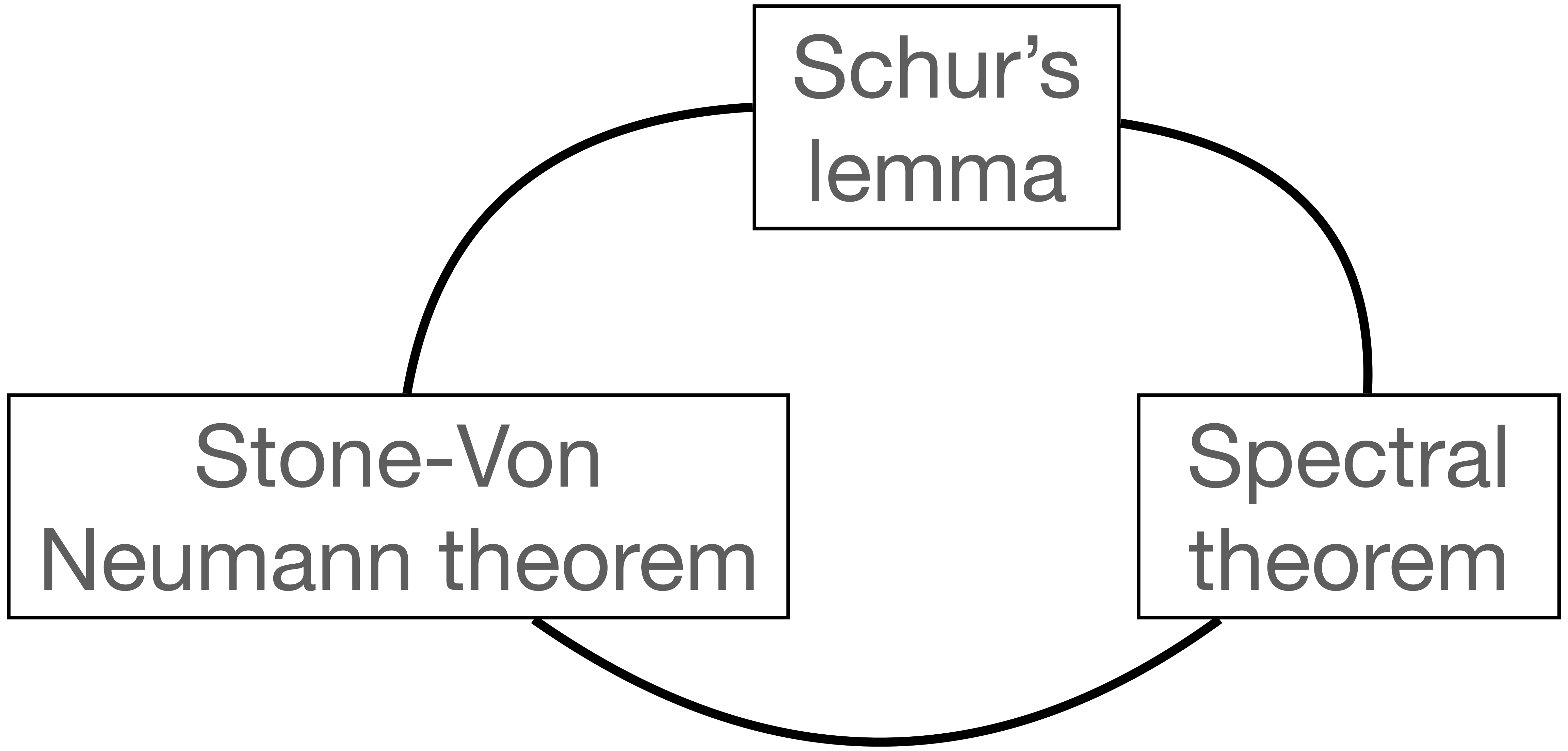
- Co-occurence complex in Math research paper [Salikov et al, 2018]

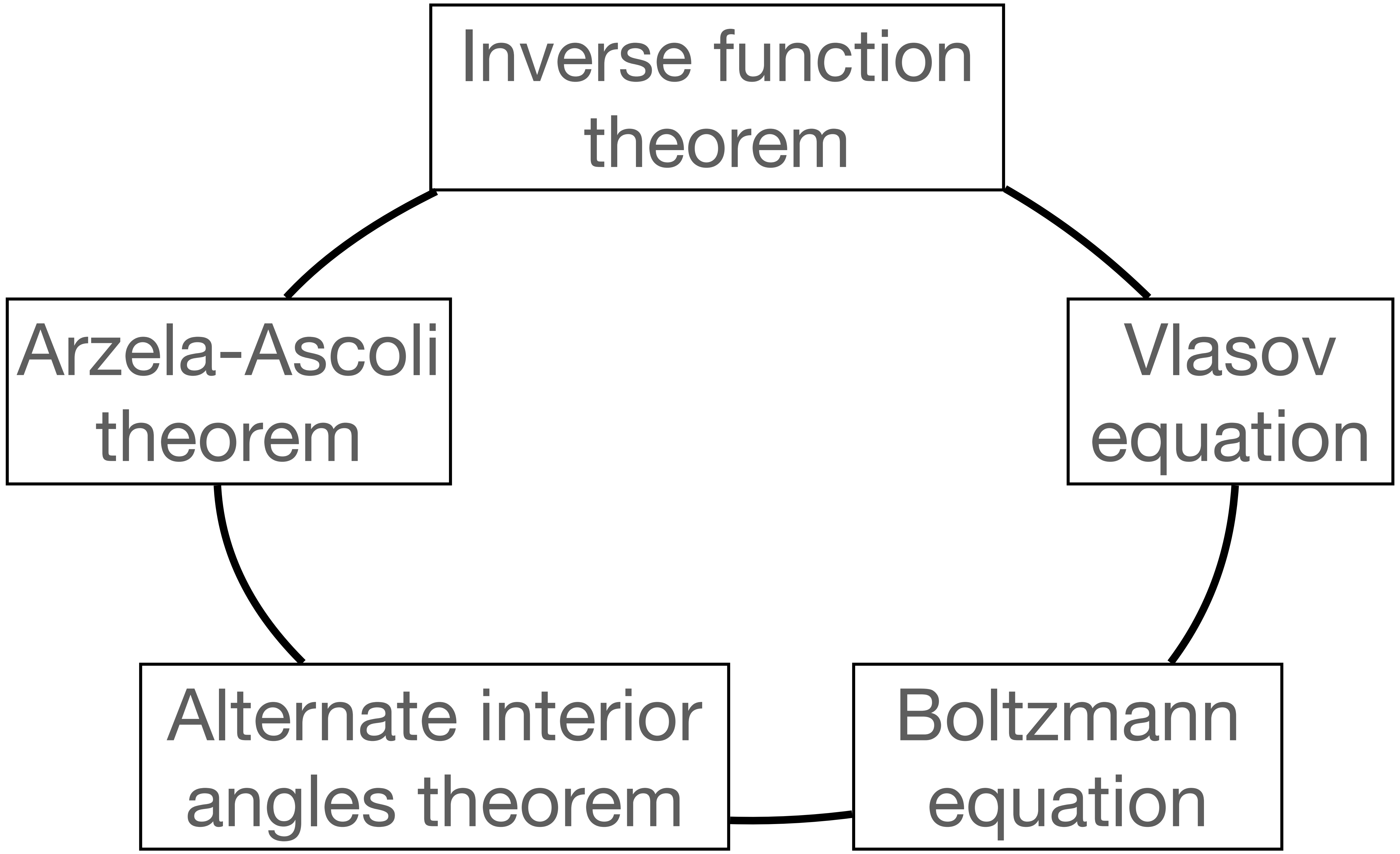

concept A
concept B
concept C


concept A
paper 1
concept B
concept C


concept A
paper 2
concept B
concept C

Gap in Understanding





Benchmark of Comparison?

II. Preferential Attachment

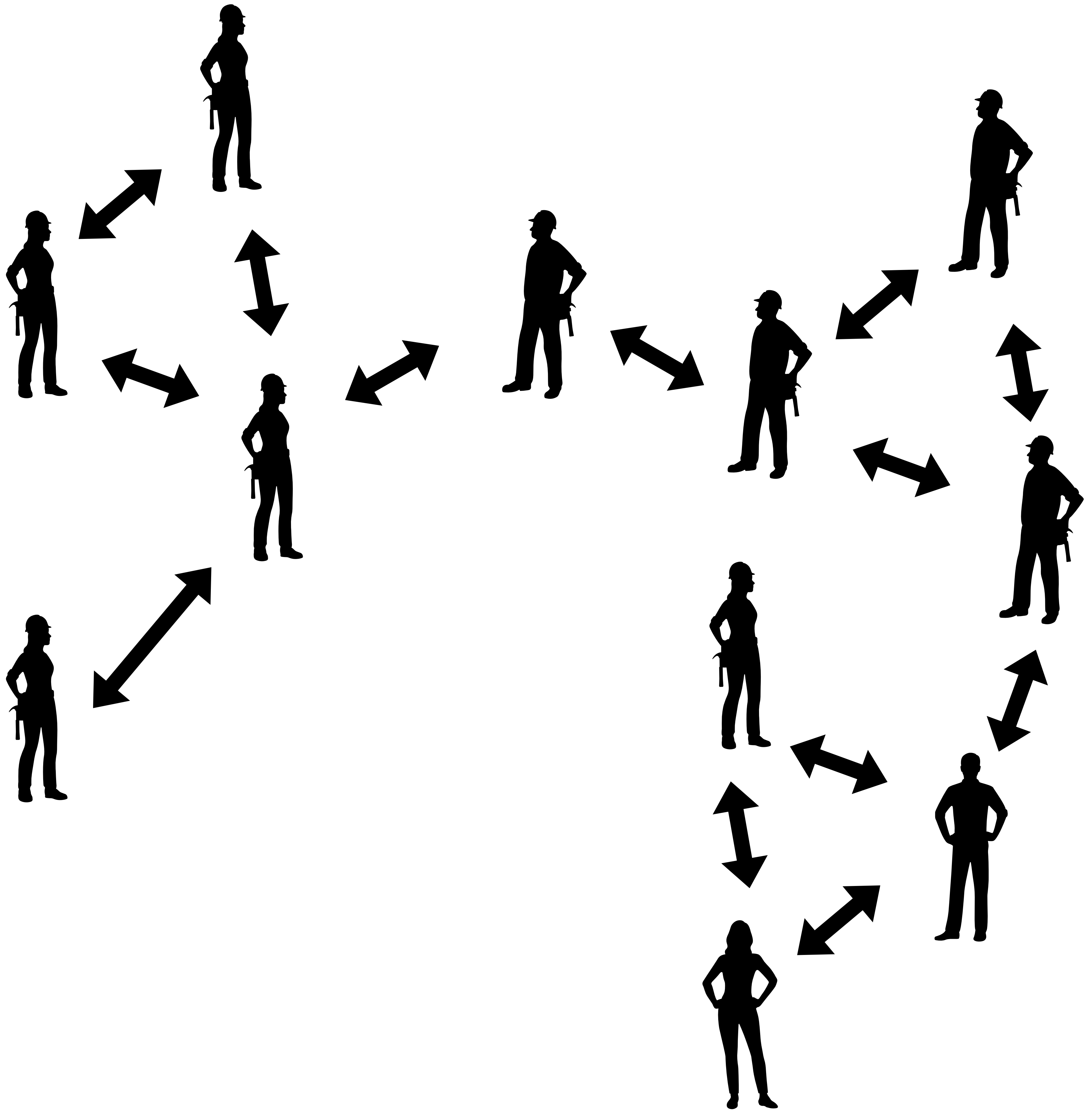
Towards a random model

Preferential Attachment
[Albert and Barabasi 1999]

(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Preferential Attachment

[Albert and Barabasi 1999]



Preferential Attachment

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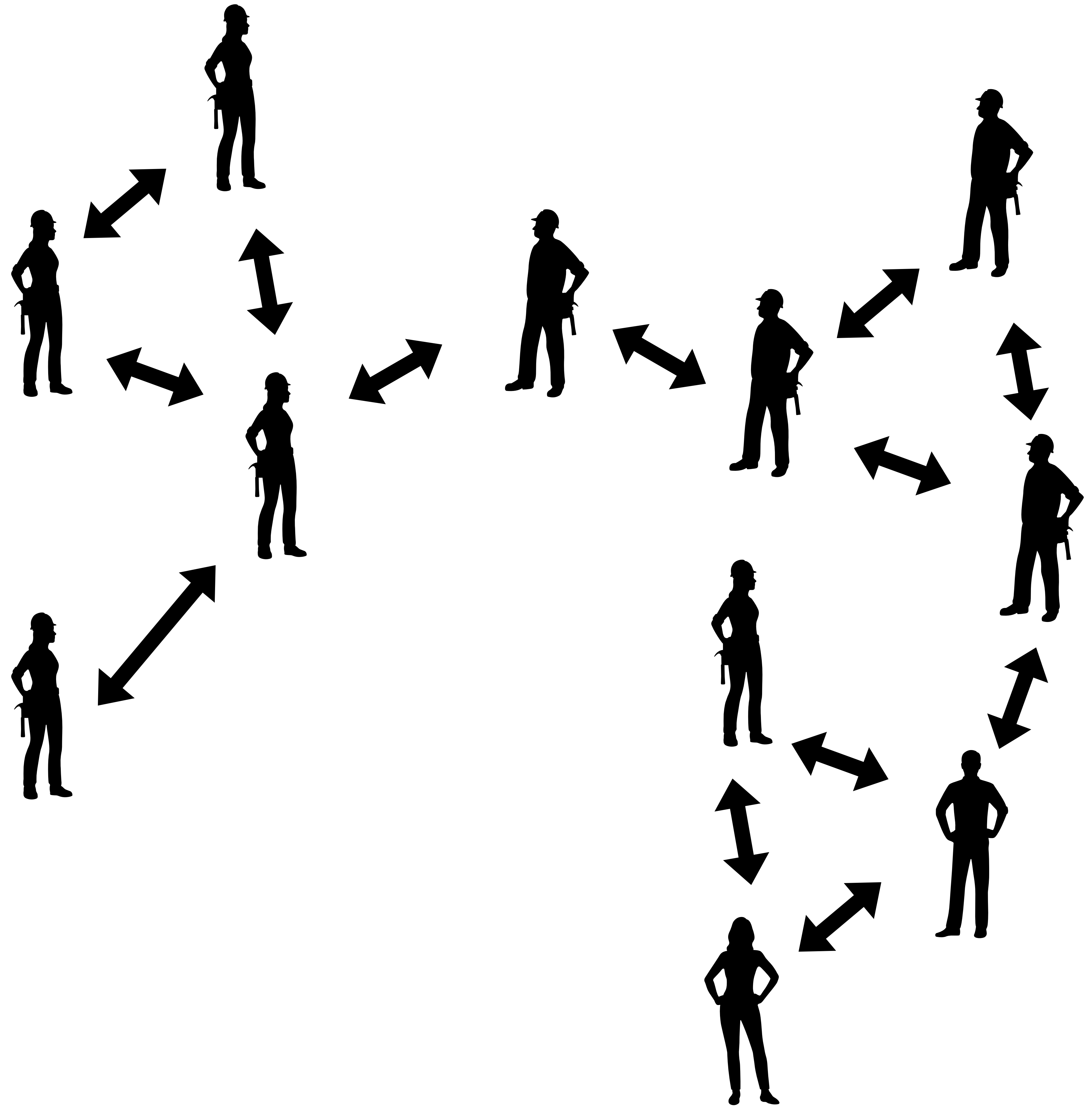

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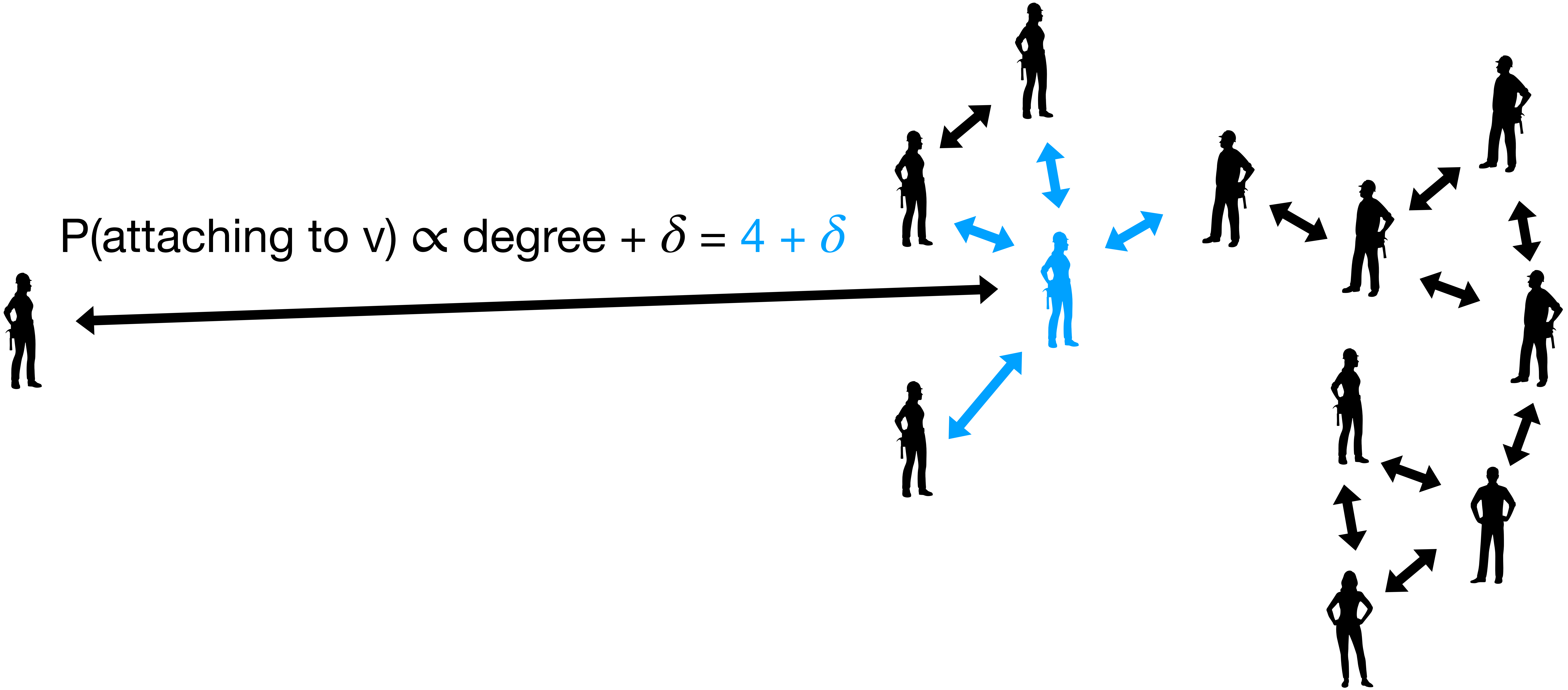
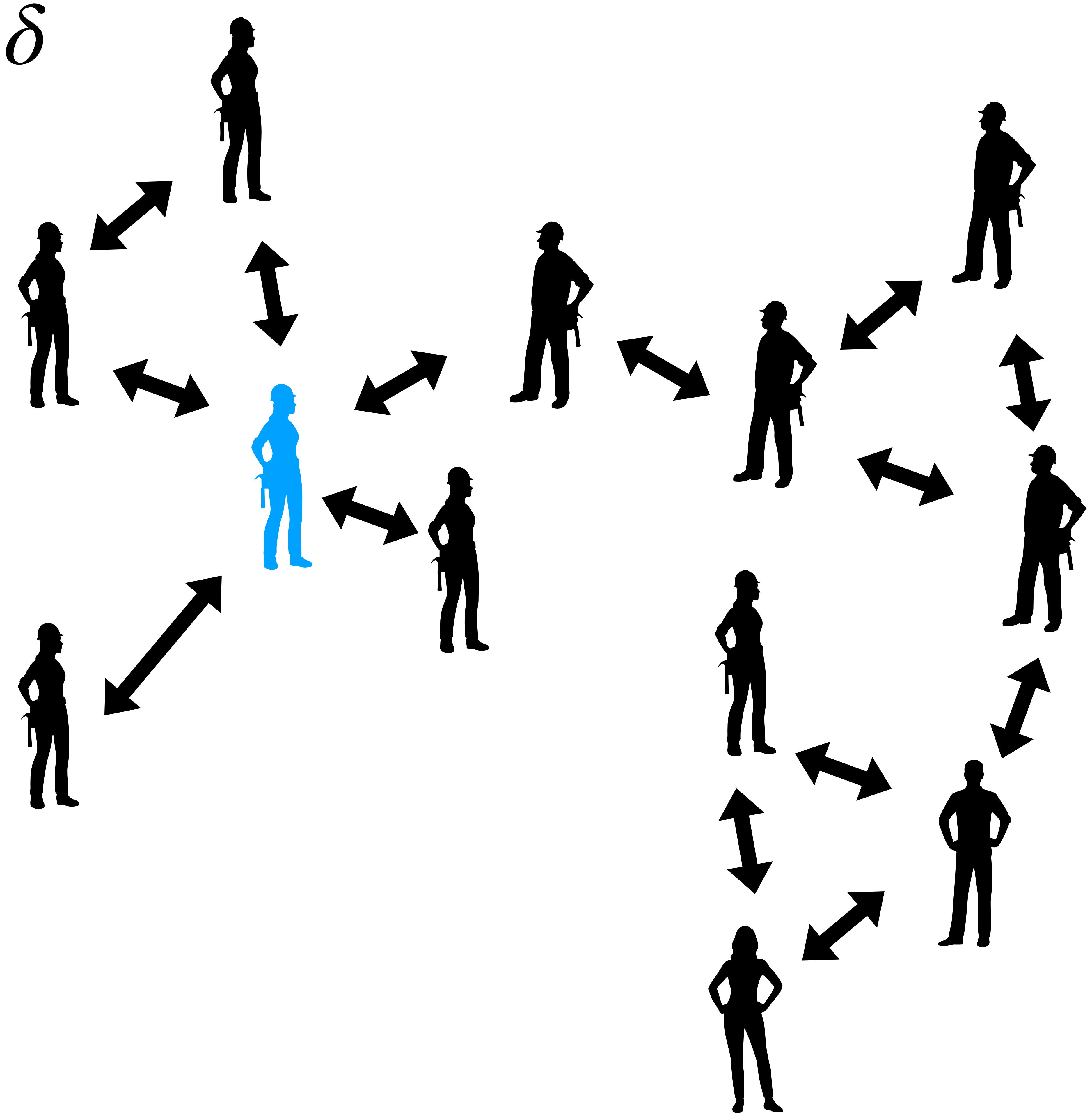


Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) ∝ degree + a tuning parameter δ



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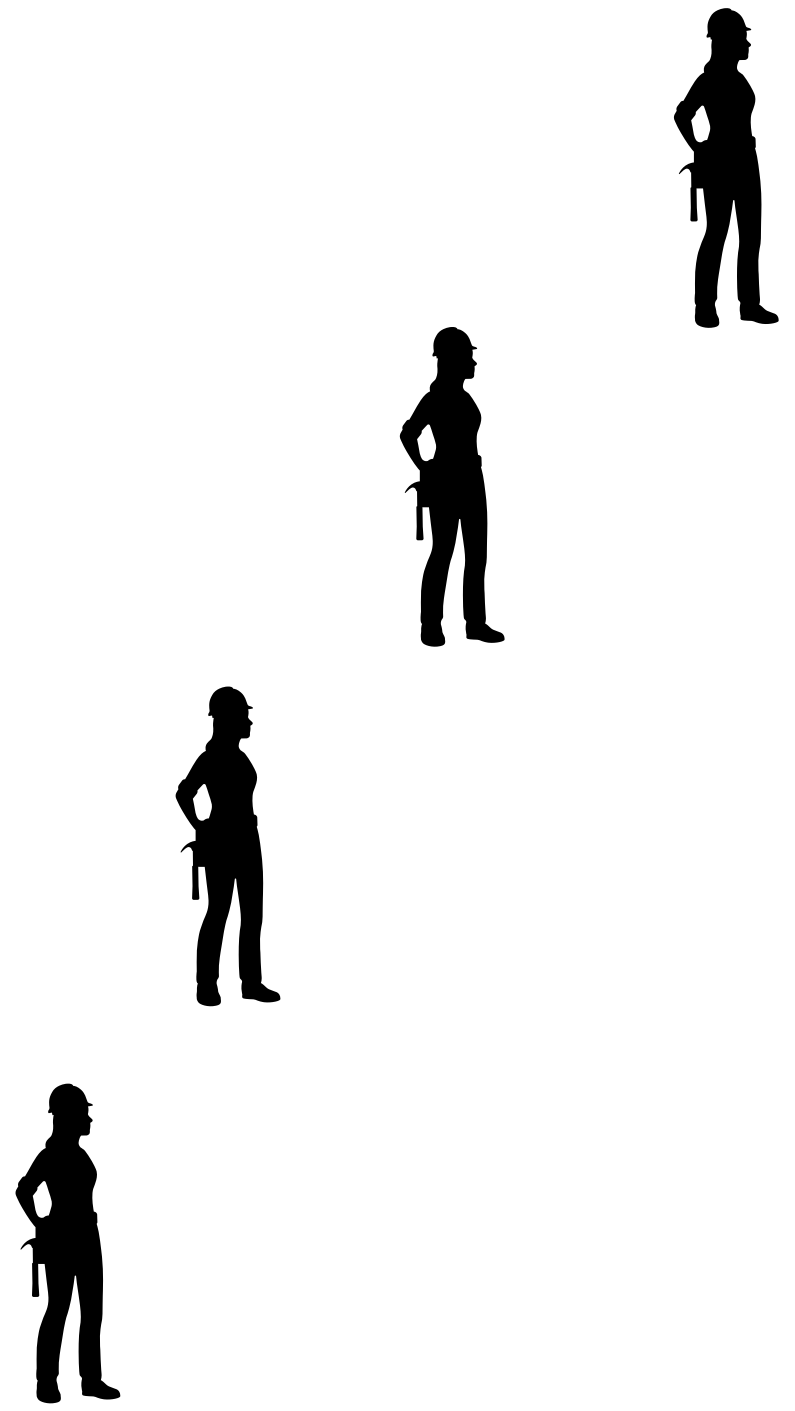

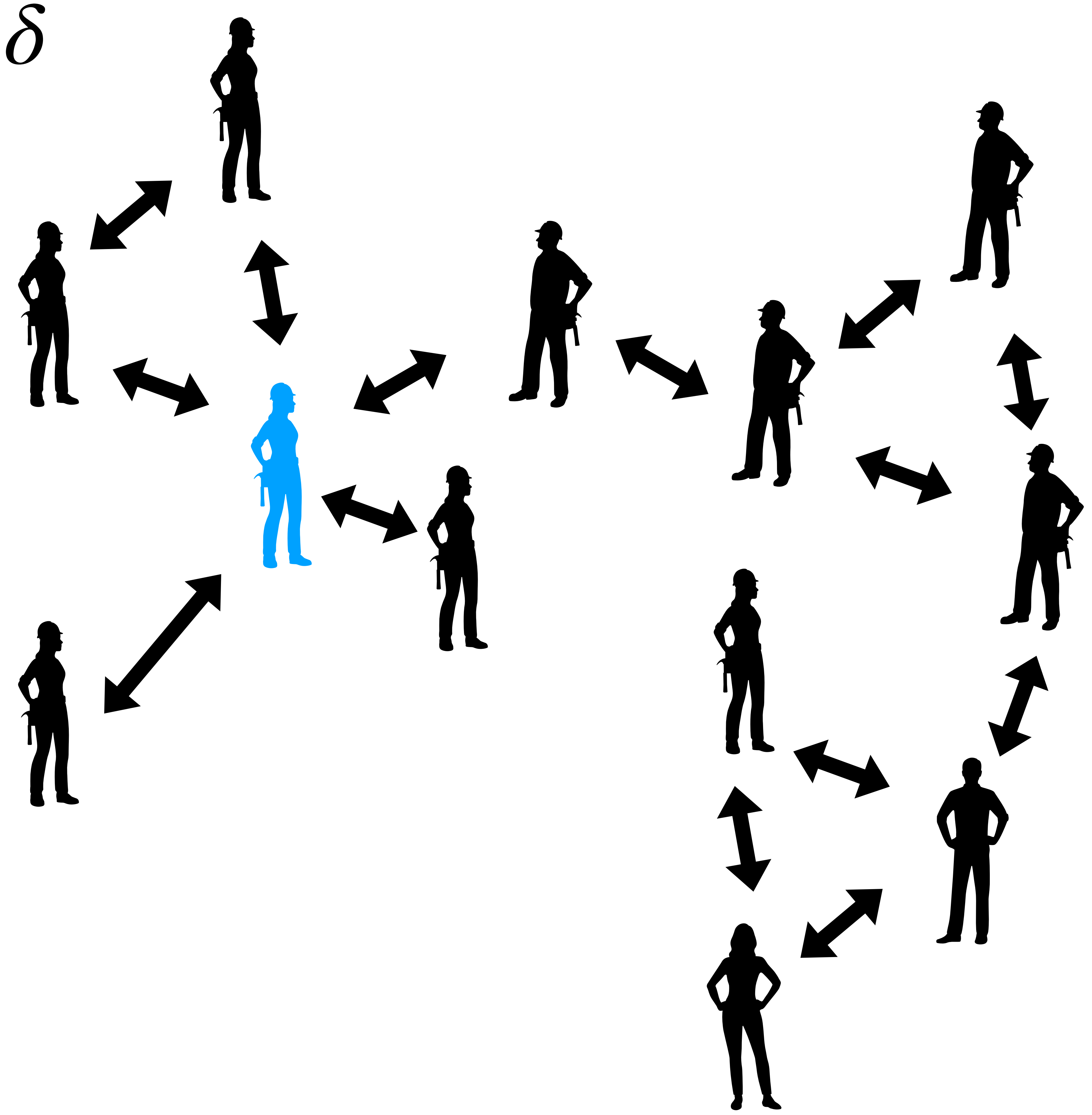

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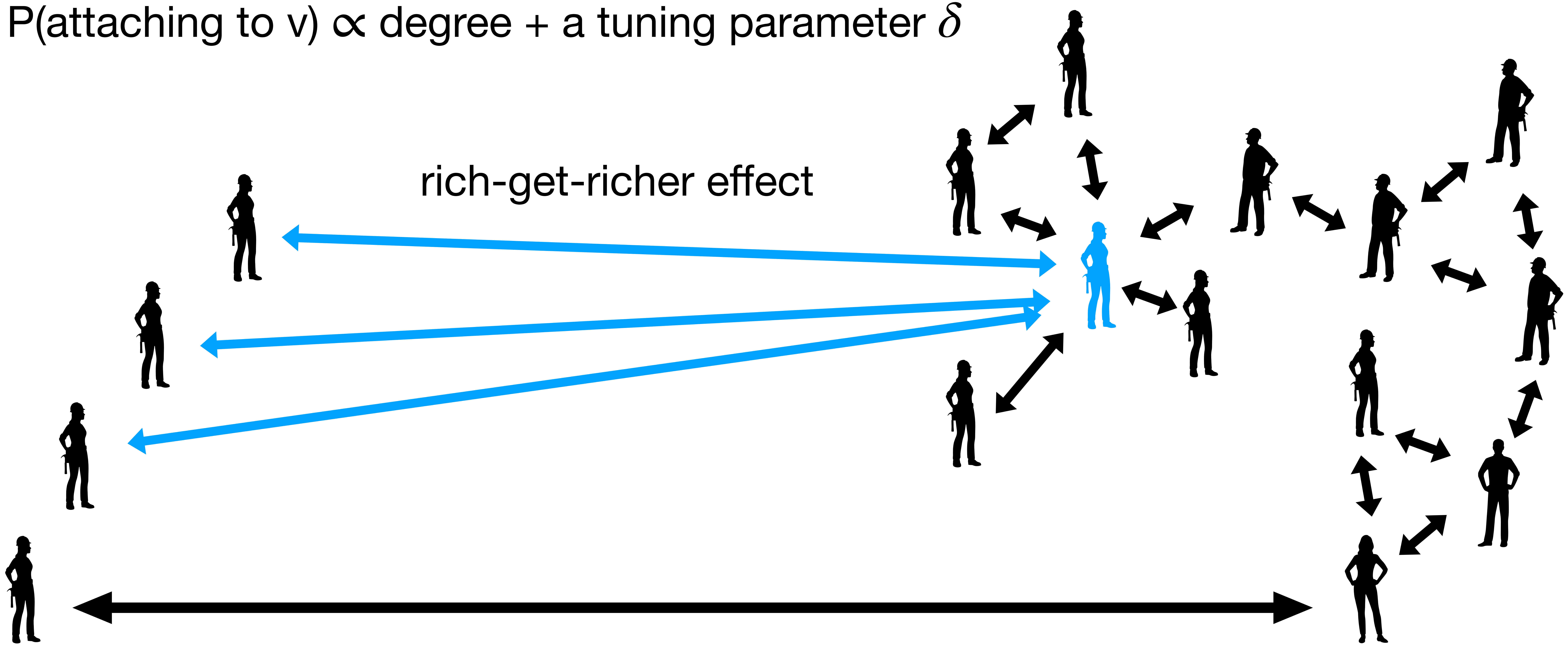

What do we know?

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- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]

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- subgraph counts [Garavaglia and Steghuis 2019]

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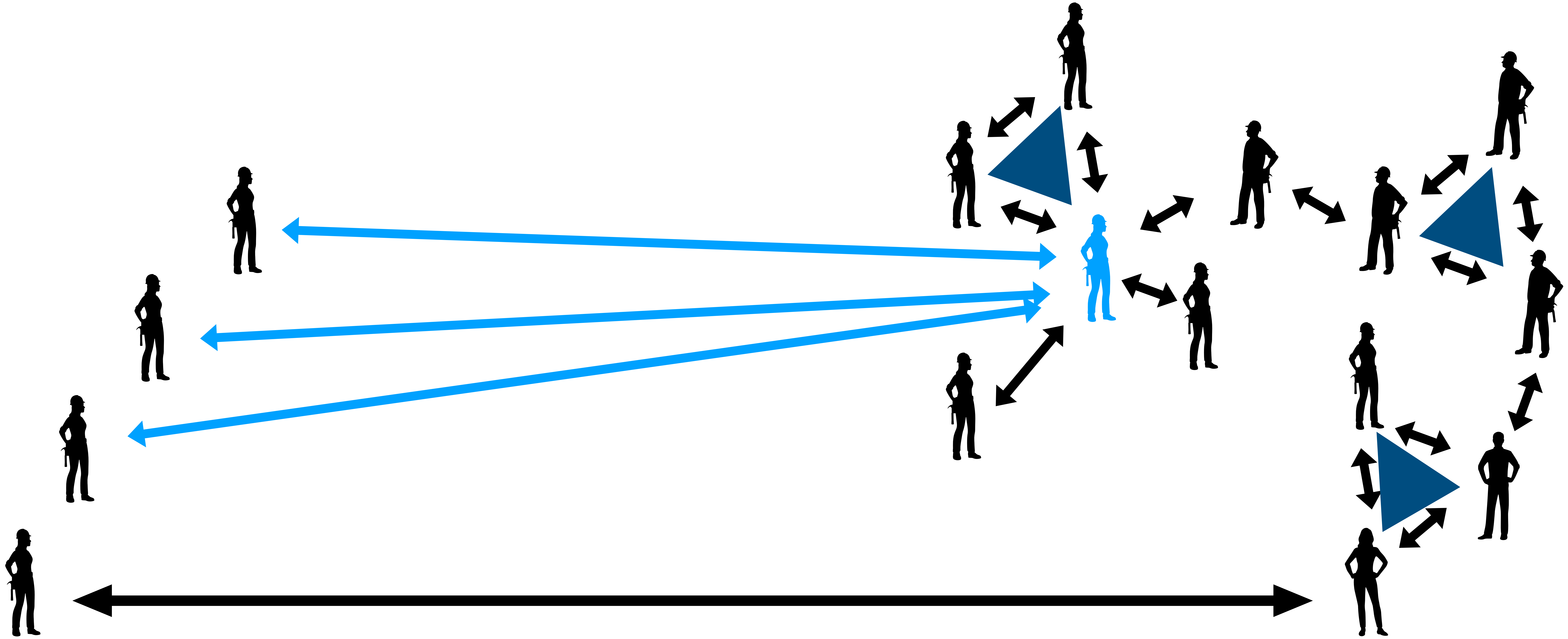
- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]
- and more…

