The Topology of Preferential Attachment

The Asymptotics of the Expected Betti Numbers of Preferential Attachment Clique Complexes

Chunyin Siu
Cornell University
cs2323@cornell.edu

Waving through the window

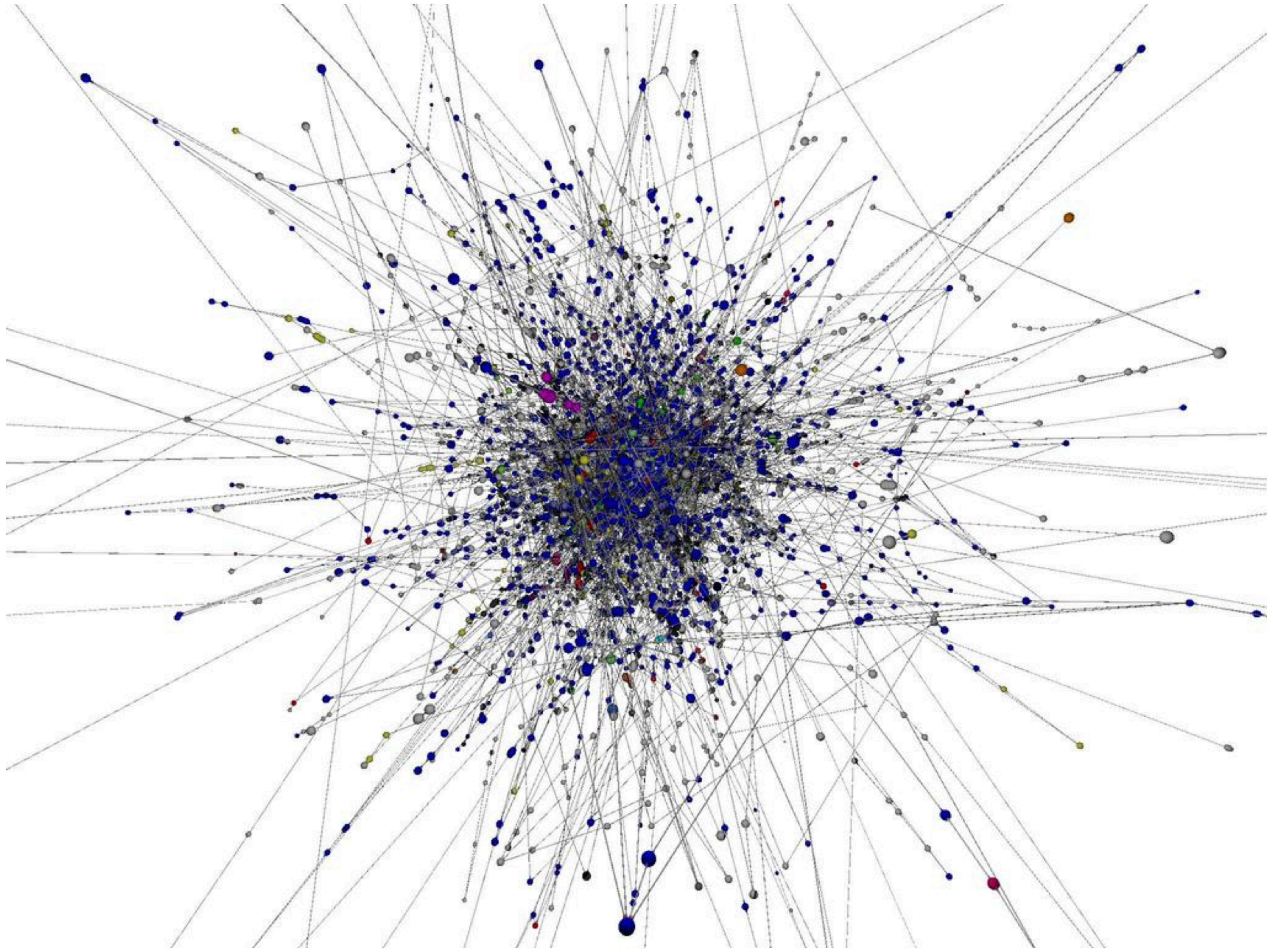
postdoc for 24/25

The Topology of Preferential Attachment

The Asymptotics of the Expected Betti Numbers of Preferential Attachment Clique Complexes

Chunyin Siu
Cornell University
cs2323@cornell.edu

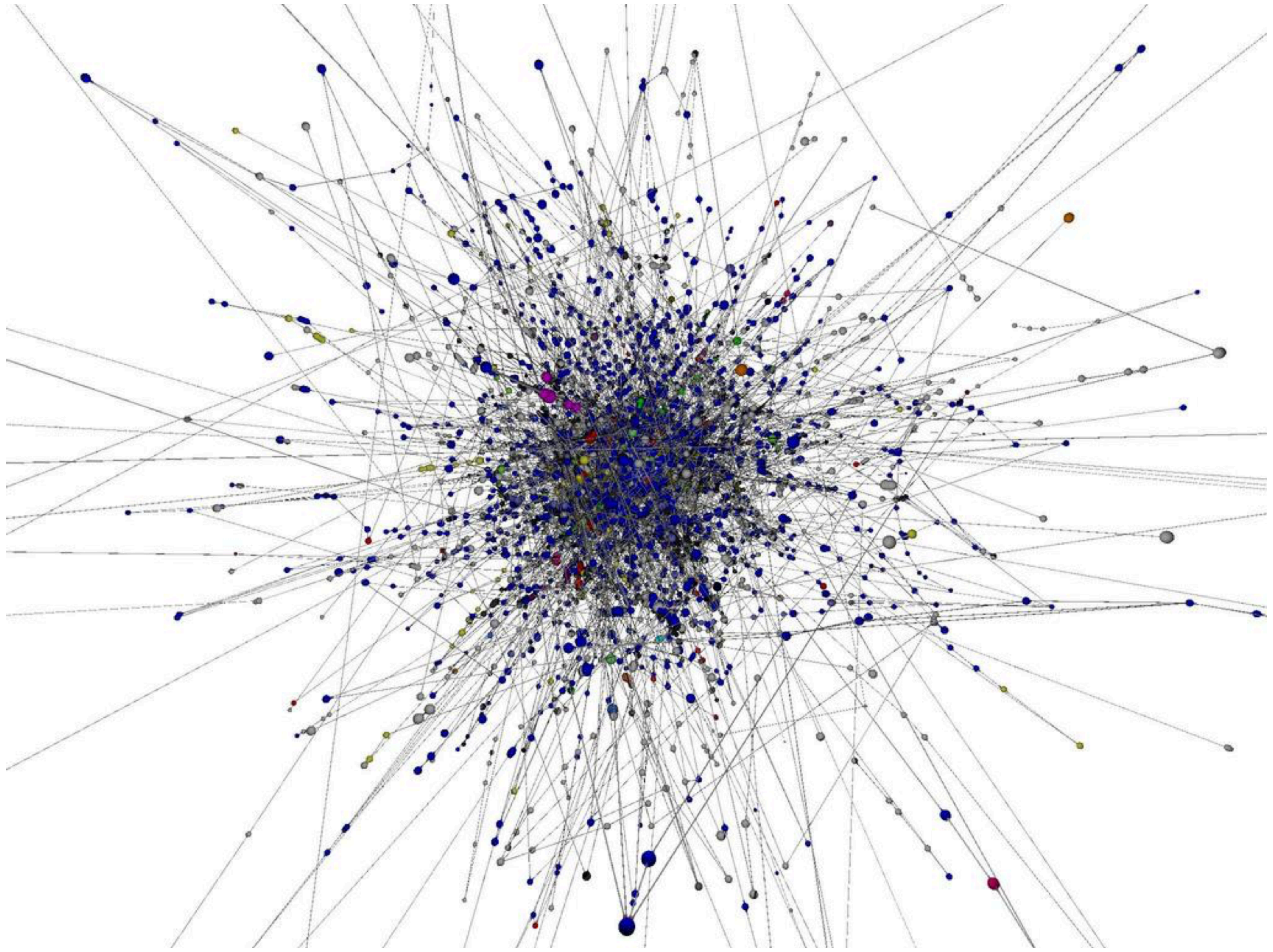
So, preferential attachment…



(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

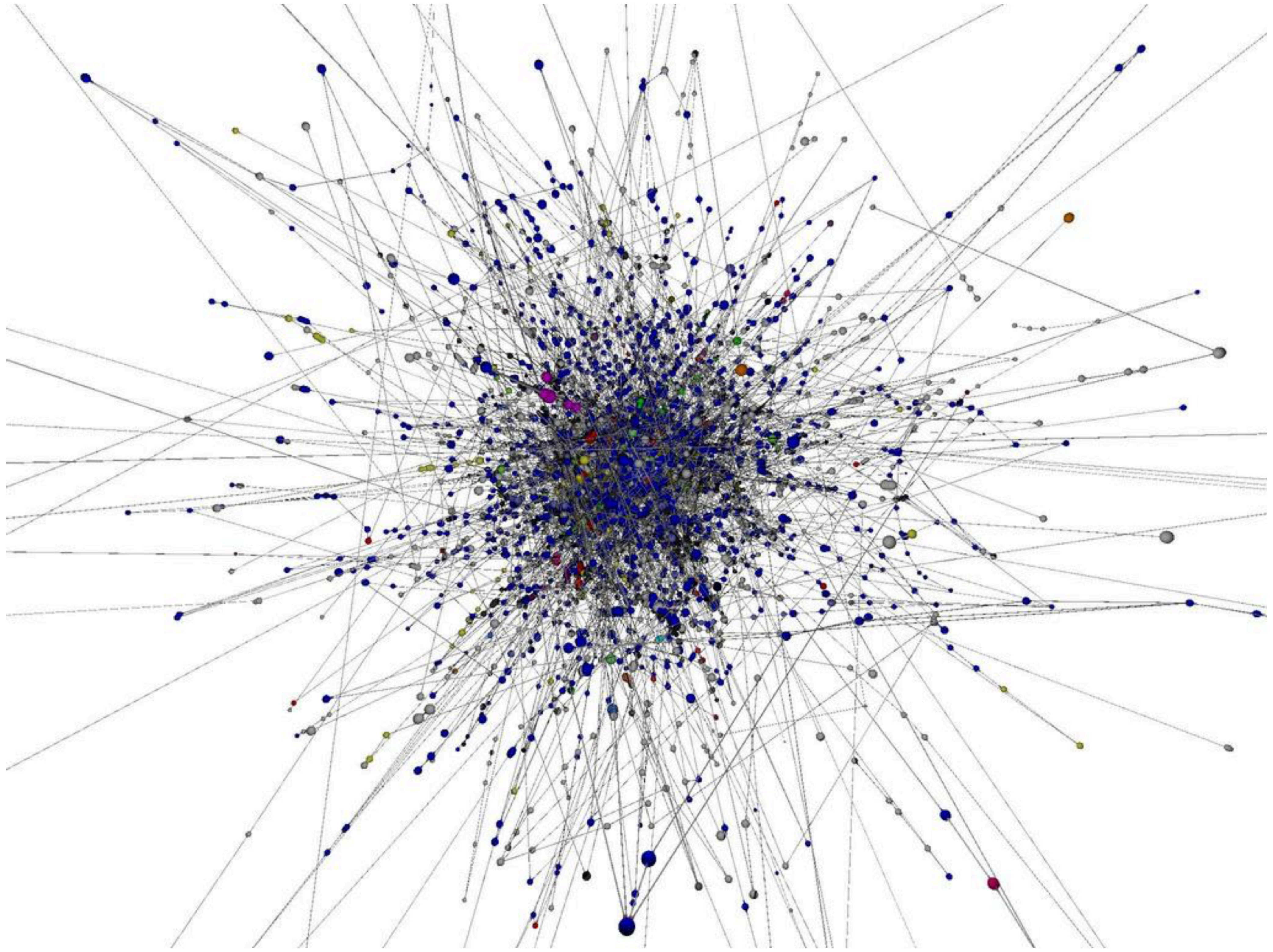
- Just a bouquet of circles?



(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Just a bouquet of circles?
- What is intrinsic and what is just random fluctuation?



(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

So, preferential attachment…

- Just a bouquet of circles?
- What is intrinsic and what is just random fluctuation?
- —> random topology

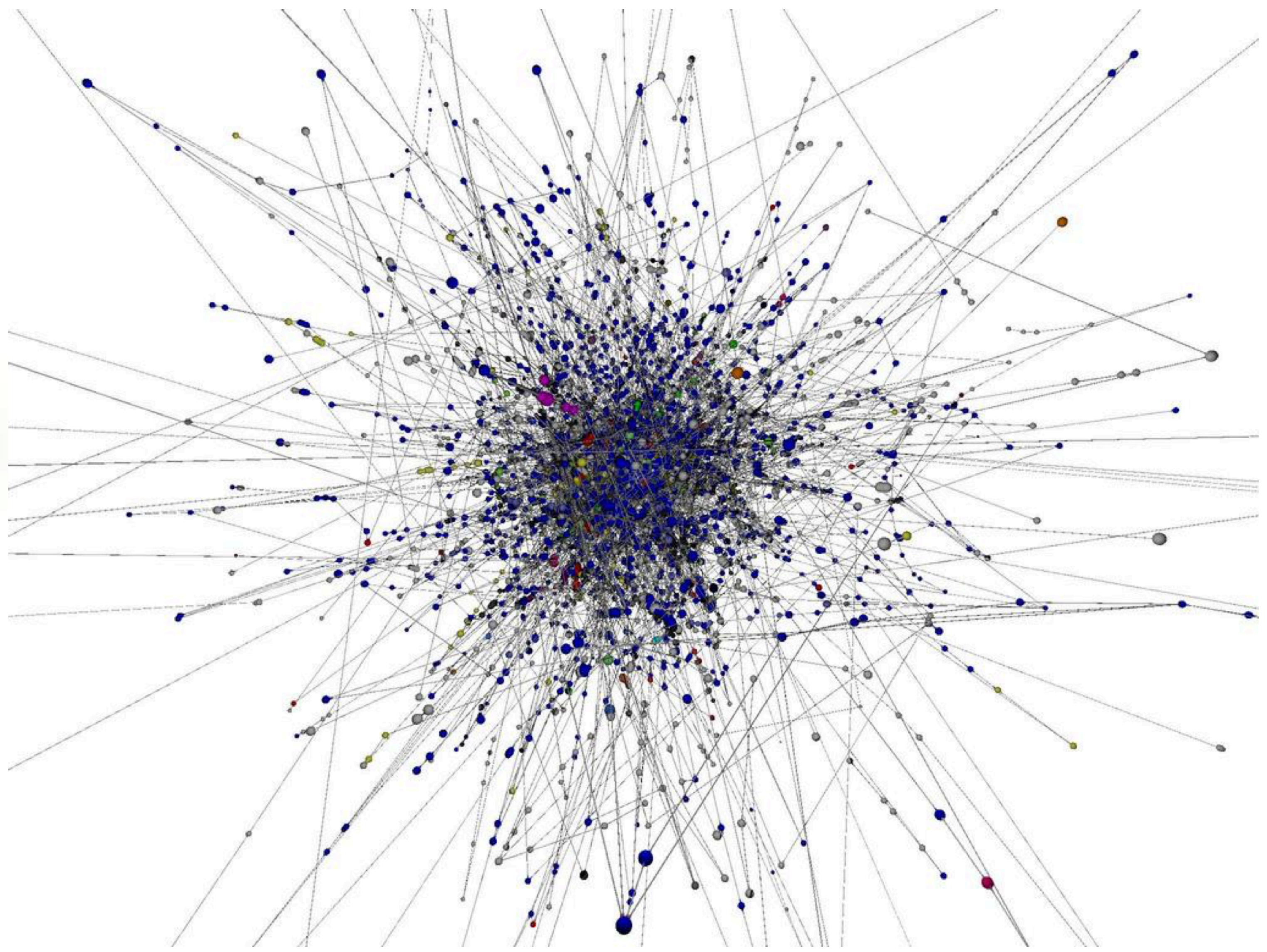
(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