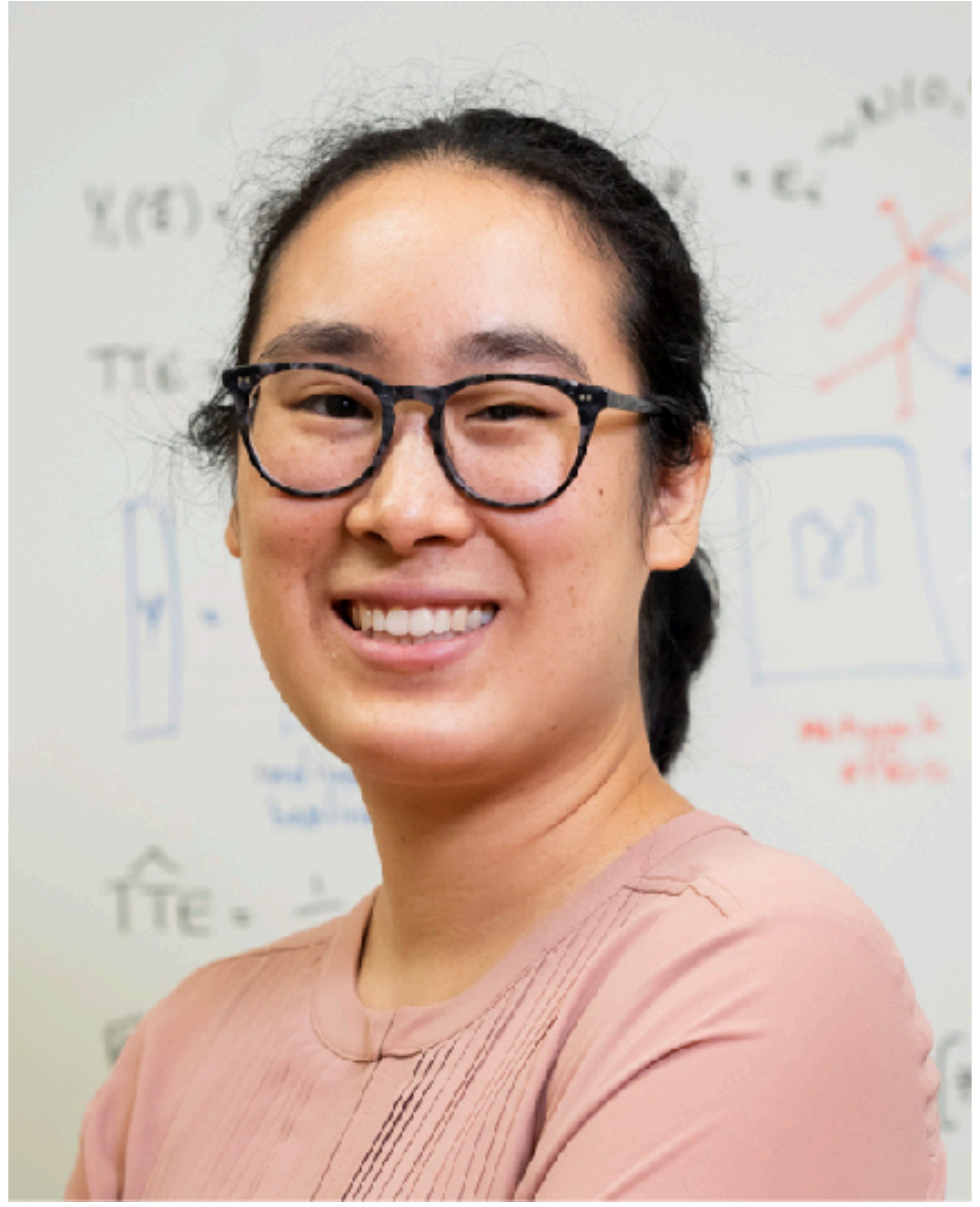
Clique Complex

aka Flag Complex



III Topology of Preferential Attachment

My Lovely Collaborators



Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

Expected Betti Number $E[\beta_q]$

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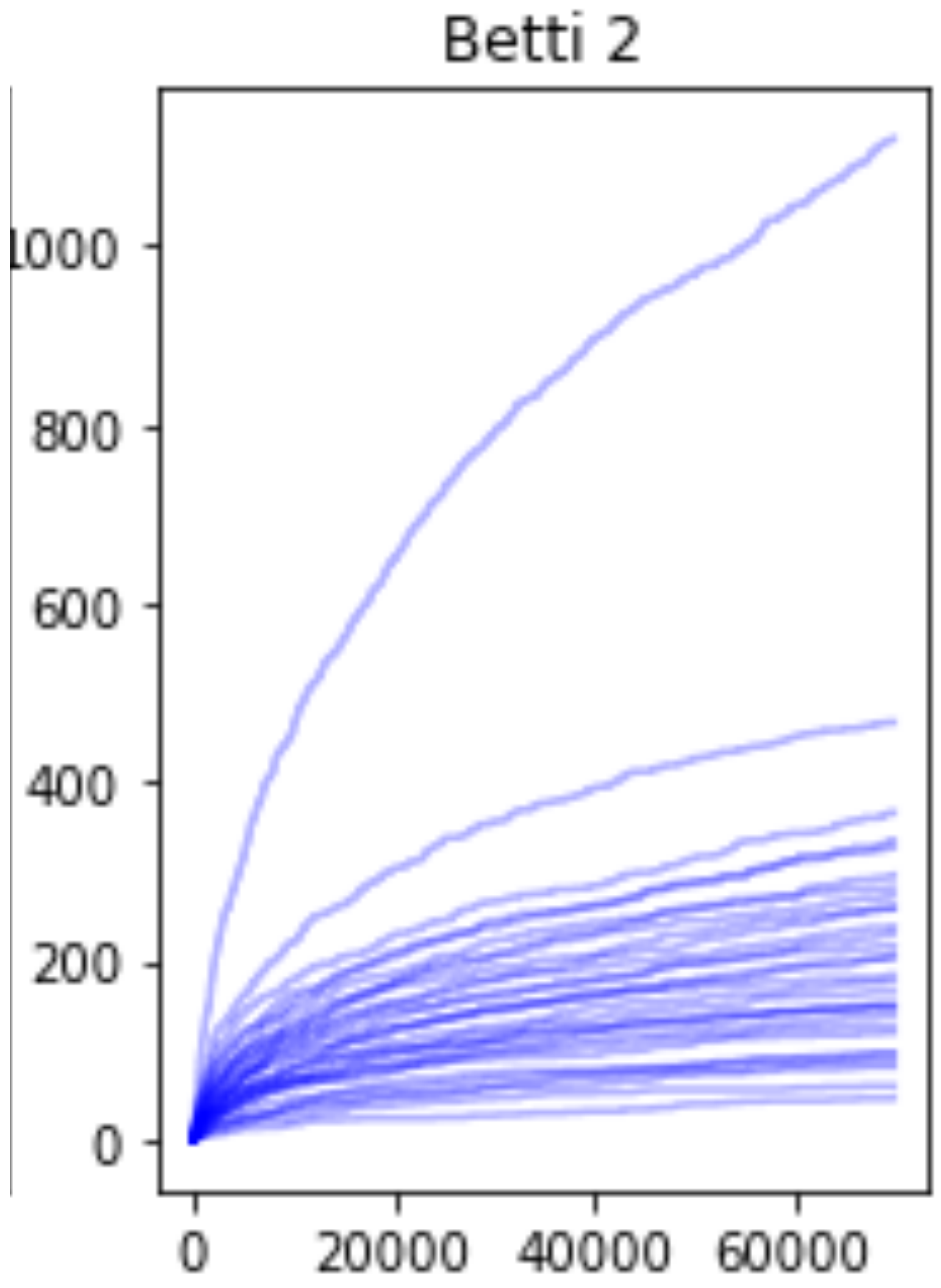
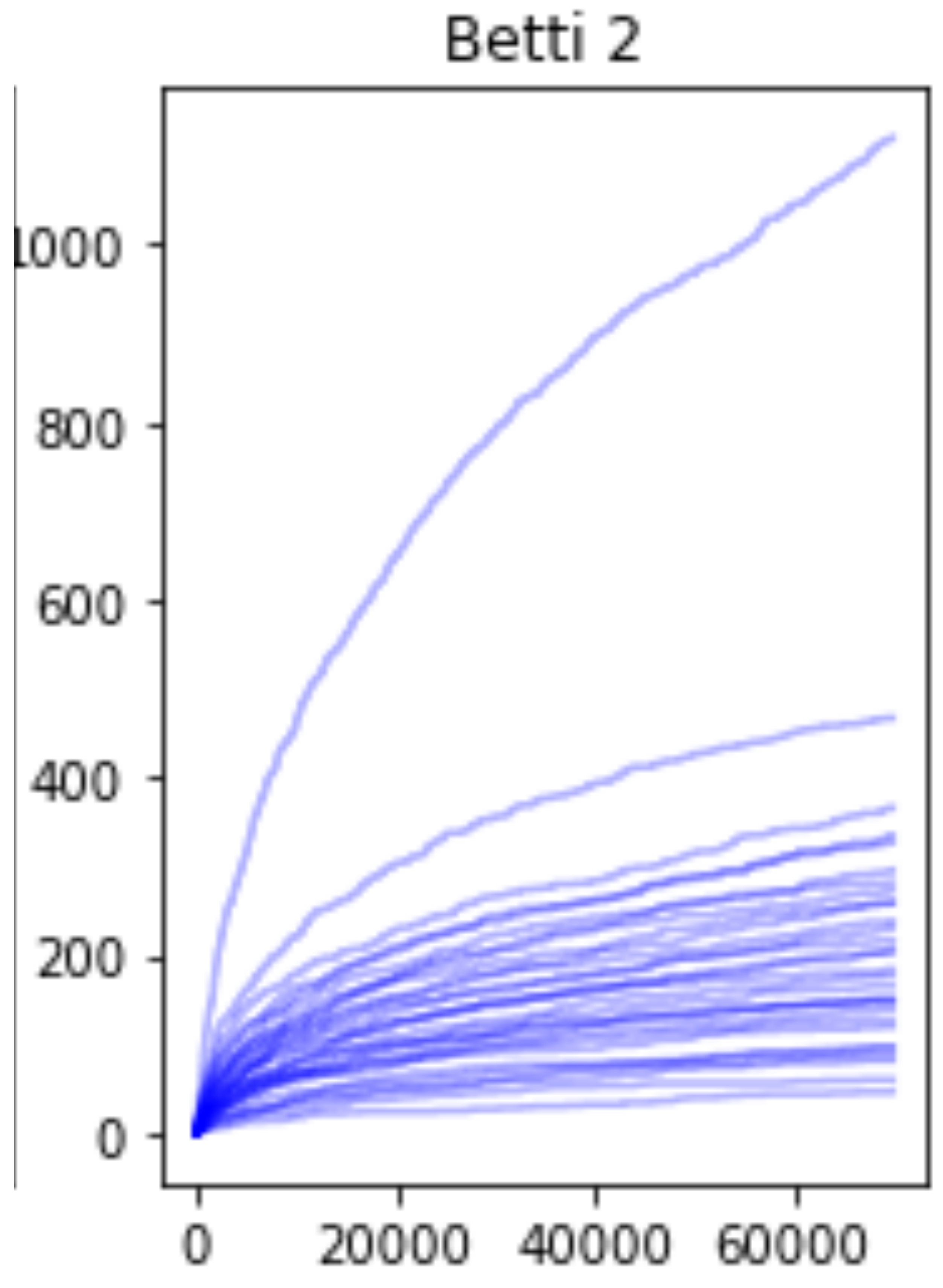


Different curves, different random seeds.
All curves have the same model parameters.

Expected Betti Number $E[\beta_q]$

- increasing trend



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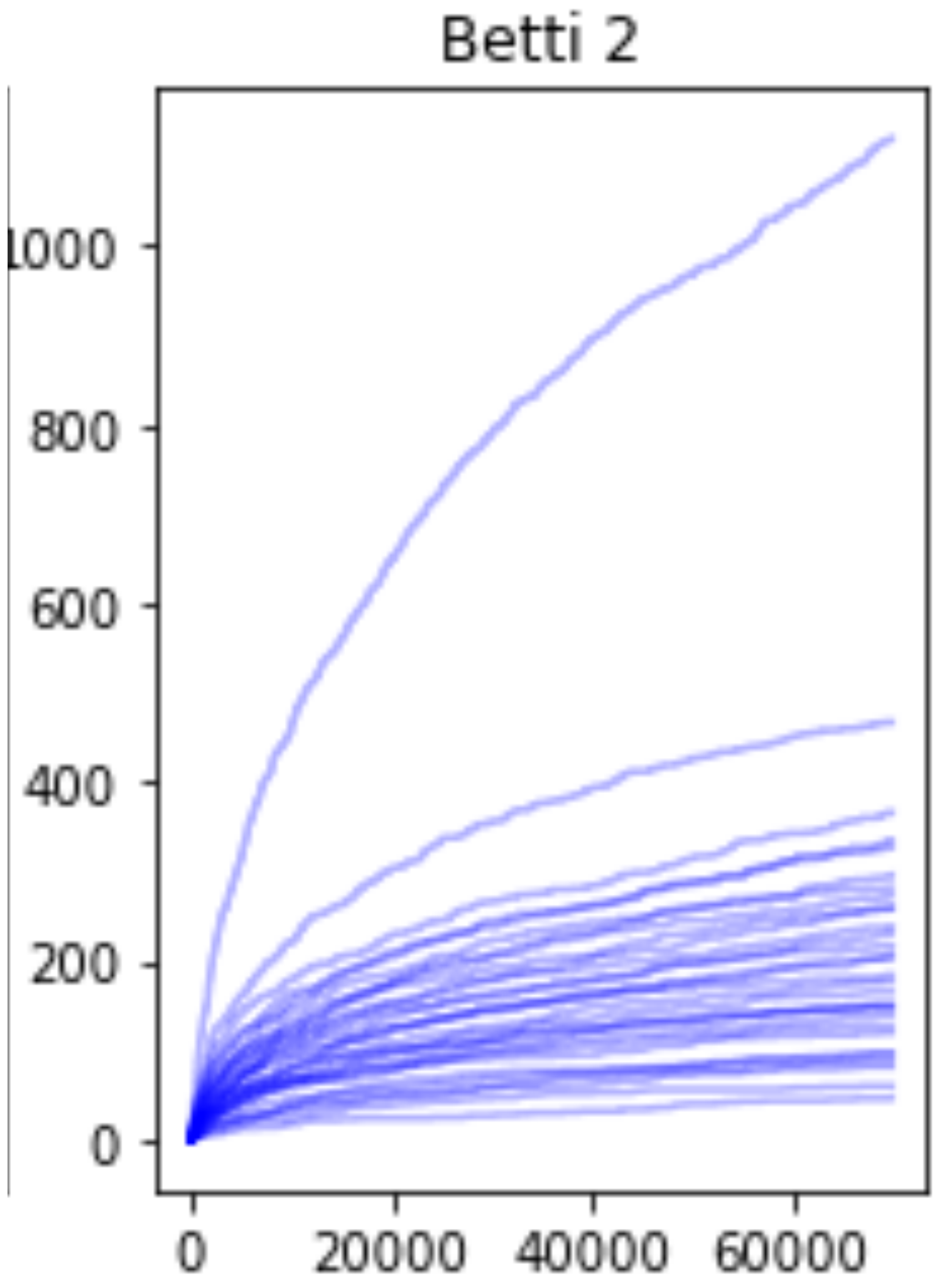
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth



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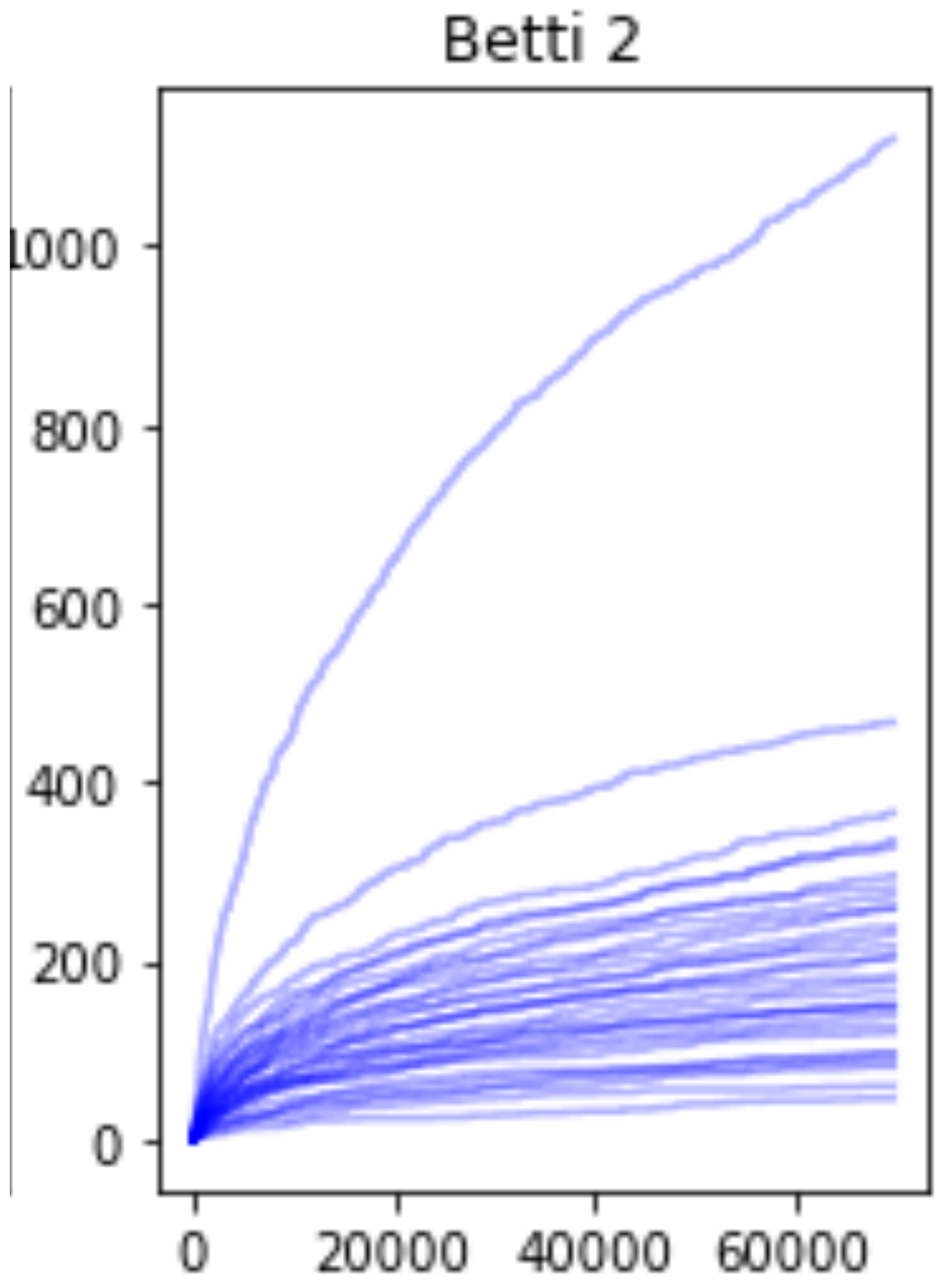
Expected Betti Number $E[\beta_q]$

- increasing trend
- concave growth
- outlier



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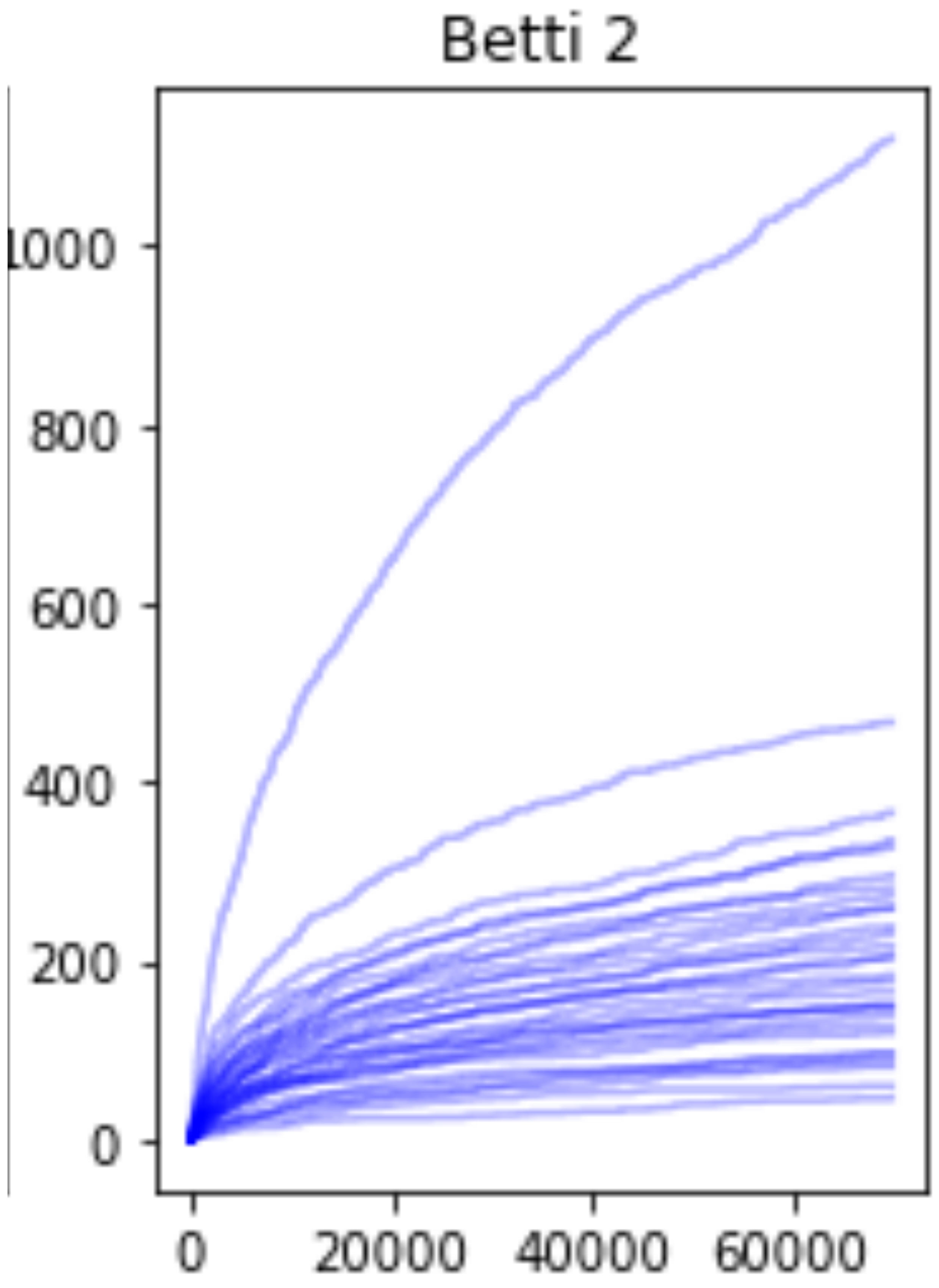
Expected Betti Number $E[\beta_q]$

- $c(\text{num of nodes}^{1-4x}) \leq E[\beta_2] \leq C(\text{num of nodes}^{1-4x})$
  under mild assumptions
  - $x \in (0,1/2)$ depends on the preferential attachment strength.



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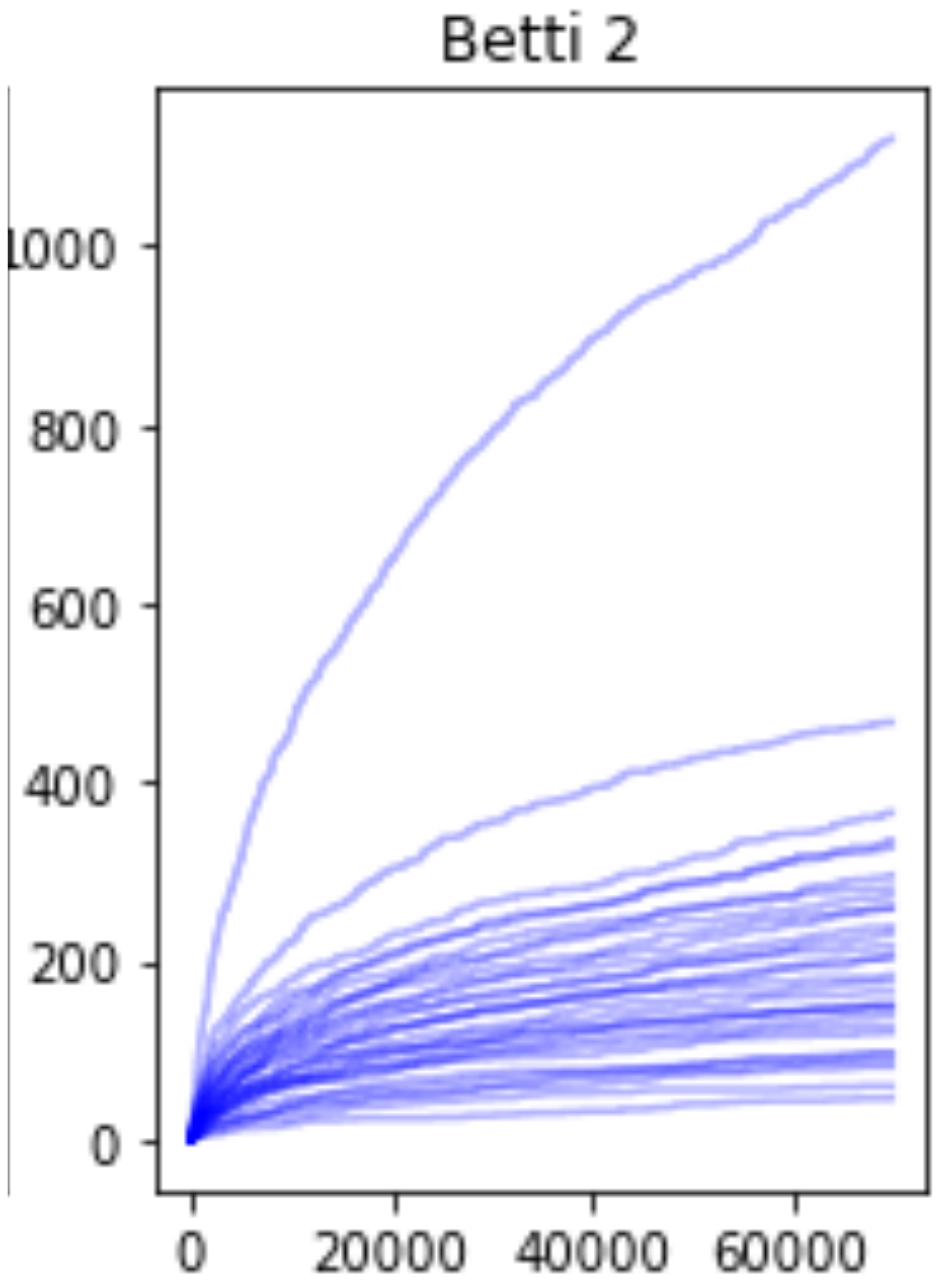
  - If $1 - 4x < 0$, then $E[\beta_2] \leq C$.



Expected Betti Number $E[\beta_q]$

- c(num of nodes^{1-4x}) $\leq E[\beta_2] \leq C$(num of nodes^{1-4x})
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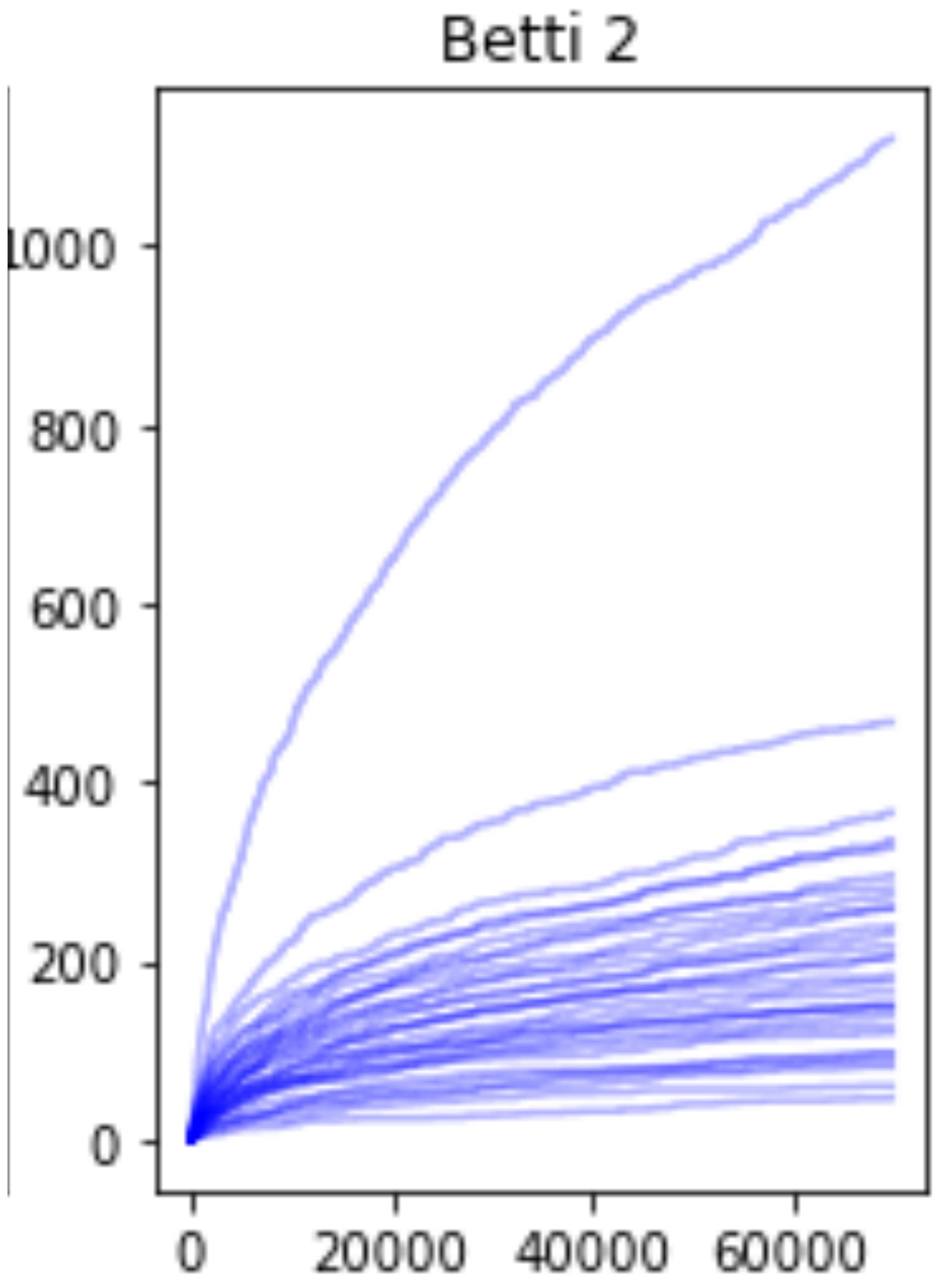
- c(num of nodes^{1-2qx}) $\leq E[\beta_q] \leq C$(num of nodes^{1-2qx}) for $q \geq 2$.

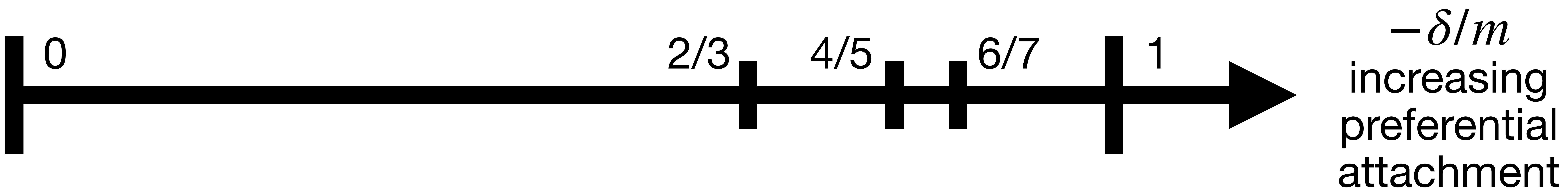


Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



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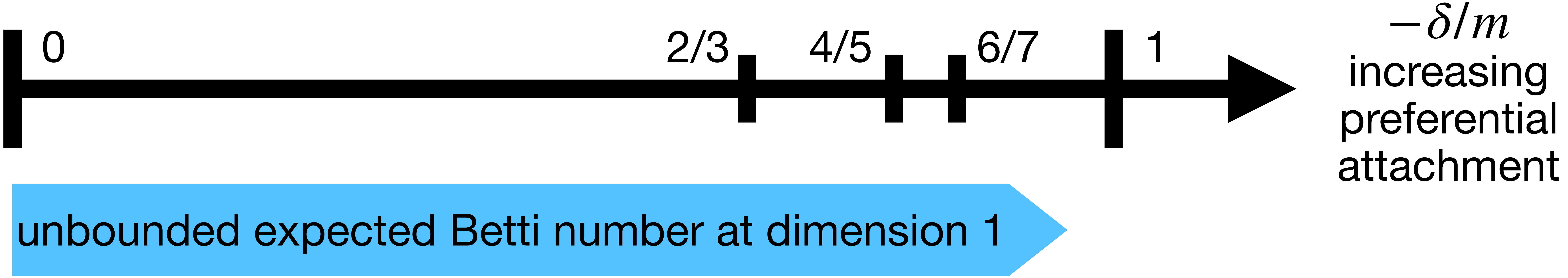

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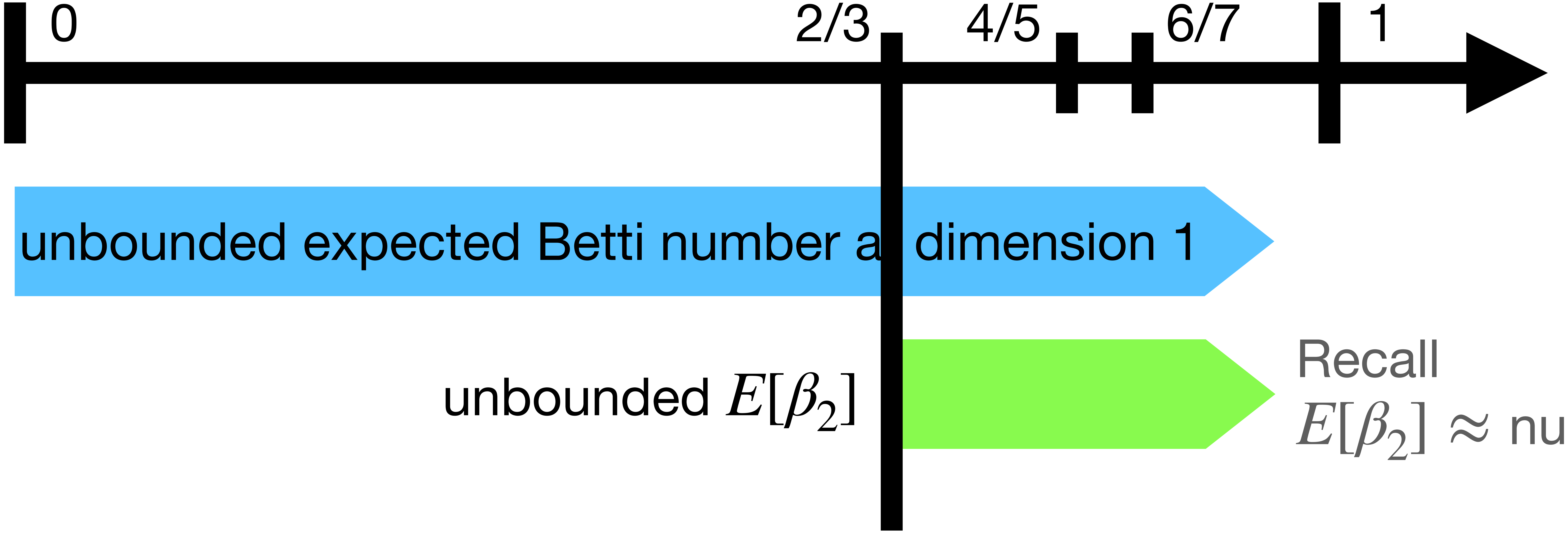

$-\delta/m$

increasing preferential attachment

Recall

$E[\beta_2] \approx$ num of nodes^{1-4x}

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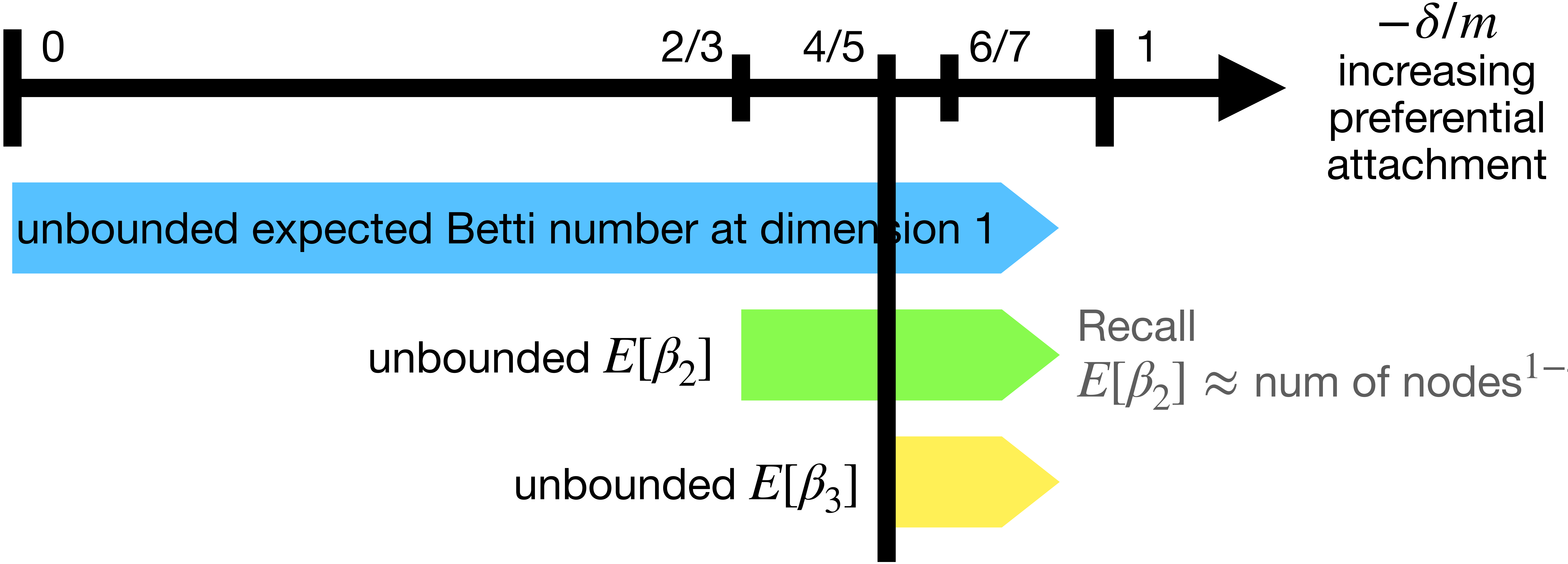

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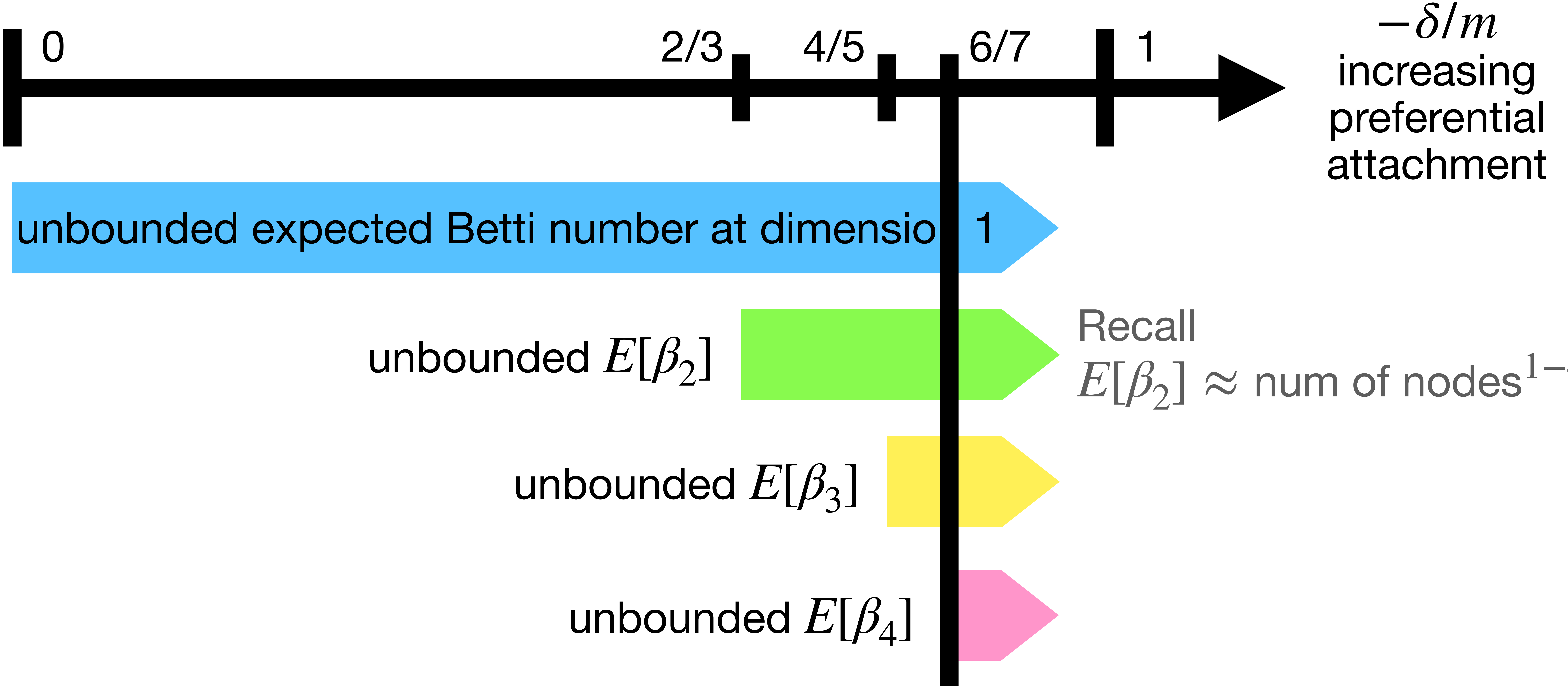

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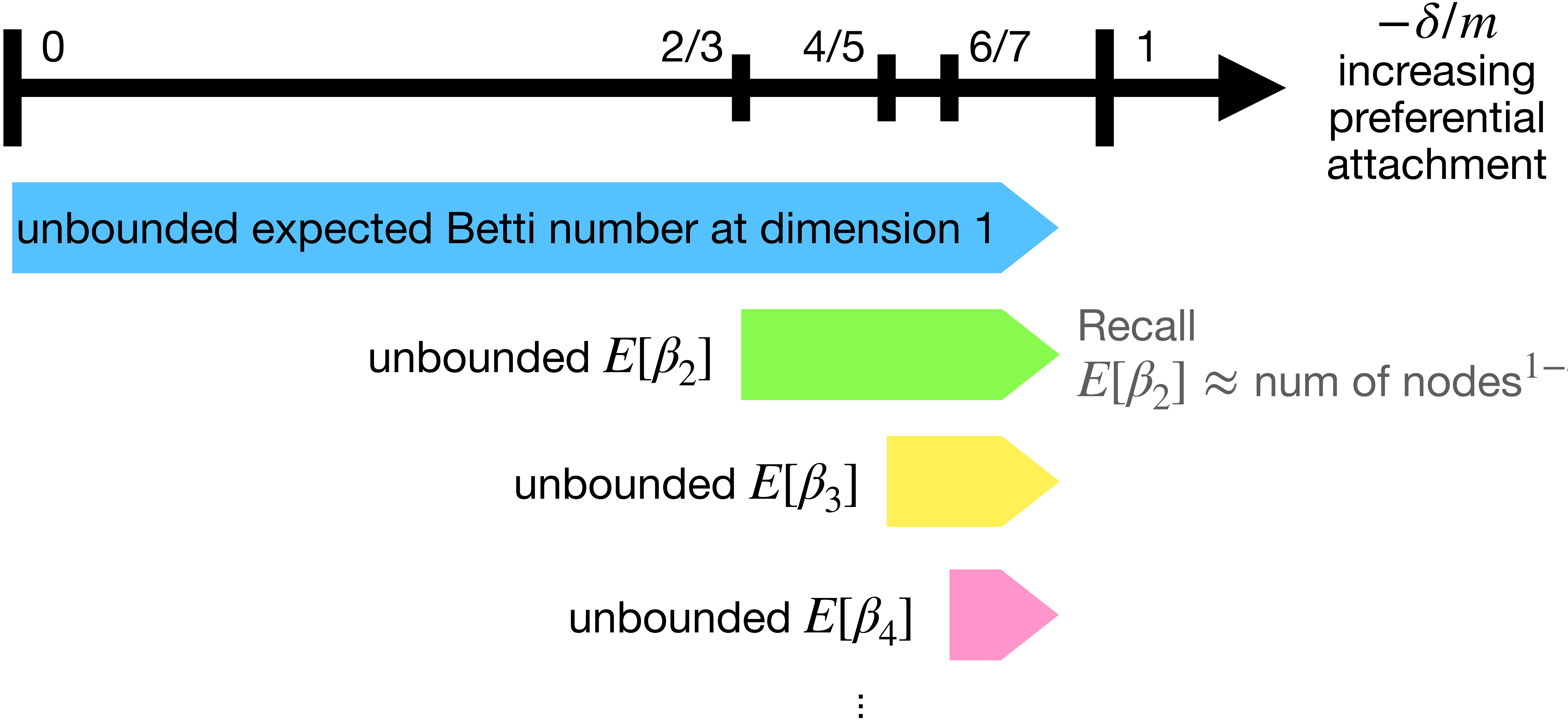


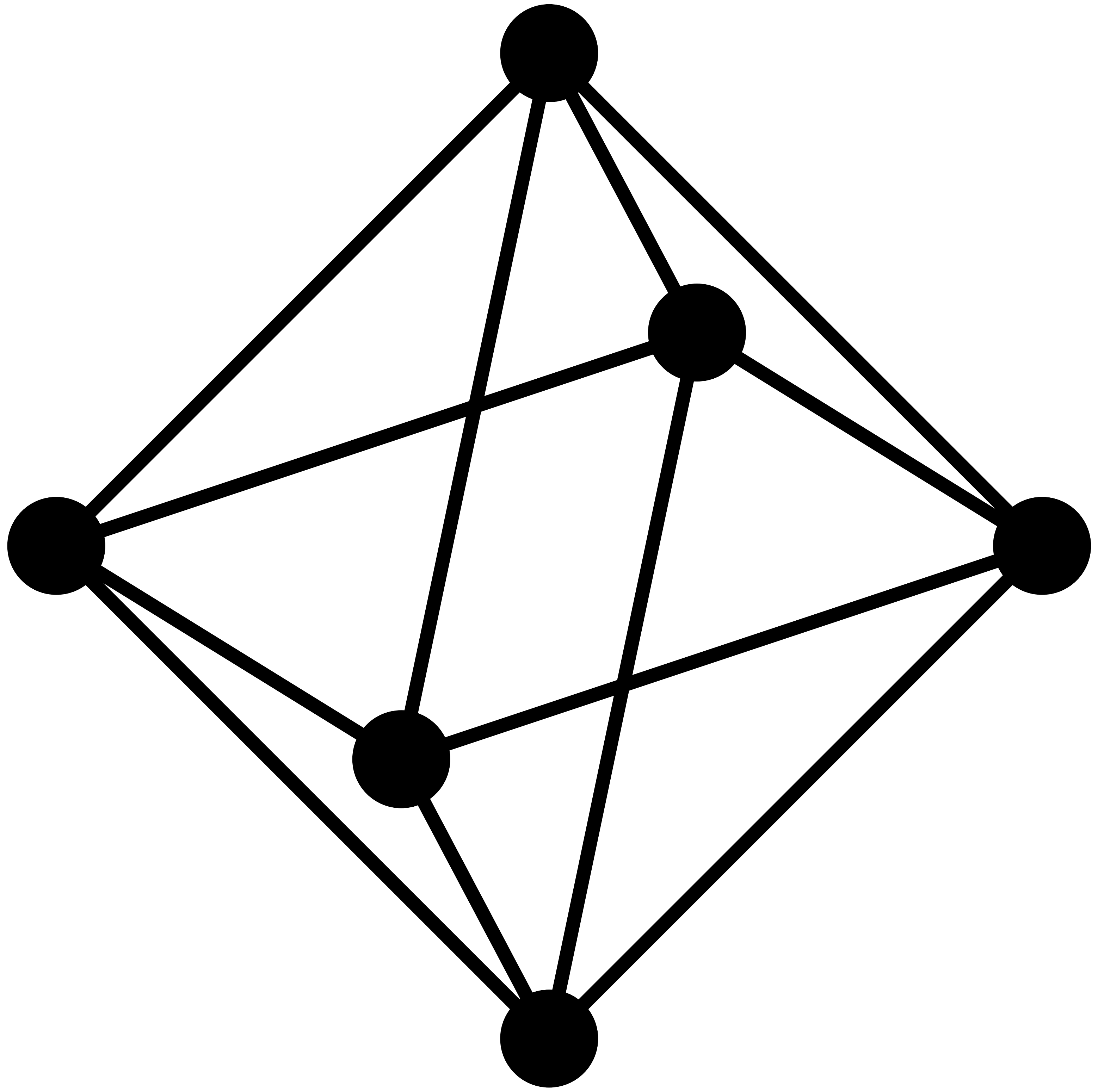
Recall

$E[\beta_2] \approx$ num of nodes$^{1-4\chi}$

Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

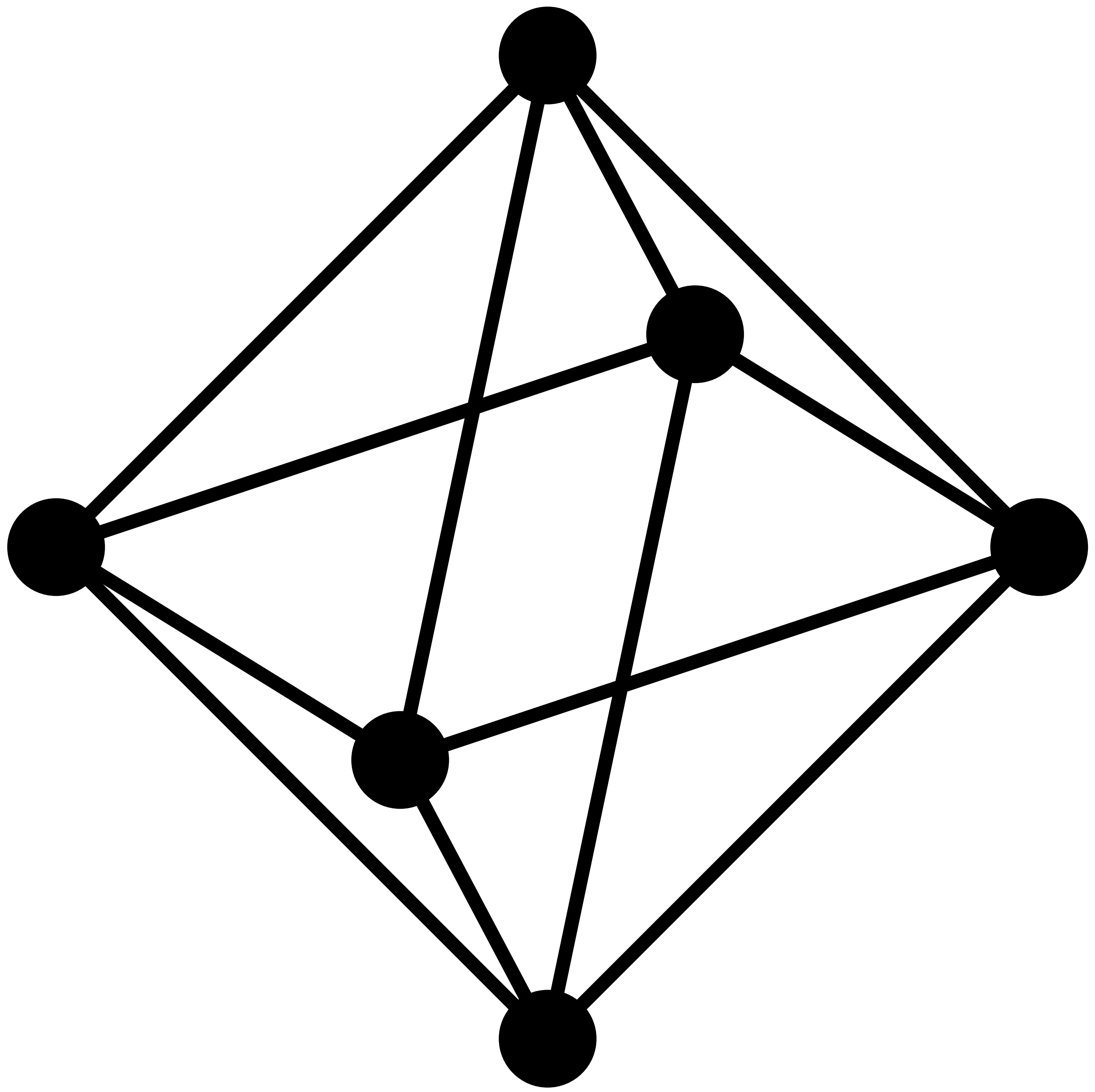
Proof?