Agenda

random topology

Agenda

random topology

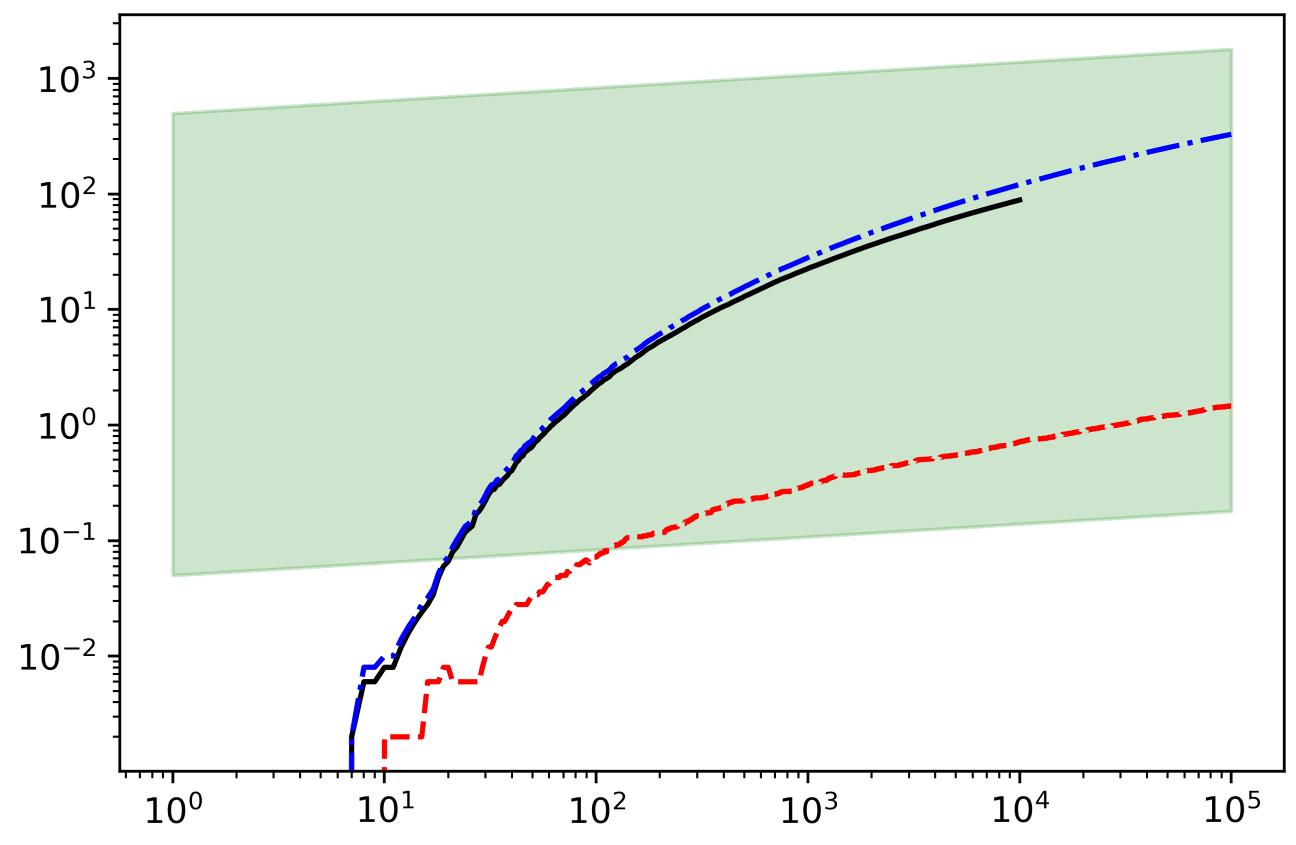


preferential attachment

Agenda

random topology

preferential attachment

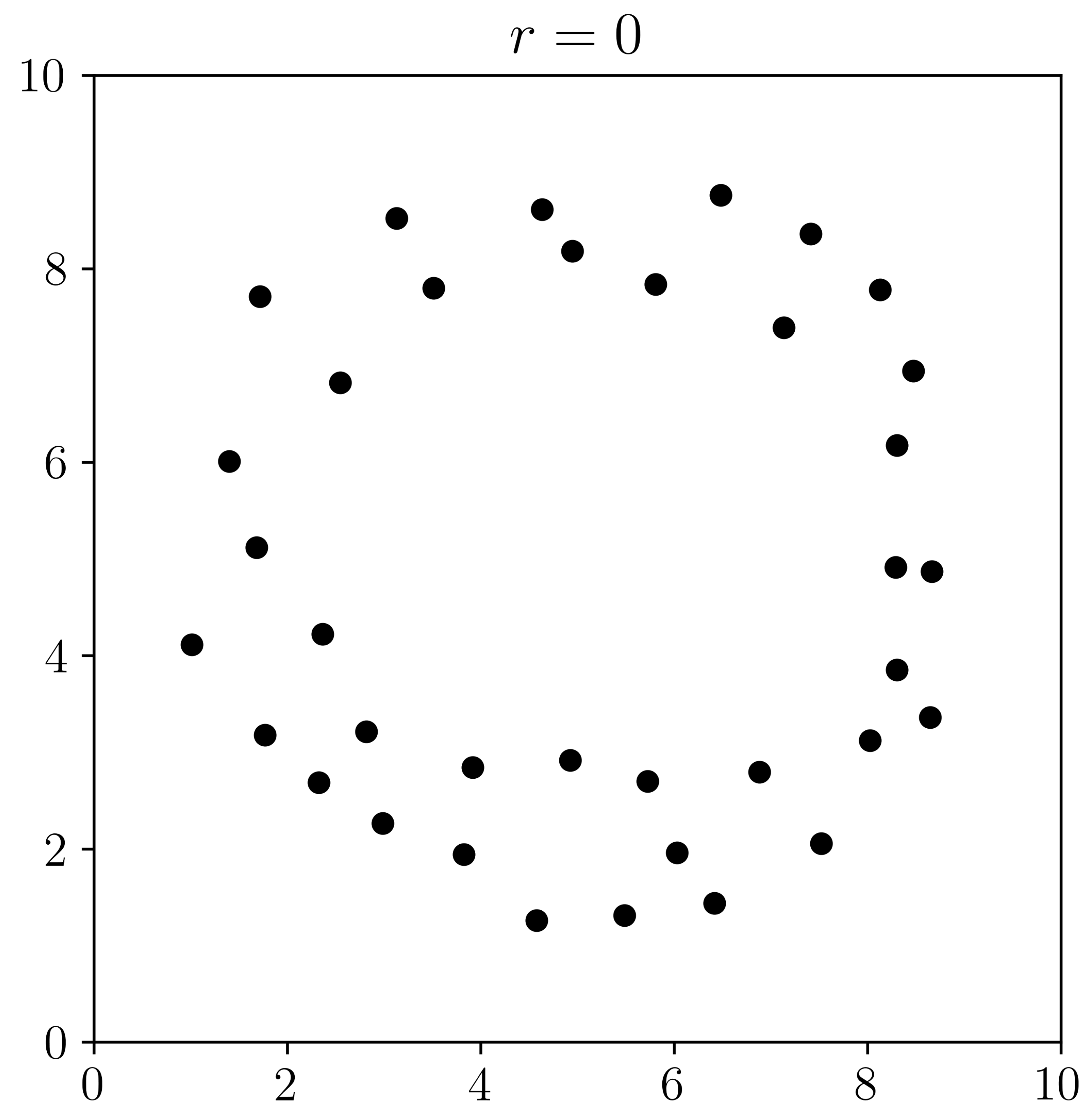


our result

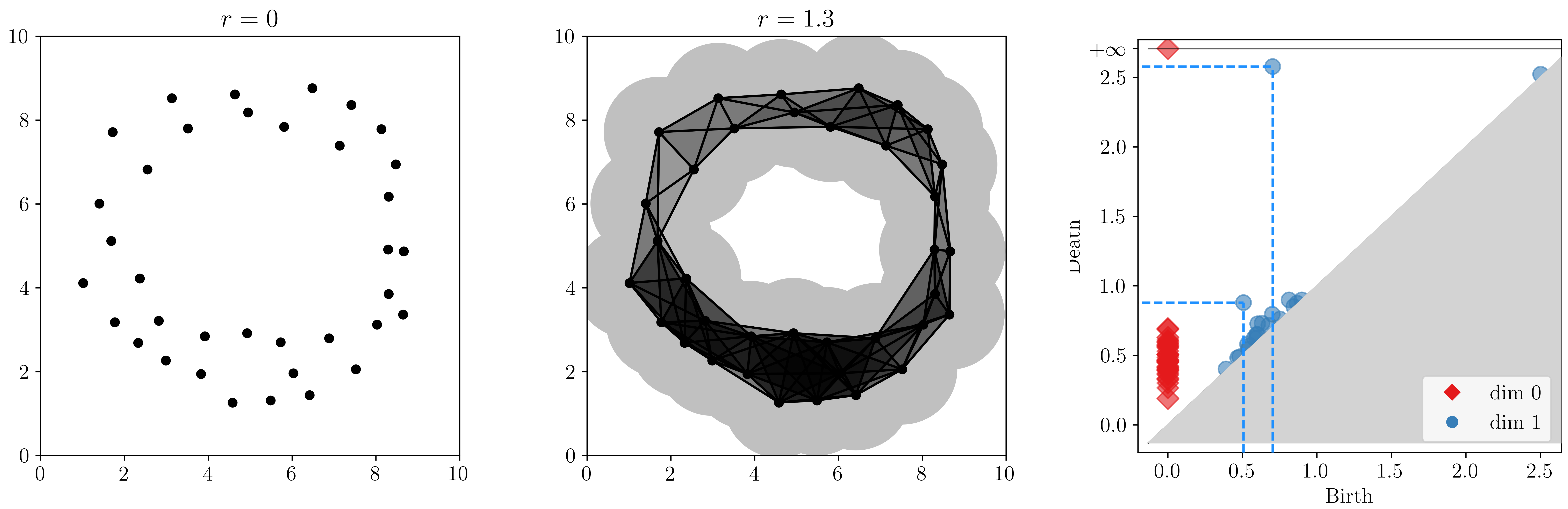
Yell at me whenever

I. A Probabilist’s Apology

Why Random Topology and What we Know

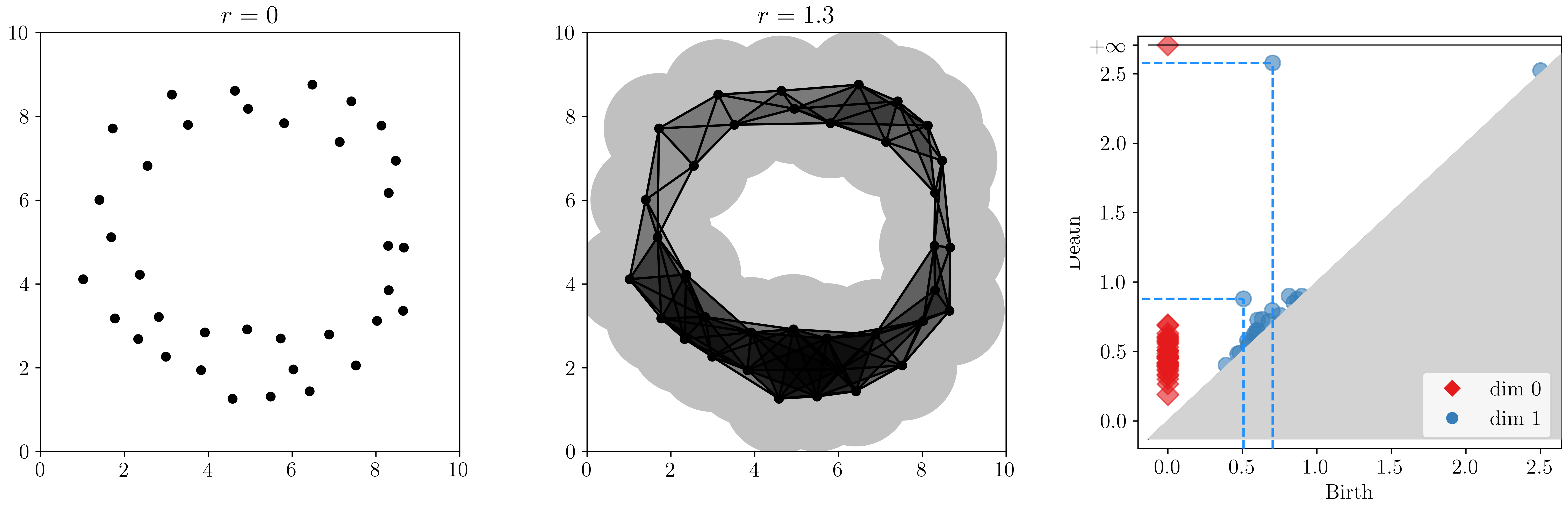


$r = 0$
0
2
4
6
8
10
0
2
4
6
8
10

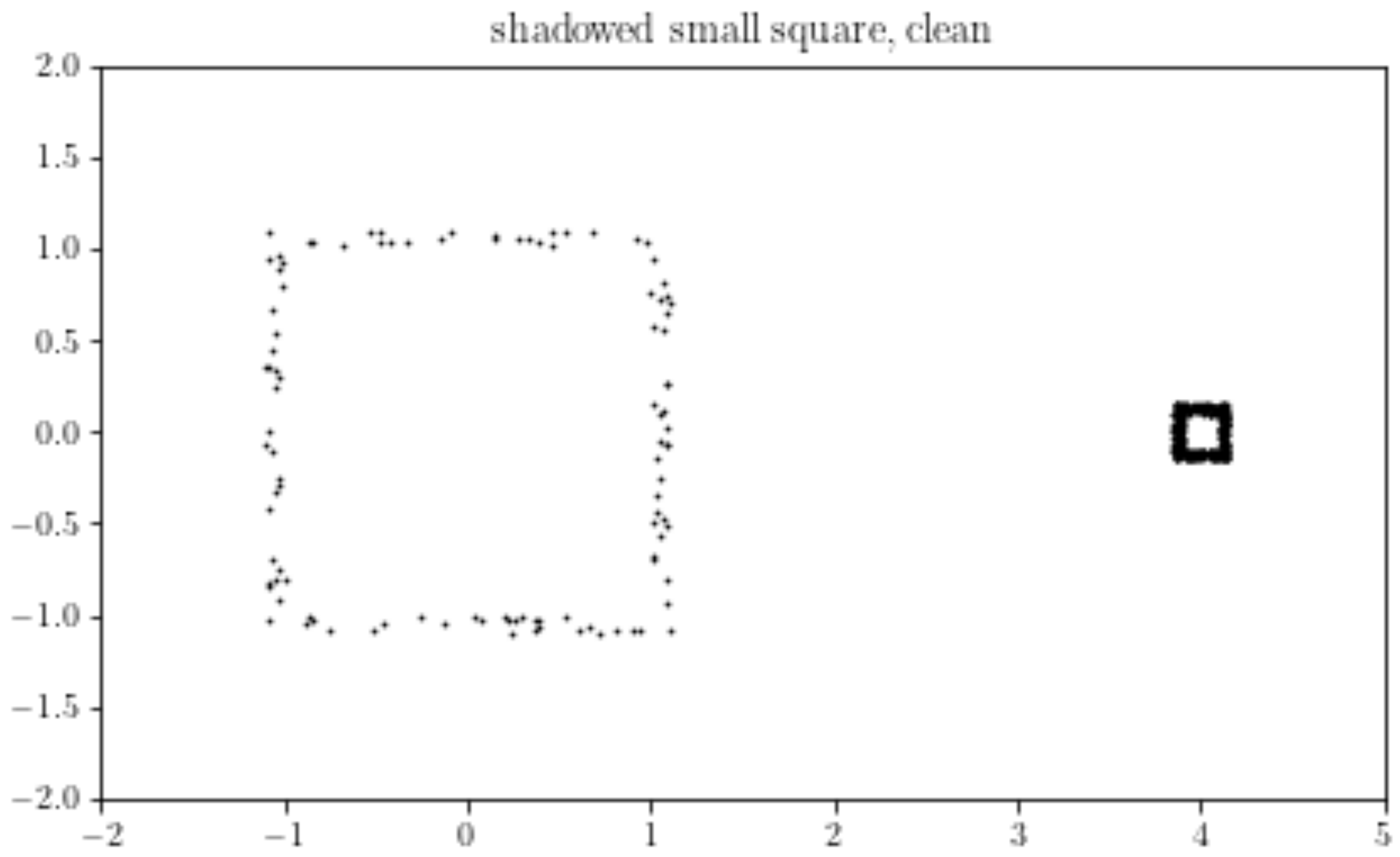


plots generated by Andrey Yao

Size is Signal

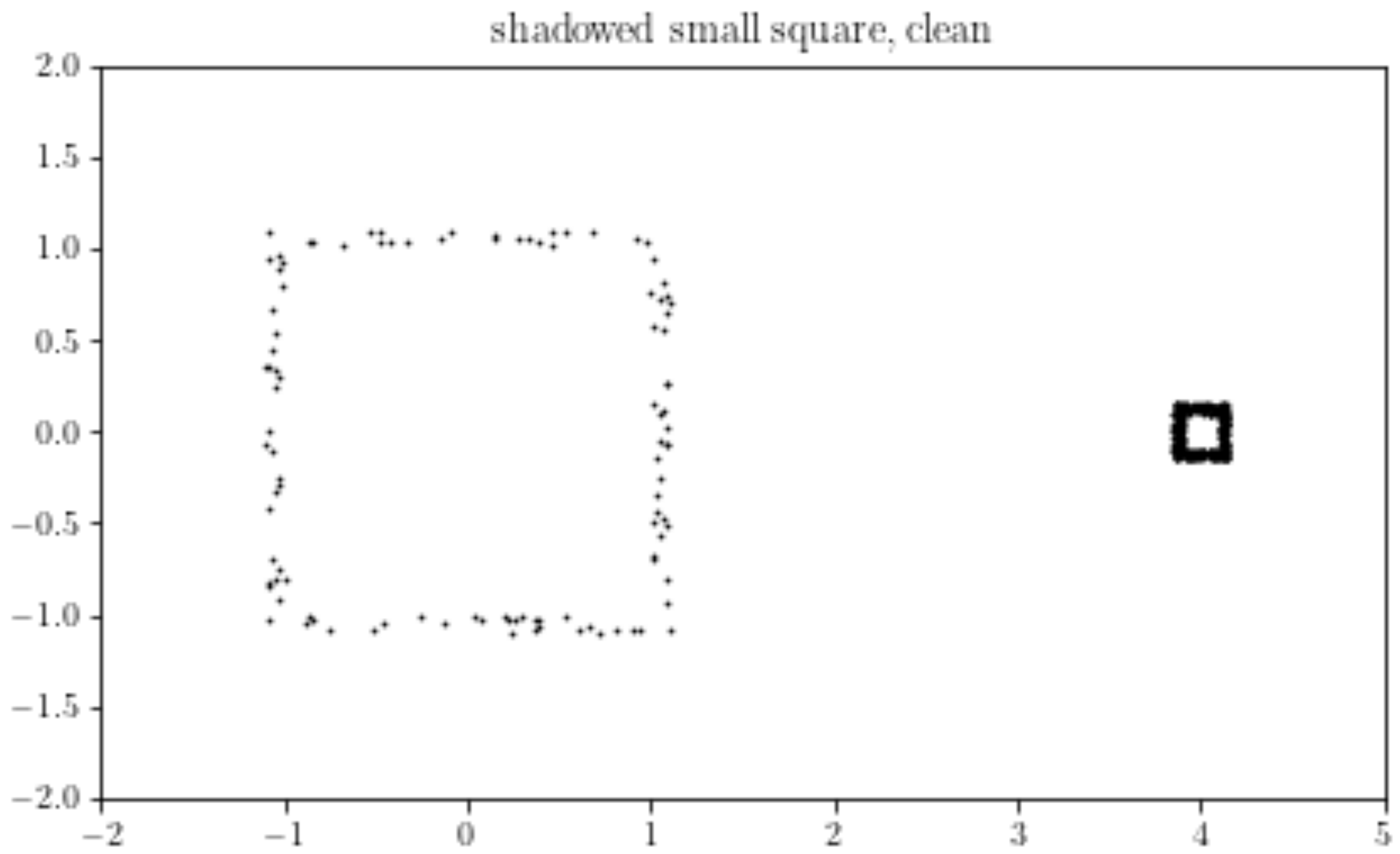


Or is it?

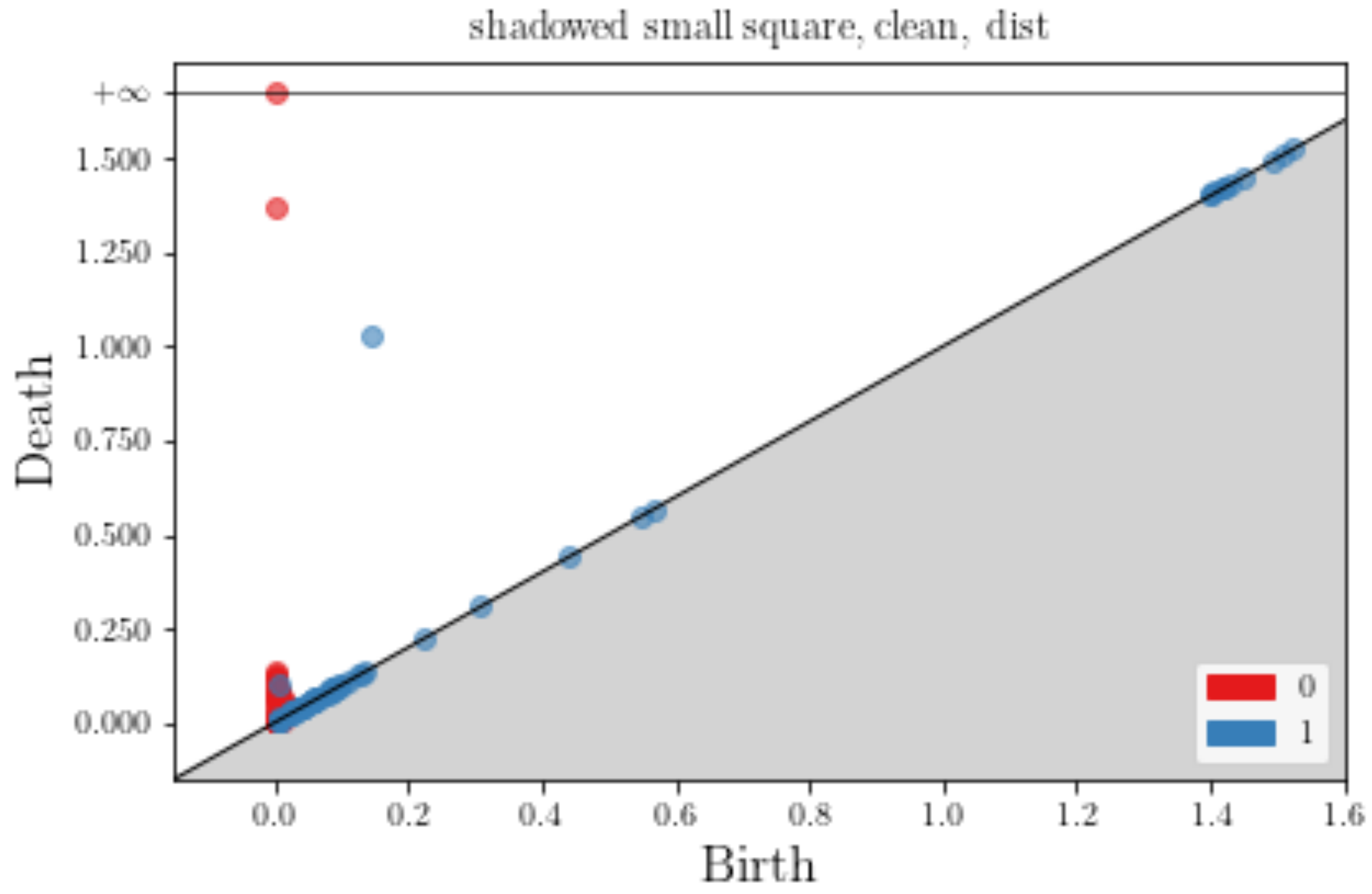


shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5

Or is it?



shadowed small square, clean
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2
−1
0
1
2
3
4
5



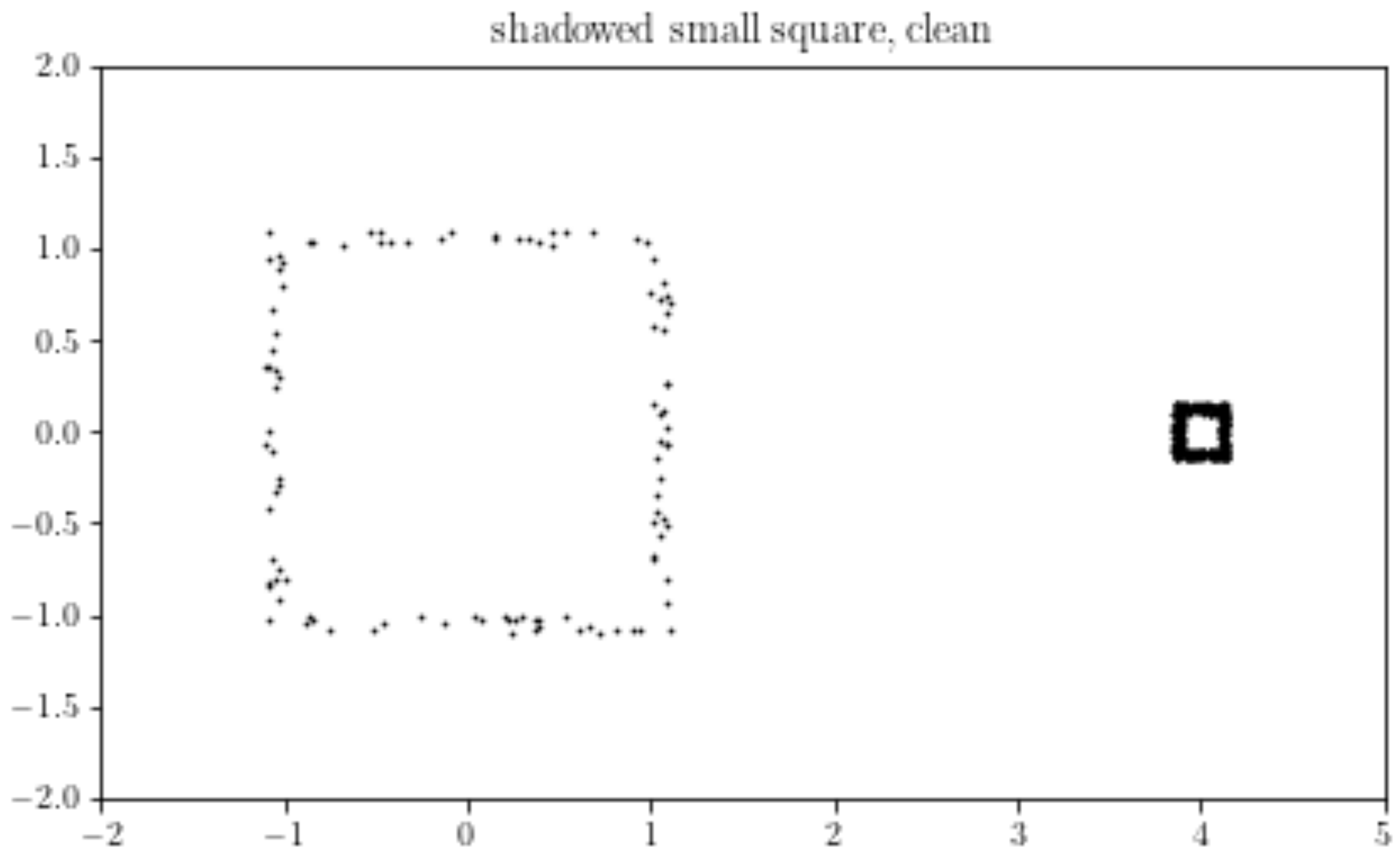
shadowed small square, clean, dist
+∞
1.500
1.250
1.000
0.750
0.500
0.250
0.000
Death
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
Birth
0
1

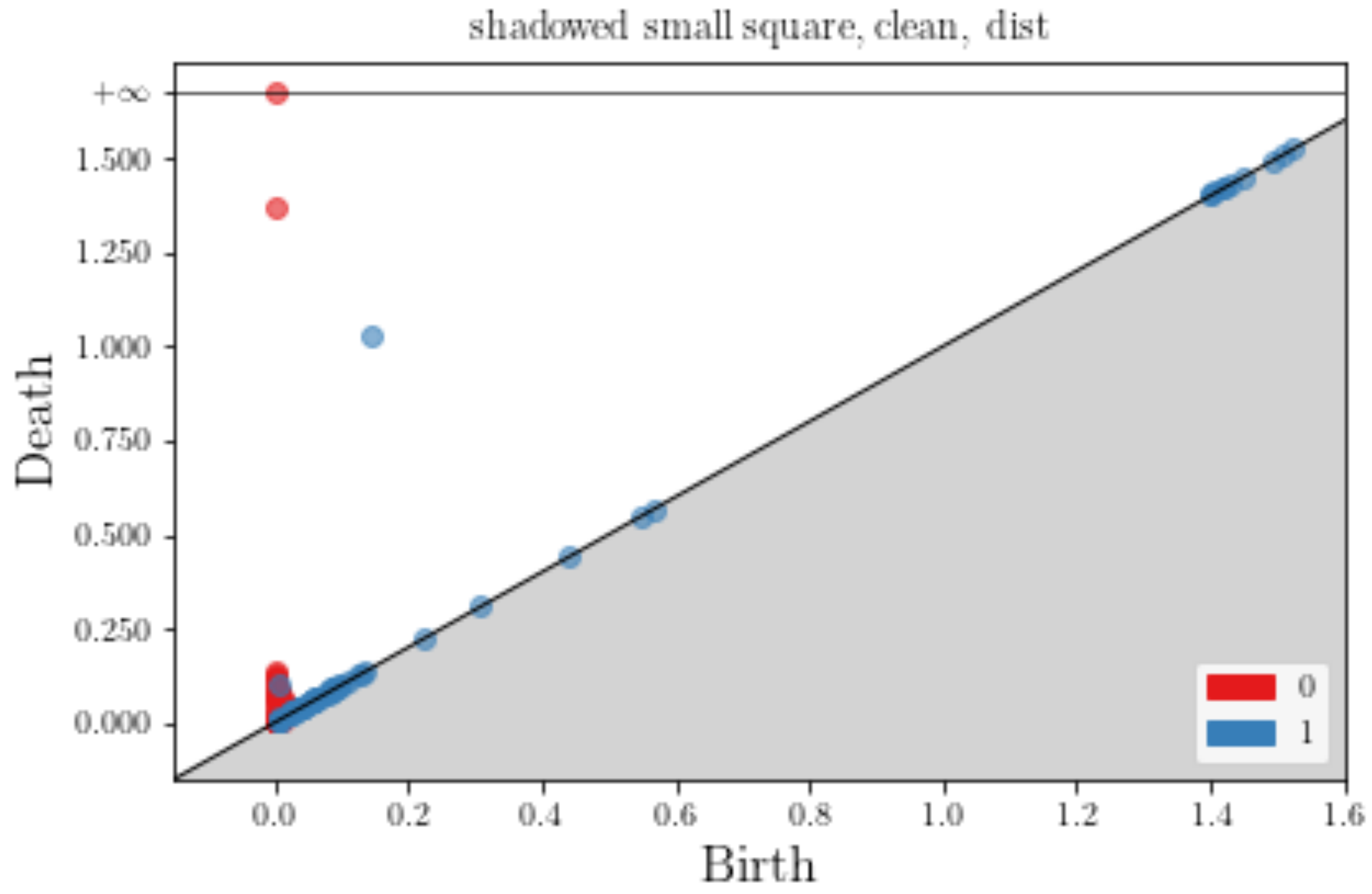
Size is Signal?

Surprise

~~Size~~ is Signal.

Random points don't do that.

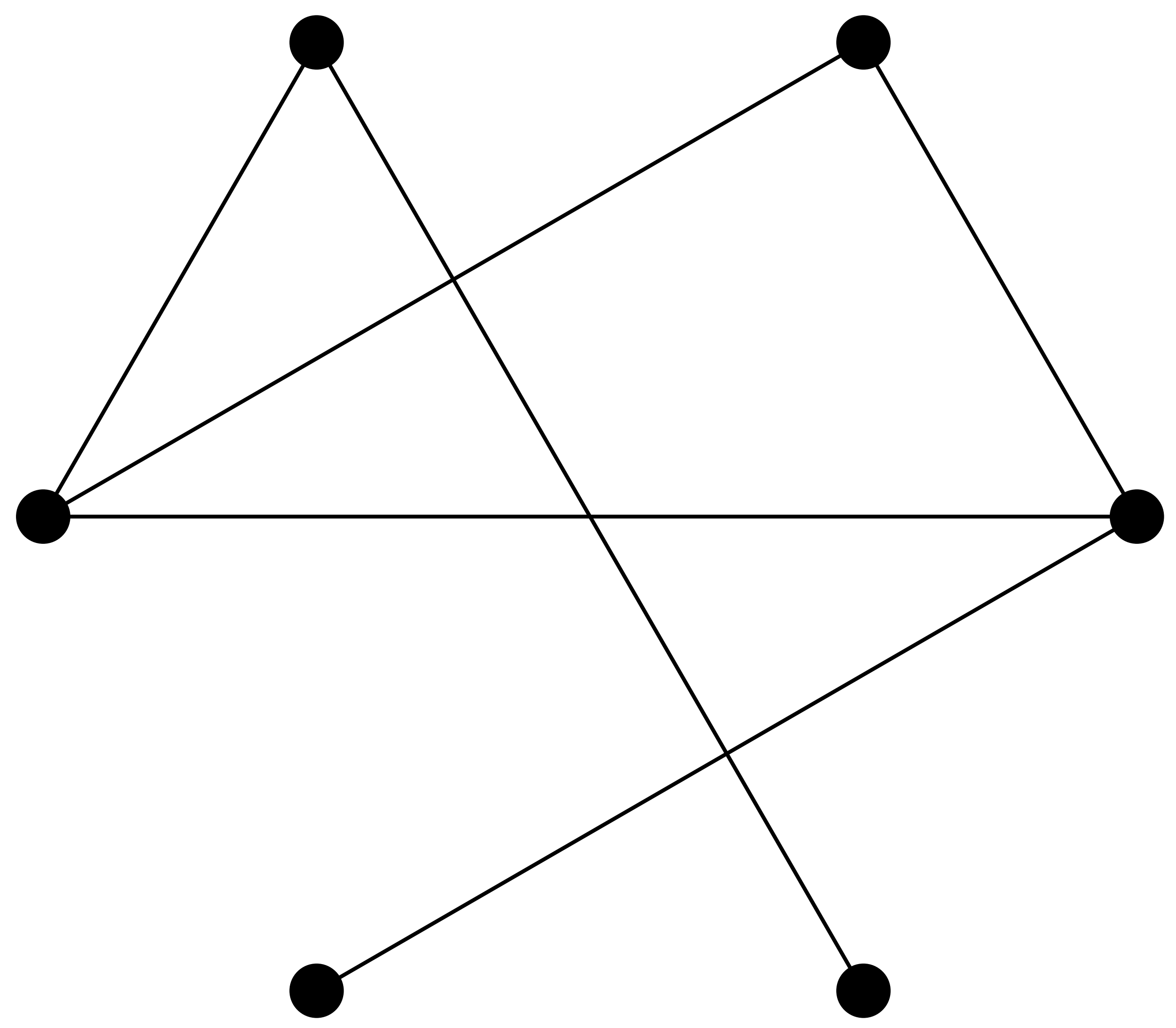




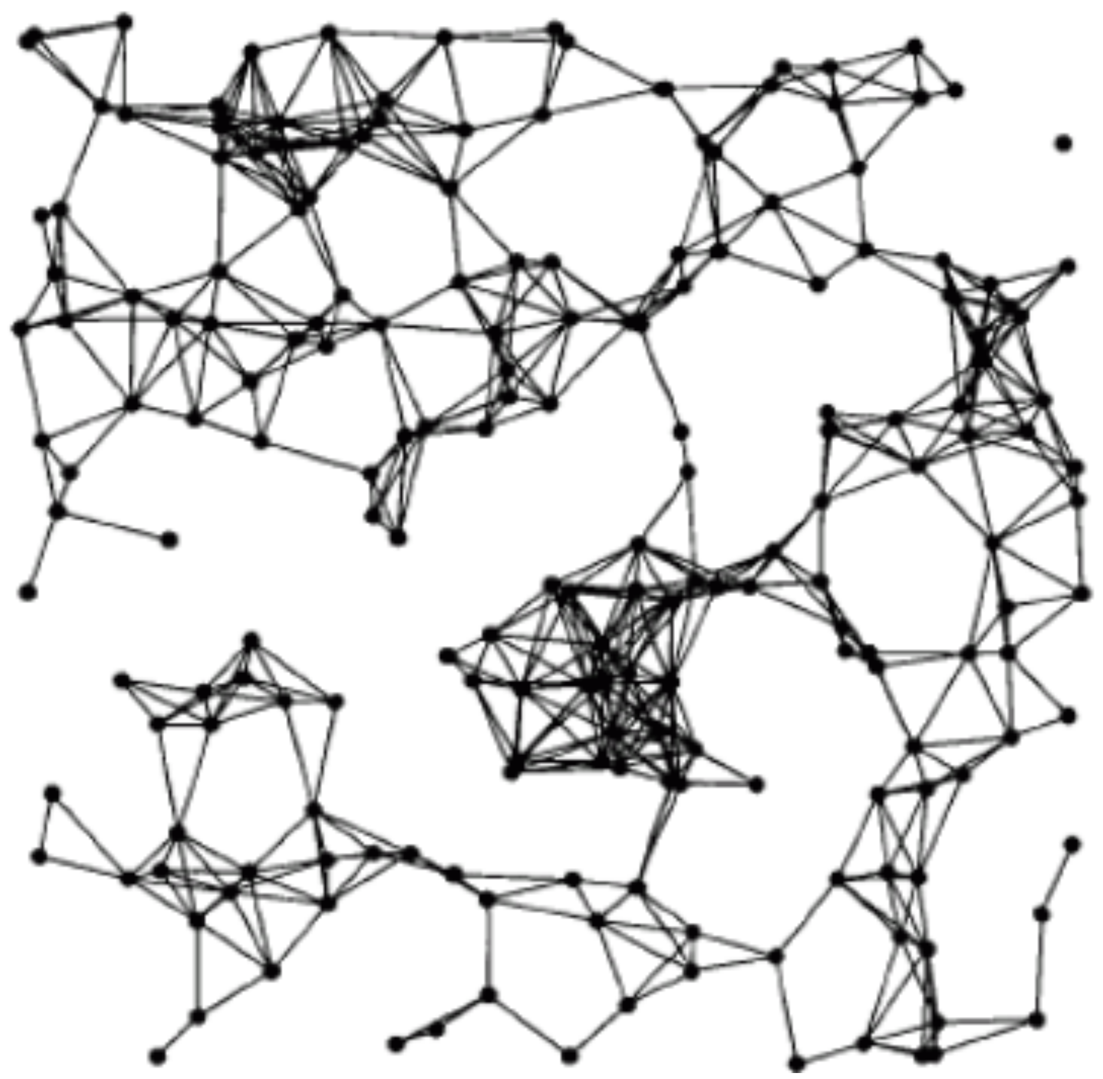
Signal is what is not random.