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



$\beta_2 = 1$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



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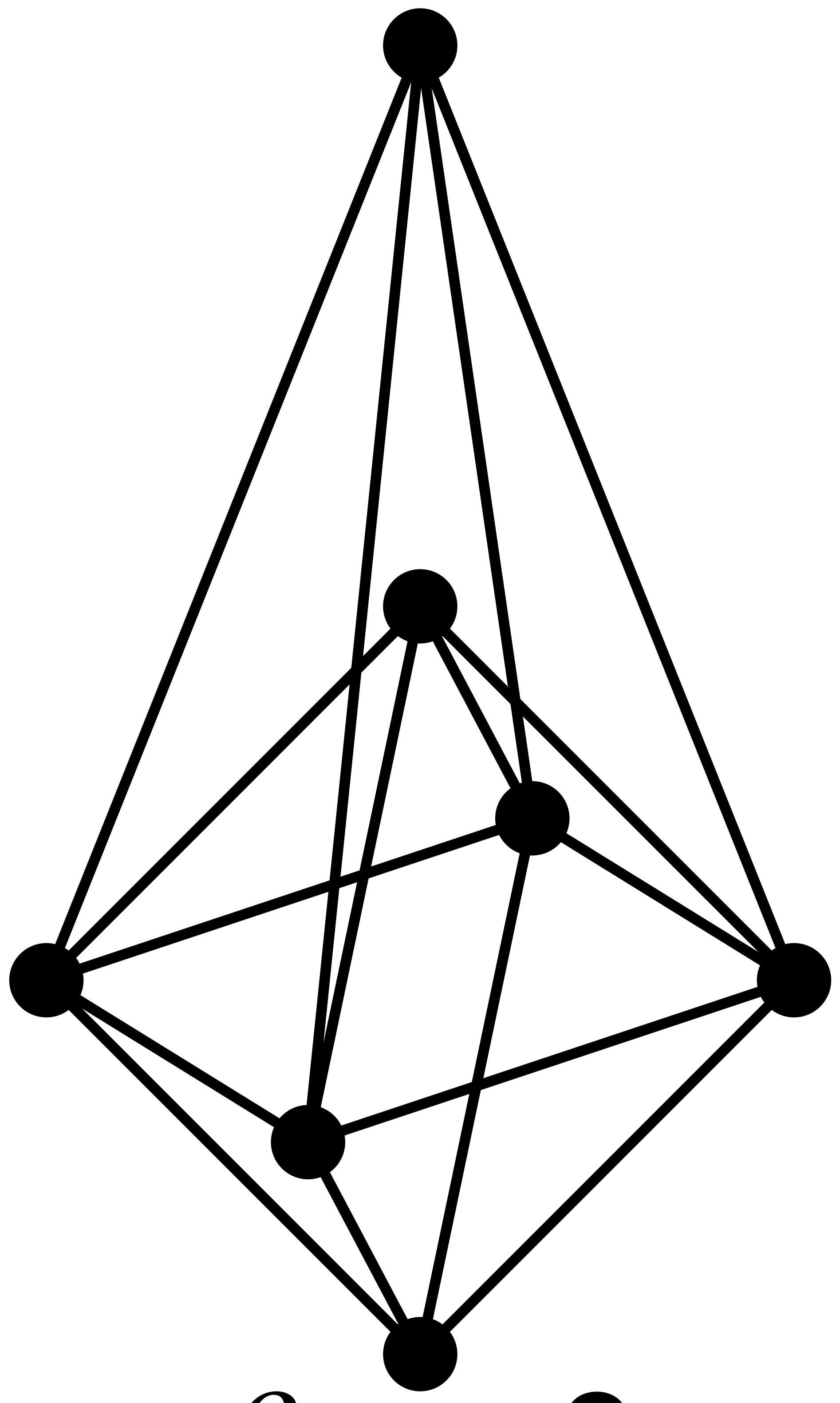
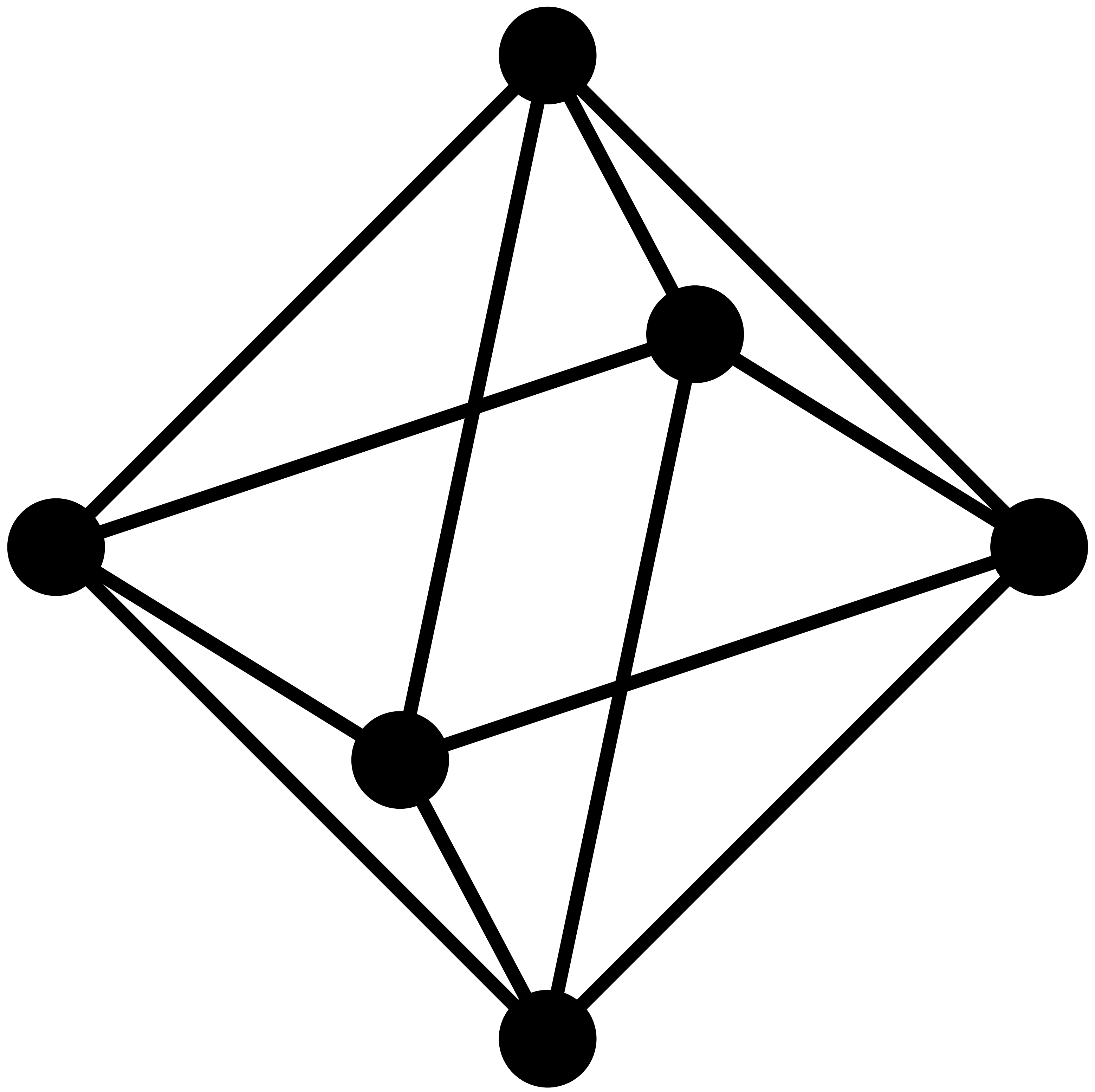
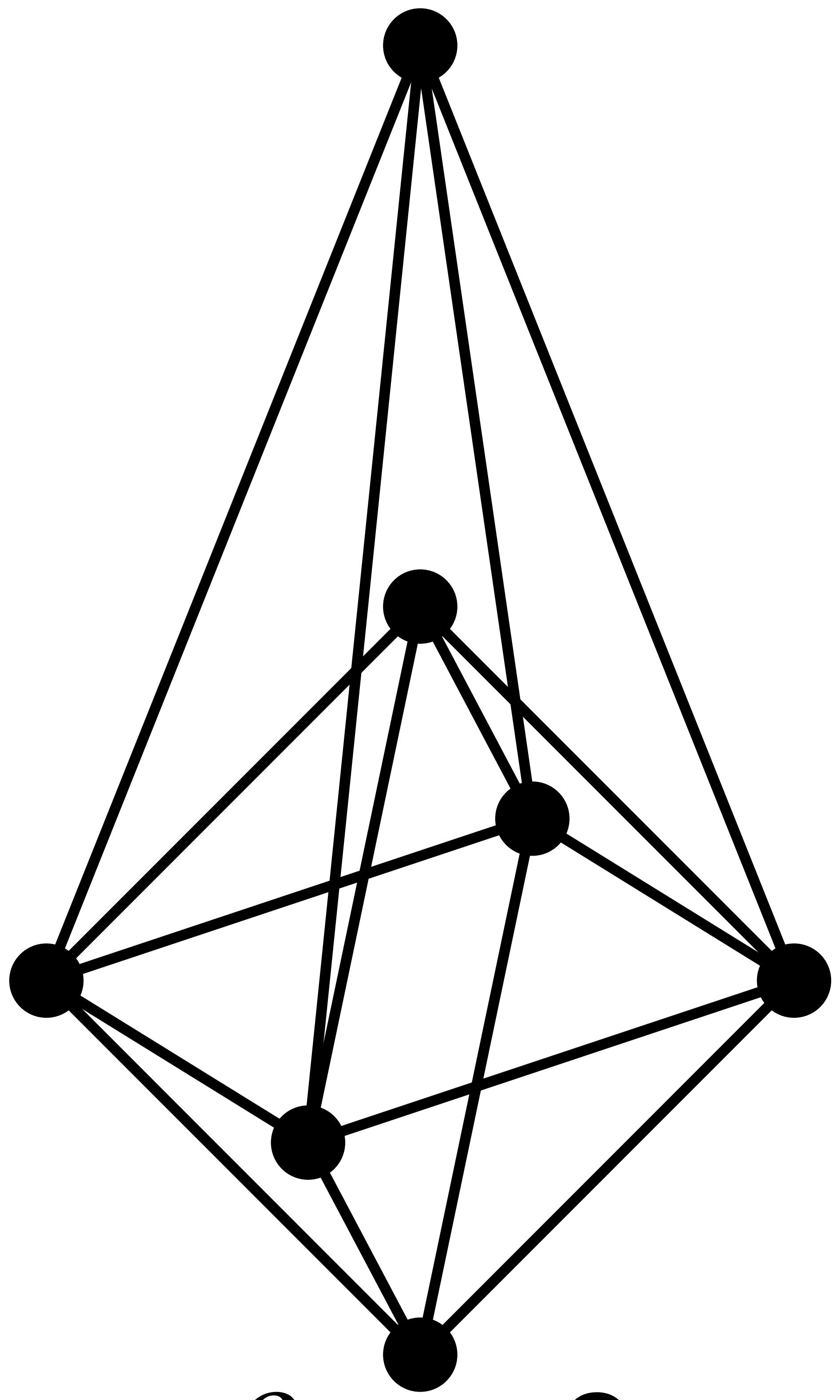


$\beta_2 = 2$

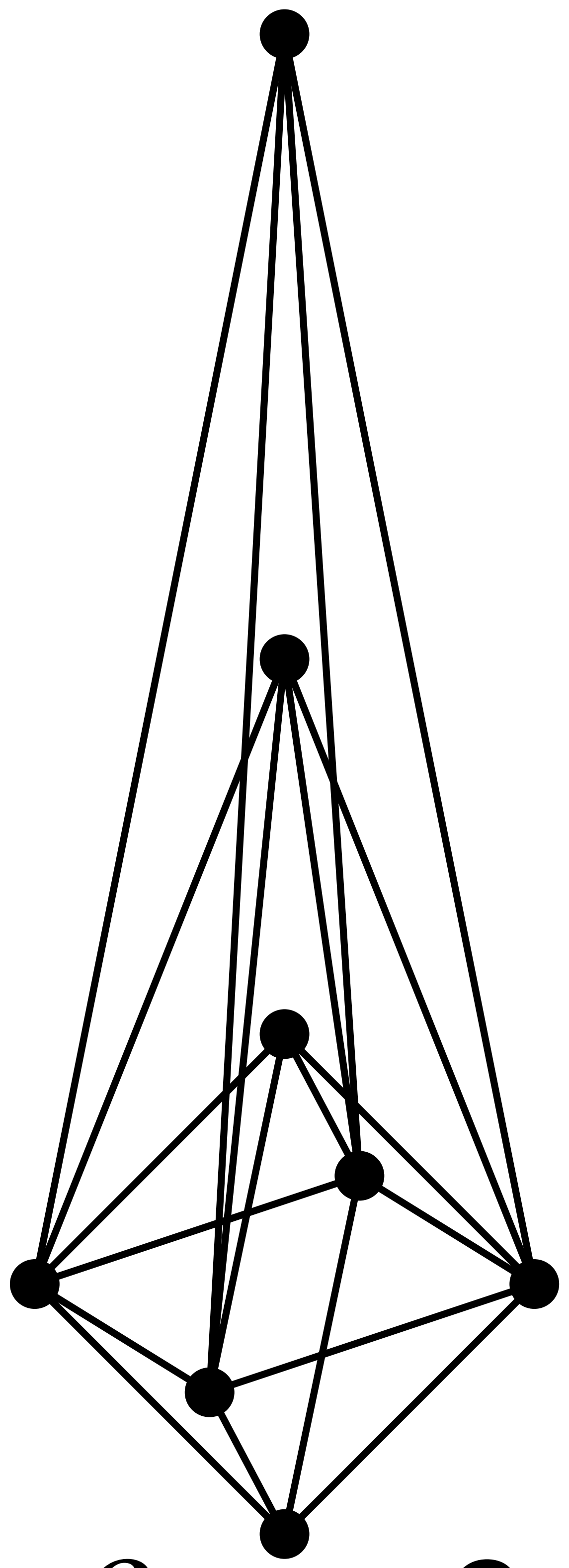
Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



$\beta_2 = 1$



$\beta_2 = 2$



$\beta_2 = 3$

Subtleties

- Need homological algebra to relate Betti numbers with counts

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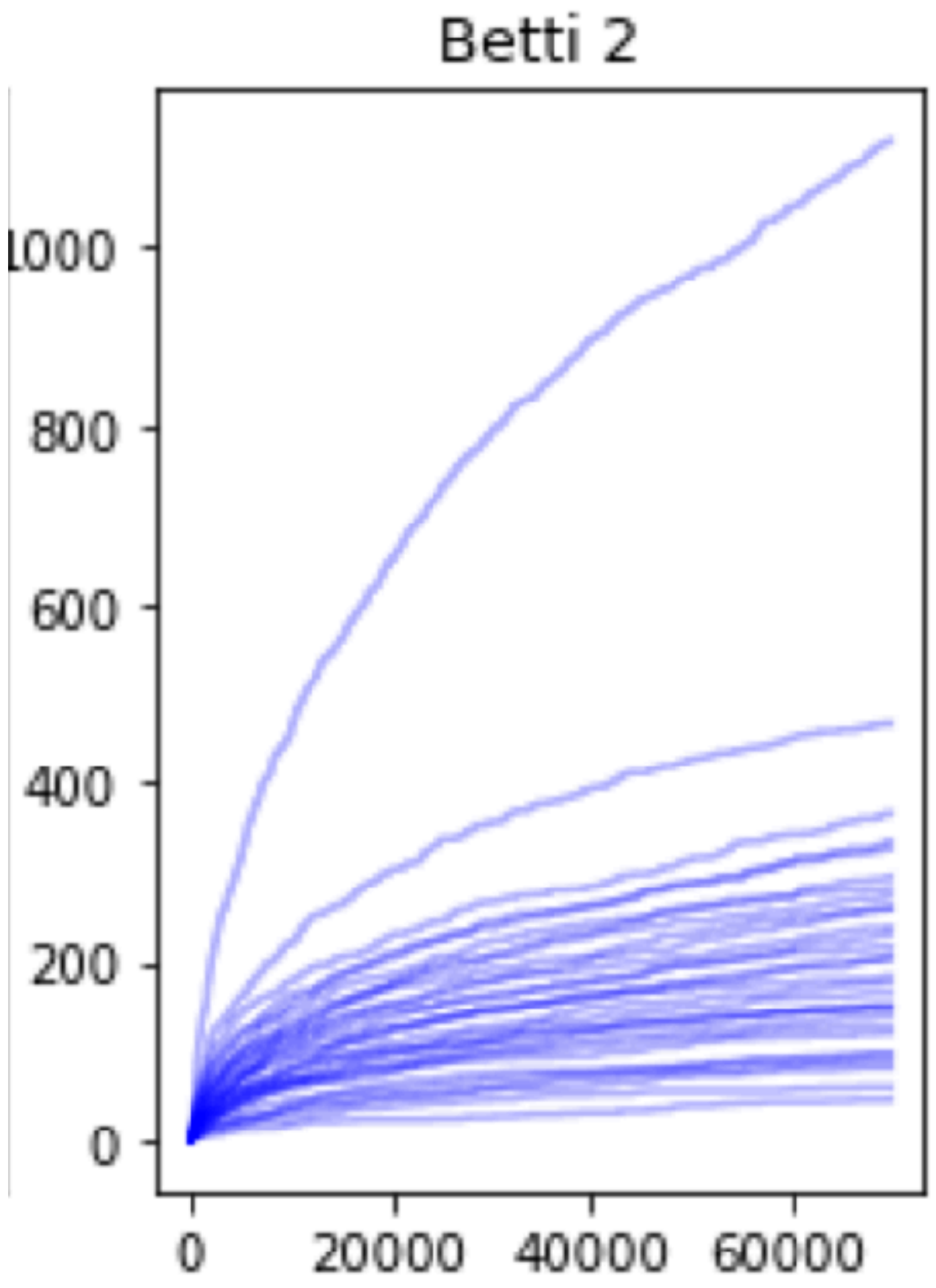
Subtleties

- Need homological algebra to relate Betti numbers with counts
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results in the language of homological algebra
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs

.

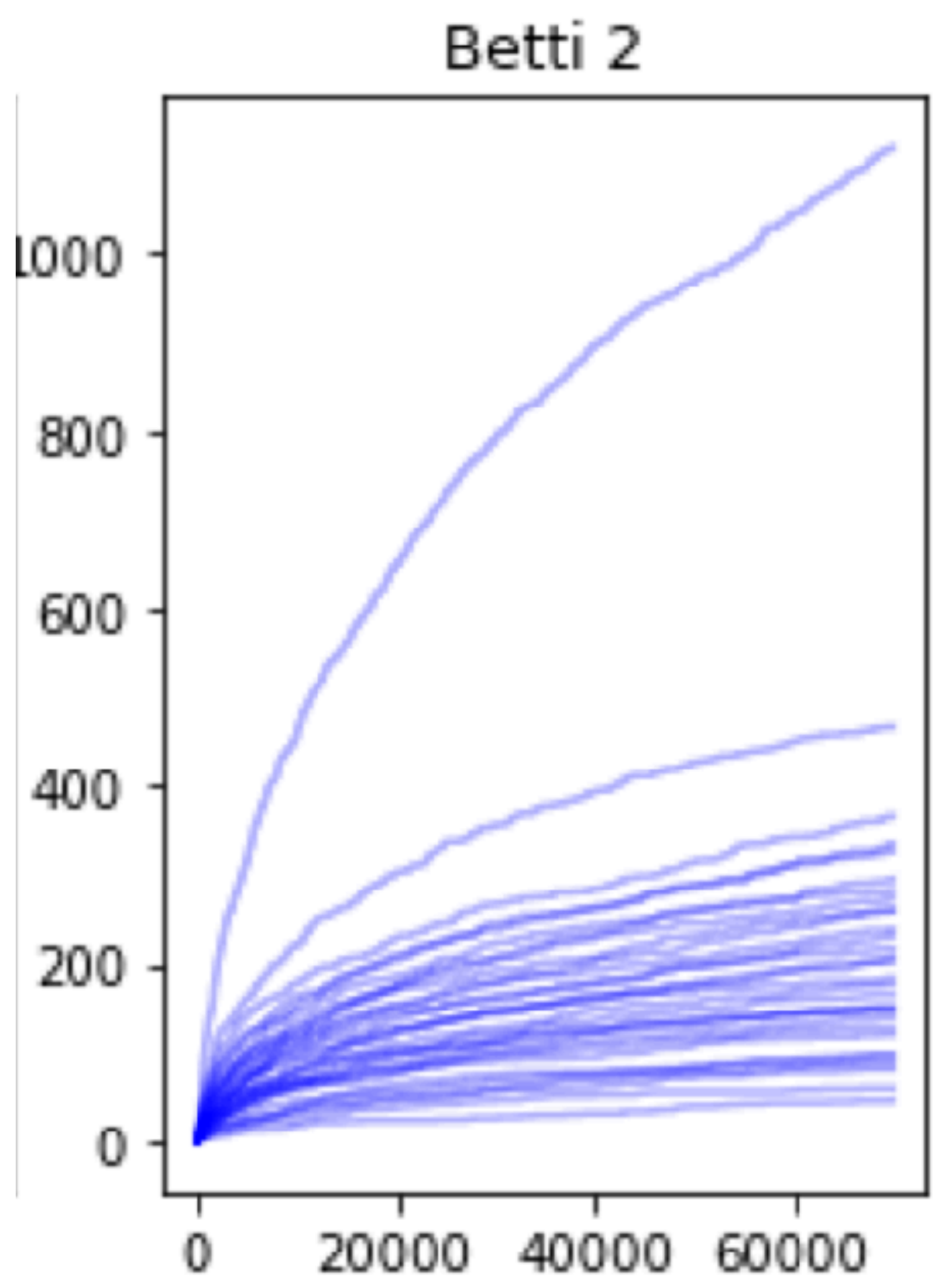
Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}
In practice???

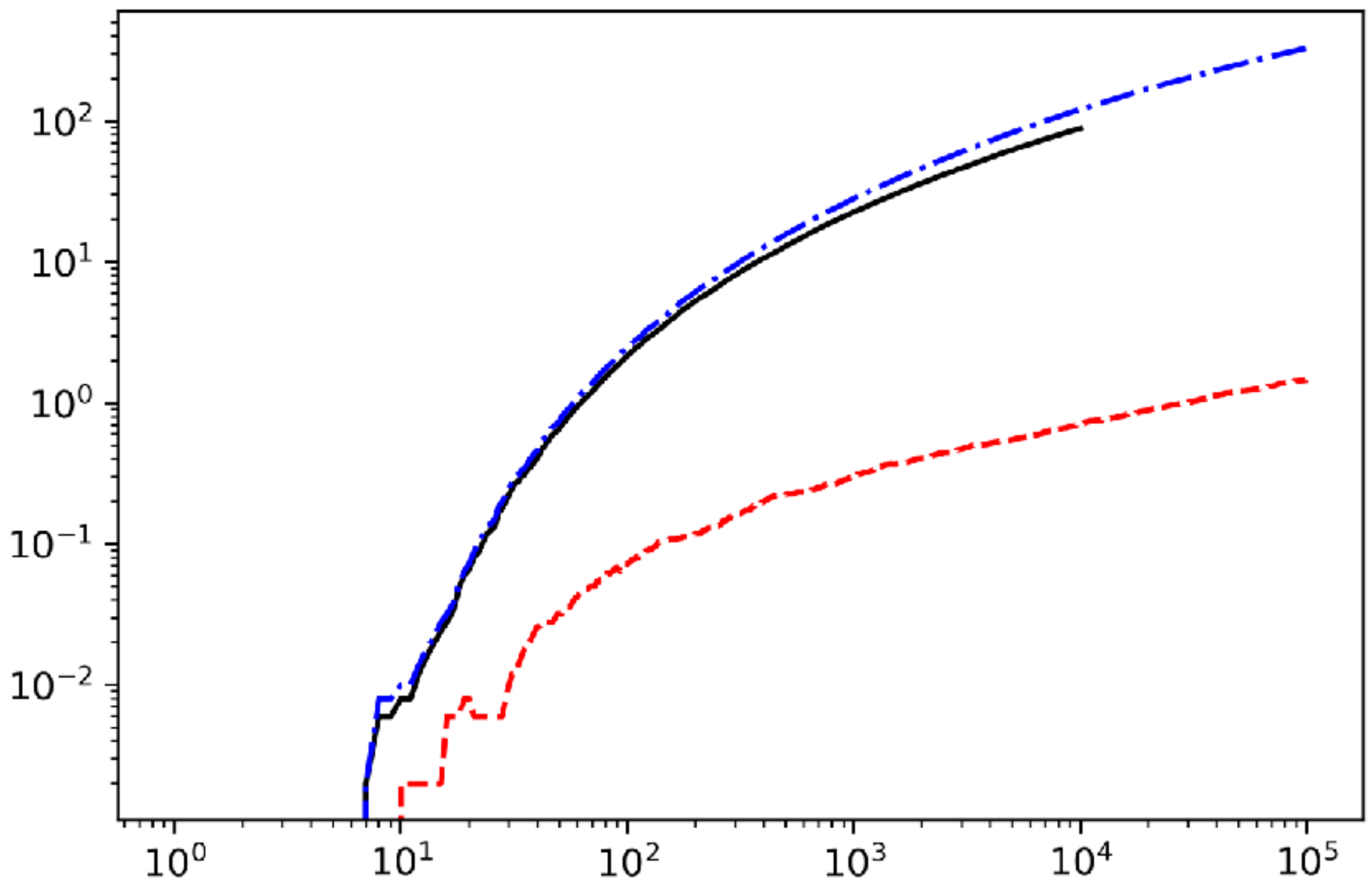
$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$



$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

$$\log E[\beta_2] \approx (1-4x)\log(\textbf{num of nodes})$$





$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

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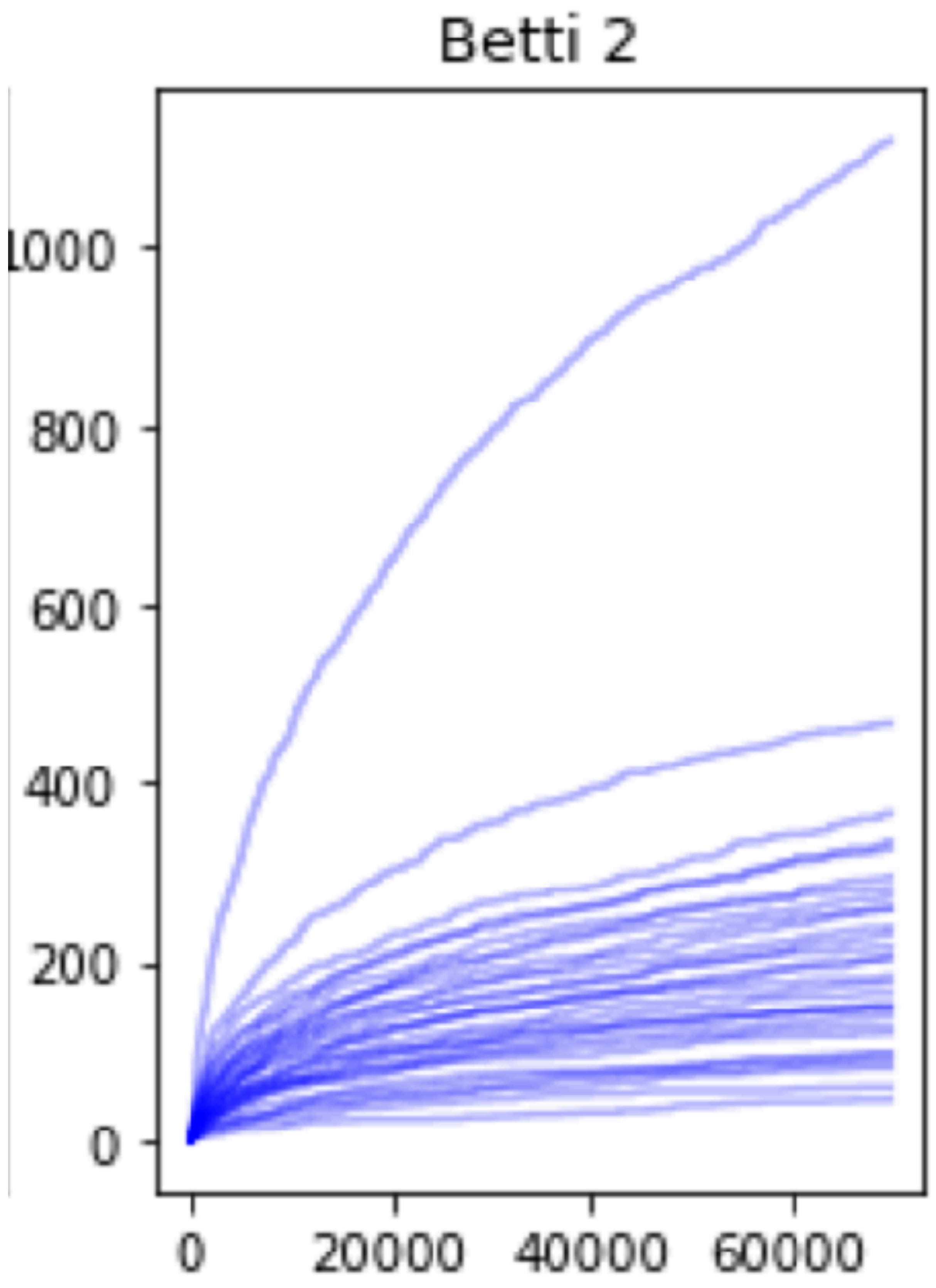

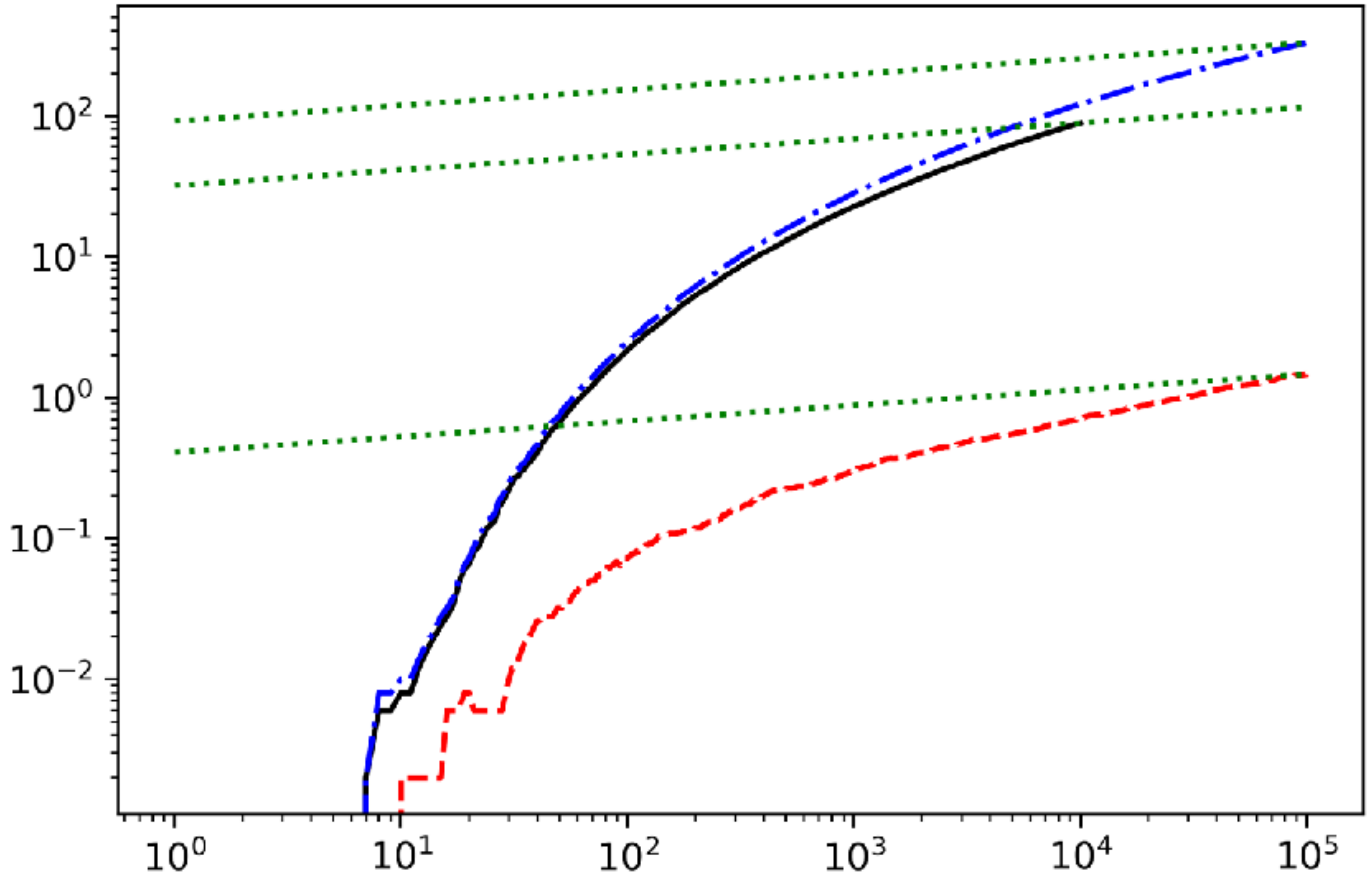

V. What lies ahead

order of magnitude of
expected Betti numbers

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

parameter estimation?

homotopy connectedness
of the infinite complex?

order of magnitude of
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homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

simplicial preferential attachment?

parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

simplicial preferential attachment?

other non-homogeneous complexes?

What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

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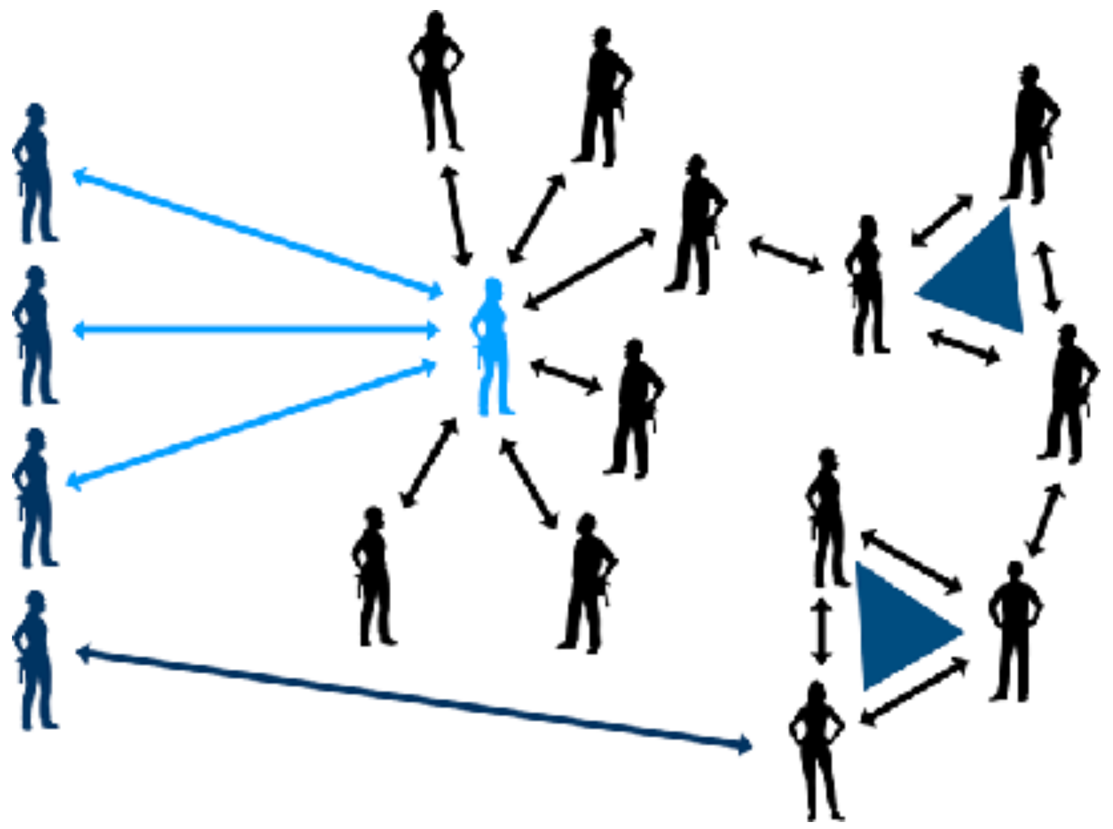
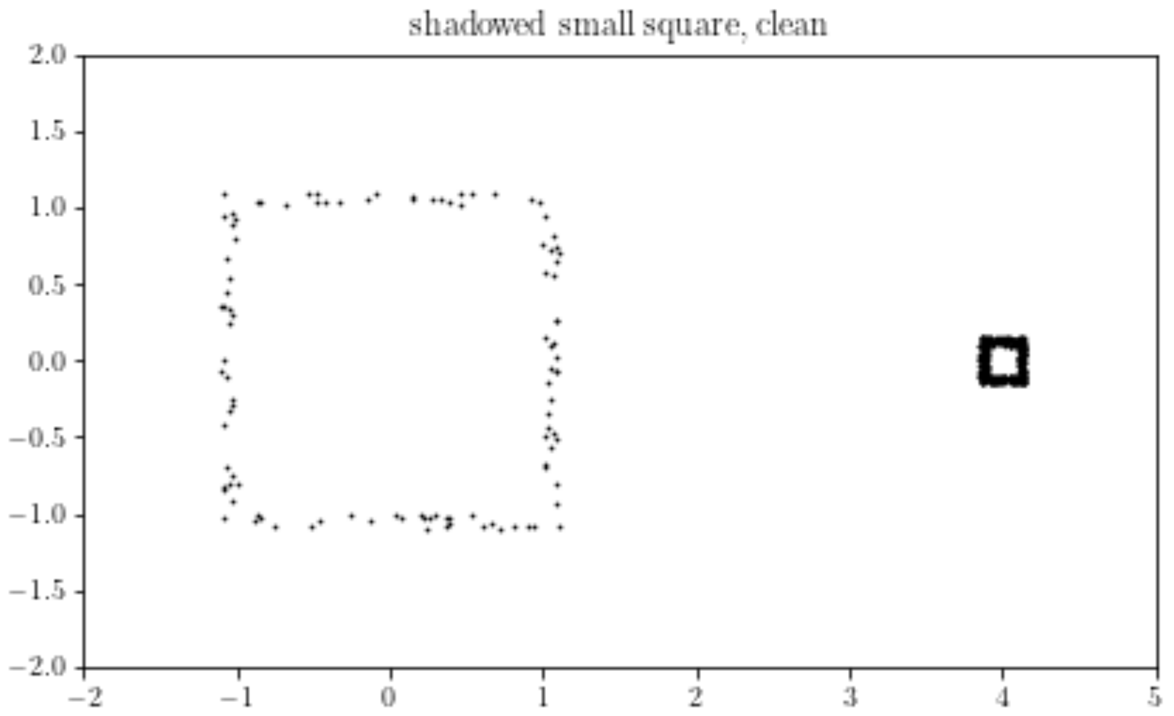


arxiv paper



my video about small holes

Thank you!