Signal is what is not **random.**
So what is **random?**

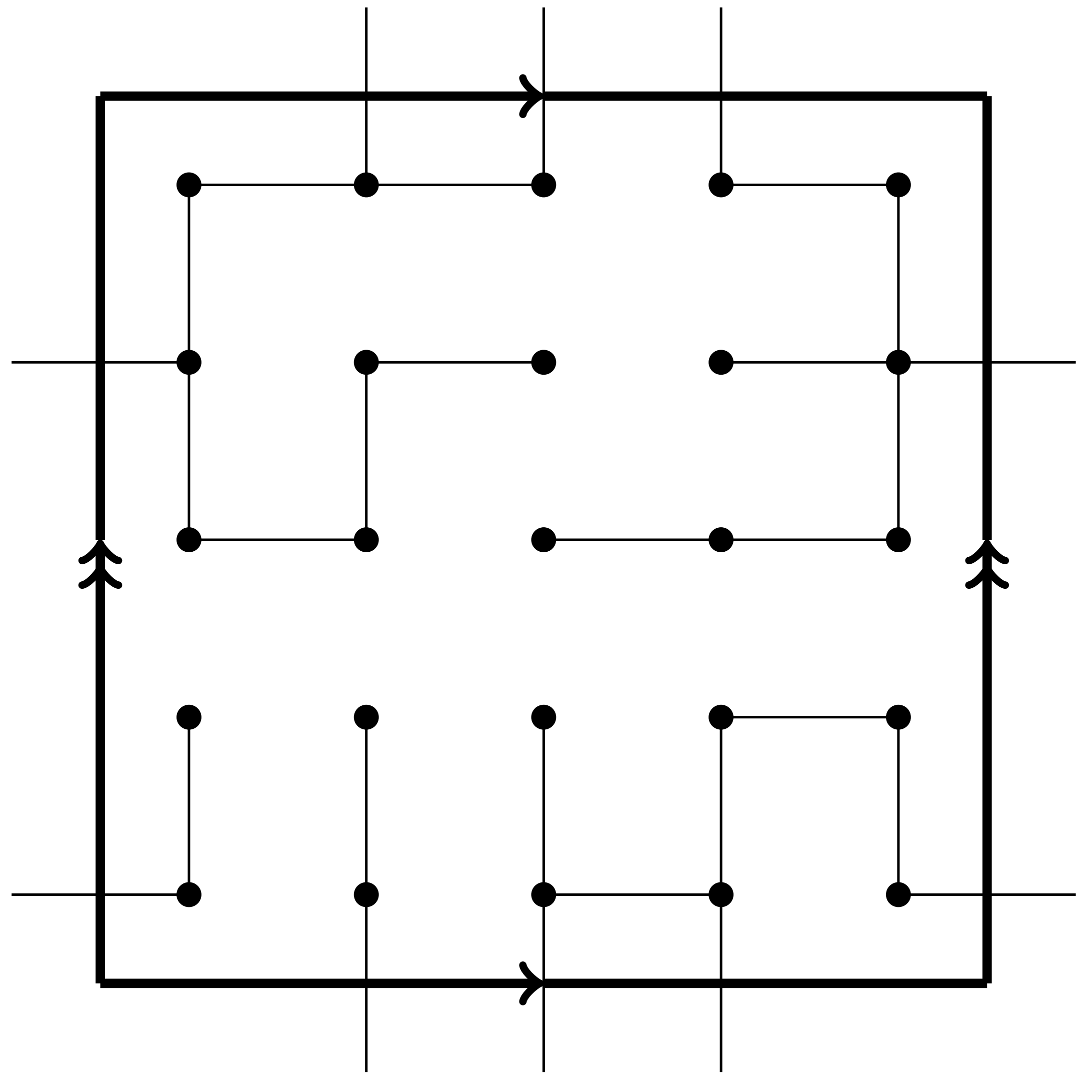
Tapas of Random Topology



Erdo-Renyi Complexes

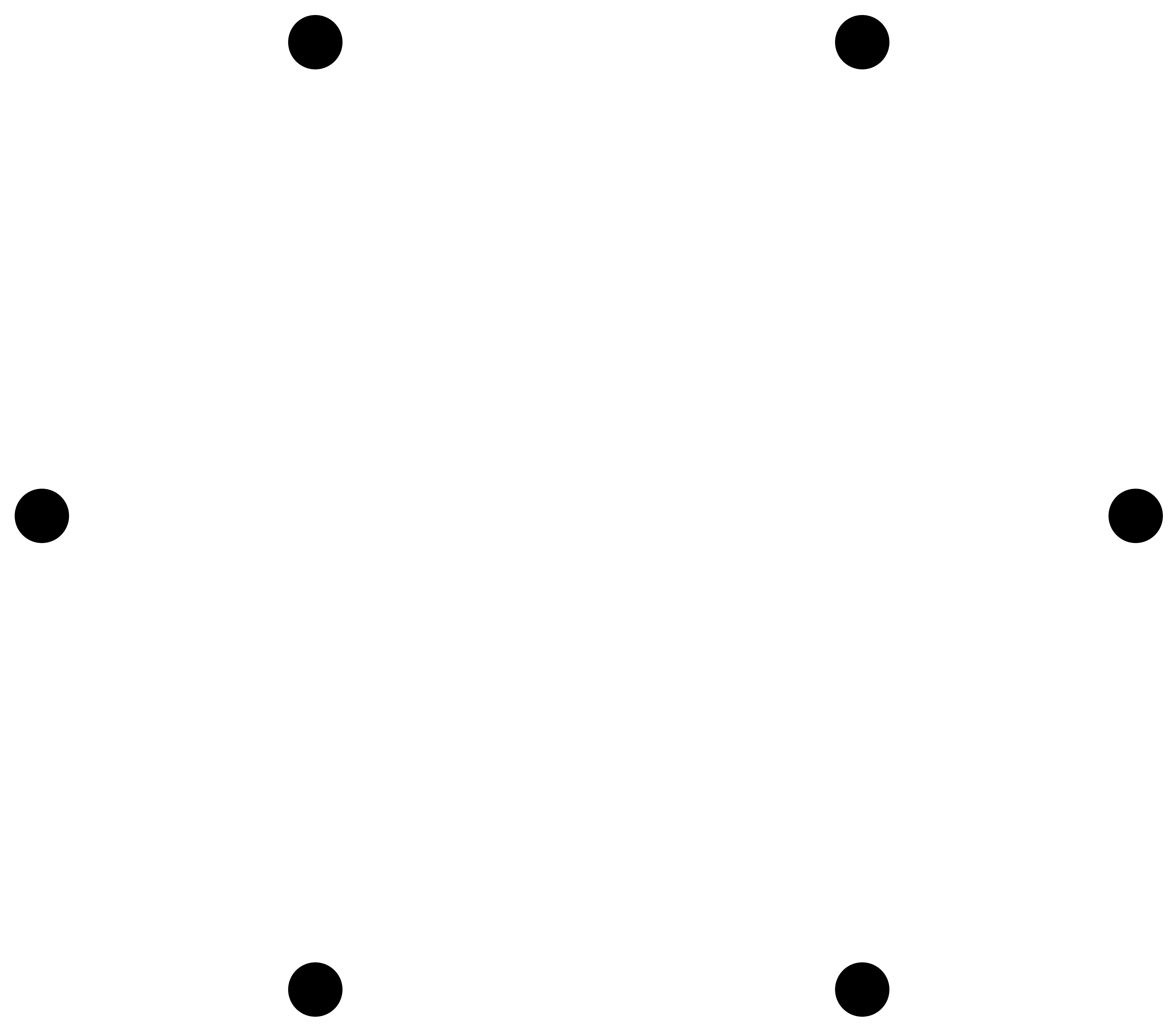


Geometric Complexes

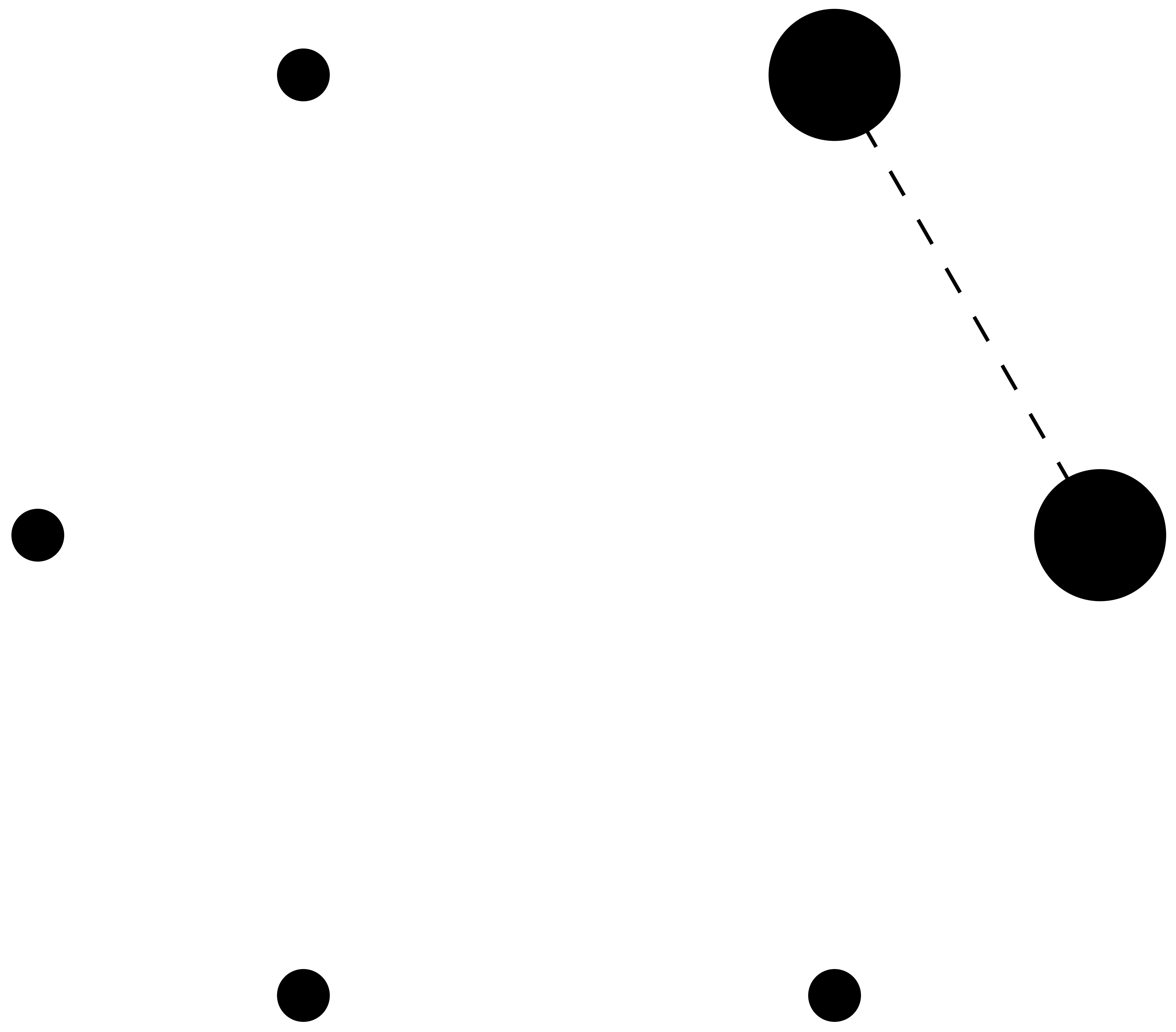


Topological Percolation

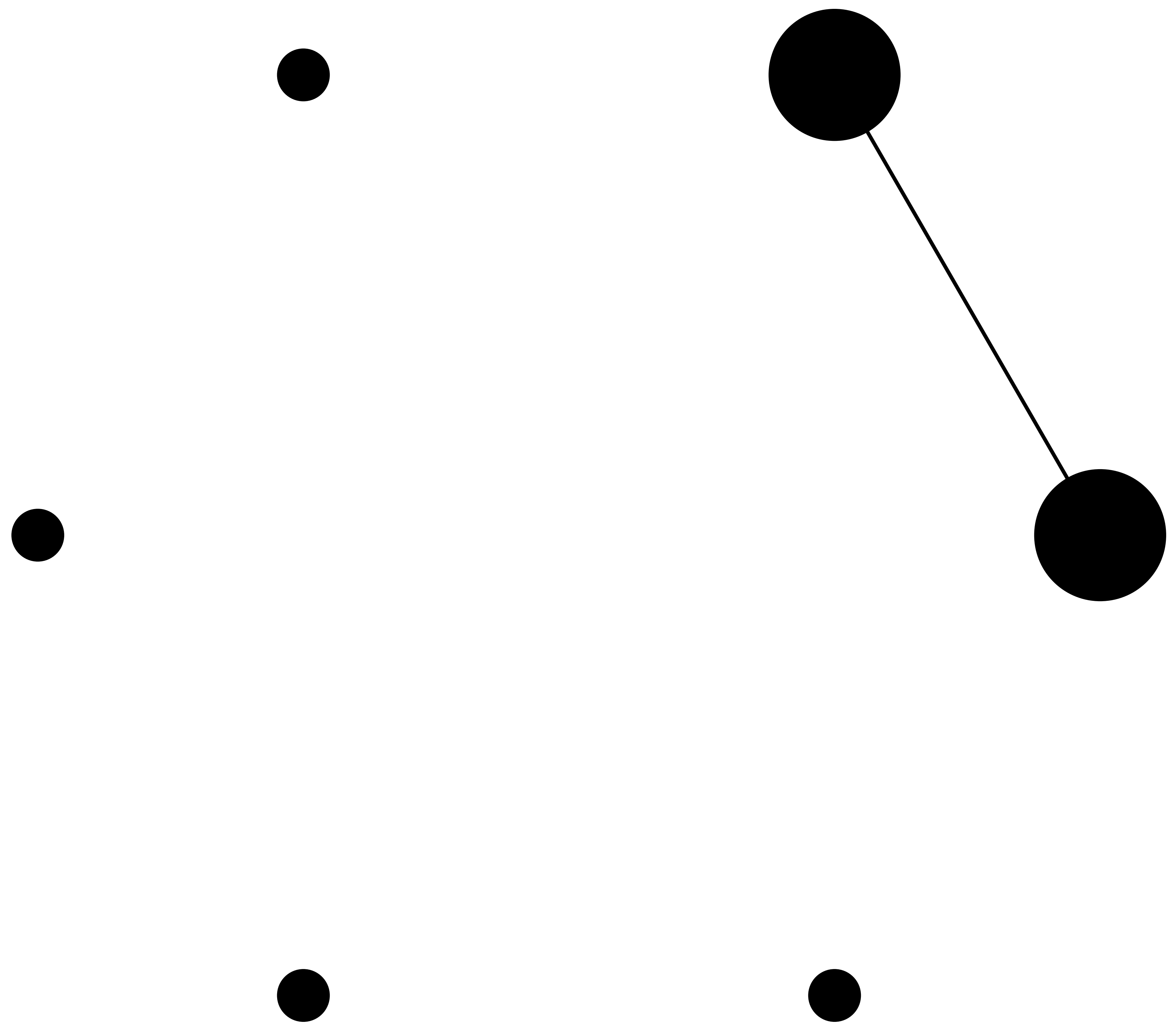
Erdos-Renyi graphs



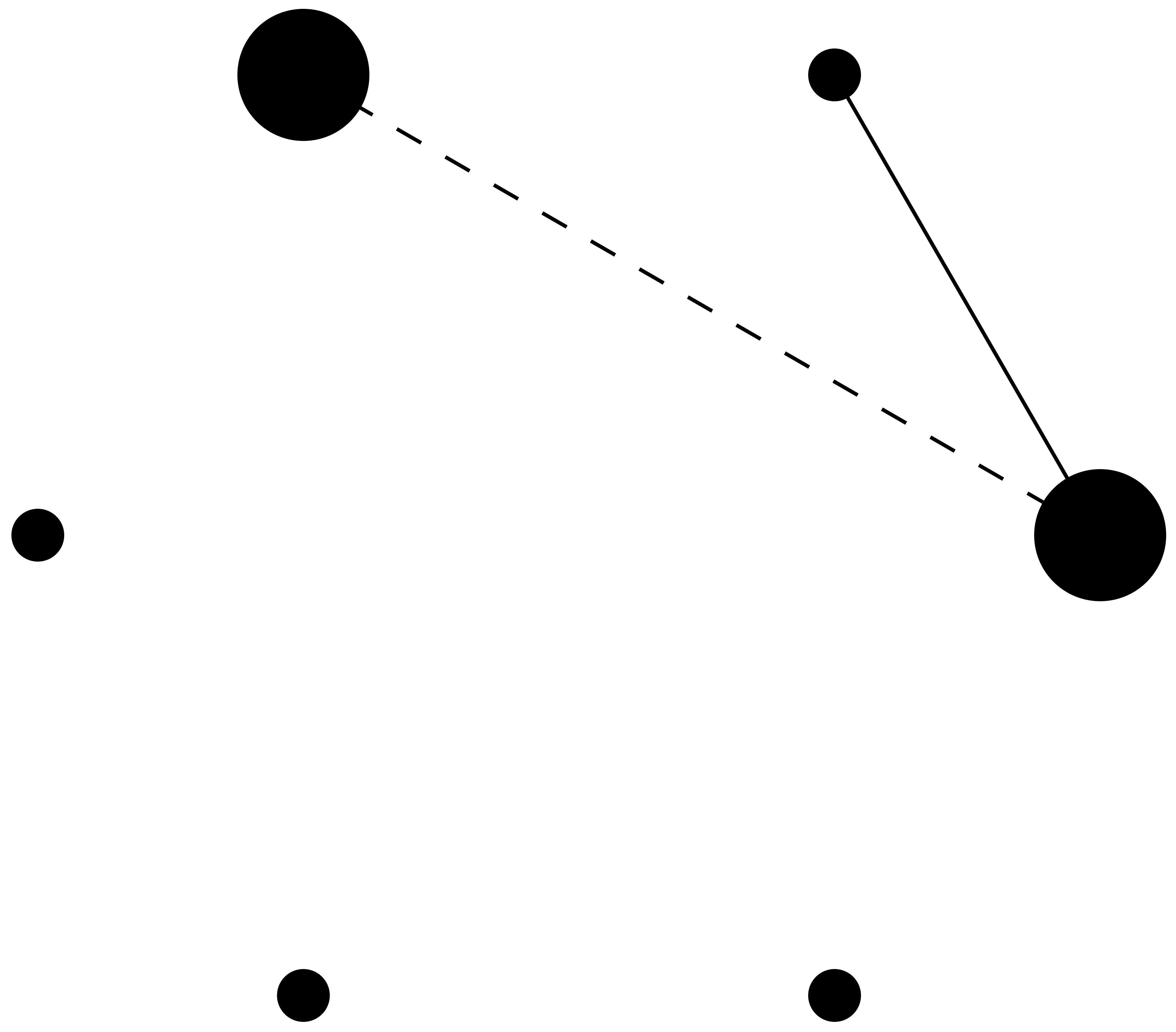
Erdos-Renyi graphs



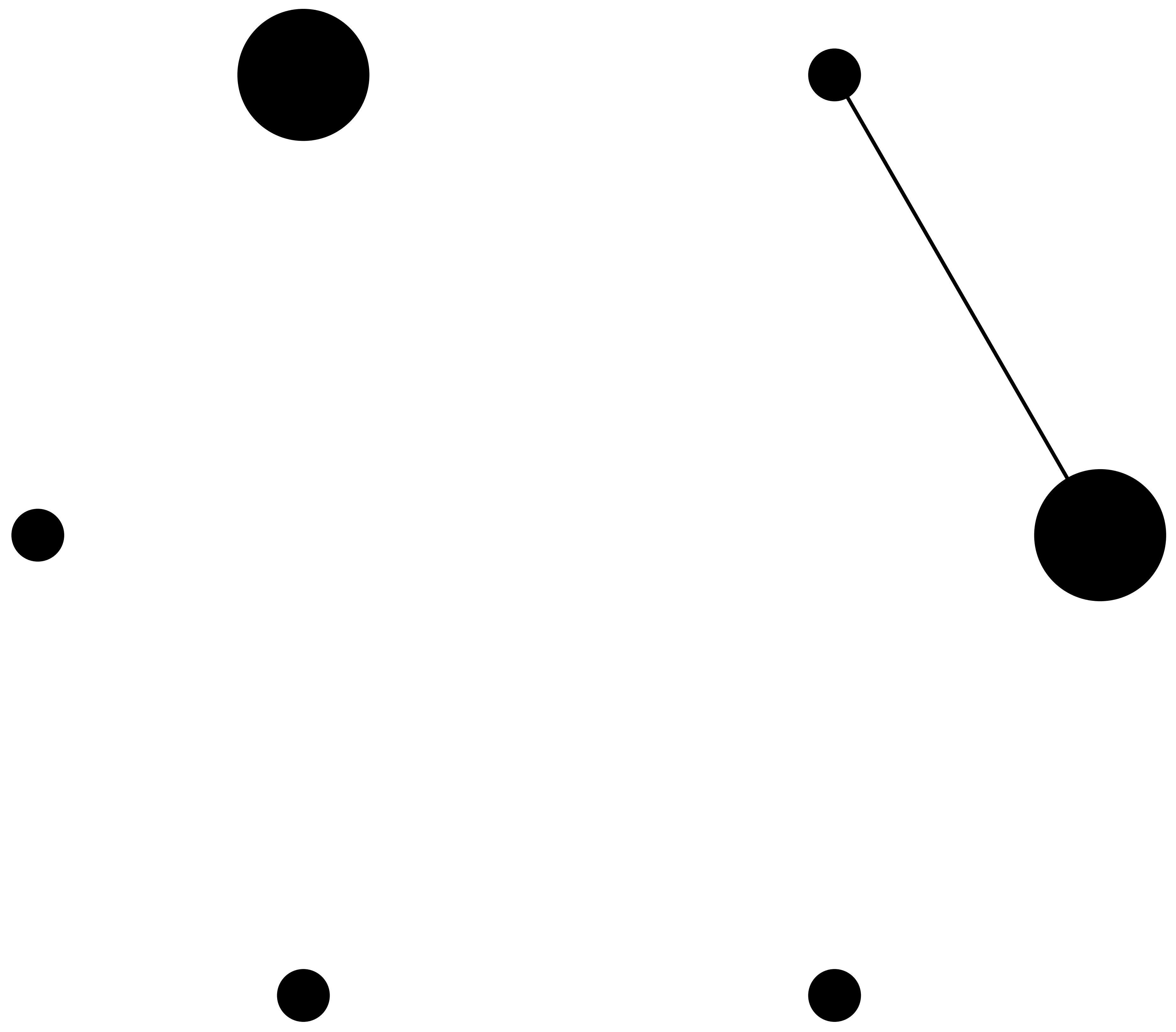
Erdos-Renyi graphs



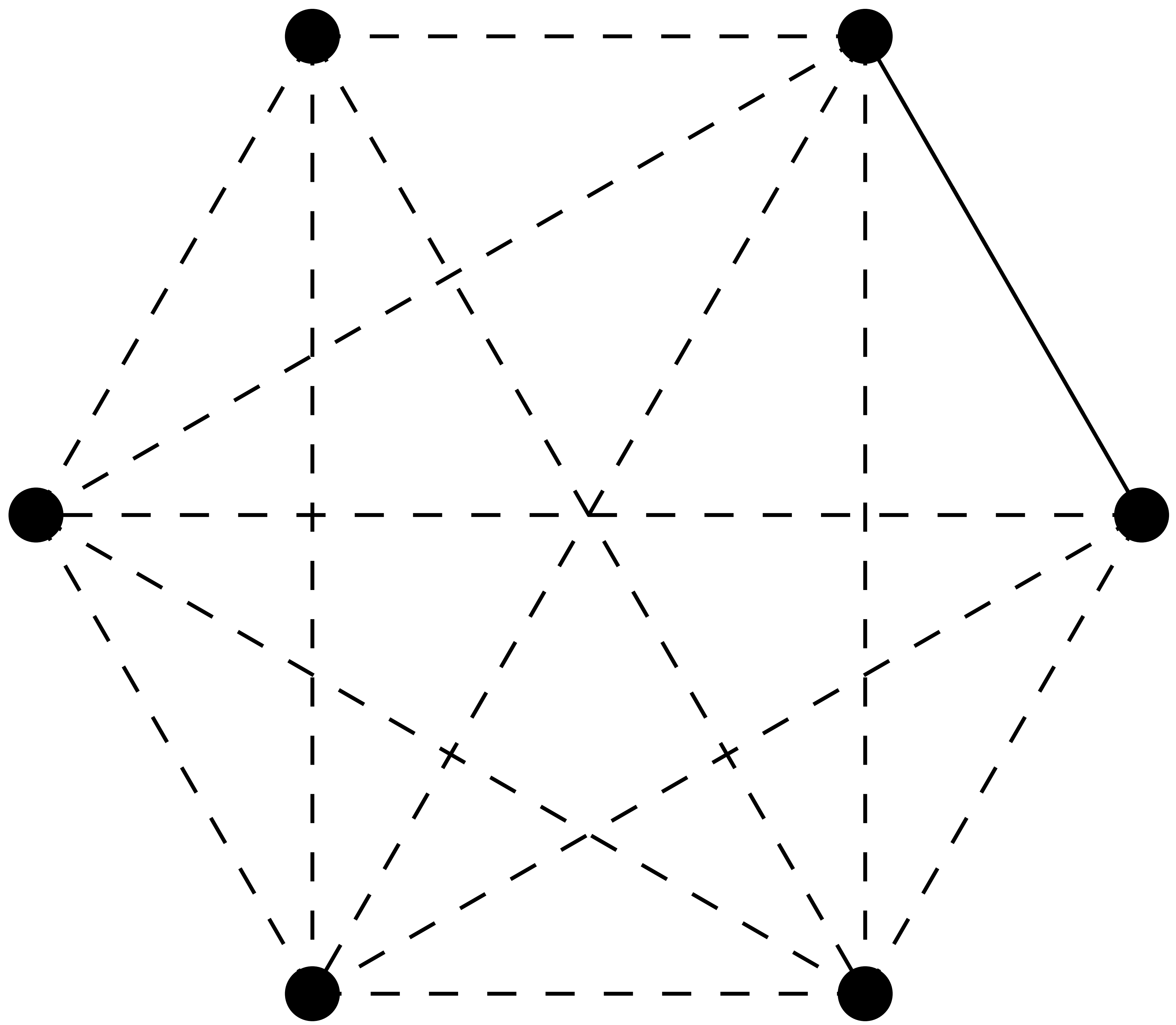
Erdos-Renyi graphs



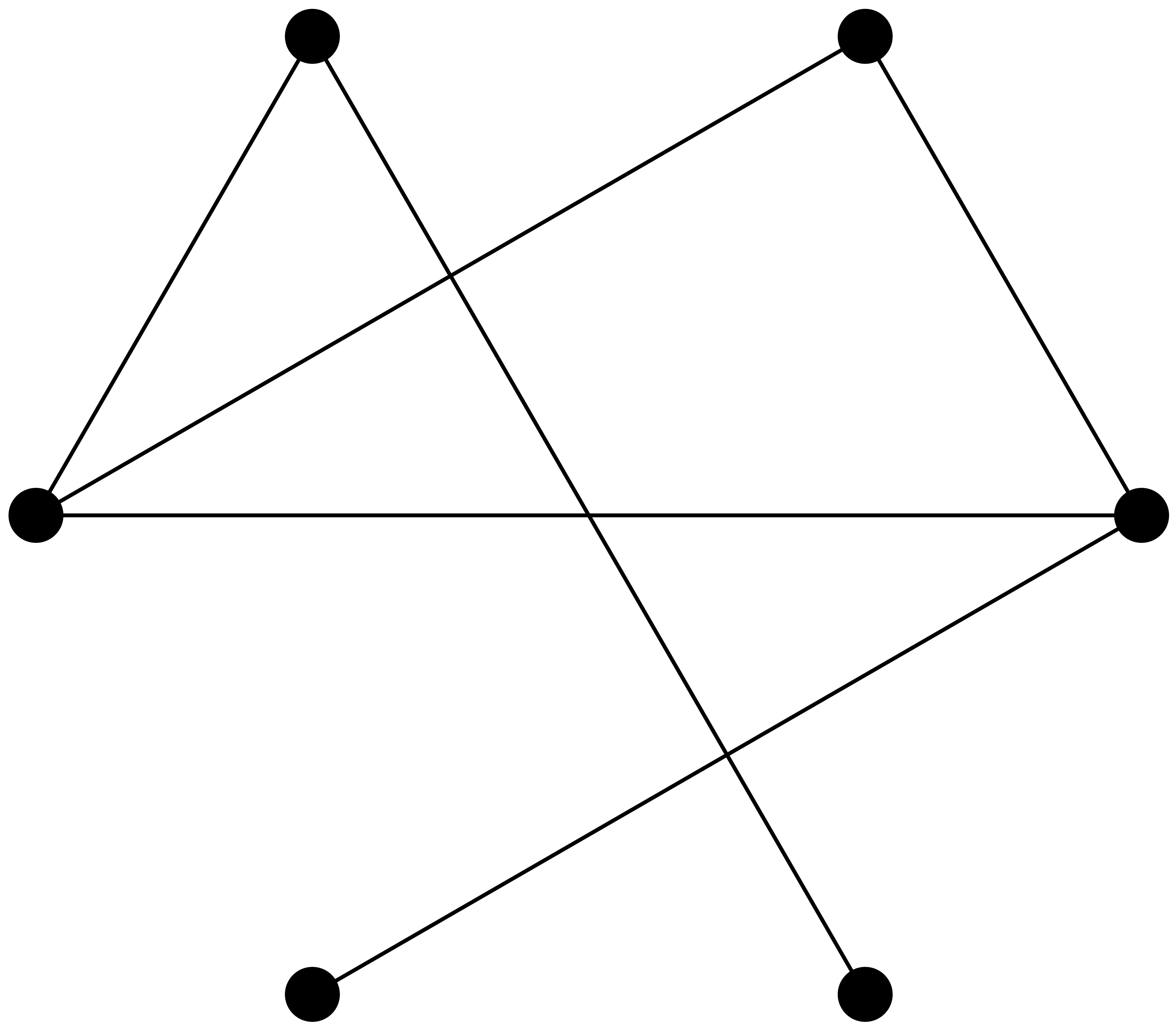
Erdos-Renyi graphs



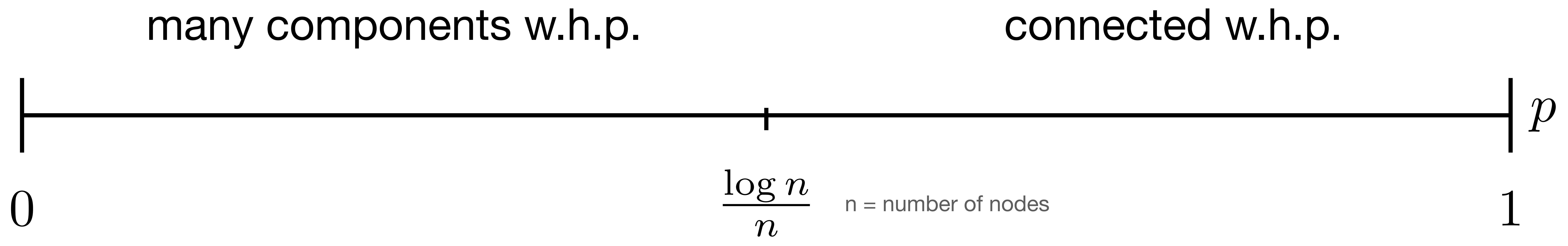
Erdos-Renyi graphs



Erdos-Renyi graphs

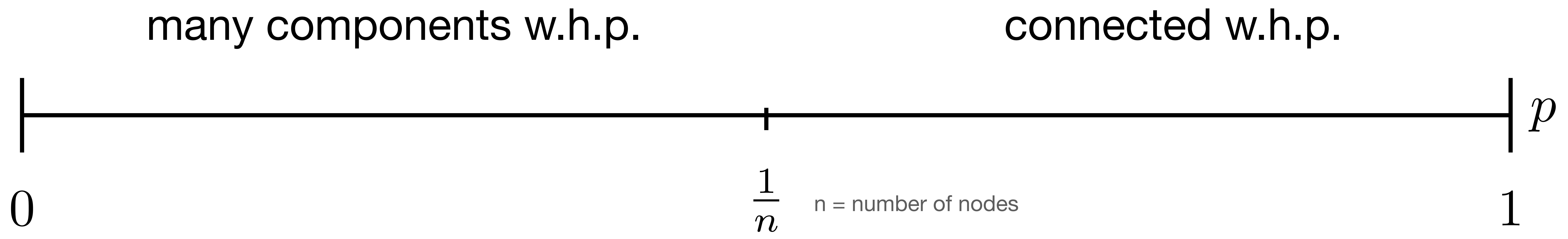


Phase Transition
[Erdos-Renyi 1960]



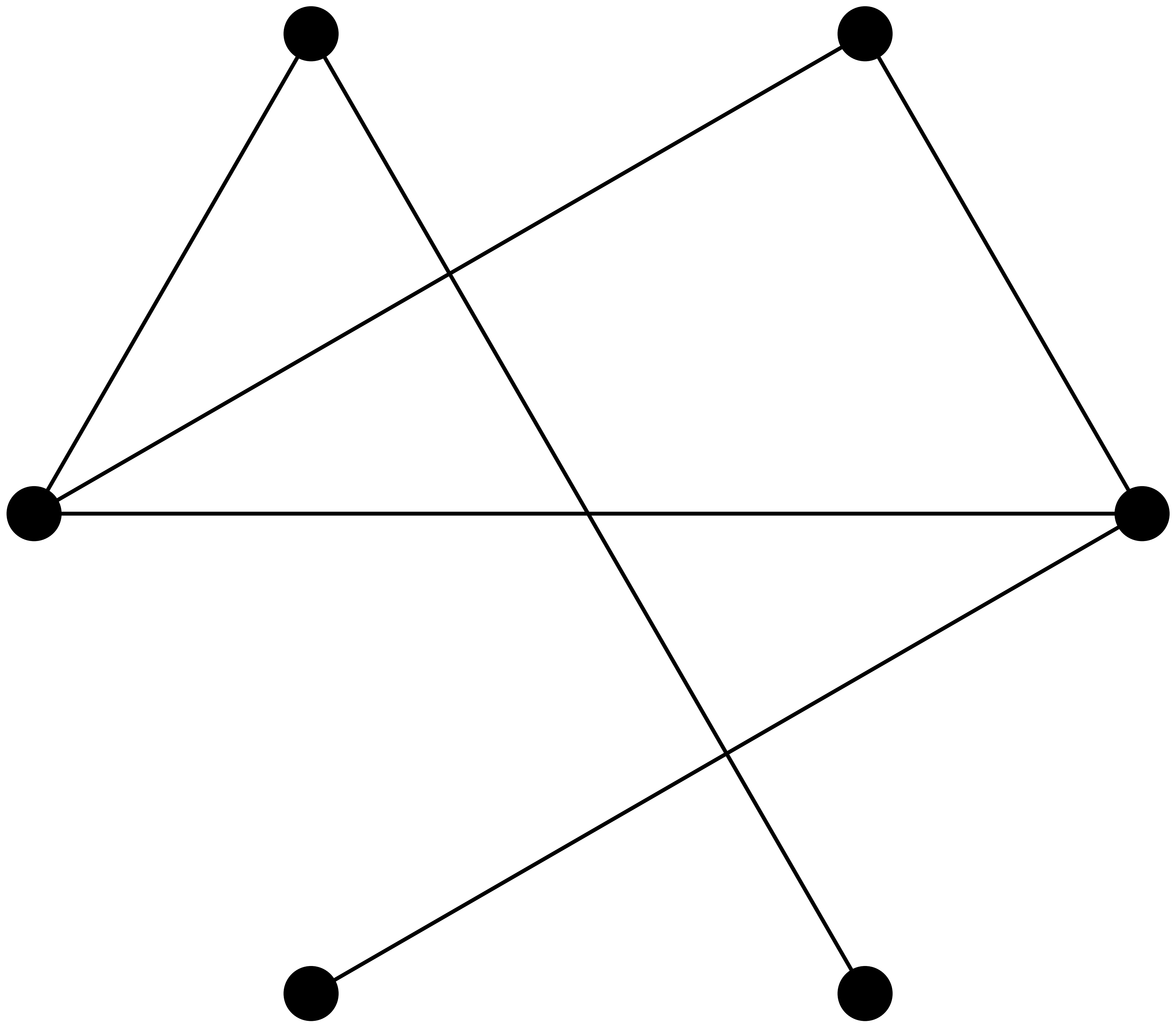
all log terms and constants forgone

Phase Transition
[Erdos-Renyi 1960]

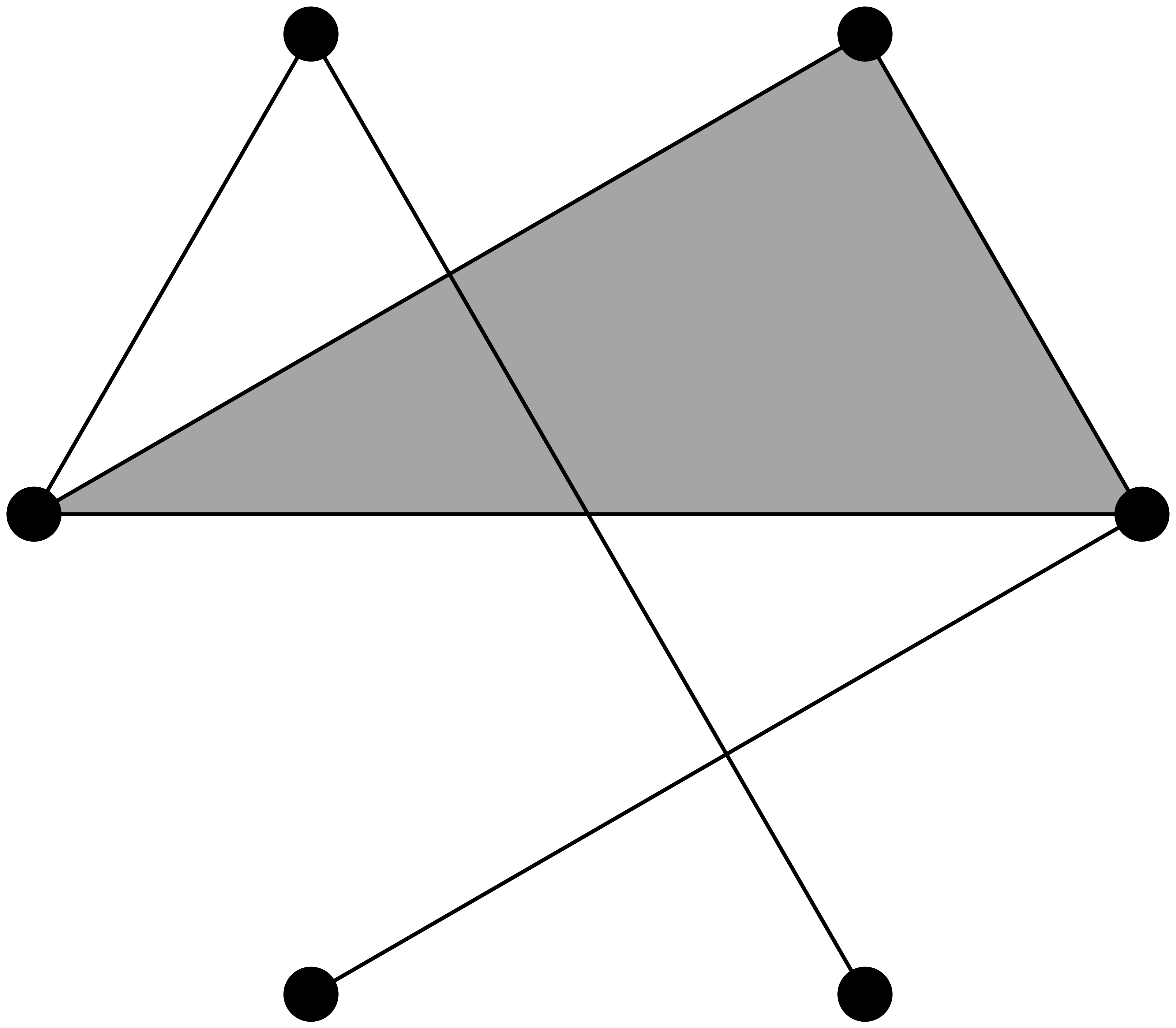


all log terms and constants forgone

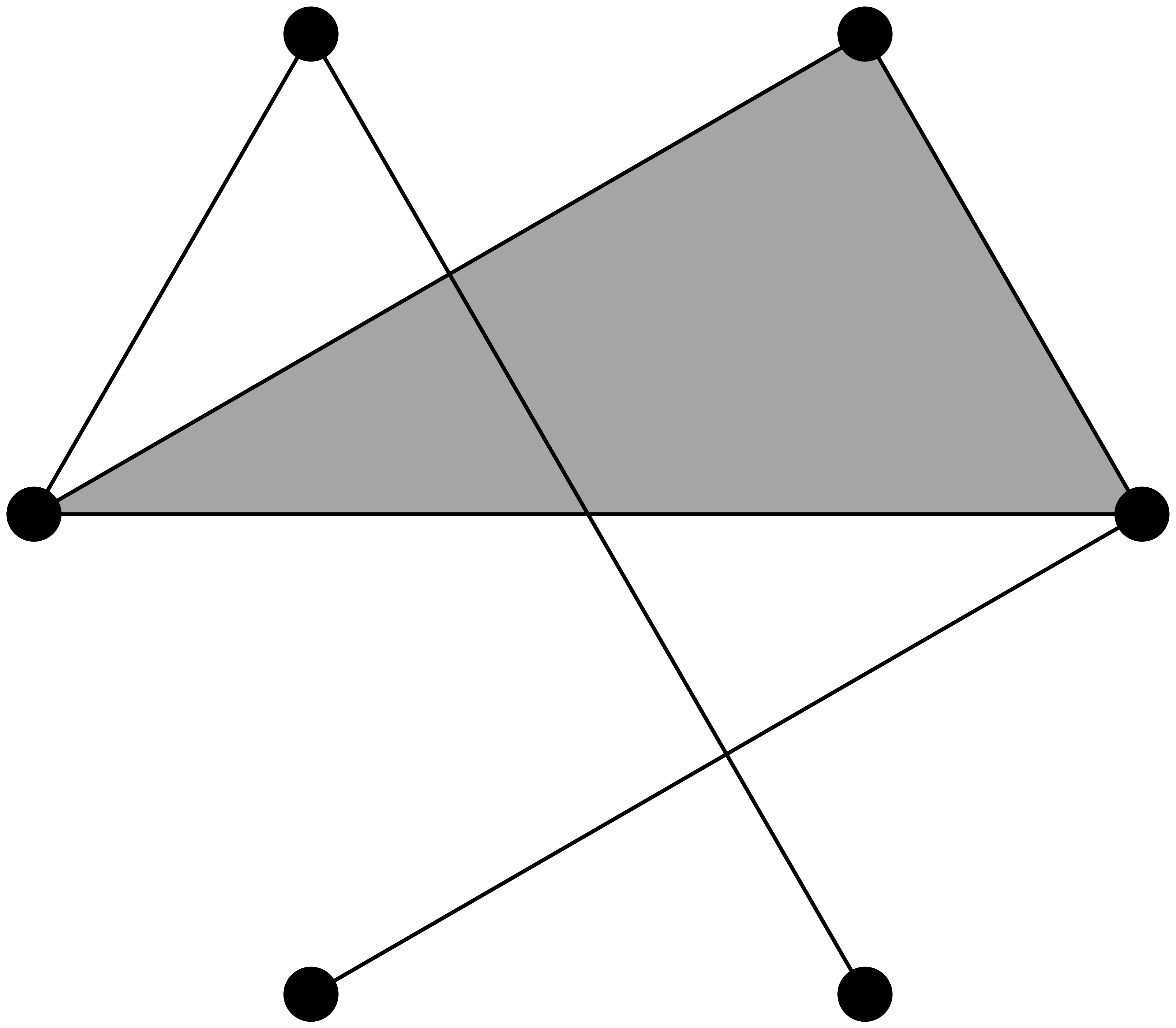
Erdos-Renyi Clique Complex

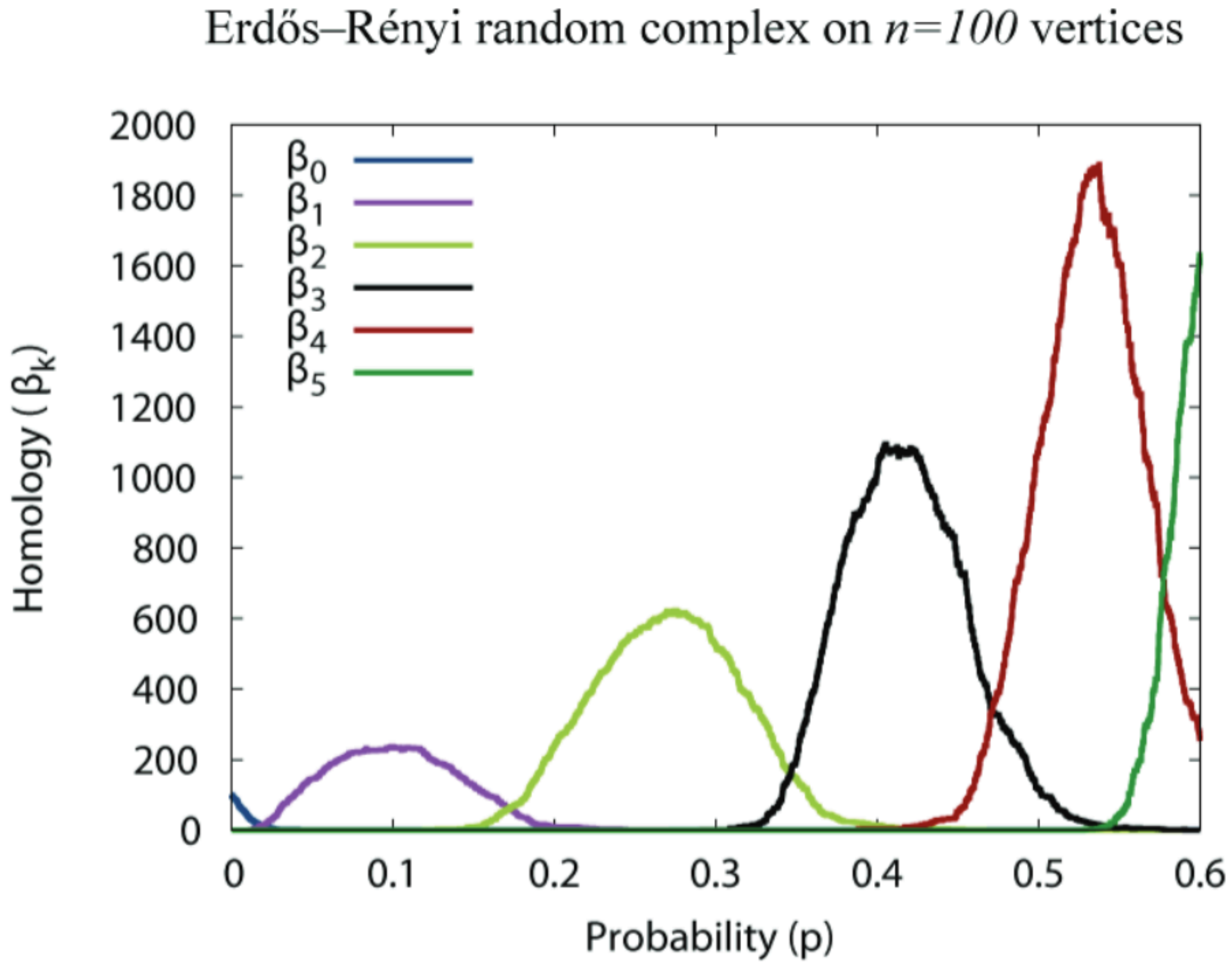


Erdos-Renyi Clique Complex



Betti Numbers

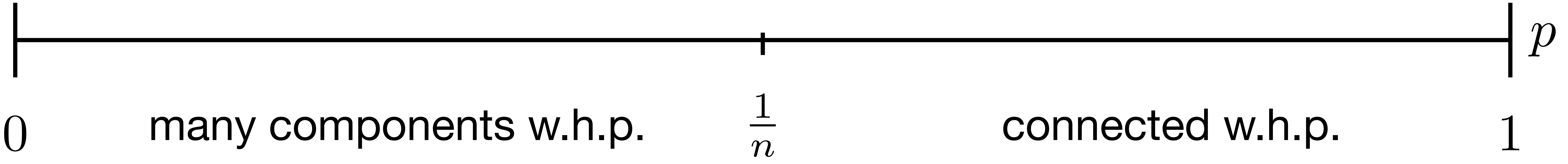




computation and plotting done by Zomorodian

Phase Transition

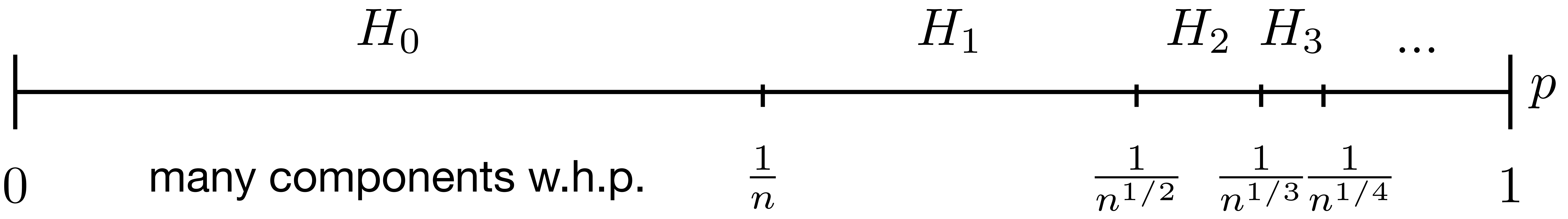
[Erdos-Renyi 1960]



n = number of nodes
all log terms and constants forgone

Phase Transition

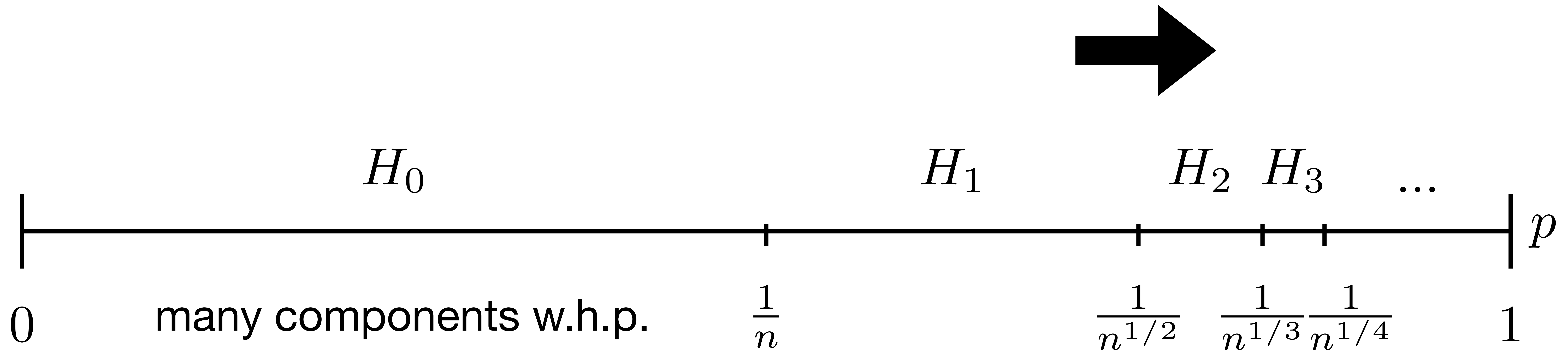
[Kahle 2009, 2014]



n = number of nodes
all log terms and constants forgone

Phase Transition
[Kahle 2009, 2014]

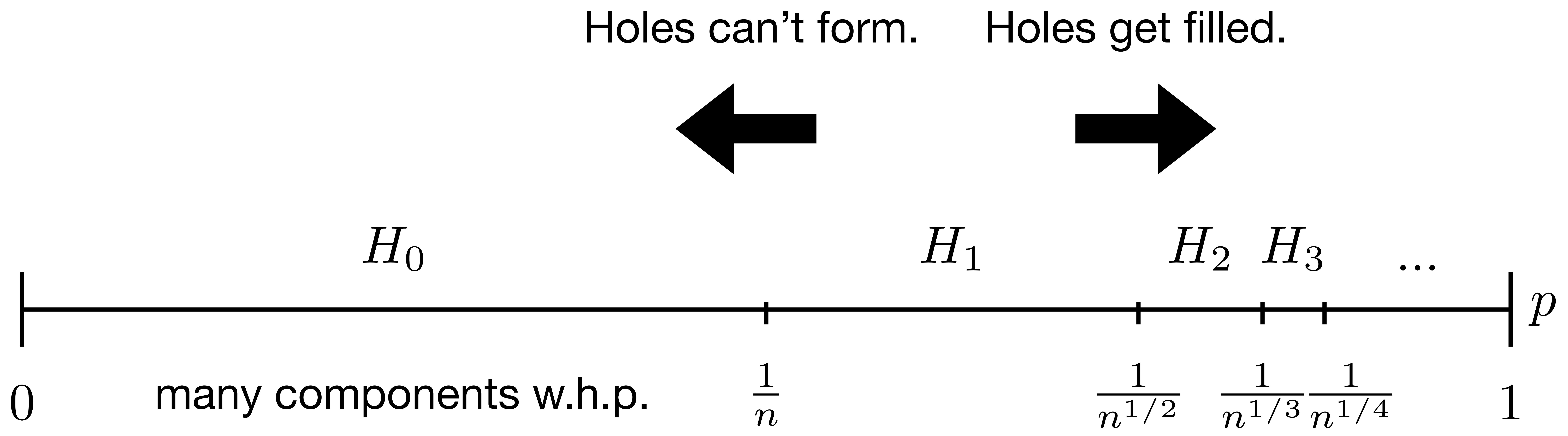
Holes get filled.



n = number of nodes
all log terms and constants forgone

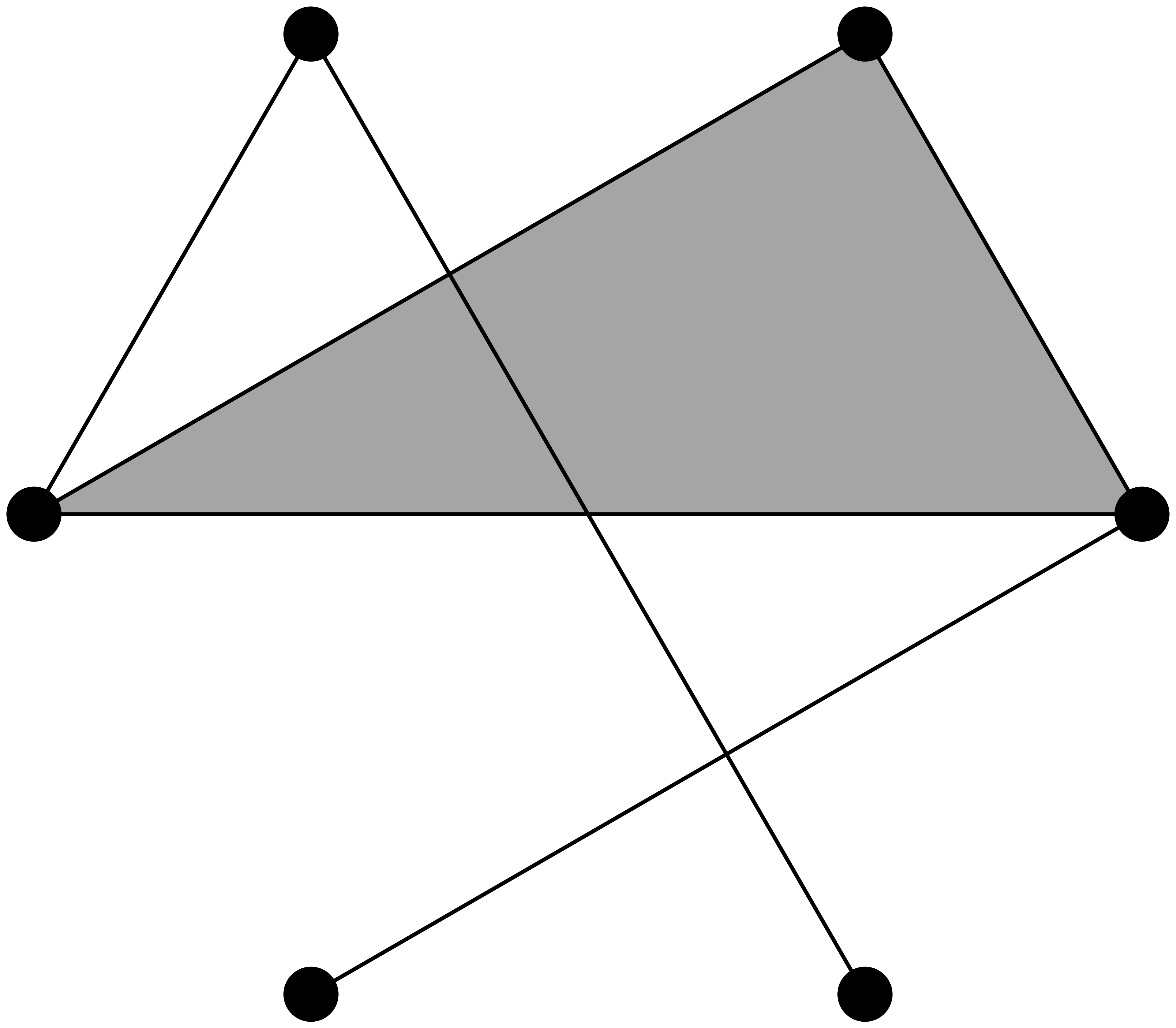
Phase Transition

[Kahle 2009, 2014]



n = number of nodes
all log terms and constants forgone

Erdos-Renyi Clique Complex



Geometric Complexes

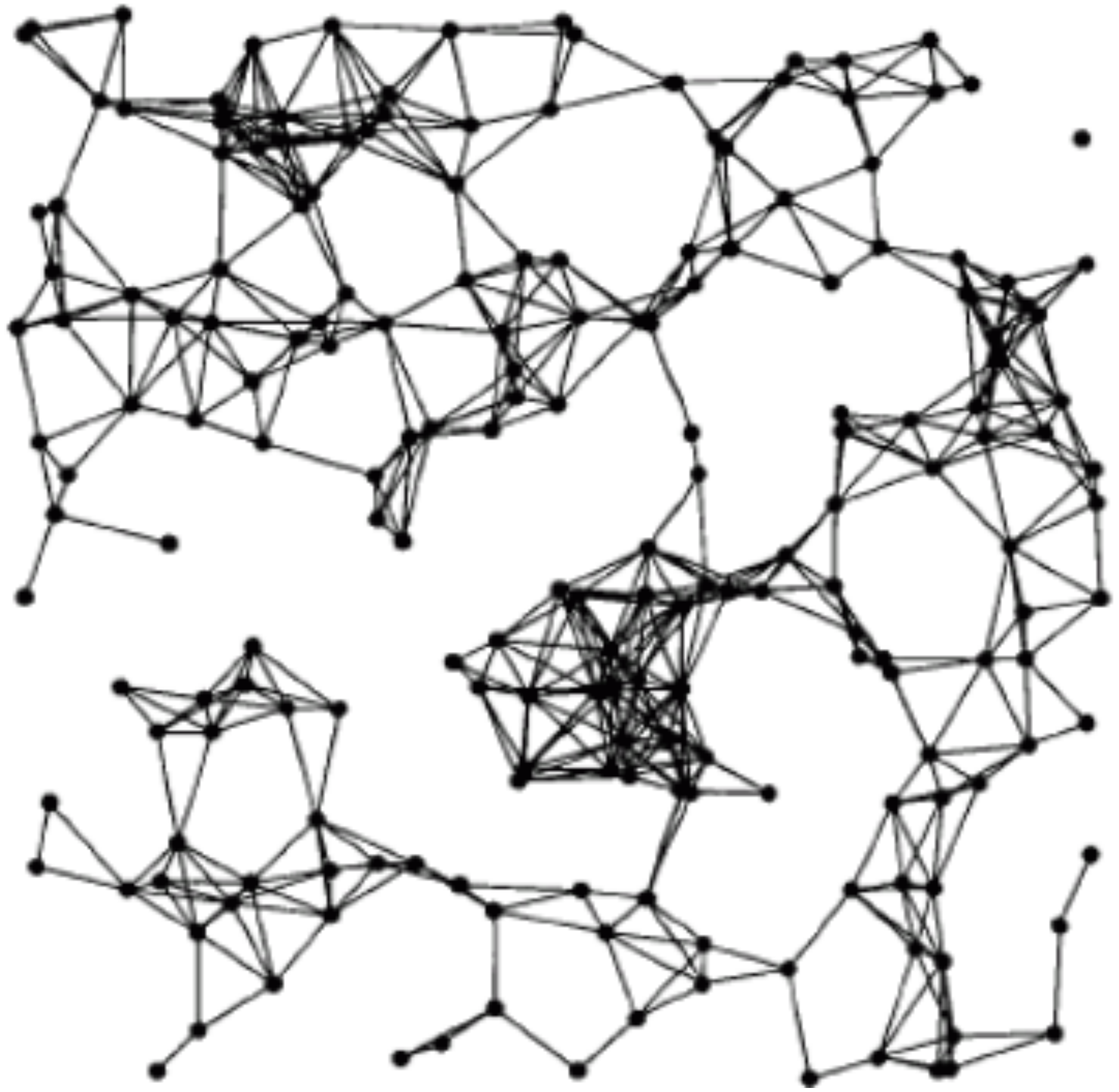


image credit: Penrose

Geometric Complexes

- Rips
- Cech

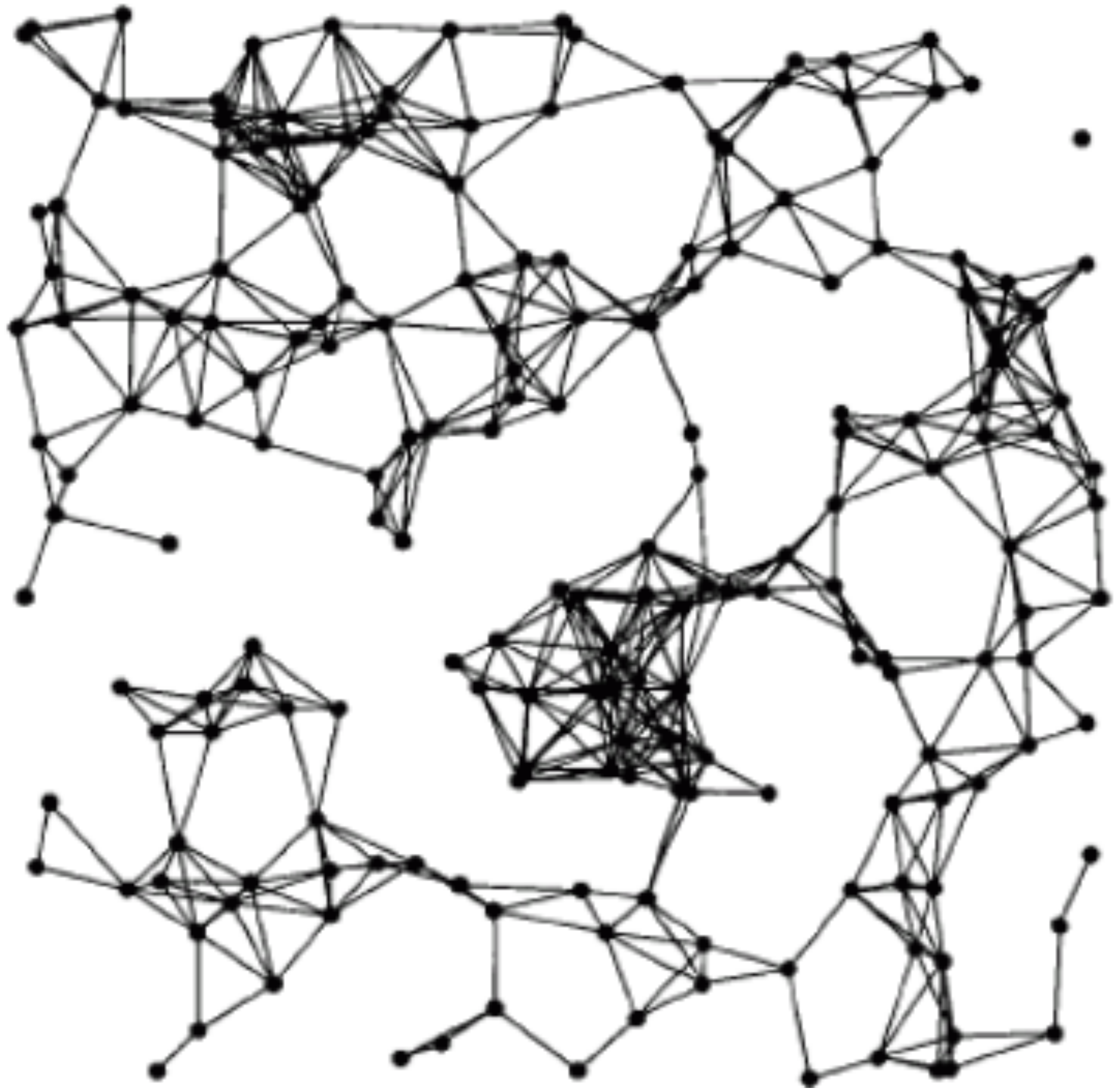


image credit: Penrose

Expected Betti numbers at dimension k

[Kahle 2011]

Expected Betti numbers at dimension k
[Kahle 2011]

- n, the number of points

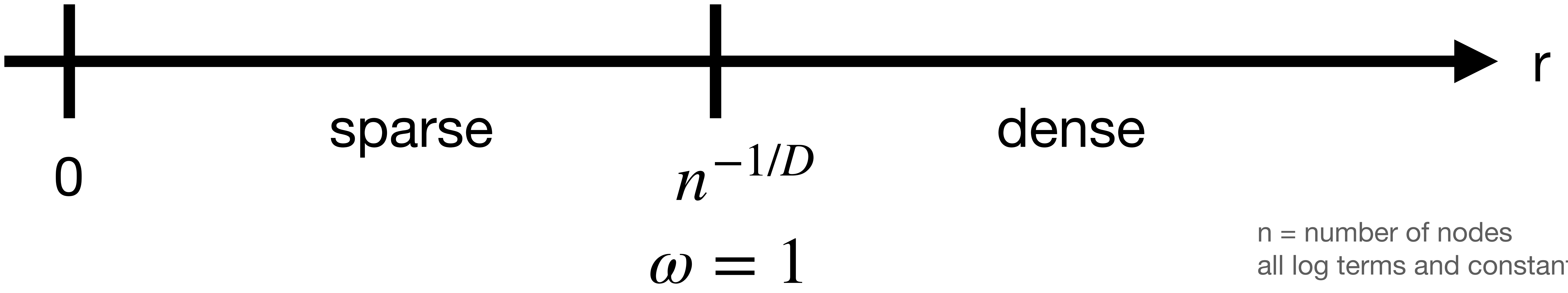
Expected Betti numbers at dimension k

[Kahle 2011]

- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension

Expected Betti numbers at dimension k
[Kahle 2011]

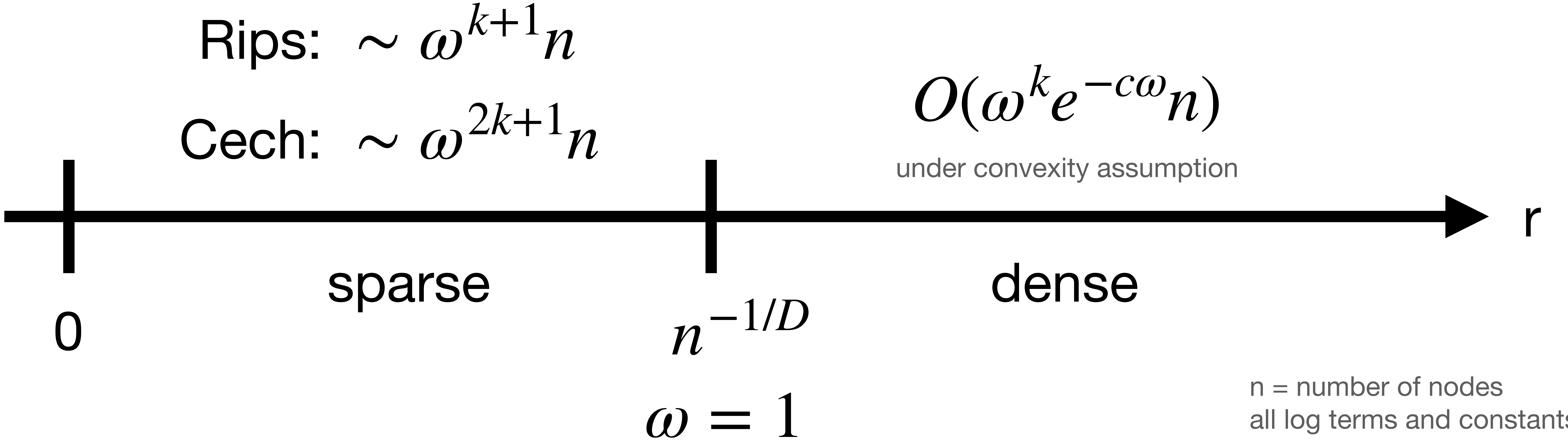
- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension



n = number of nodes
all log terms and constants forgone

Expected Betti numbers at dimension k
[Kahle 2011]

- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension

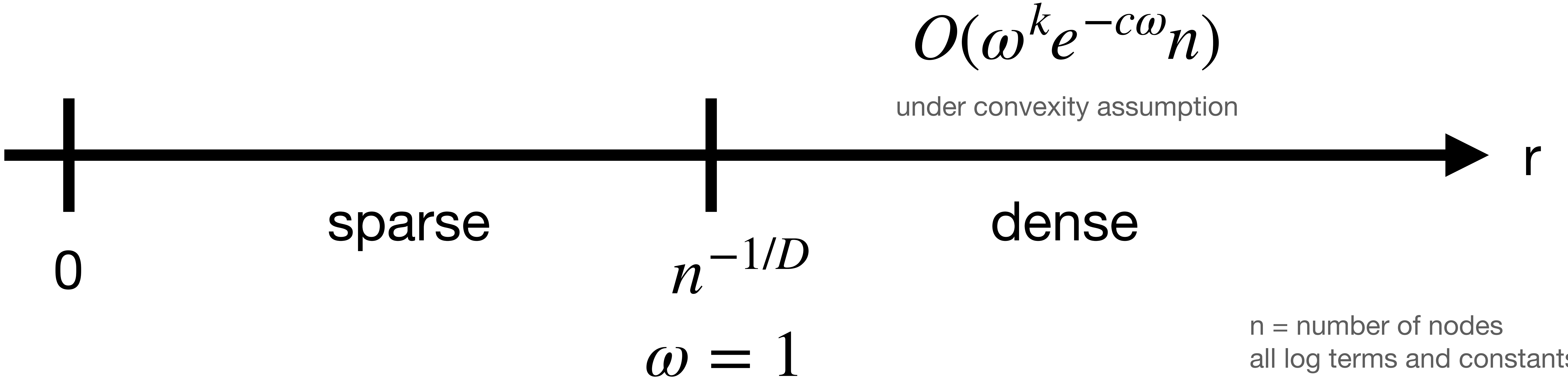


n = number of nodes
all log terms and constants forgone

Expected Betti numbers at dimension k

[Kahle 2011]

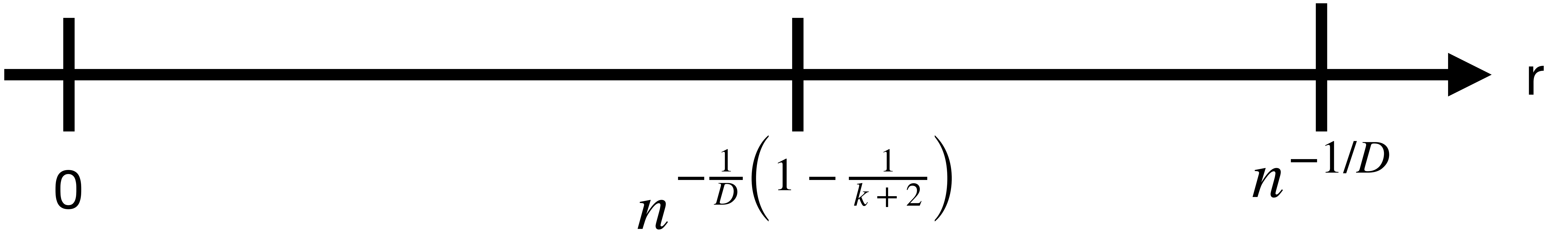
- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension
- $E\beta_k(\text{Cech}) \sim \omega^{2k+1}n$



n = number of nodes
all log terms and constants forgone

Expected Betti numbers at dimension k
[Kahle 2011]

- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension
- $E\beta_k(\text{Cech}) \sim \omega^{2k+1}n$

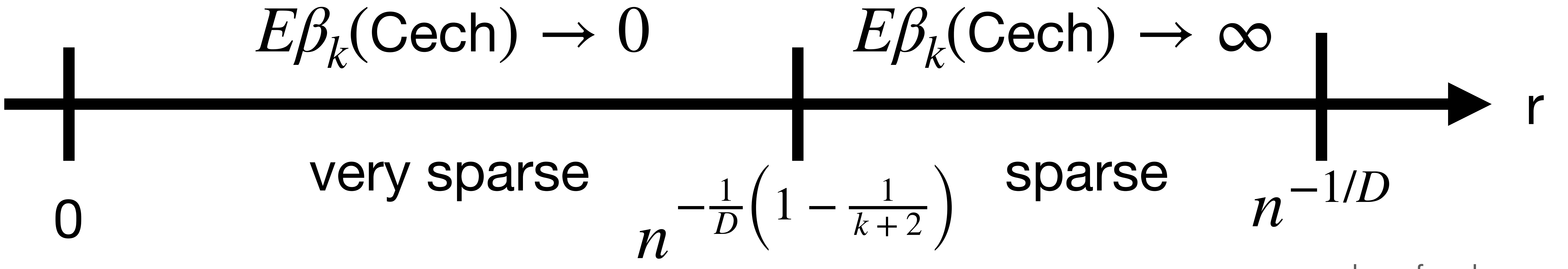


n = number of nodes
all log terms and constants forgone

Expected Betti numbers at dimension k

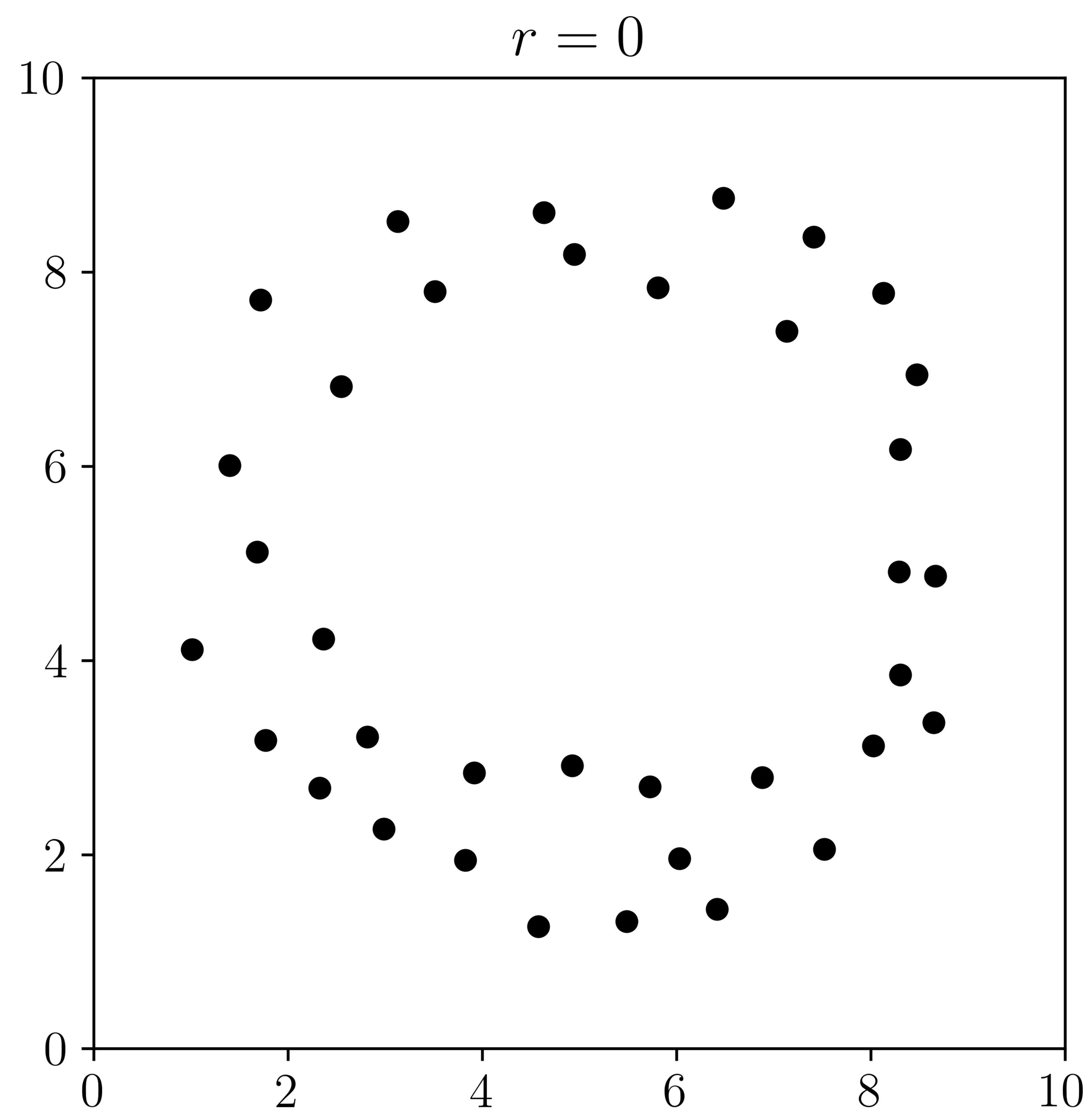
[Kahle 2011]

- n, the number of points
- $\omega = nr^D$, where D is the ambient dimension
- $E\beta_k(\text{Cech}) \sim \omega^{2k+1}n$



n = number of nodes
all log terms and constants forgone

Maximally Persistent Cycles



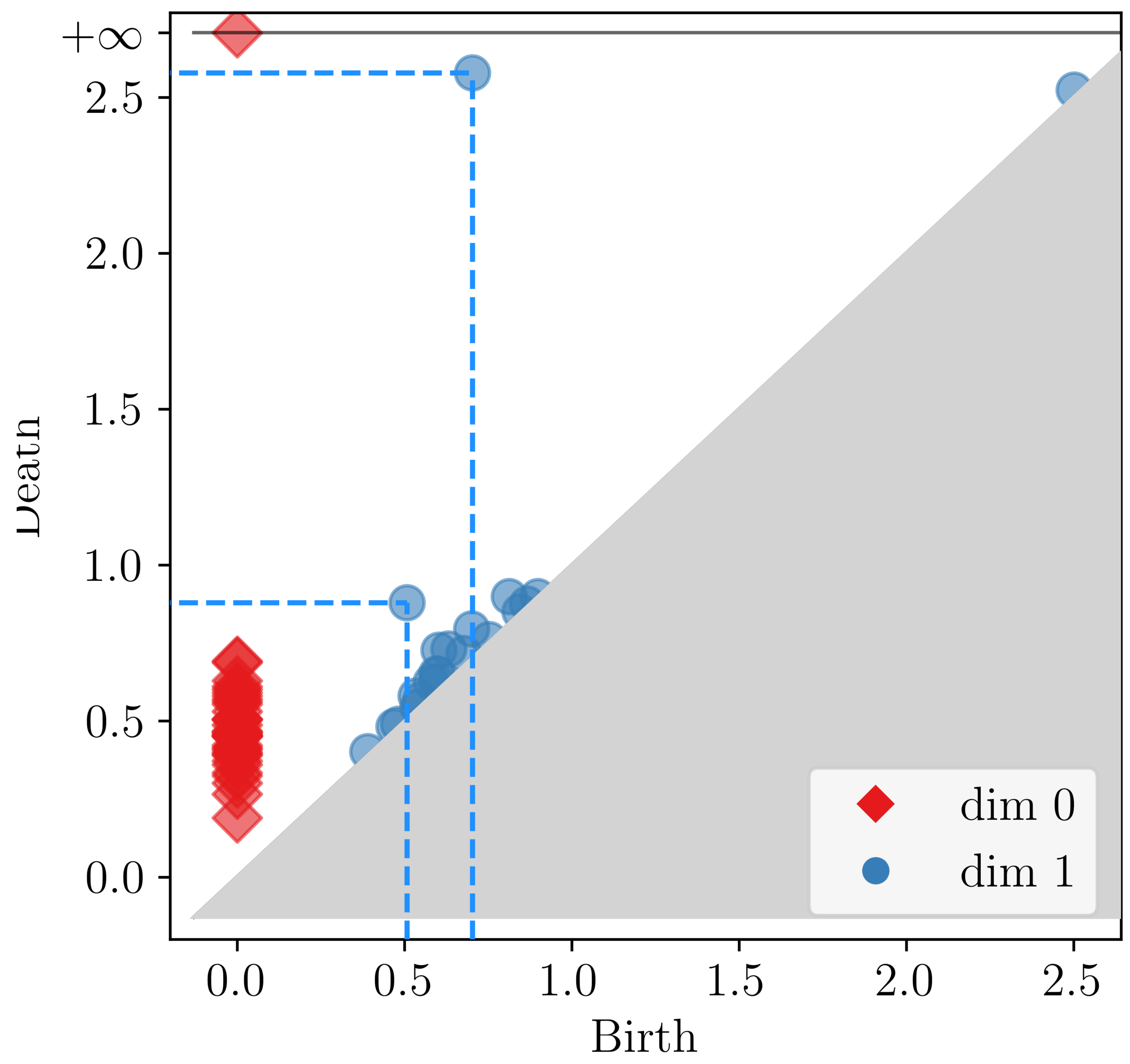


image credit: Andrey Yao

Maximally Persistent Cycles

n points in expectation

k-cycle

Maximally Persistent Cycles
[Bobrowski-Kahle-Skraba 2017]

n points in expectation

k-cycle

$$c\left(\frac{\log n}{\log\log n}\right)^{1/k} \leq \text{max persistence} \leq C\left(\frac{\log n}{\log\log n}\right)^{1/k}$$

a.a.s.

Geometric Complexes

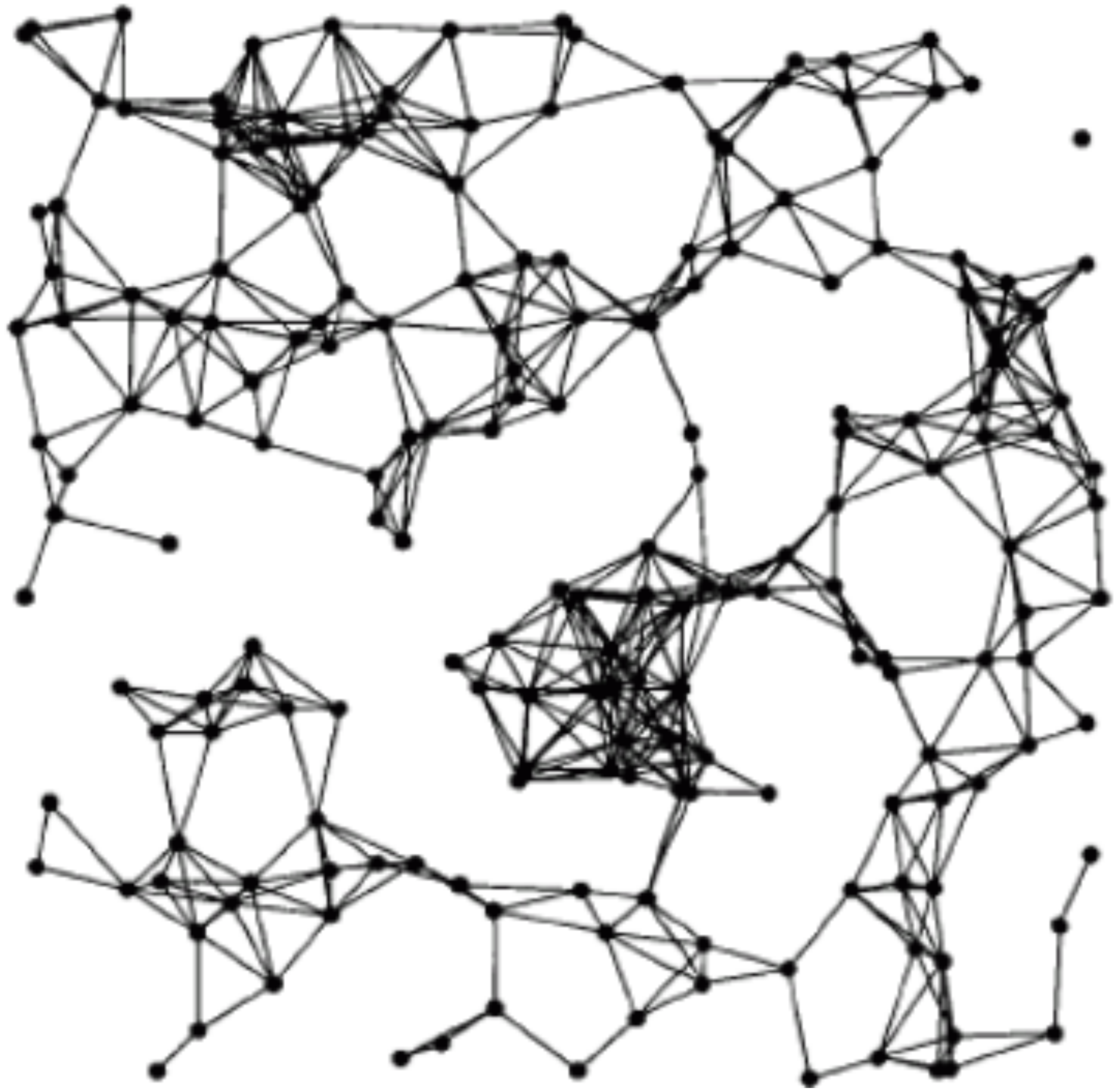
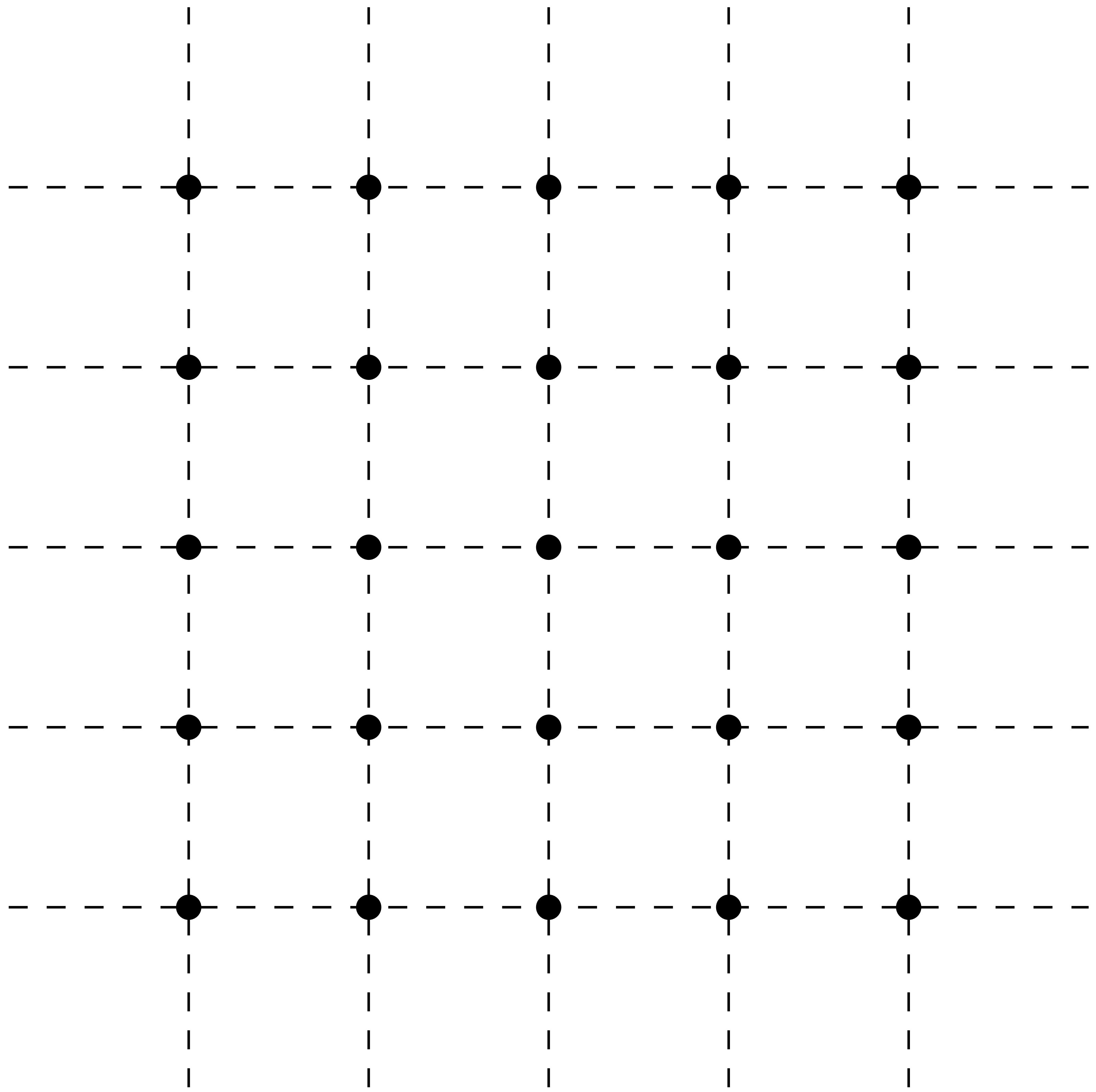
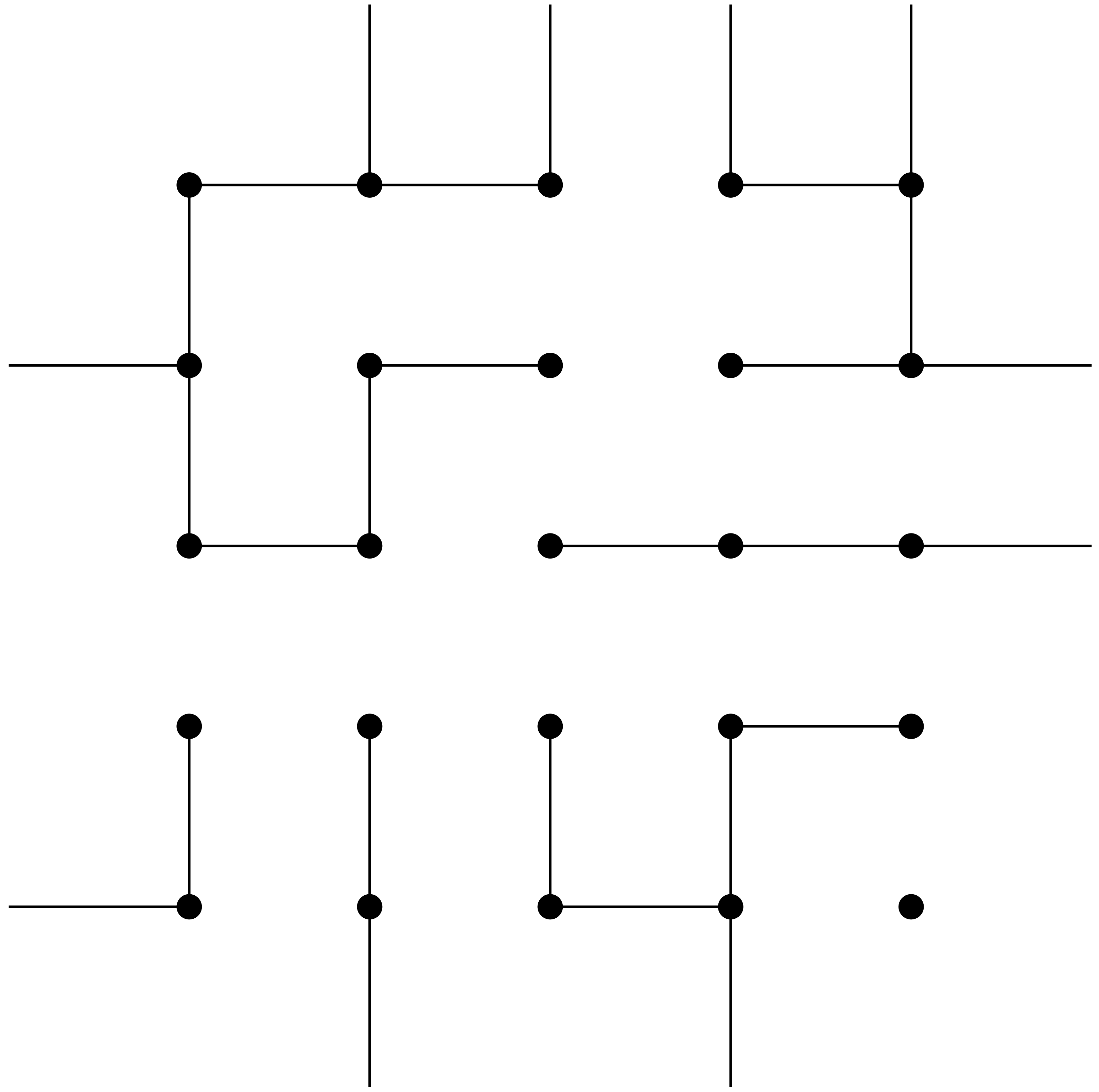


image credit: Penrose

Bernoulli Bond Percolation

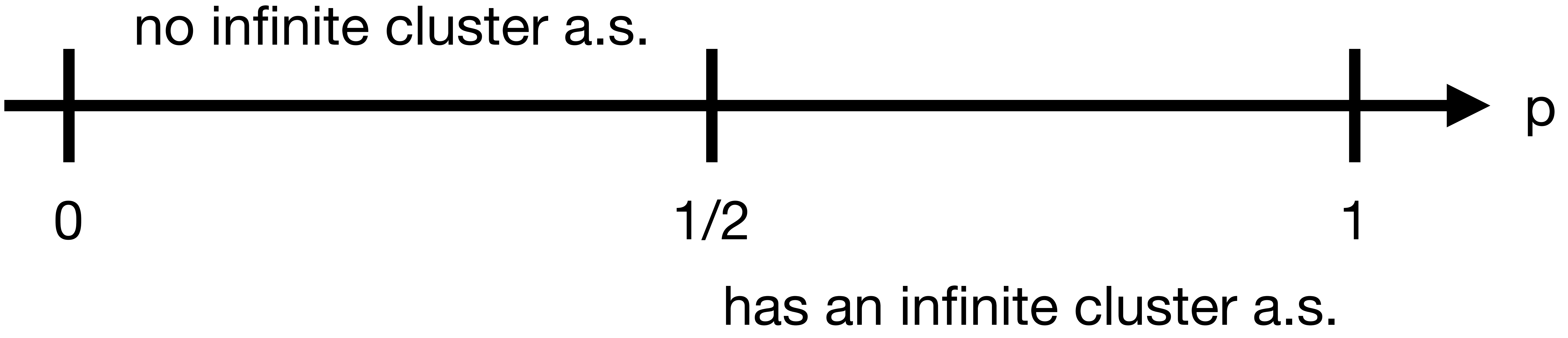


Bernoulli Bond Percolation



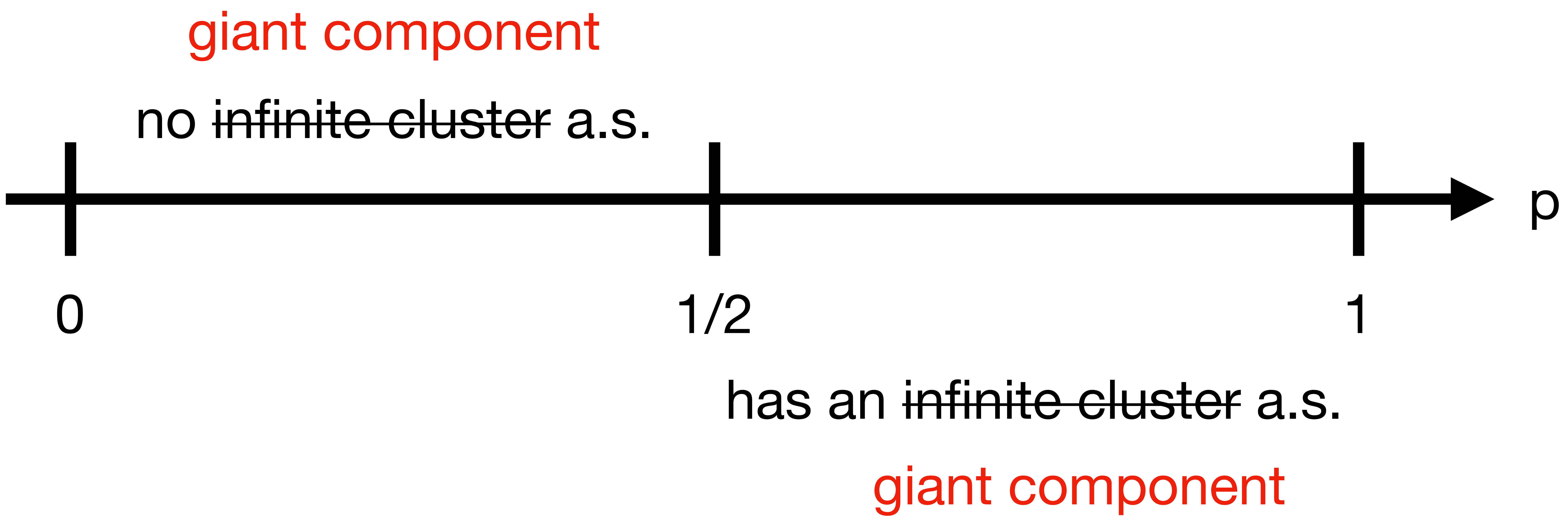
Phase Transition

[Harris 1960, Kesten 1980]



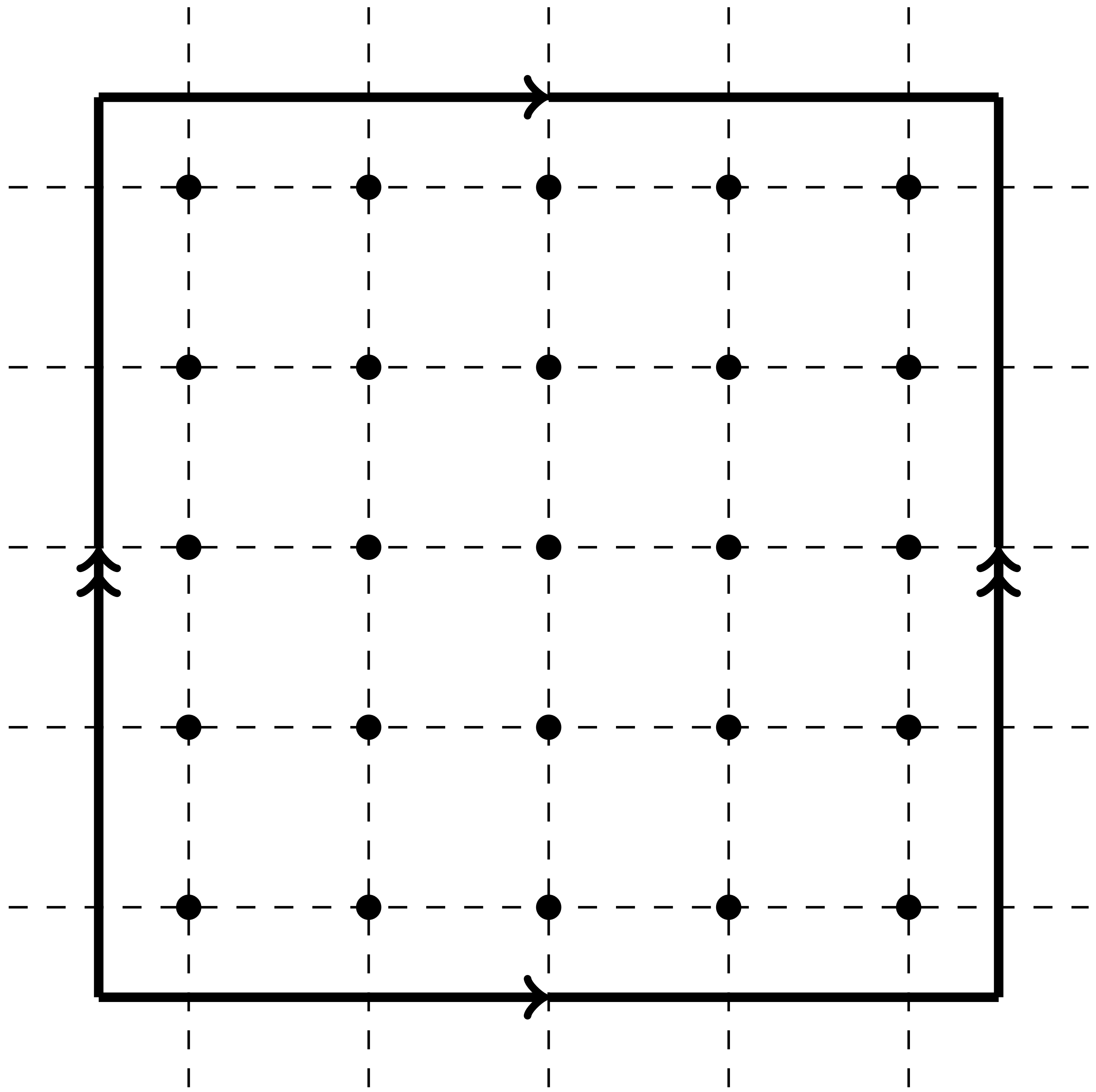
Phase Transition

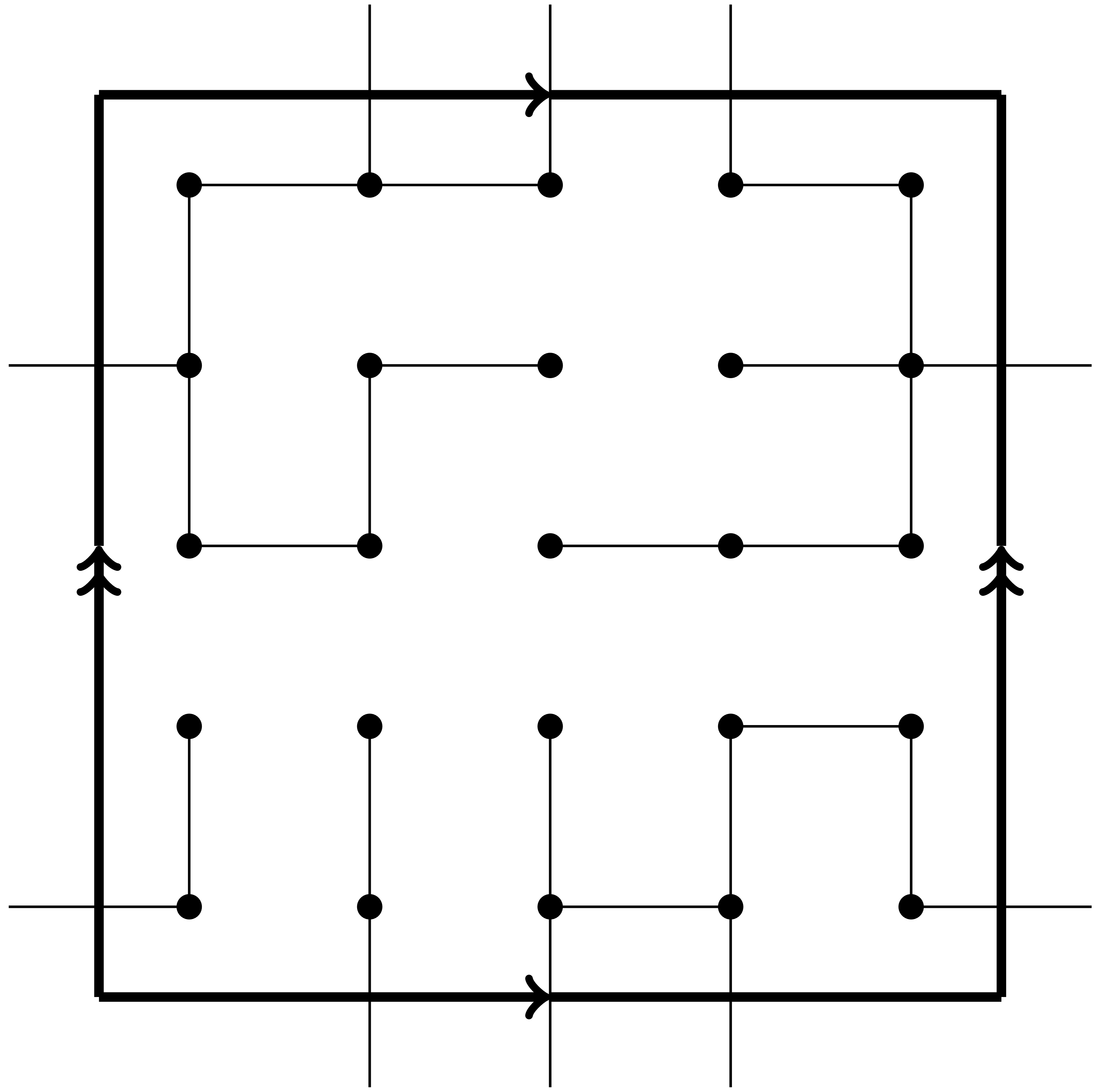
[Harris 1960, Kesten 1980]



Giant Cycles?

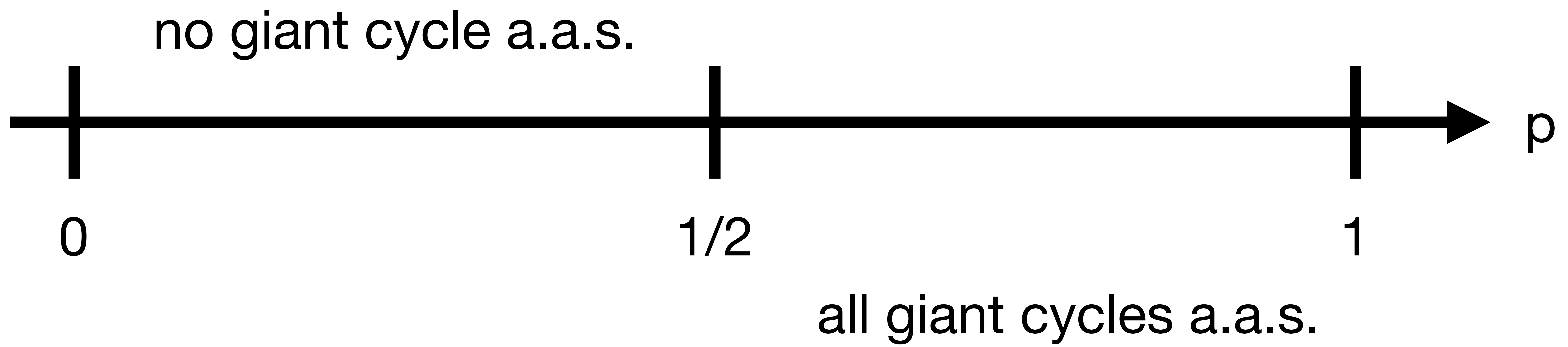
Bernoulli Bond Percolation



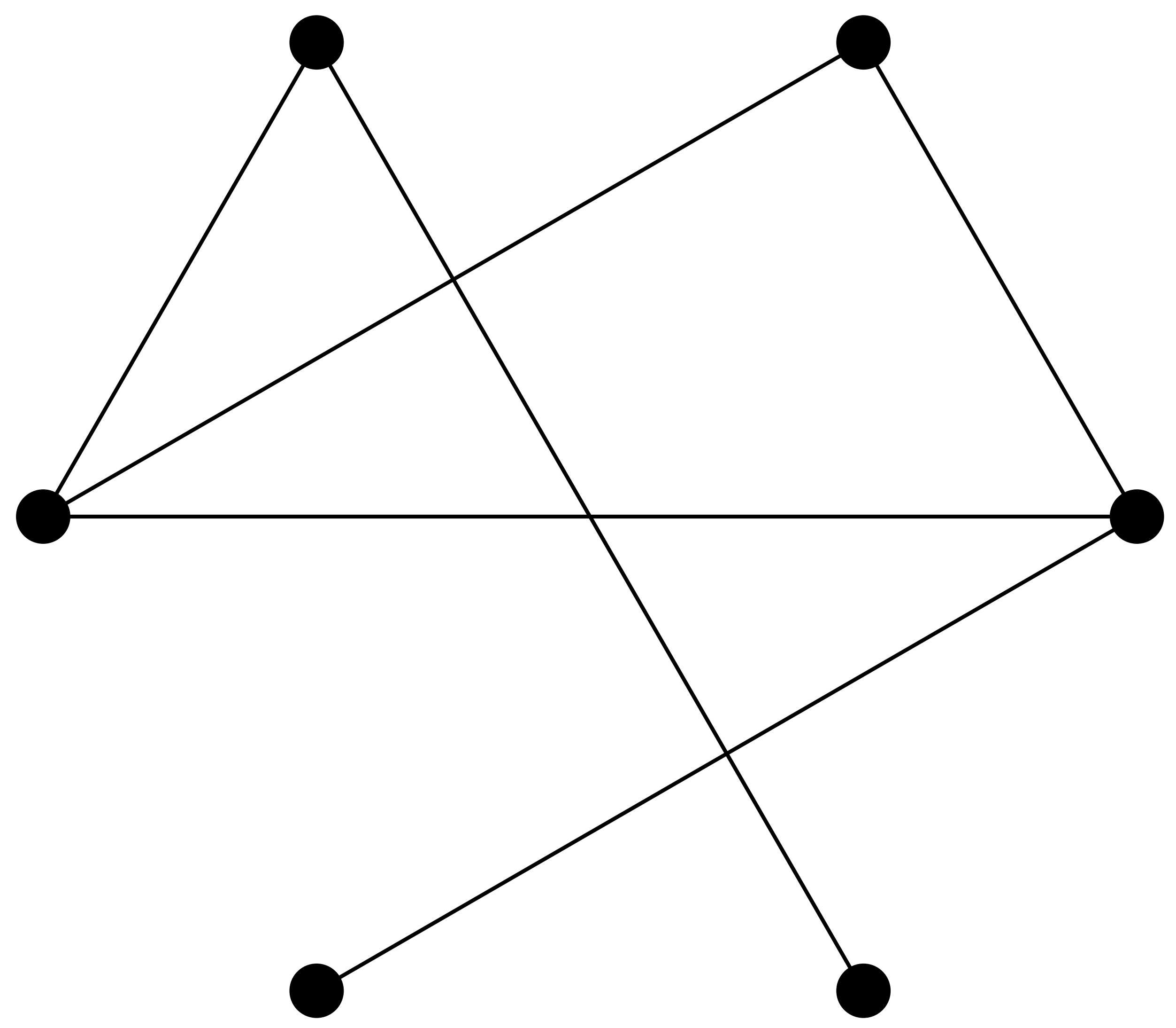


Phase Transition

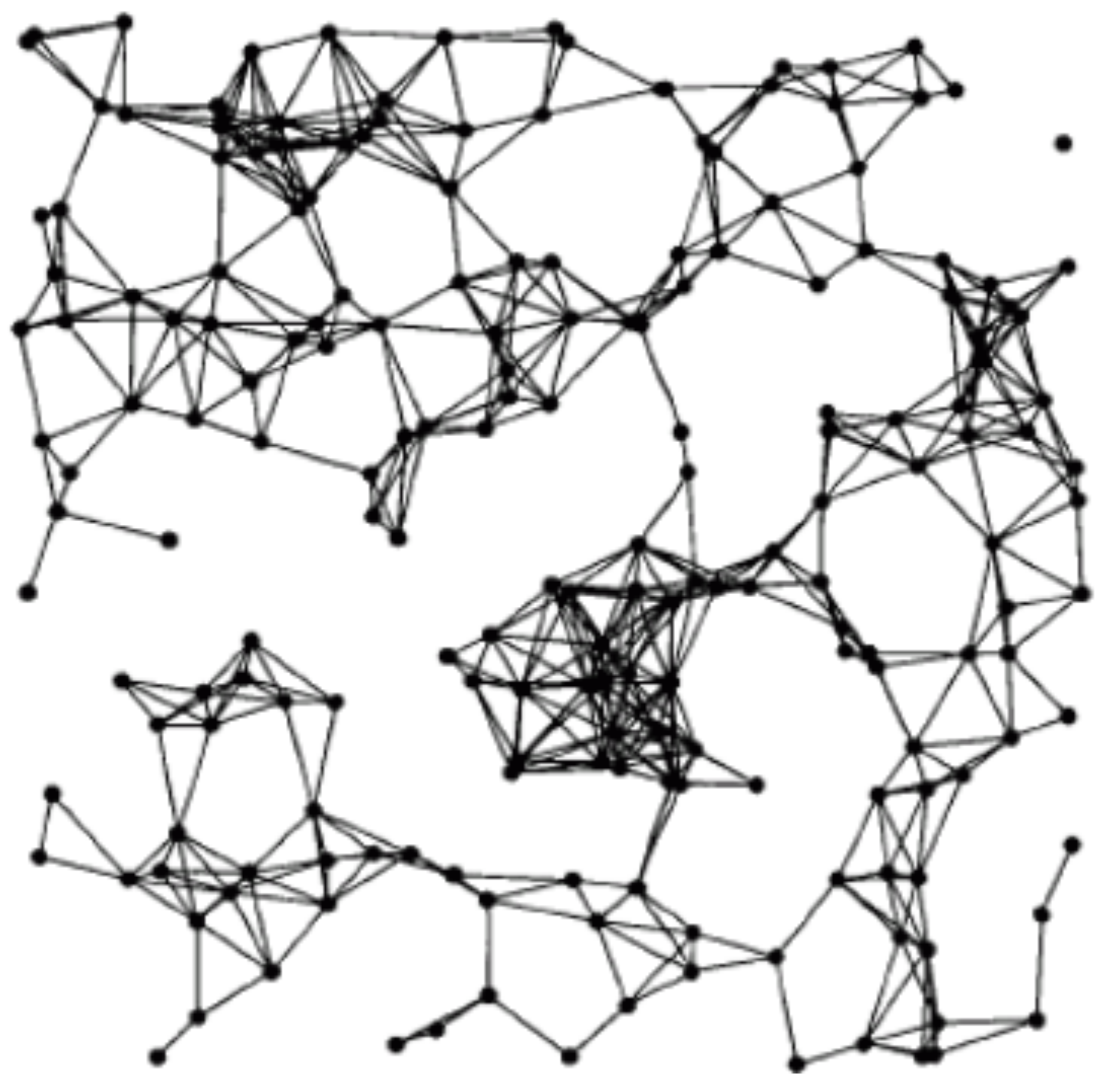
[Duncan-Kahle-Schweinhart, 2021]



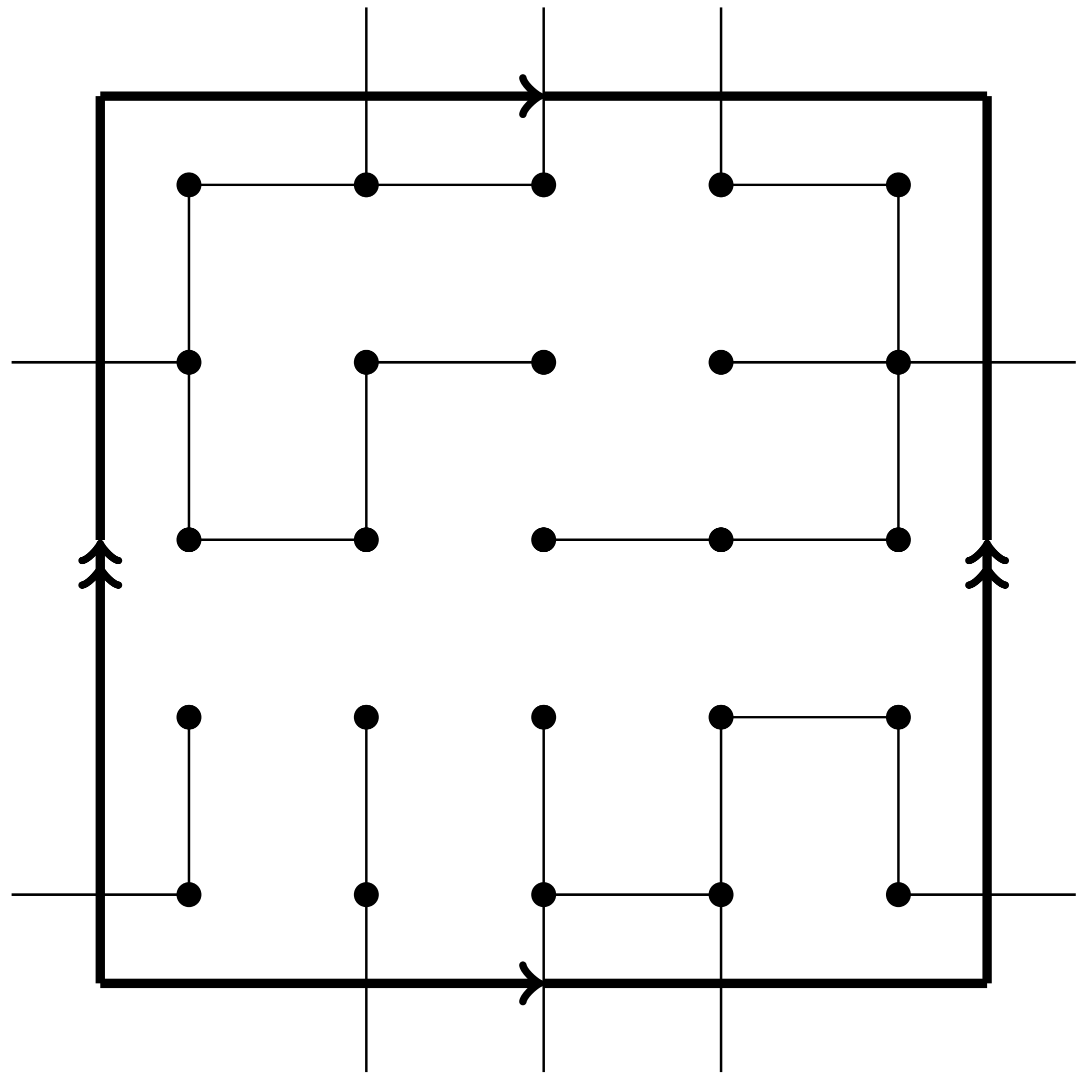
Tapas at Random Topology



Erdo-Renyi Complexes



Geometric Complexes



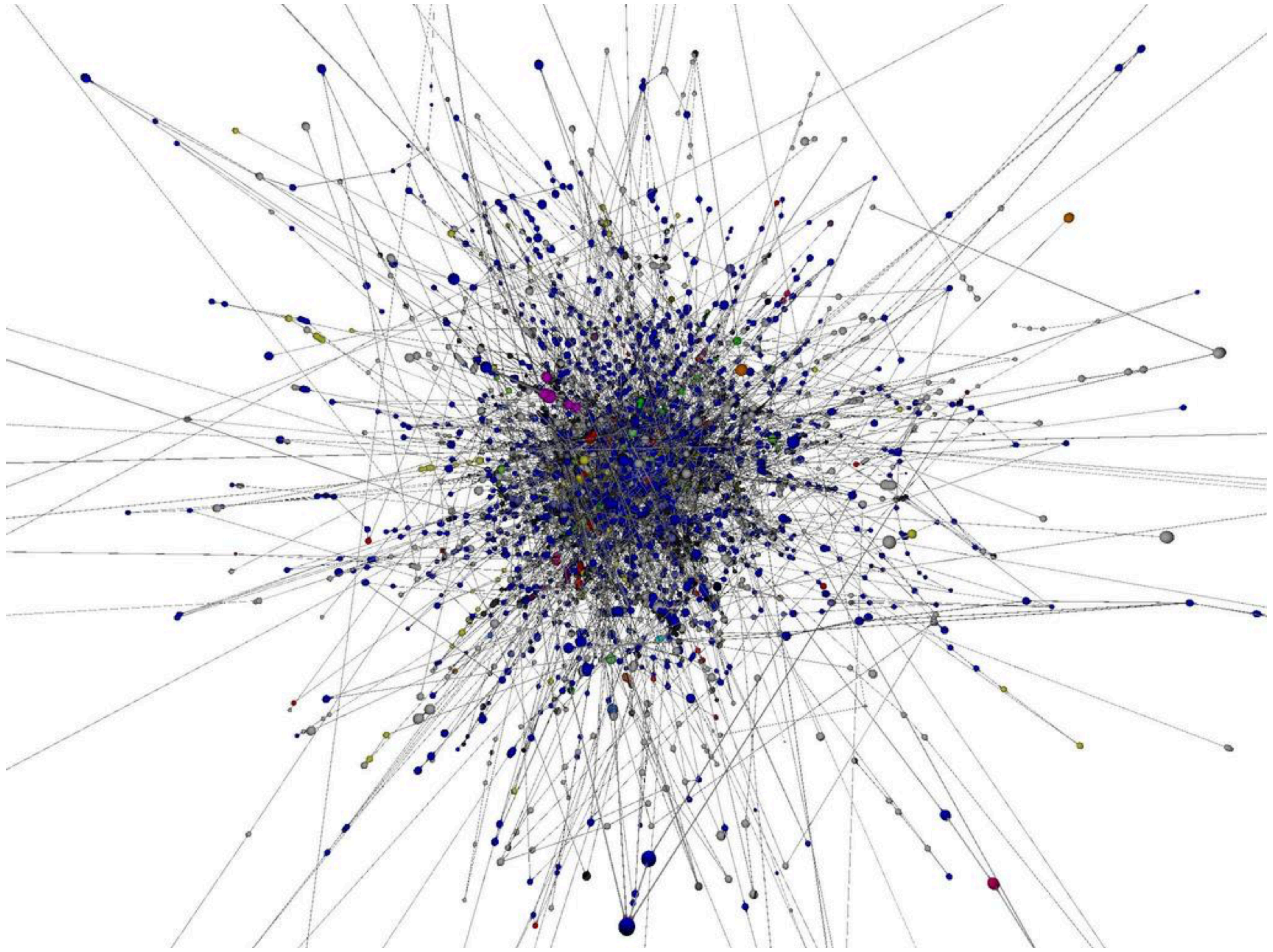
Topological Percolation

II. Preferential Attachment

Beyond independence and homogeneity

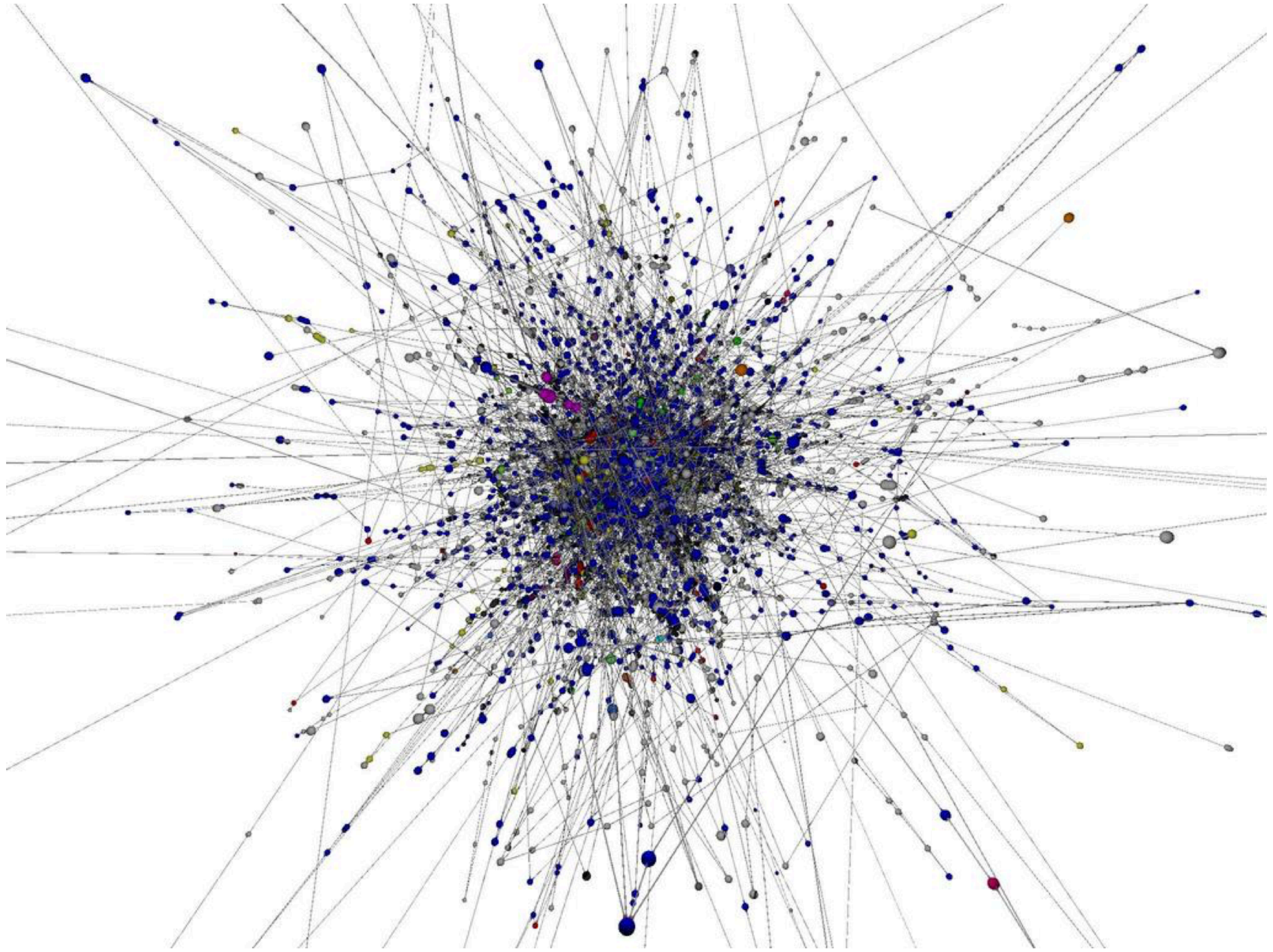
Independent and identically distributed?

Independent and identically distributed?



(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

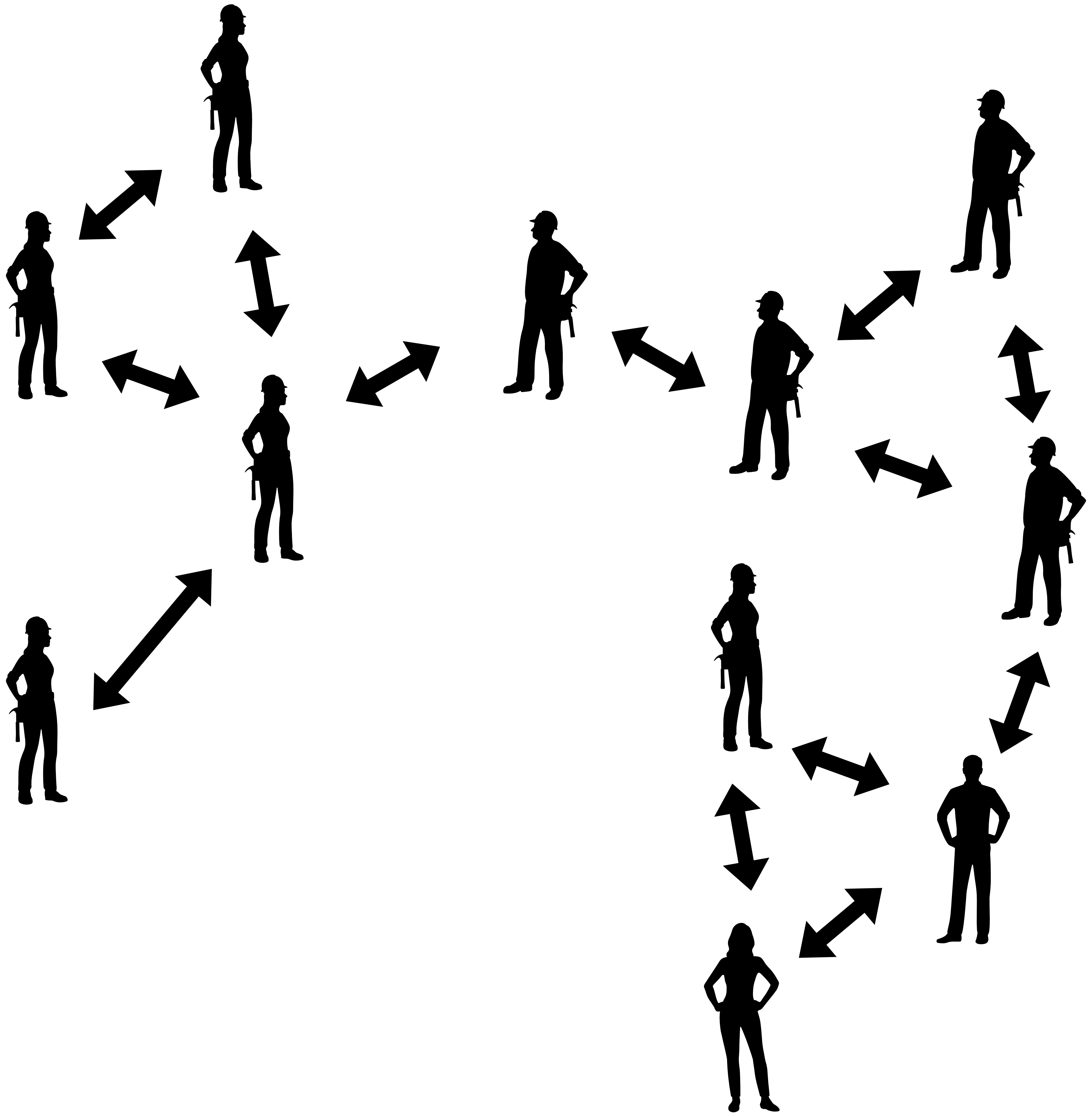
Preferential Attachment
[Albert and Barabasi 1999]



(Stephen Coast
https://www.fractalus.com/steve/stuff/ipmap/)

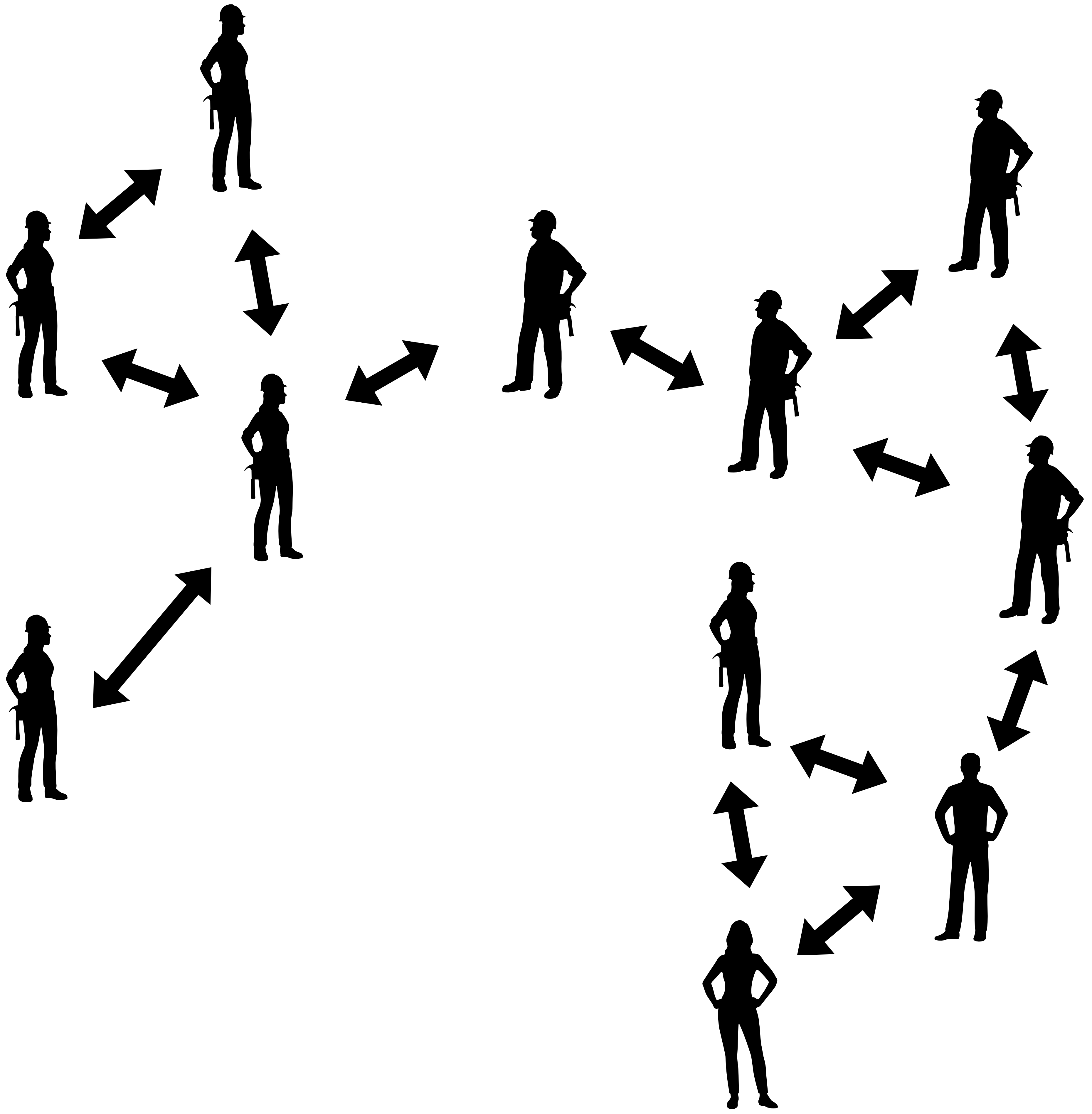
Preferential Attachment

[Albert and Barabasi 1999]



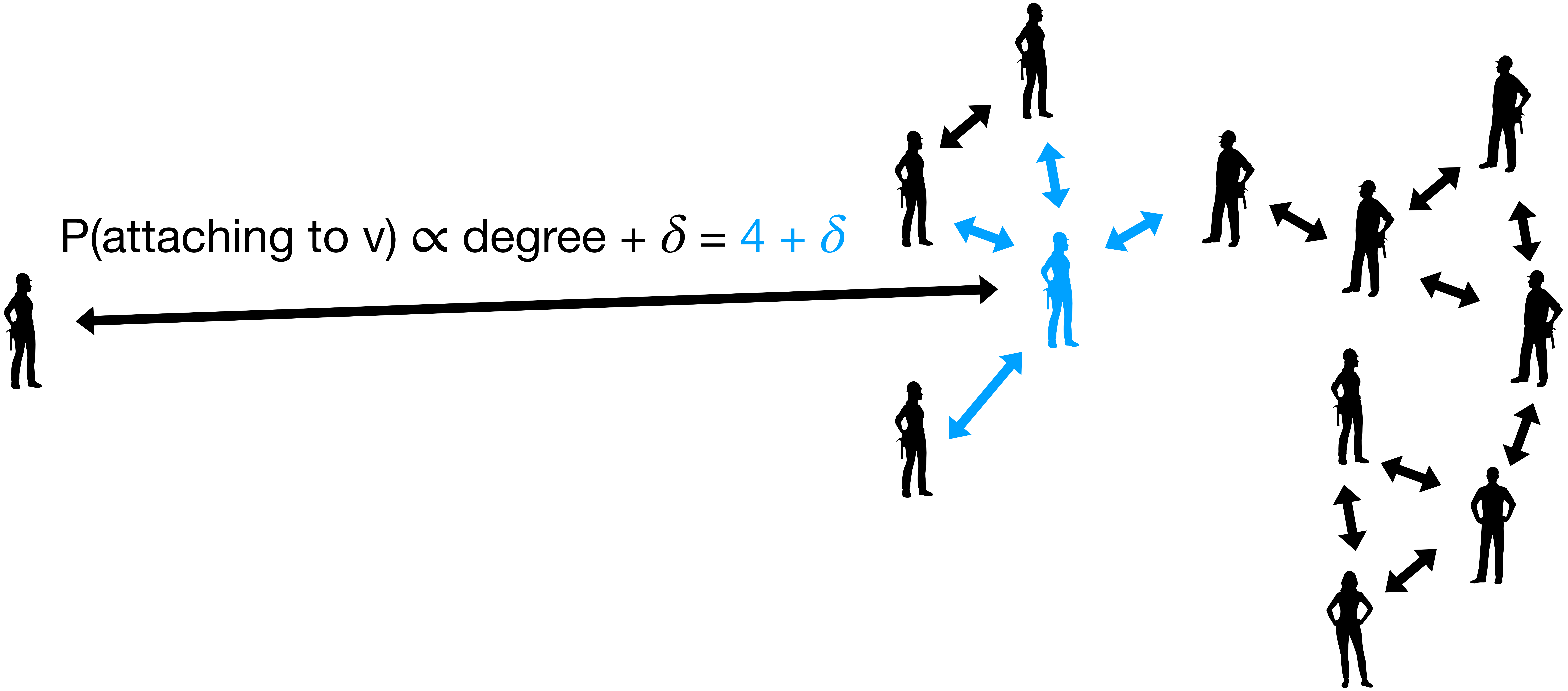
Preferential Attachment

[Albert and Barabasi 1999]



Preferential Attachment

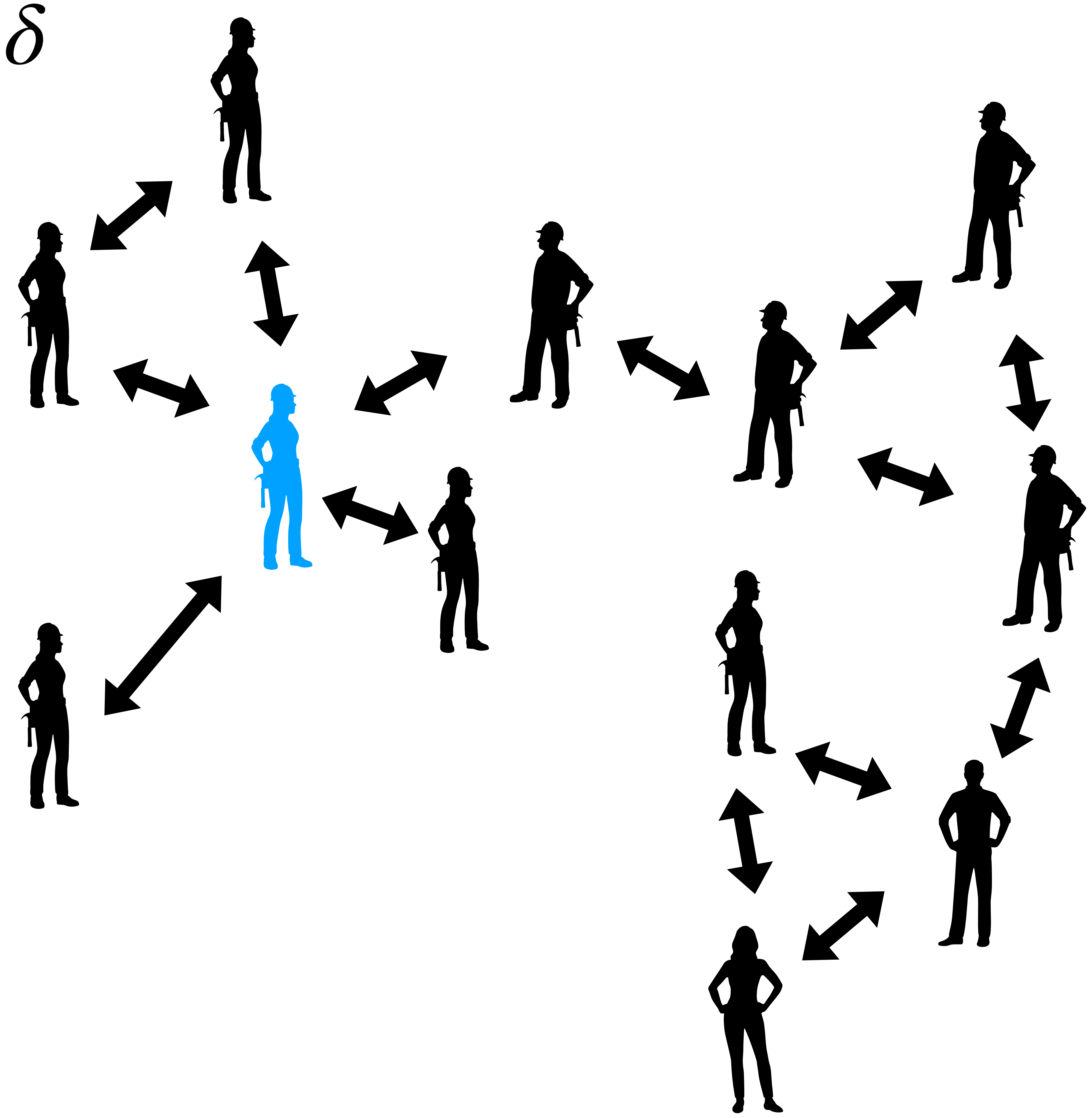
[Albert and Barabasi 1999]



Preferential Attachment

[Albert and Barabasi 1999]

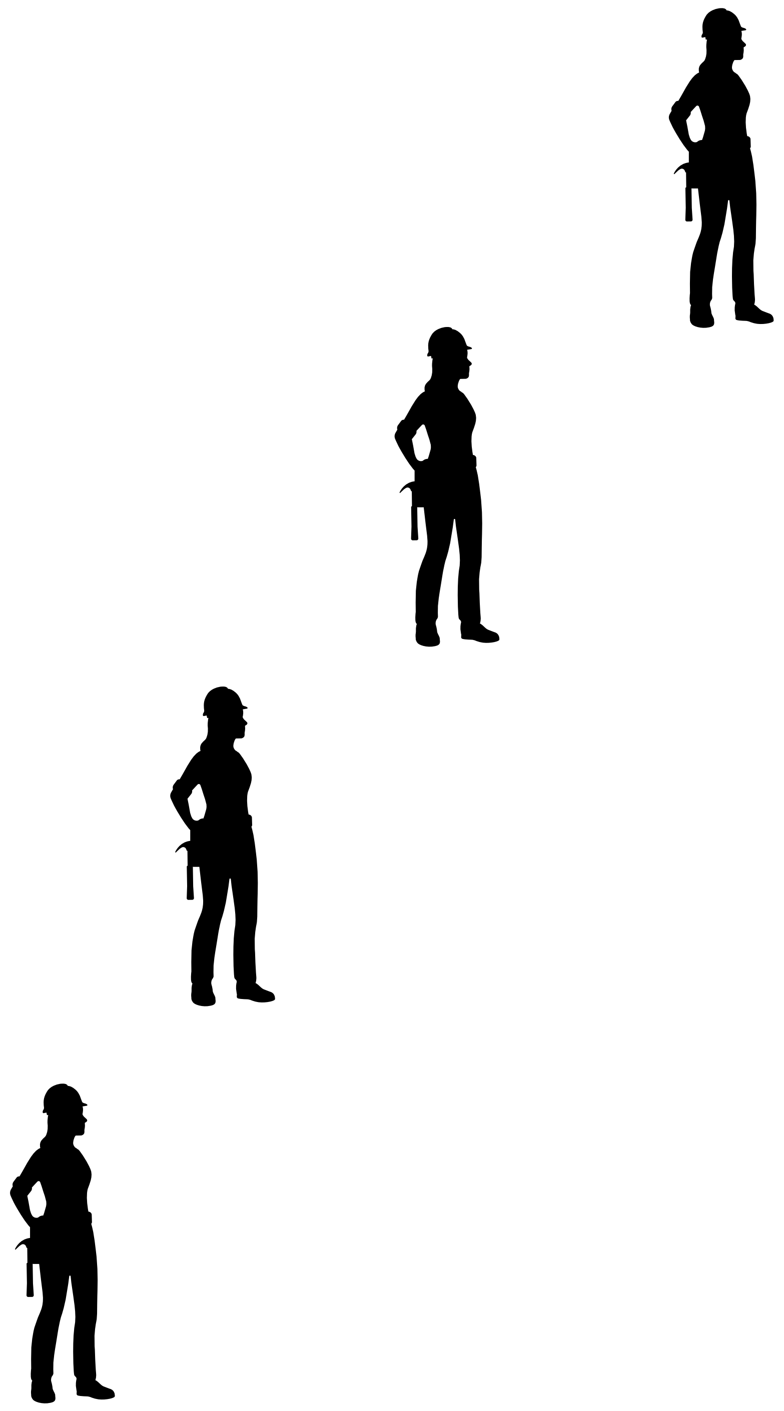
P(attaching to v) $\propto$ degree + a tuning parameter δ

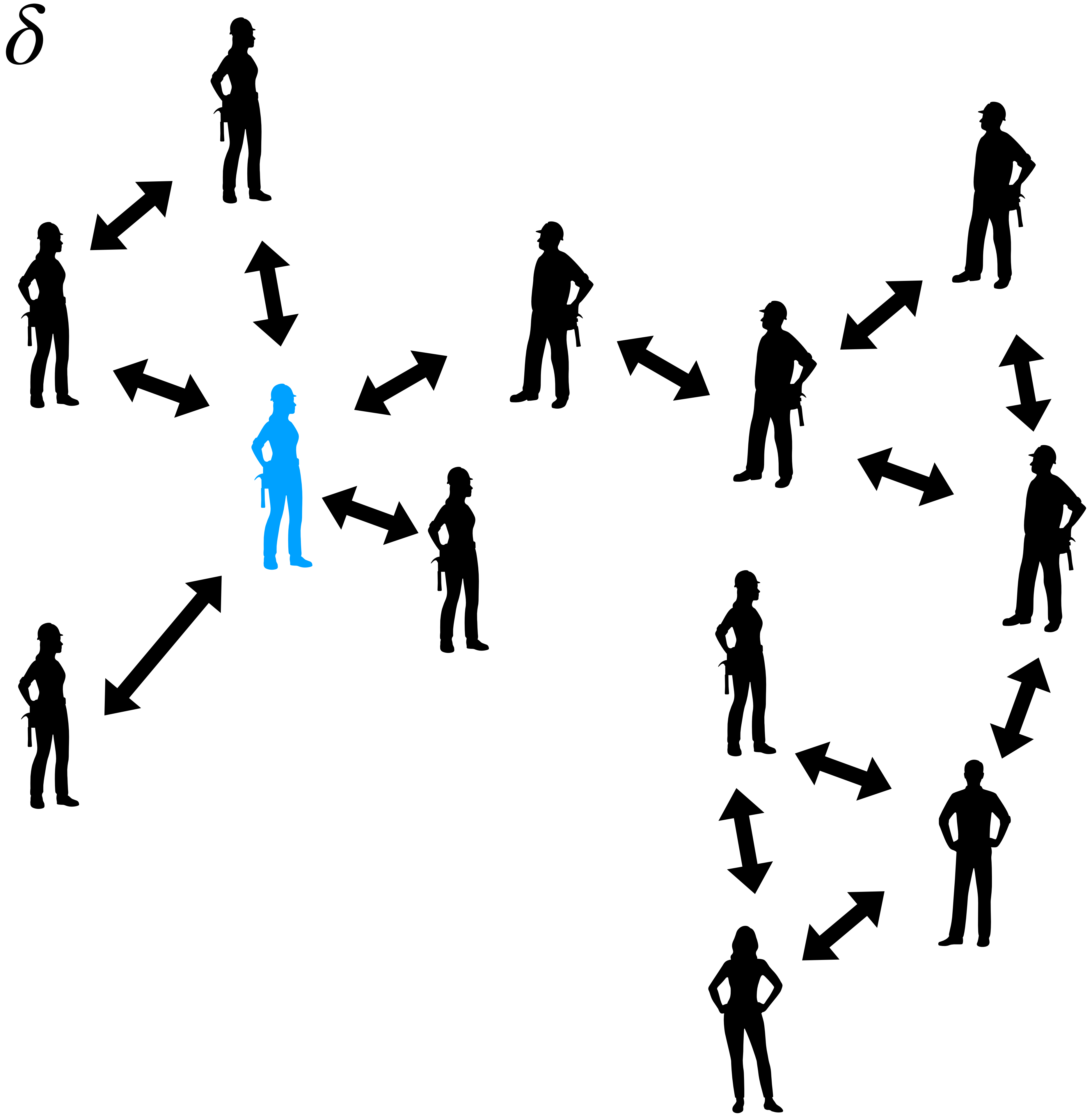


Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) $\propto$ degree + a tuning parameter δ

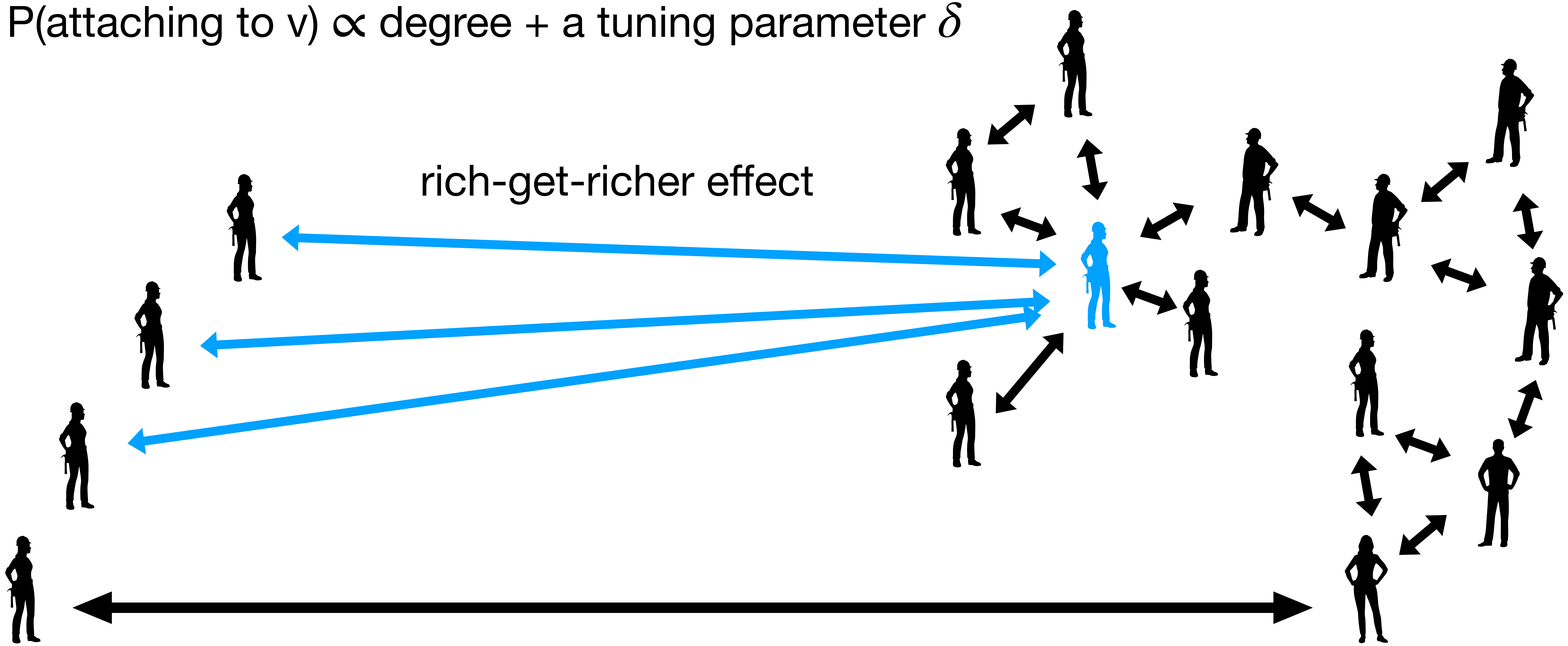




Preferential Attachment

[Albert and Barabasi 1999]

P(attaching to v) $\propto$ degree + a tuning parameter δ



What do we know?

What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]

What do we know?

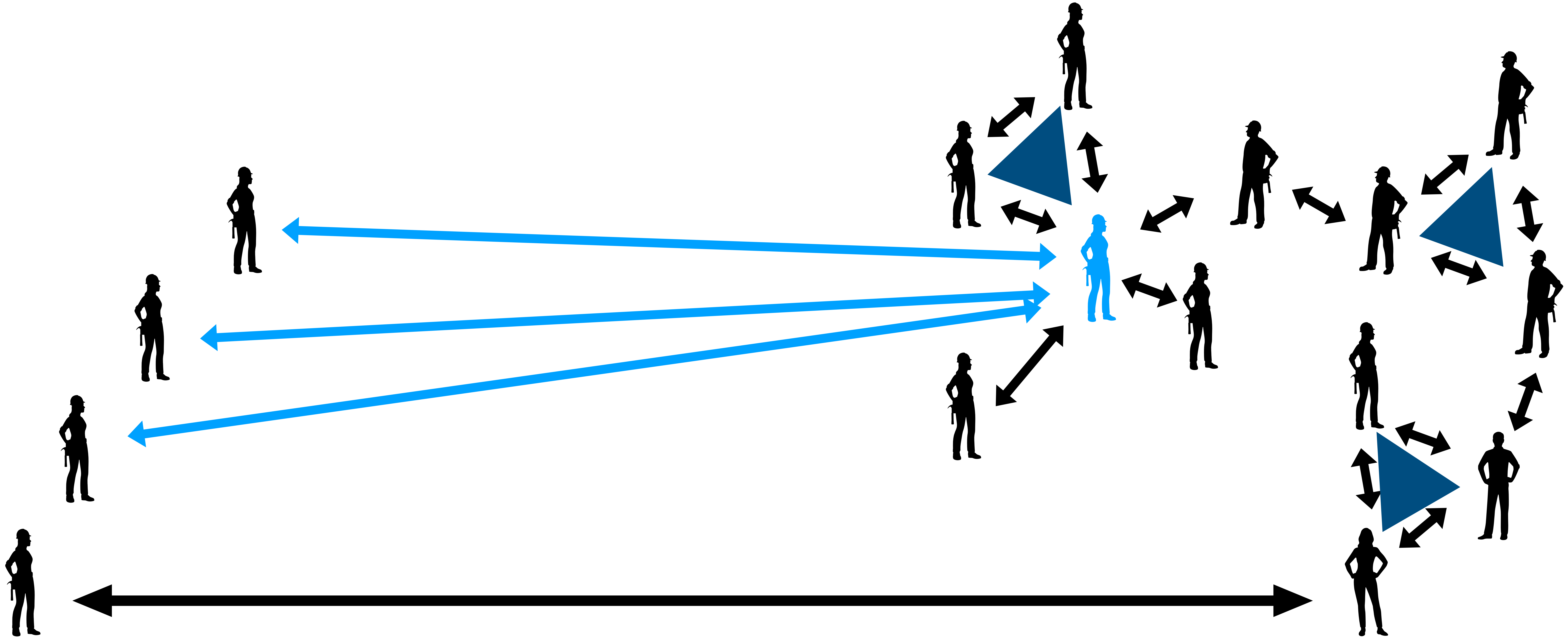
- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]

What do we know?

- triangle counts and clustering coefficient [Bollobas and Ridden 2002, Prokhorenkova et al 2013]
- subgraph counts [Garavaglia and Steghuis 2019]
- and more…

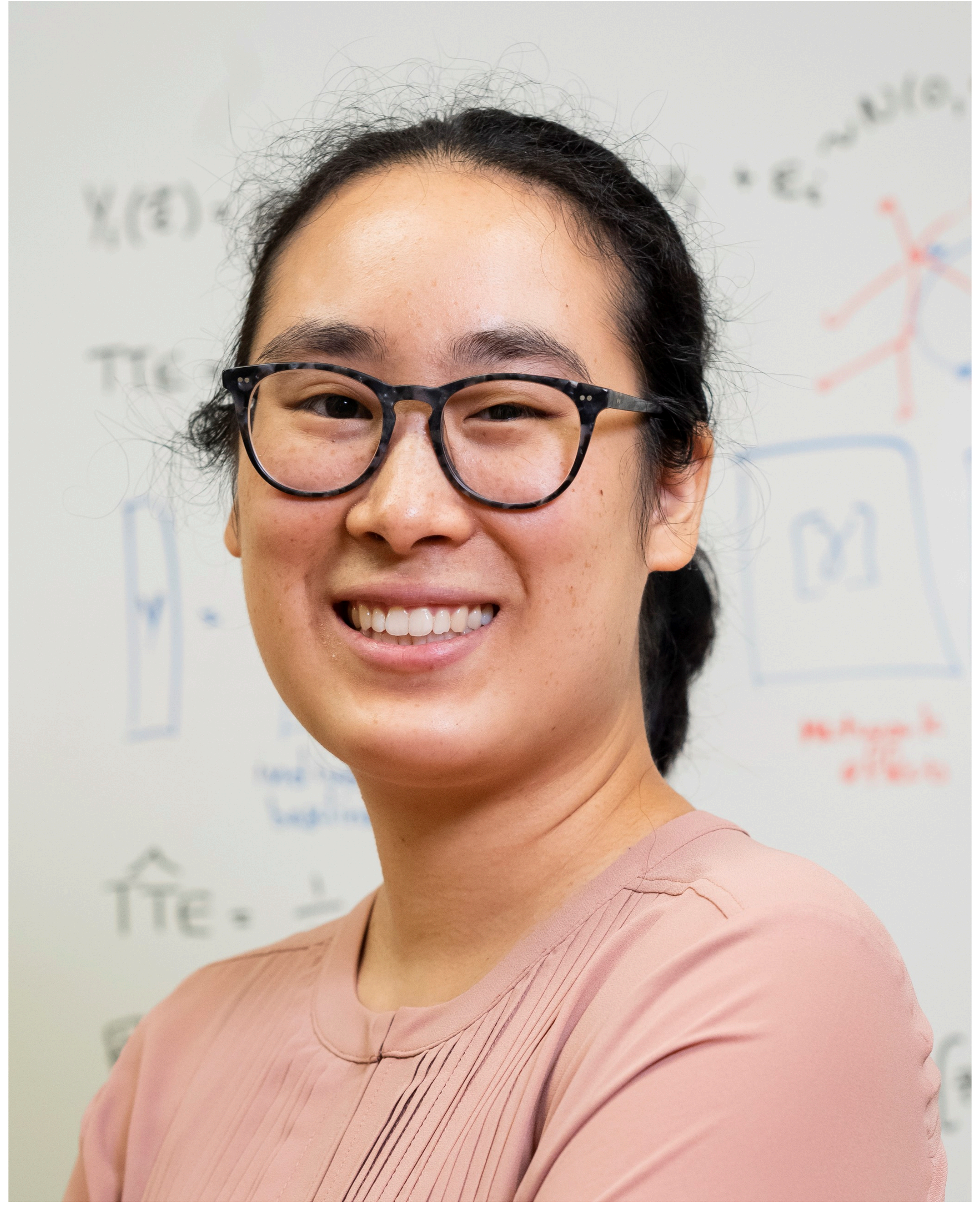
Clique Complex

aka Flag Complex



III Topology of Preferential Attachment

My Lovely Collaborators



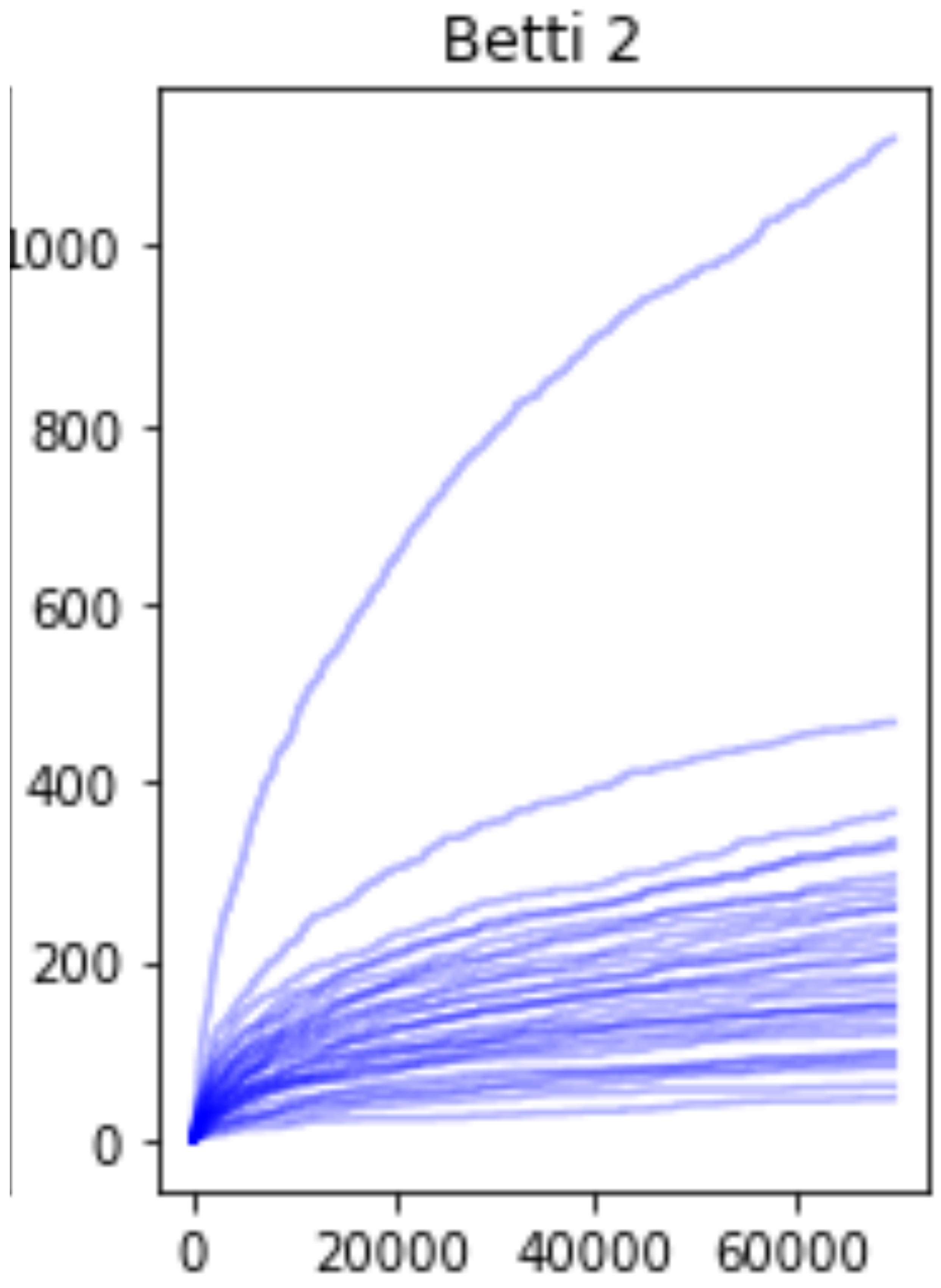
Christina Lee Yu

Gennady Samorodnitsky

Rongyi He (Caroline)

Expected Betti Number $E[\beta_q]$

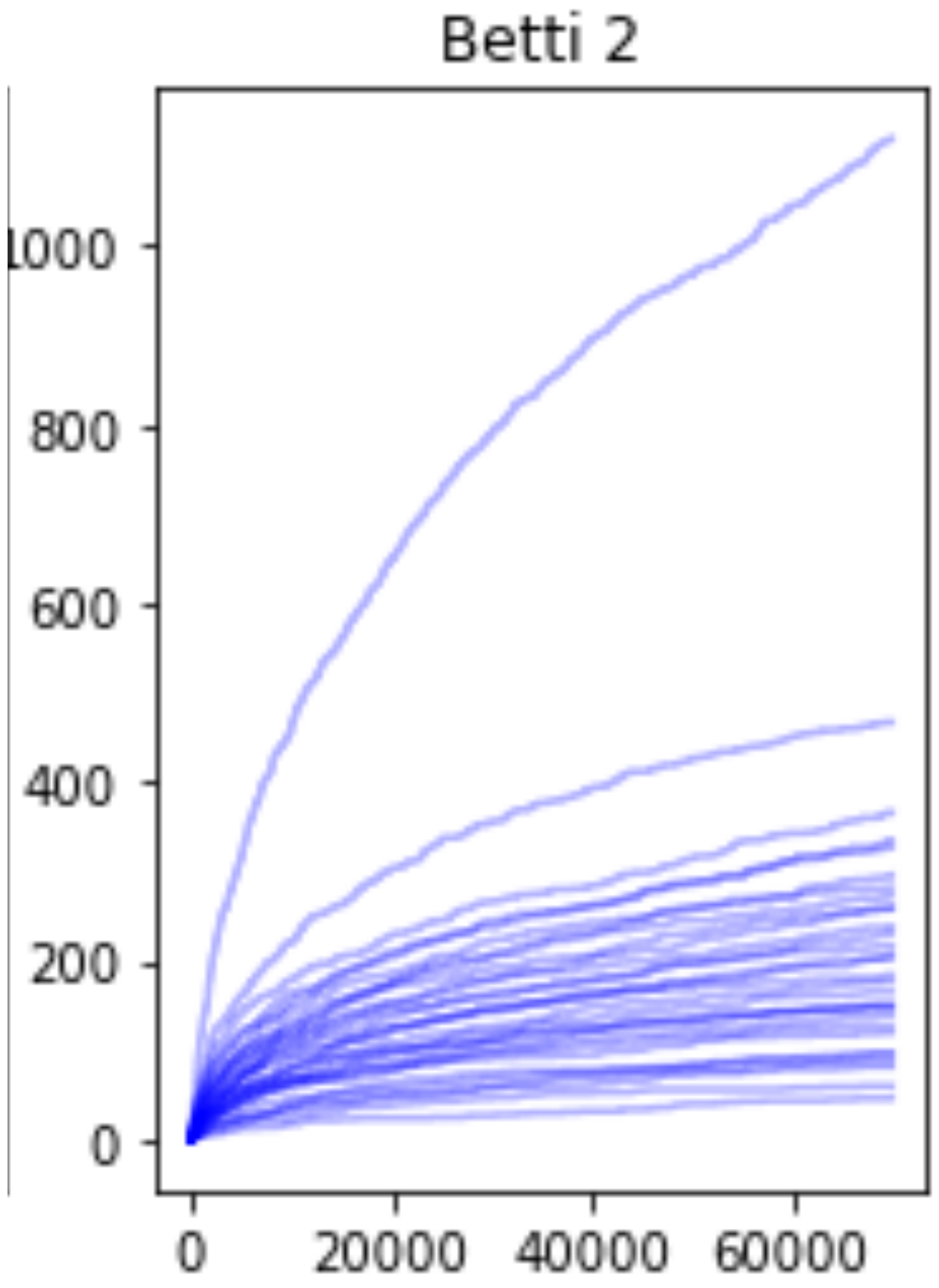
Expected Betti Number $E[\beta_q]$



Different curves, different random seeds.
All curves have the same model parameters.

Expected Betti Number $E[\beta_q]$

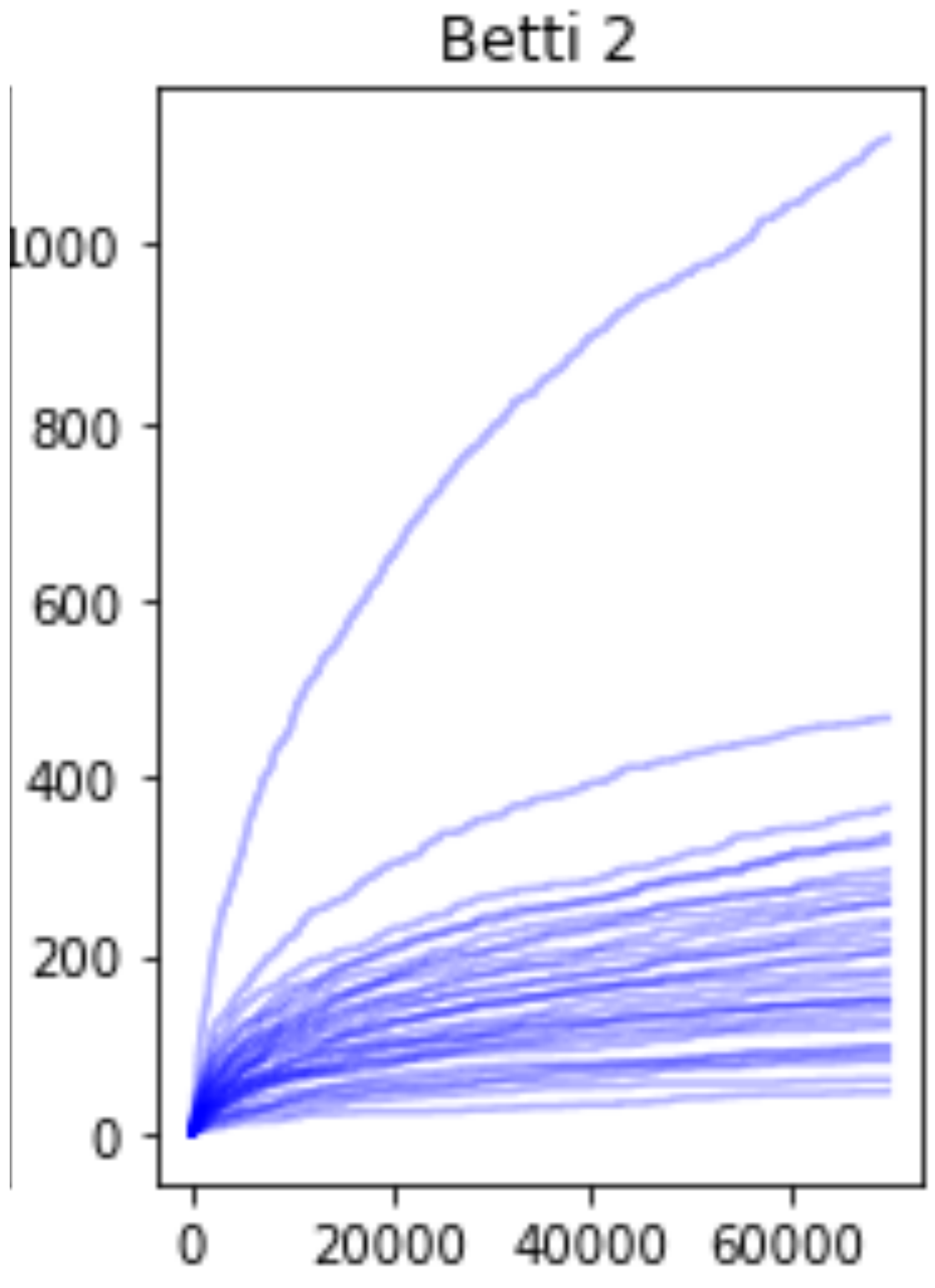
- increasing trend



Different curves, different random seeds.
All curves have the same model parameters.

Expected Betti Number $E[\beta_q]$

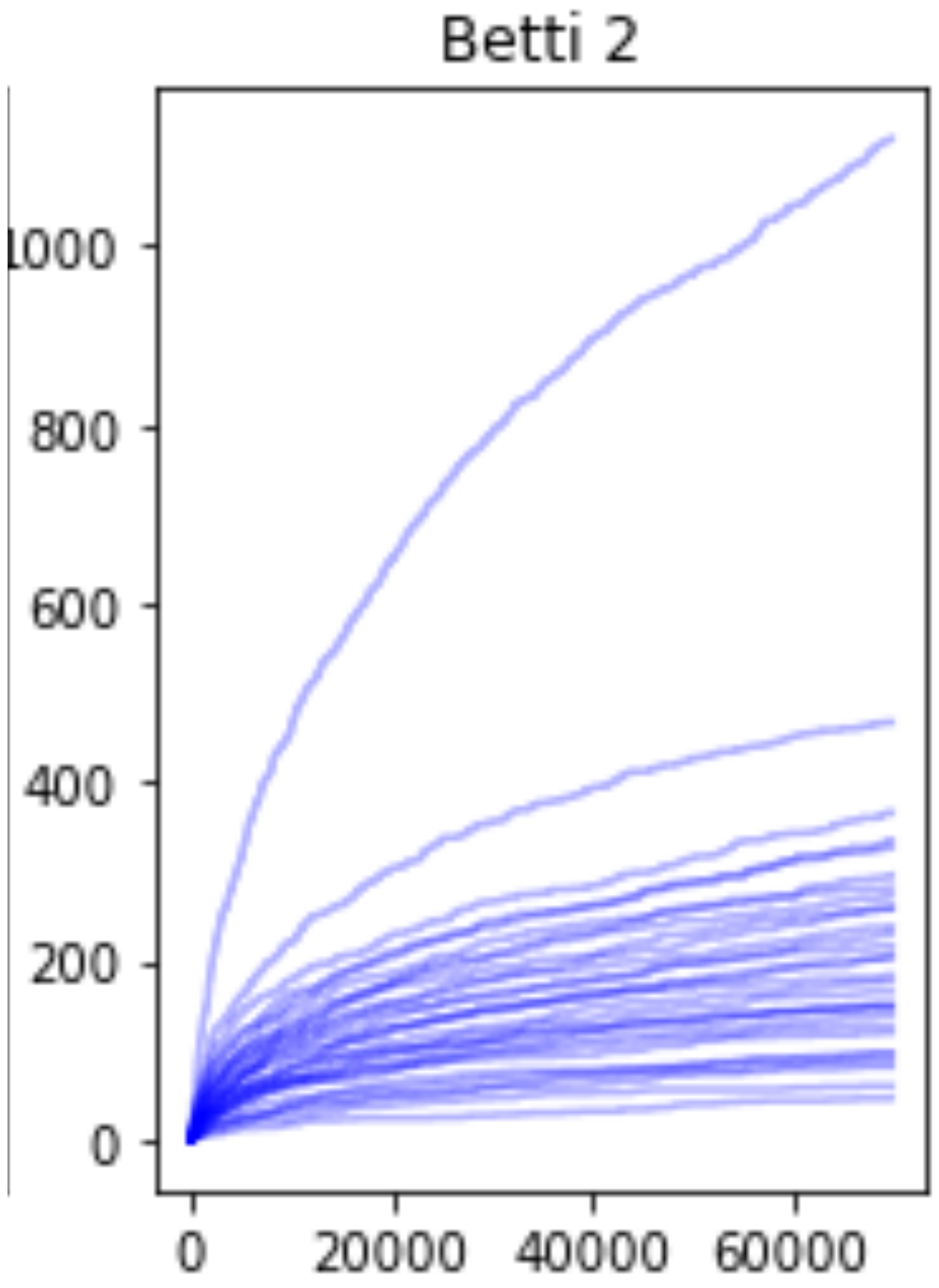
- increasing trend
- concave growth



Different curves, different random seeds.
All curves have the same model parameters.

Expected Betti Number $E[\beta_q]$

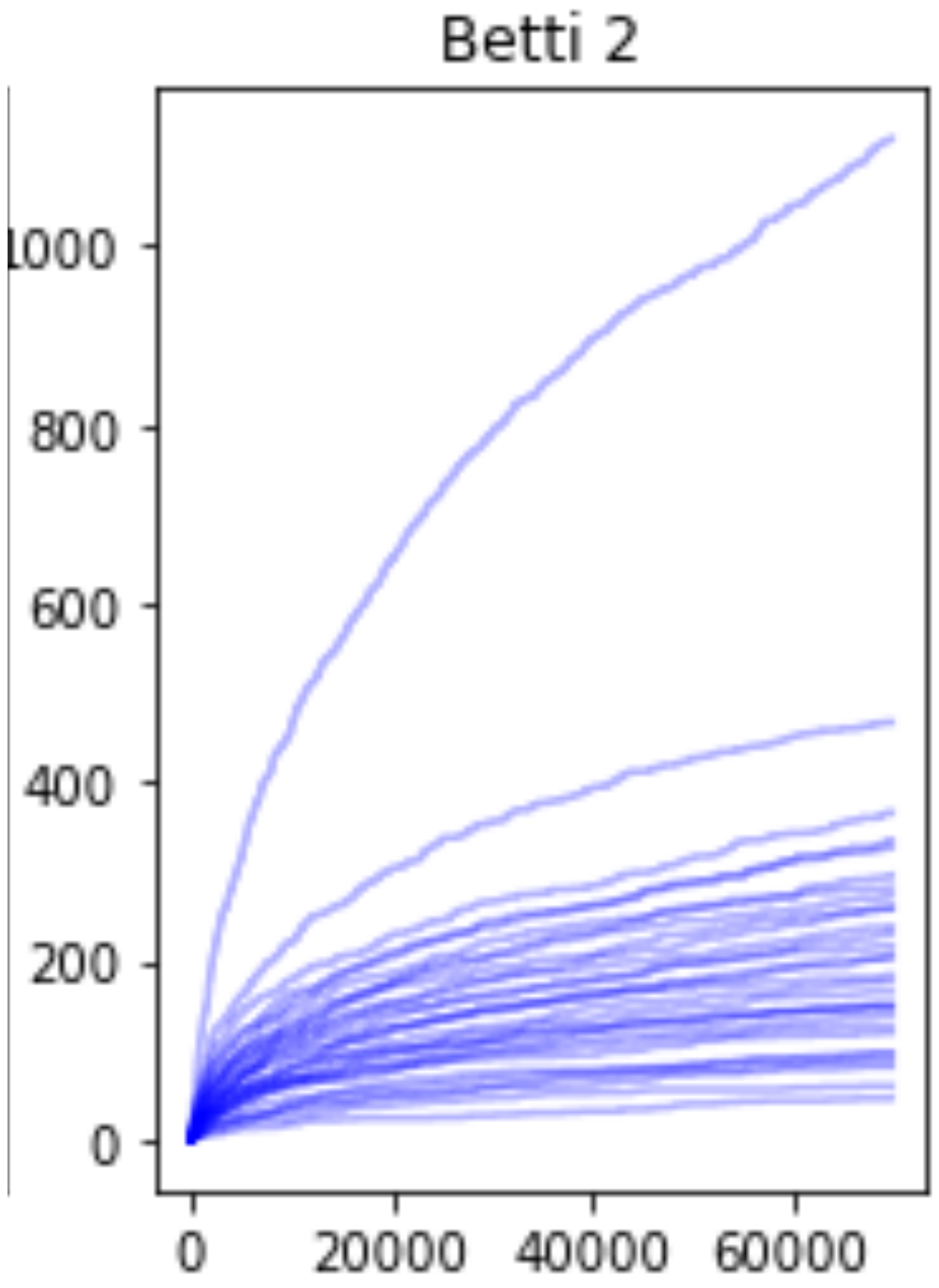
- increasing trend
- concave growth
- outlier



Different curves, different random seeds.
All curves have the same model parameters.

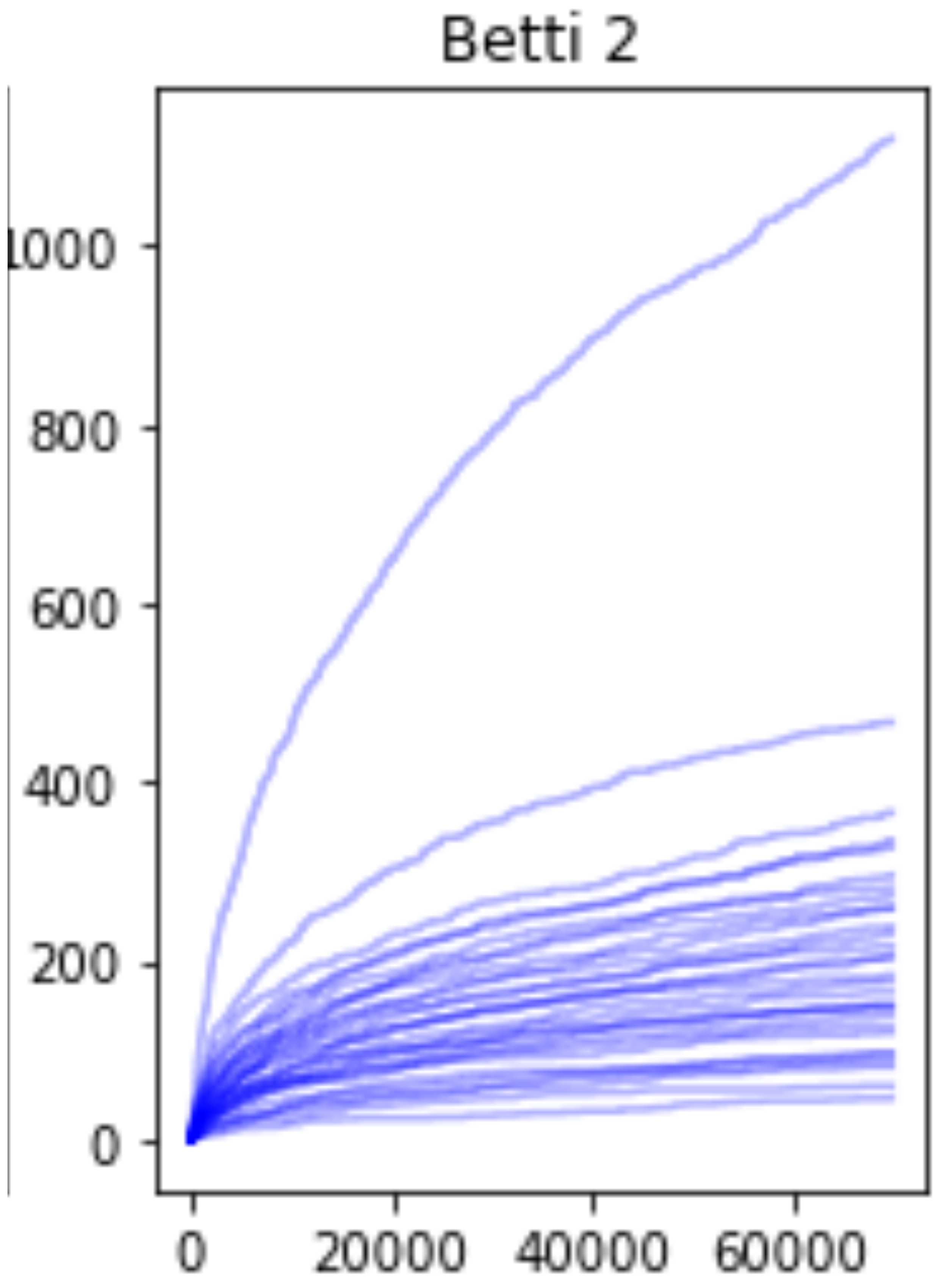
Expected Betti Number $E[\beta_q]$

- $c(\text{num of nodes}^{1-4x}) \leq E[\beta_2] \leq C(\text{num of nodes}^{1-4x})$
  under mild assumptions
  - $x \in (0,1/2)$ depends on
    the preferential attachment strength.



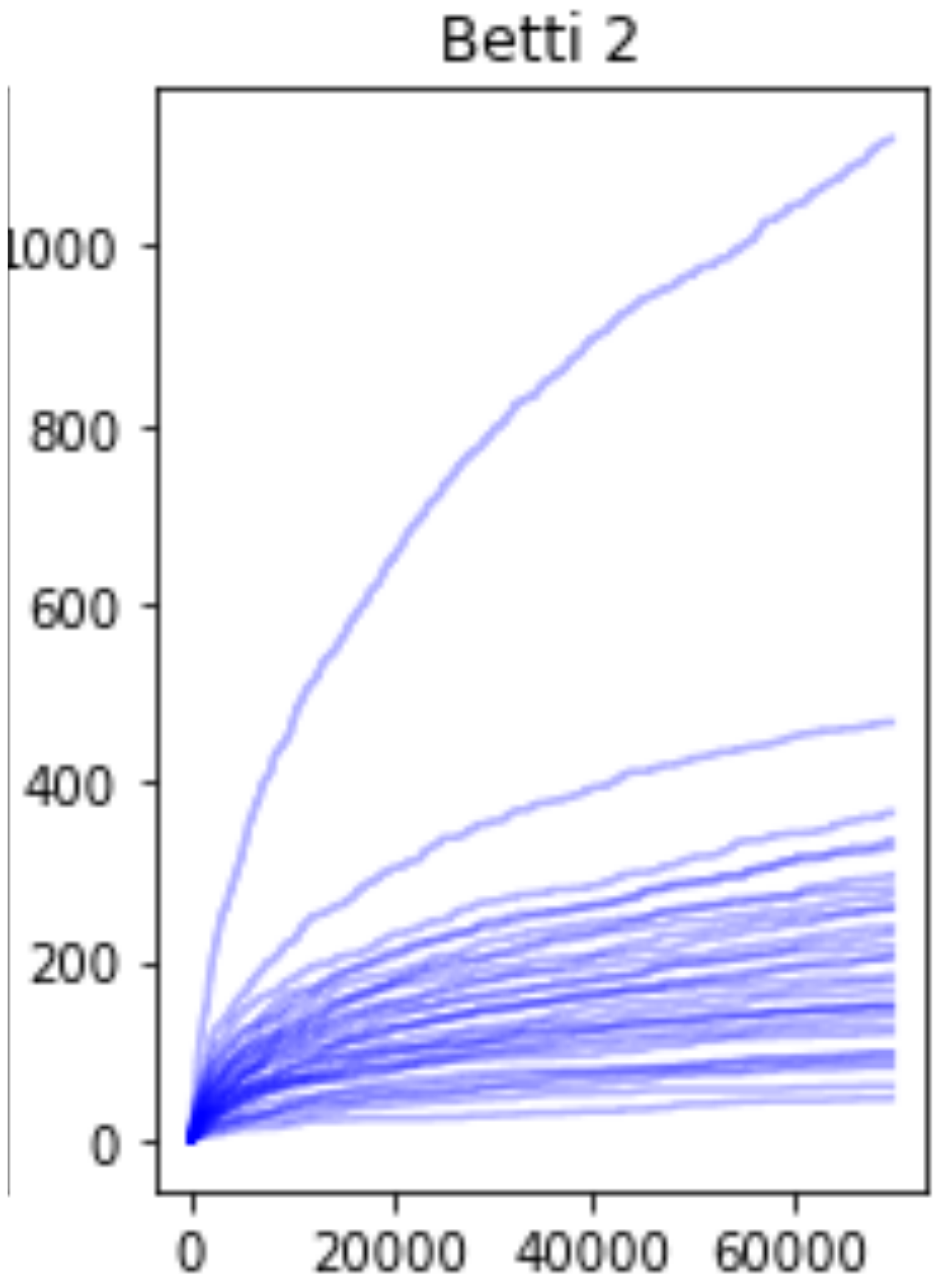
Expected Betti Number $E[\beta_q]$

- c(num of nodes^{1-4x}) $\leq E[\beta_2] \leq C$(num of nodes^{1-4x})
  under mild assumptions
  - $x \in (0,1/2)$ depends on the preferential attachment strength.
  - If $1 - 4x < 0$, then $E[\beta_2] \leq C$.



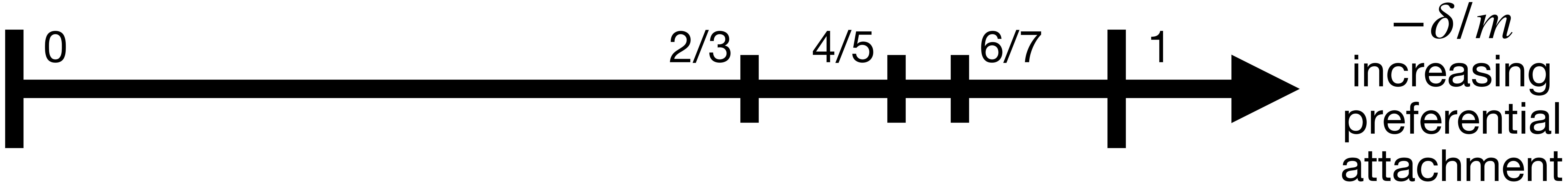
Expected Betti Number $E[\beta_q]$

- c(num of nodes^{1-4x}) $\leq E[\beta_2] \leq C$(num of nodes^{1-4x})
  under mild assumptions
  - $x \in (0,1/2)$ depends on the preferential attachment strength
  - If $1 - 4x < 0$, then $E[\beta_2] \leq C$.
- c(num of nodes^{1-2qx}) $\leq E[\beta_q] \leq C$(num of nodes^{1-2qx}) for $q \geq 2$.



Phase transition

Recall
P(attaching to v) $\propto$ degree + δ
m = number of edges per new node

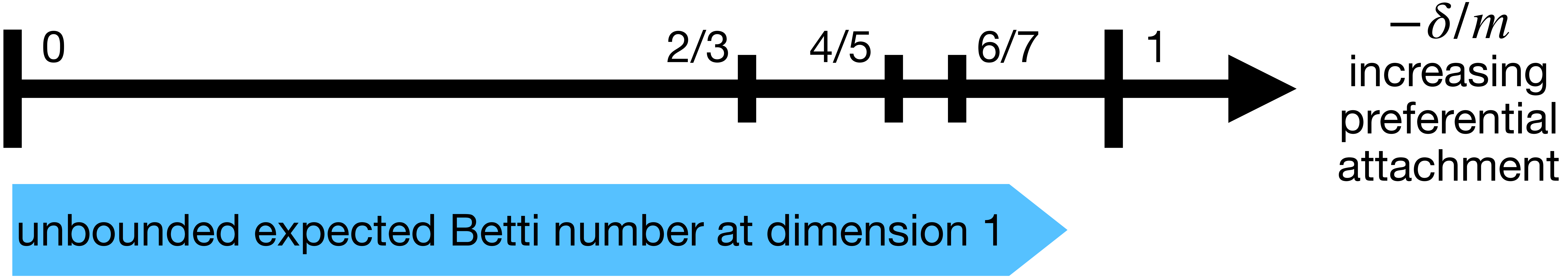


Phase transition

Recall

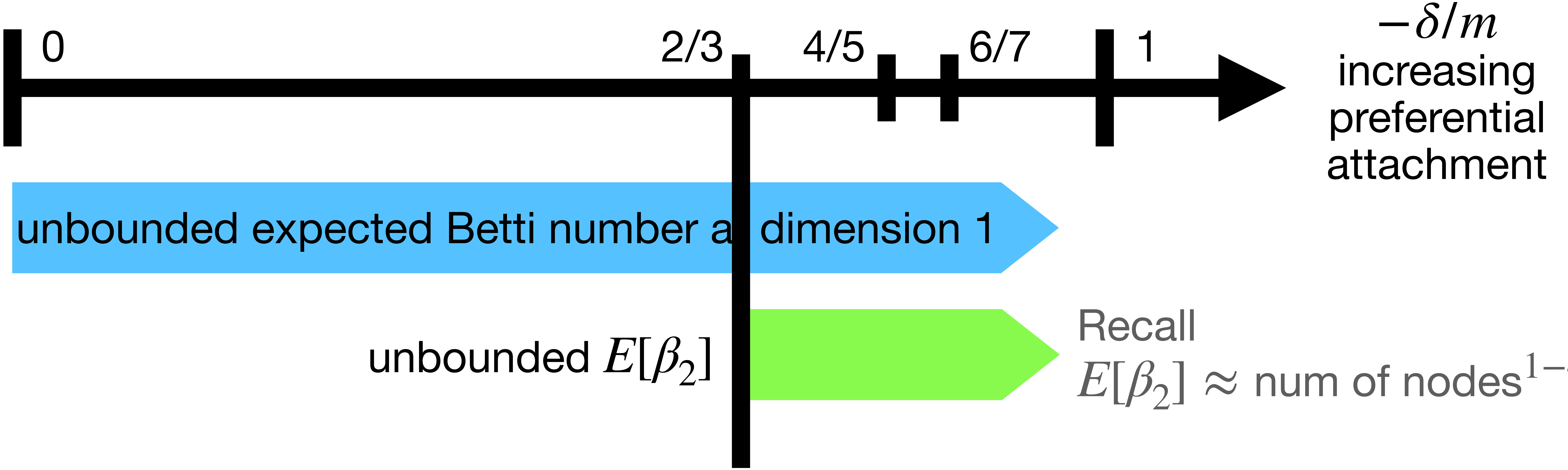
P(attaching to v) ∝ degree + δ

m = number of edges per new node



Phase transition

Recall
P(attaching to v) ∝ degree + δ
m = number of edges per new node