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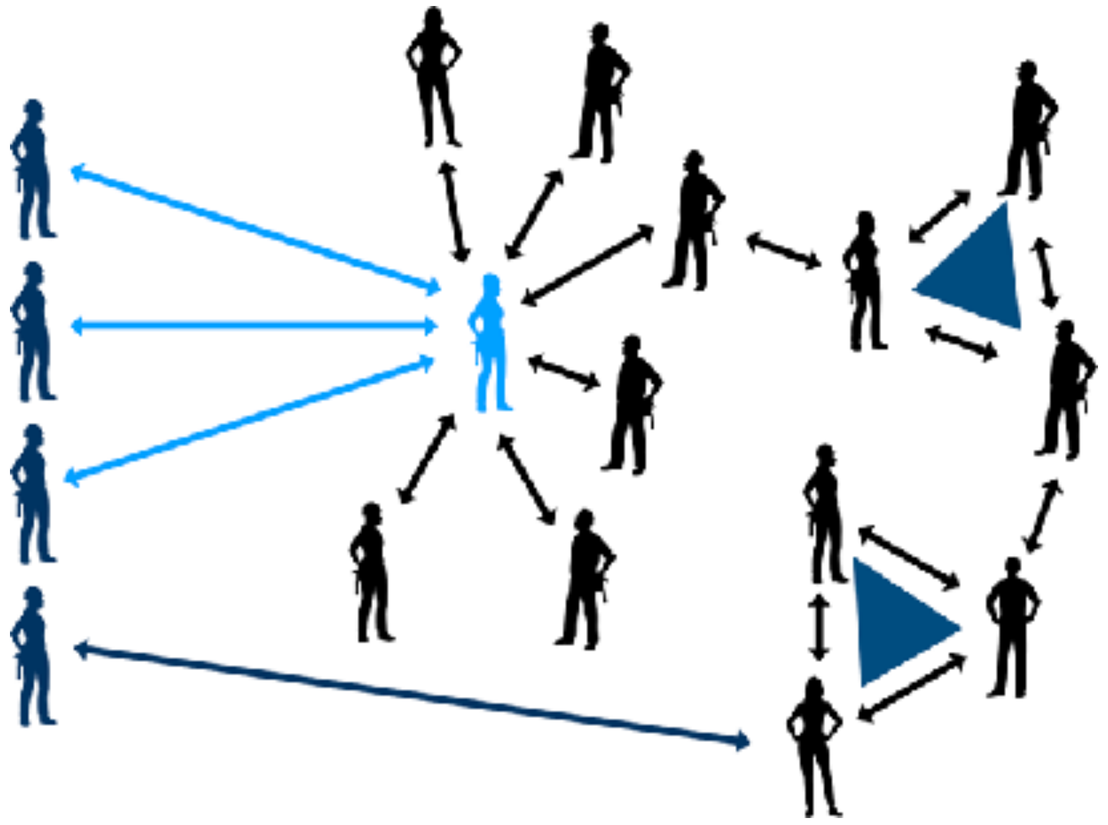
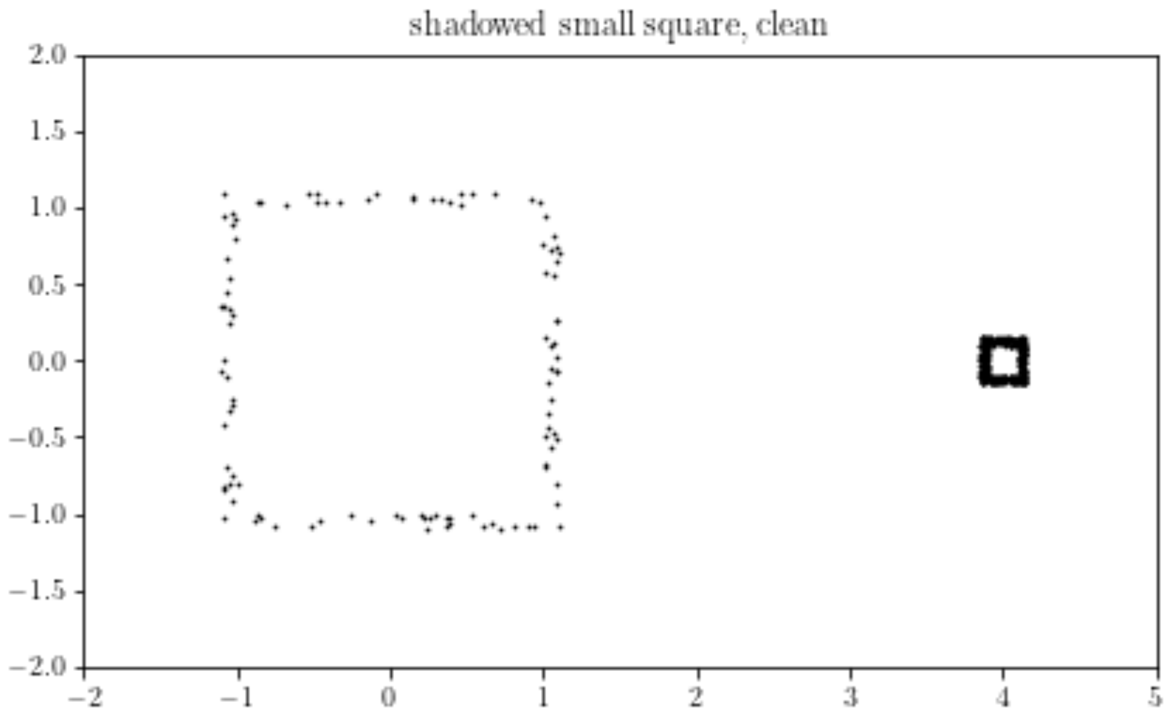


arxiv paper



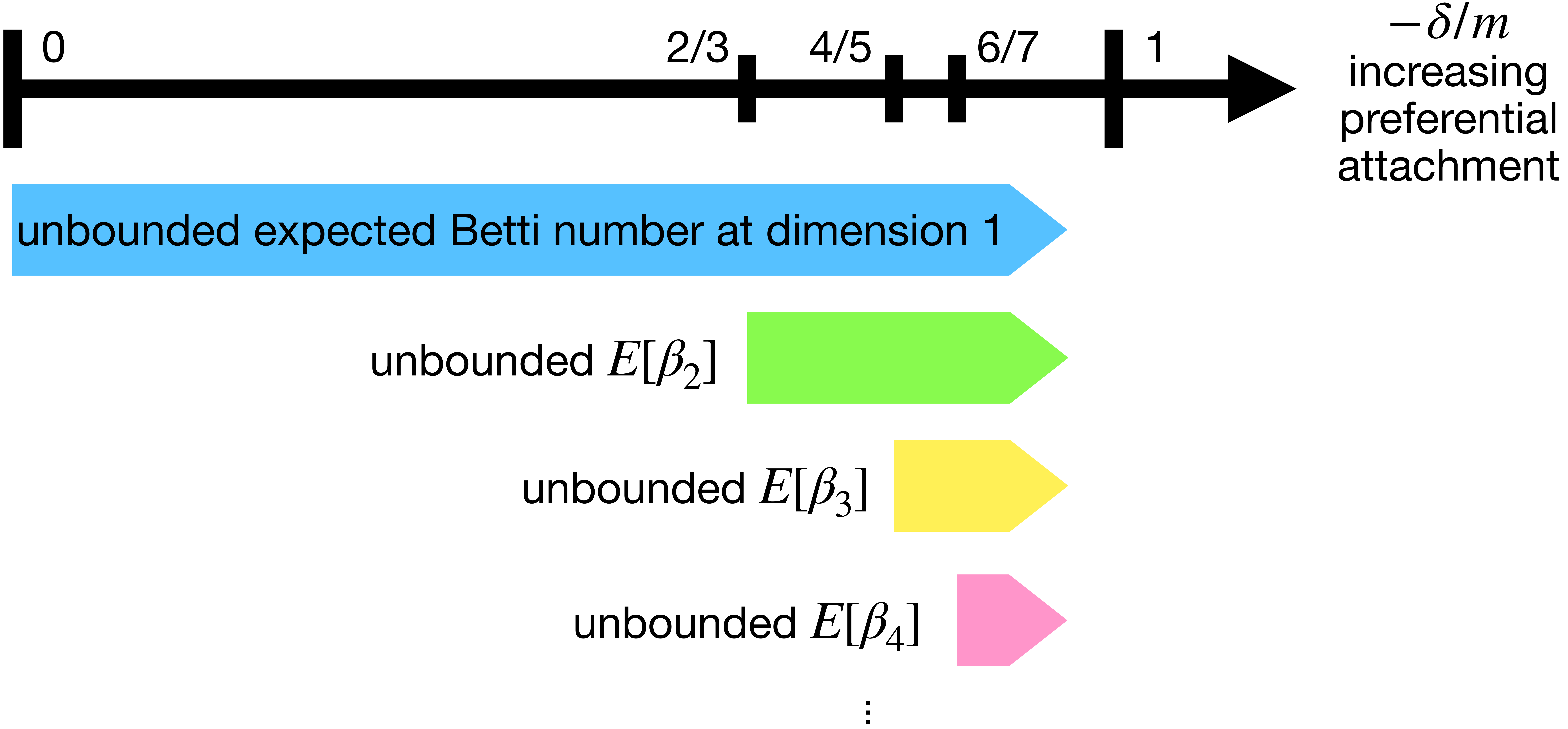
my video about small holes

Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



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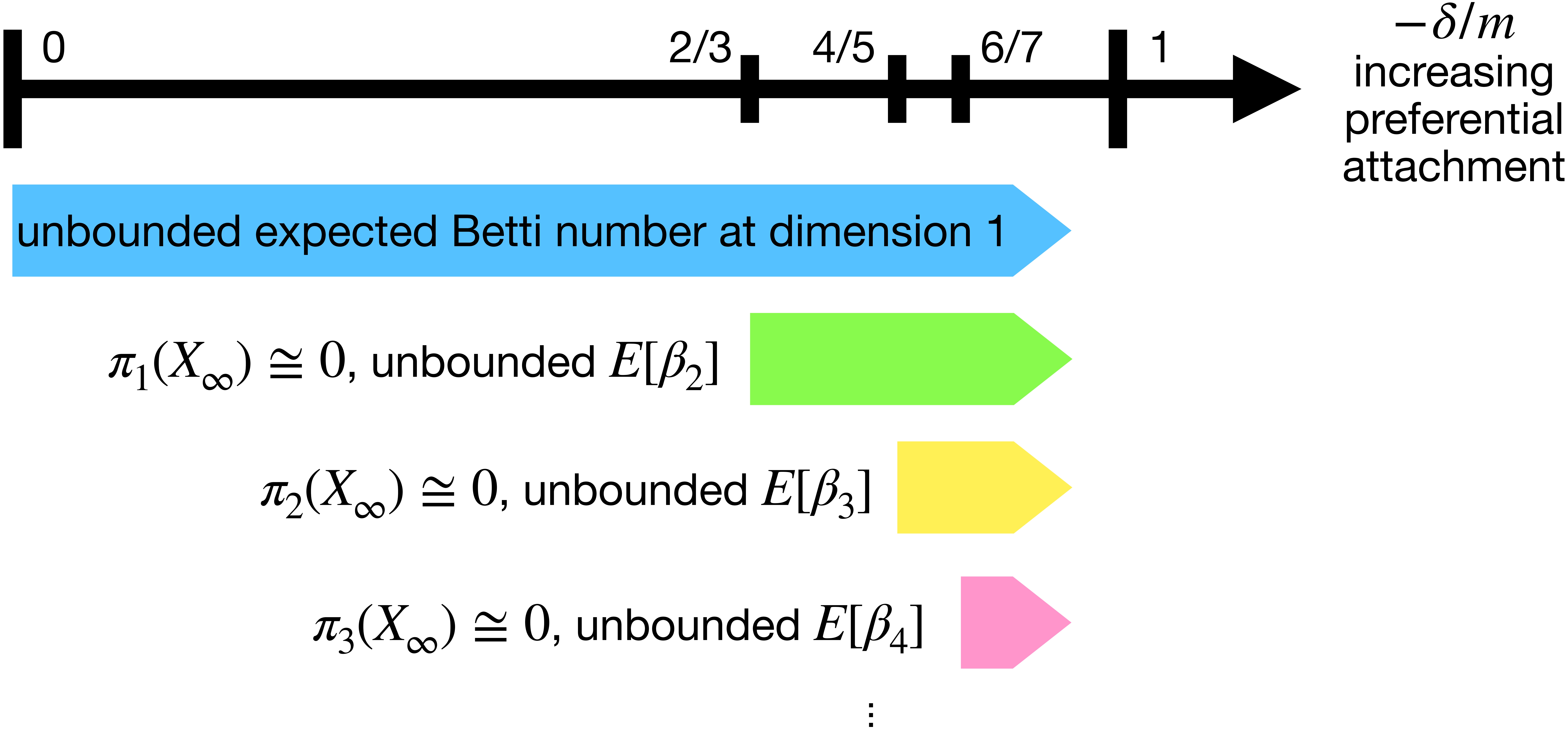

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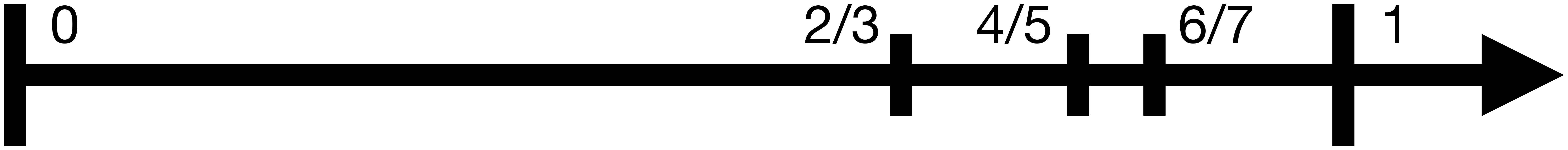

$-\delta/m$
increasing preferential attachment

unbounded expected Betti number at dimension 1

$\pi_1(X_\infty) \cong 0$, unbounded $E[\beta_2]$

$\pi_2(X_\infty) \cong 0$, unbounded $E[\beta_3]$

tight?

$\pi_3(X_\infty) \cong 0$, unbounded $E[\beta_4]$

⋮

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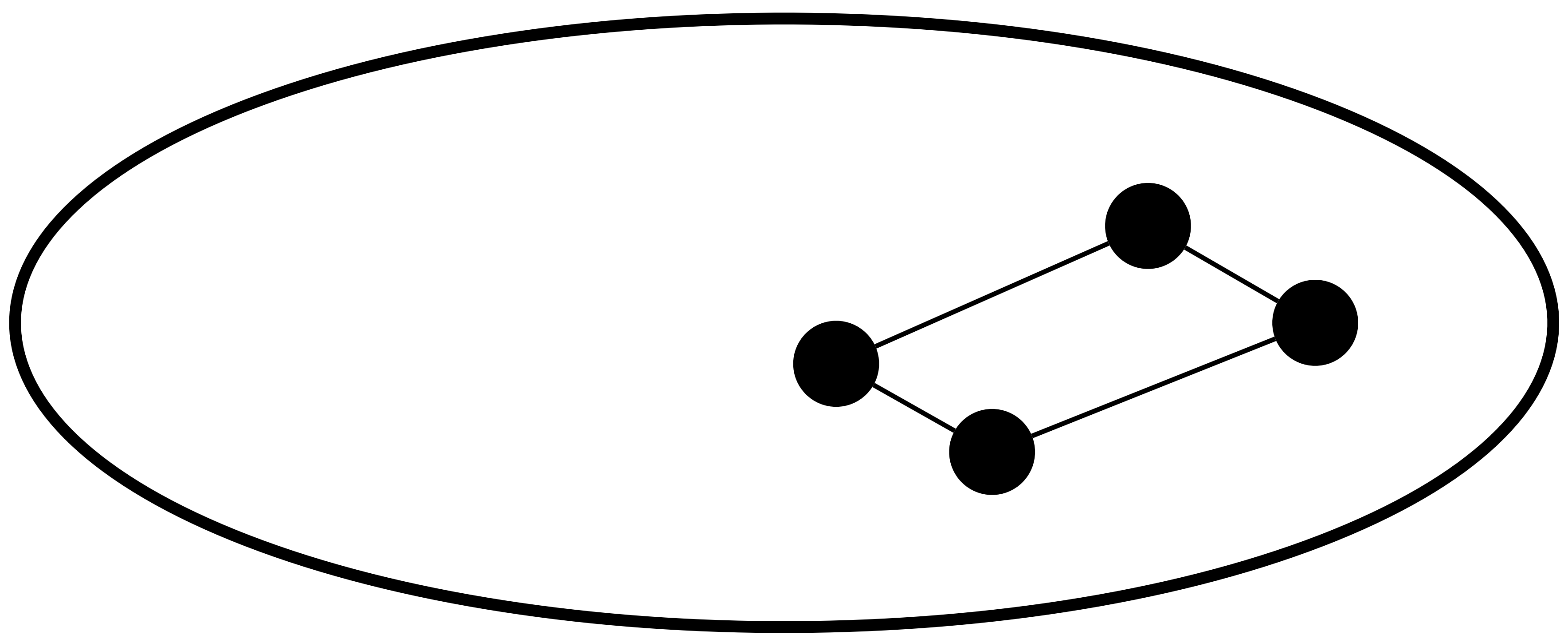
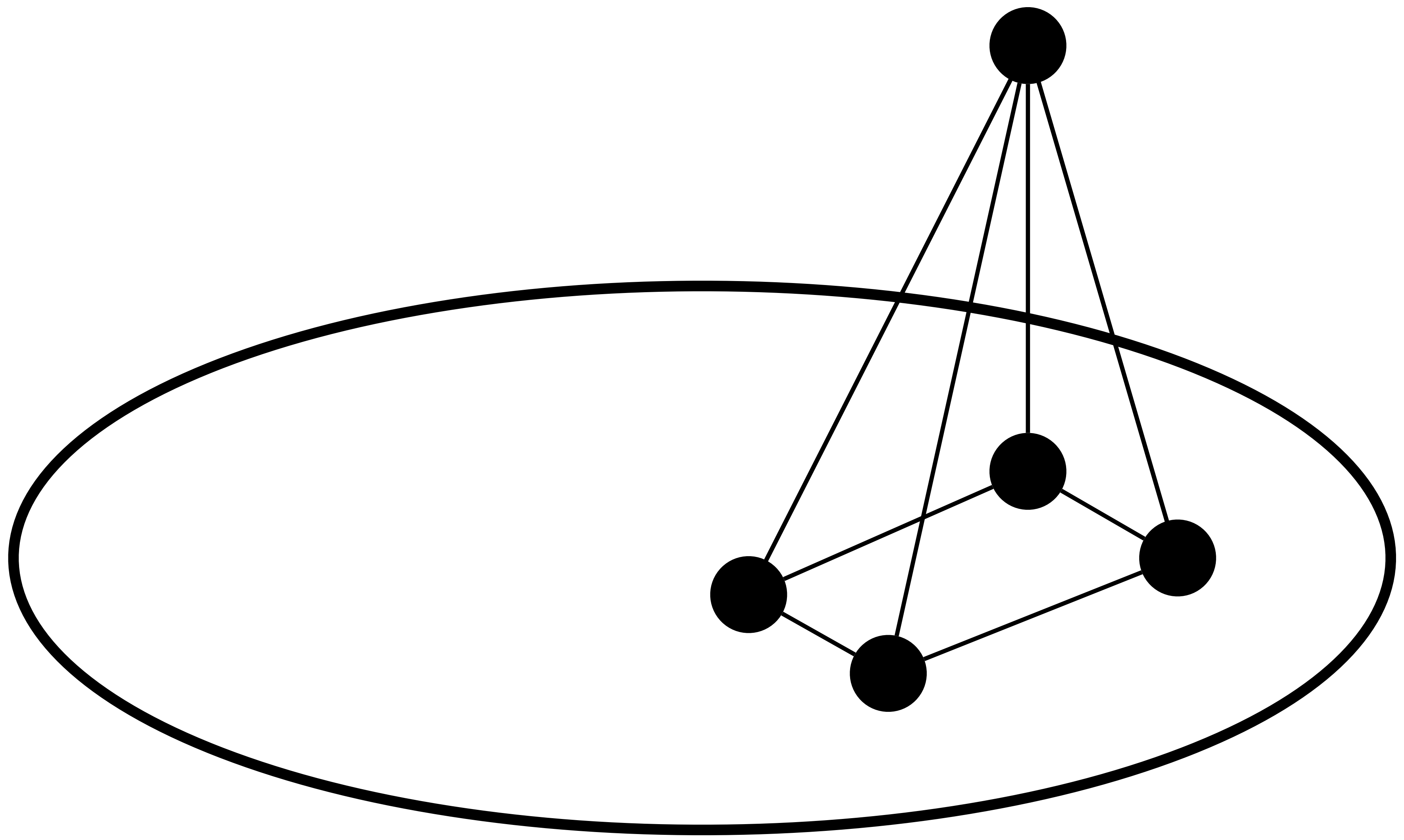
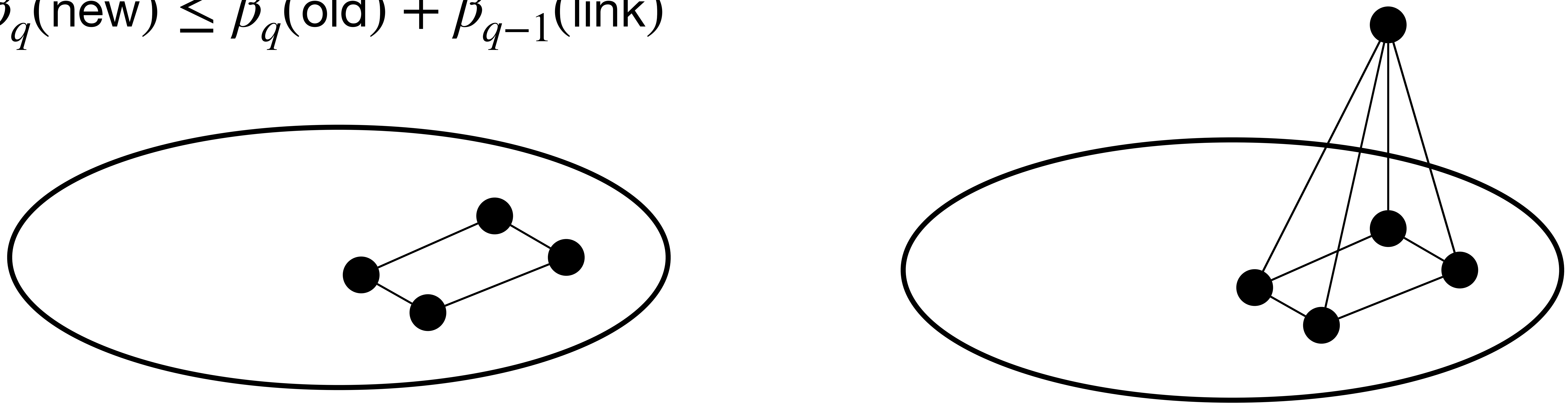




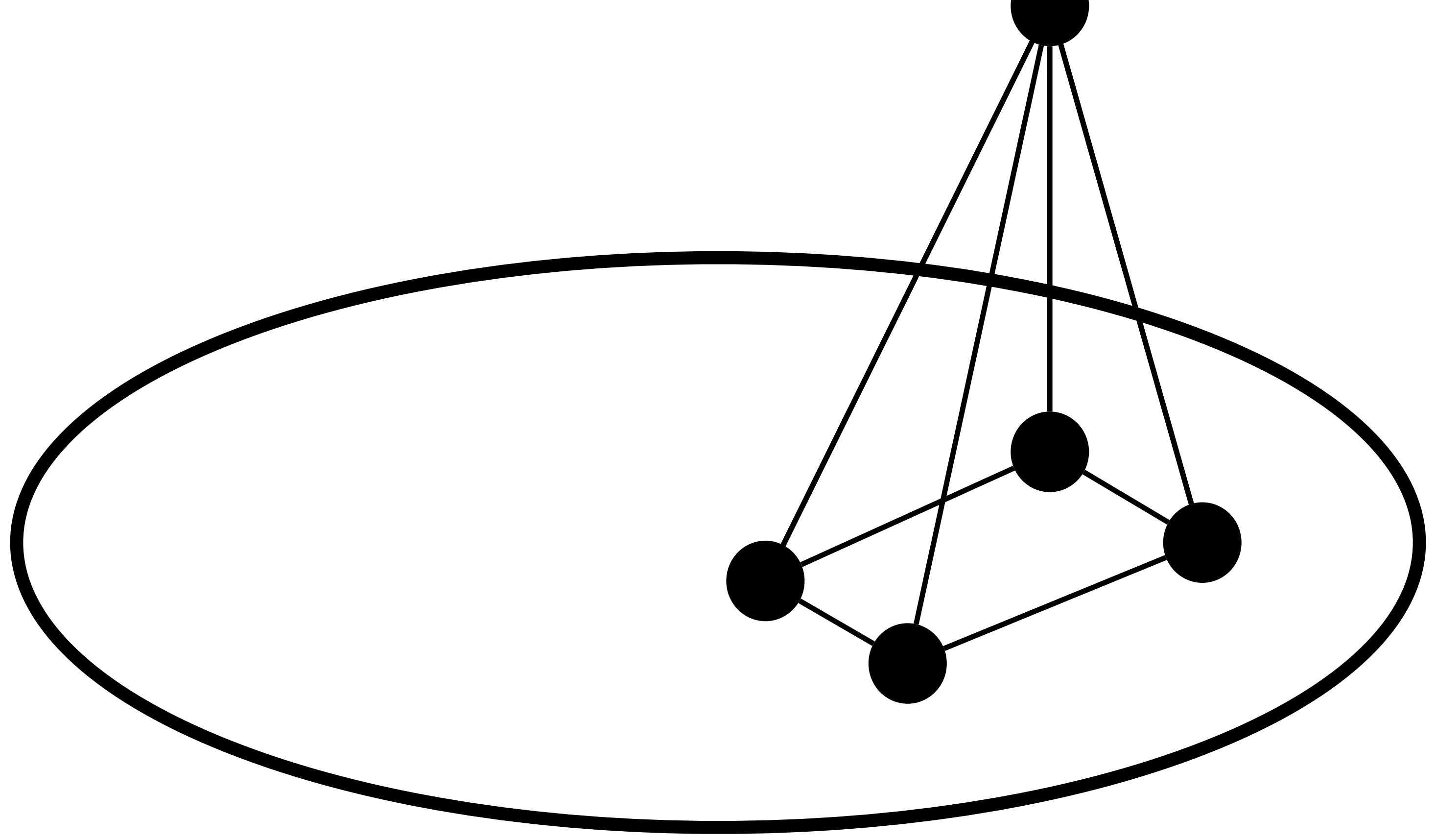
Subtleties

- Need homological algebra to relate Betti numbers with counts
  - adding a vertex = construct mapping cone
  - $\beta_q(\text{new}) \leq \beta_q(\text{old}) + \beta_{q-1}(\text{link})$



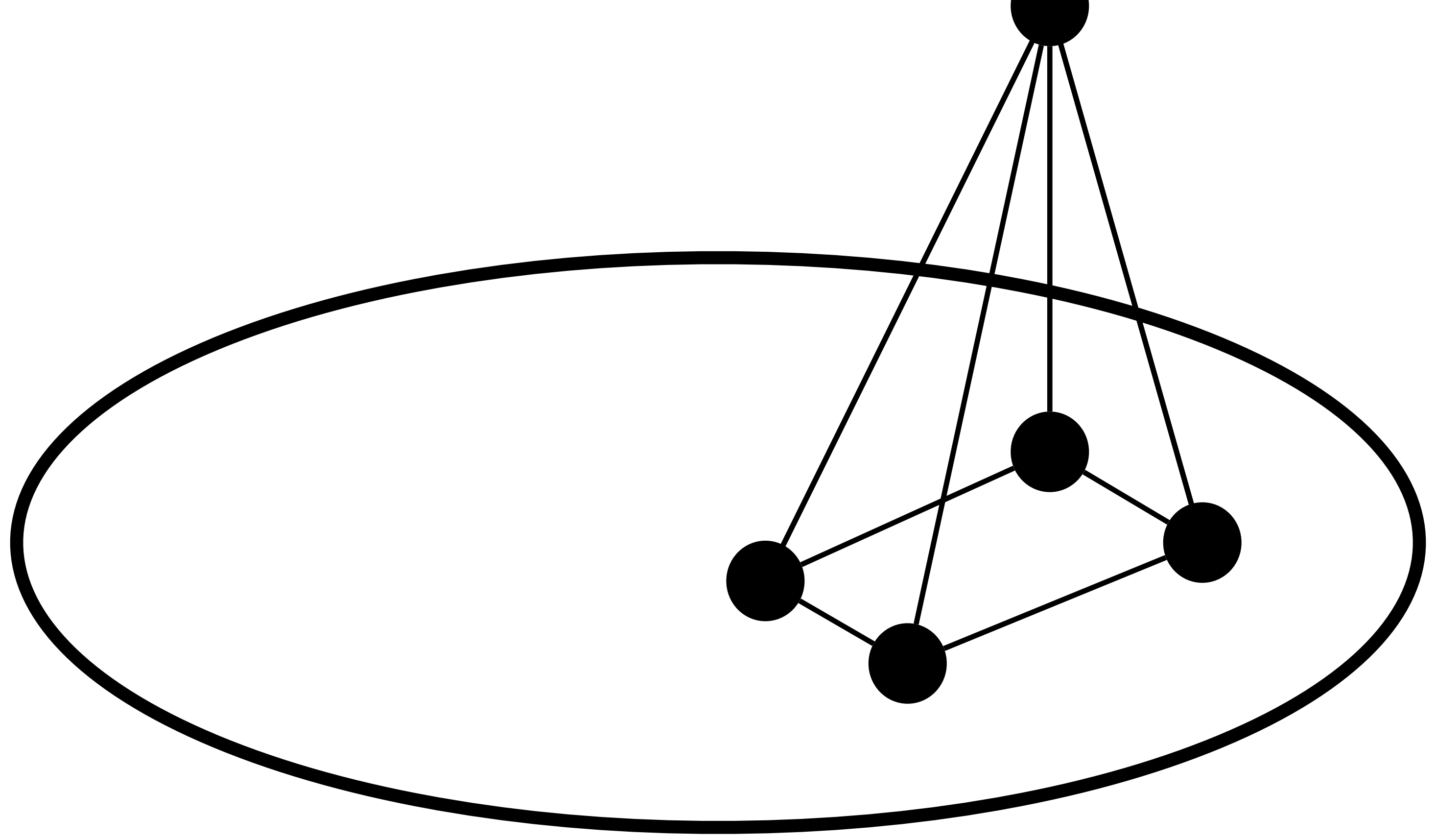
Subtleties

- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$



Subtleties

- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]



Subtleties

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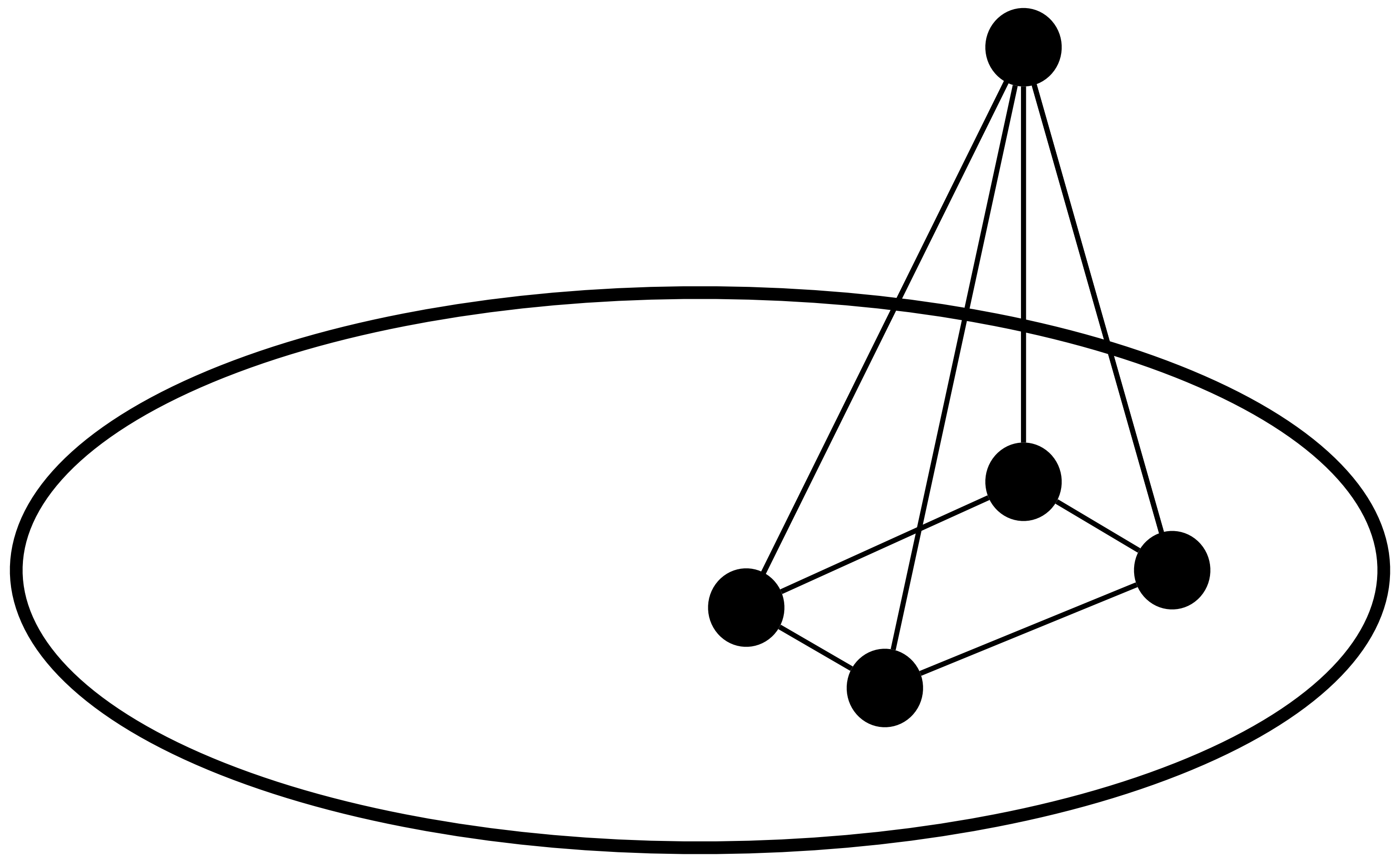
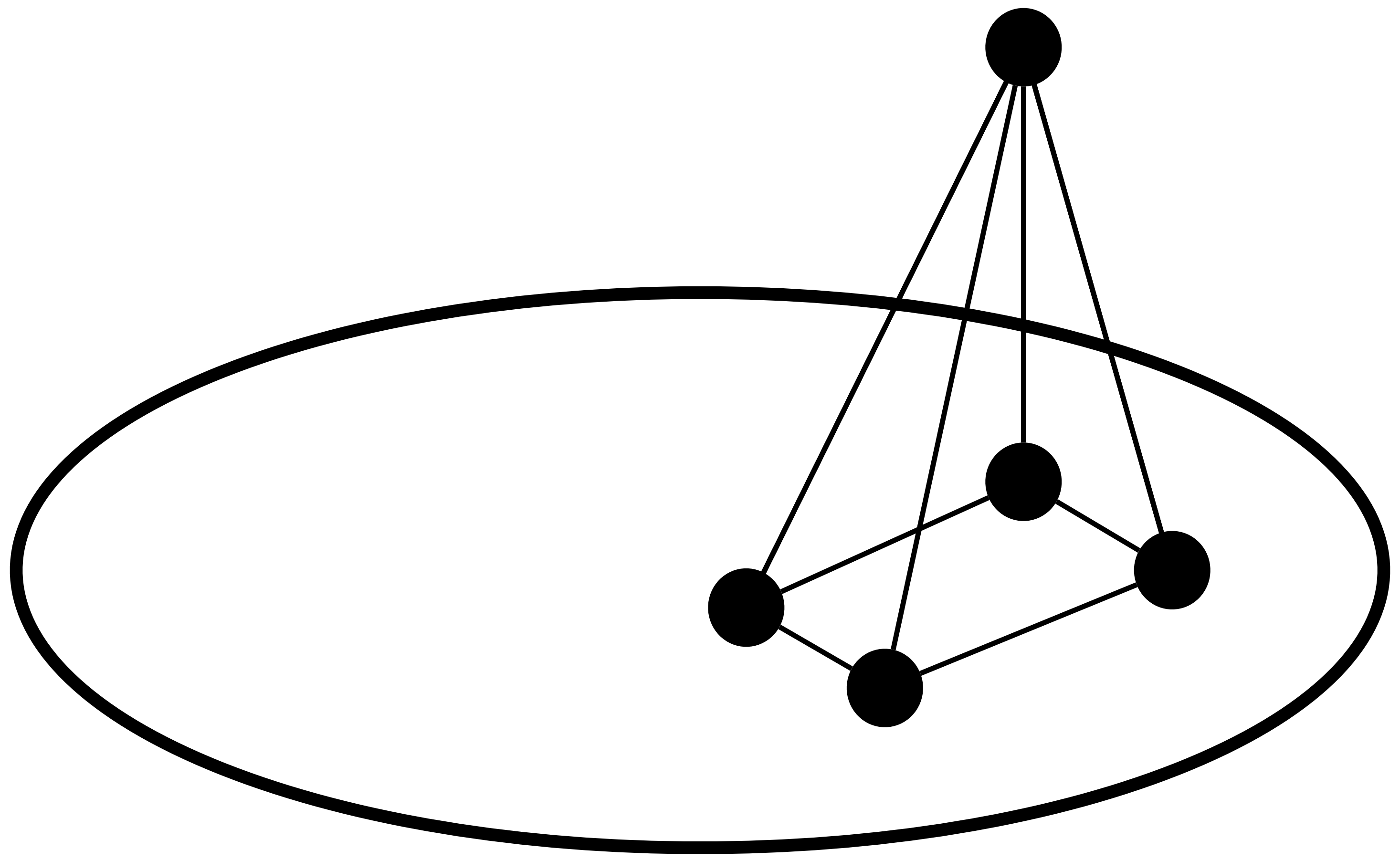
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra



Subtleties



- Need homological algebra to relate Betti numbers with counts
    - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra
    - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$

Subtleties

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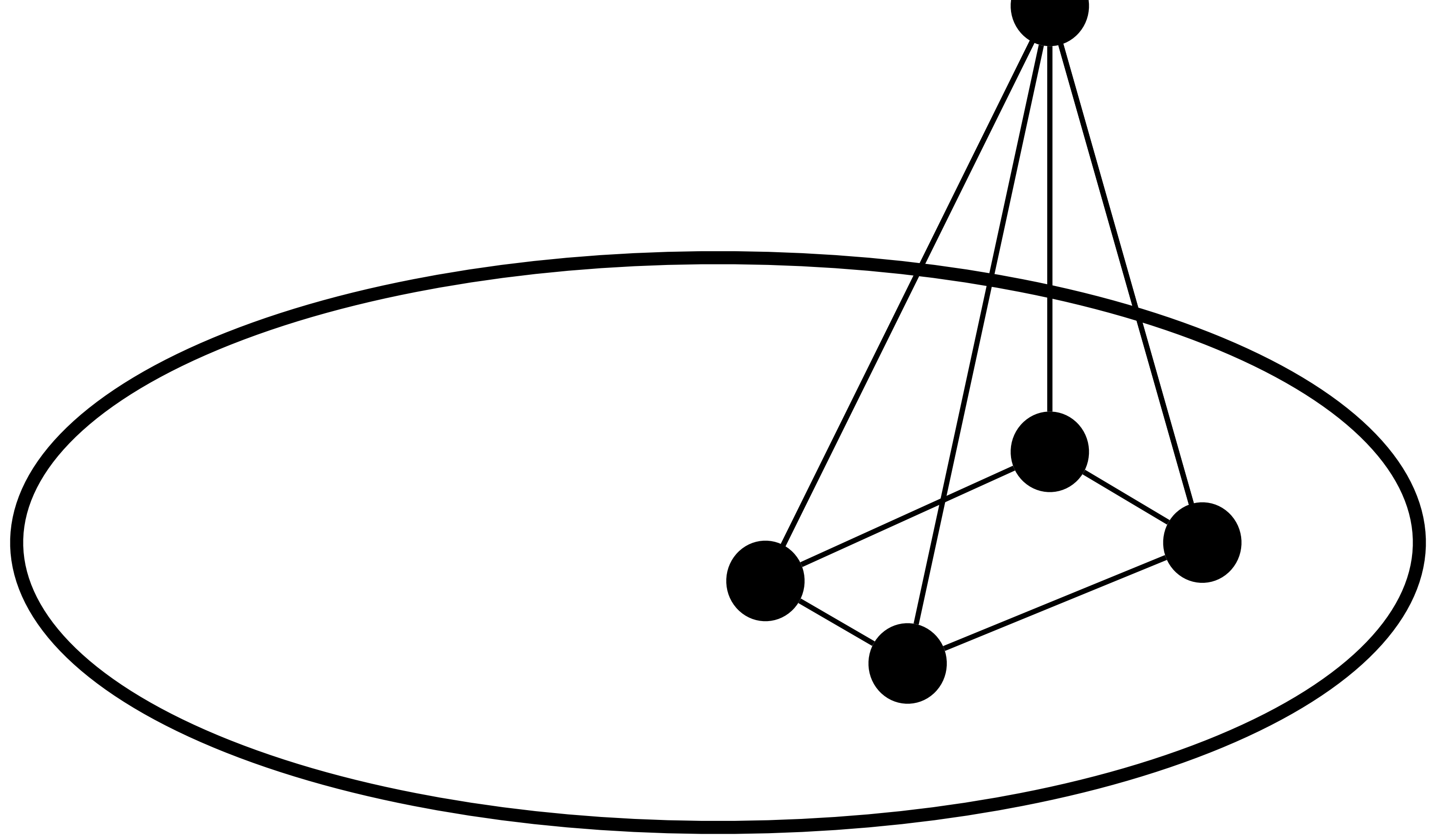

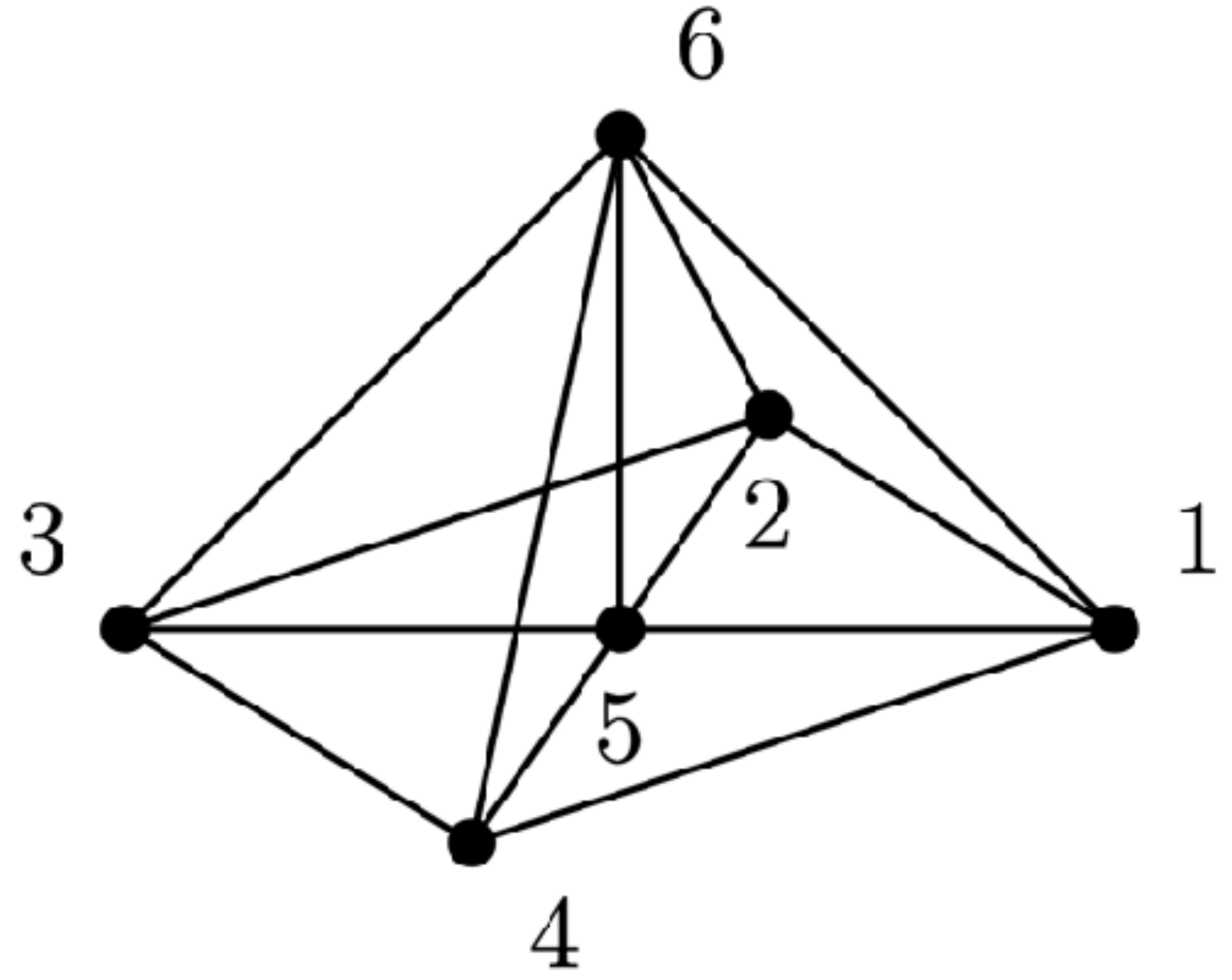

Subtleties

- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra
  - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs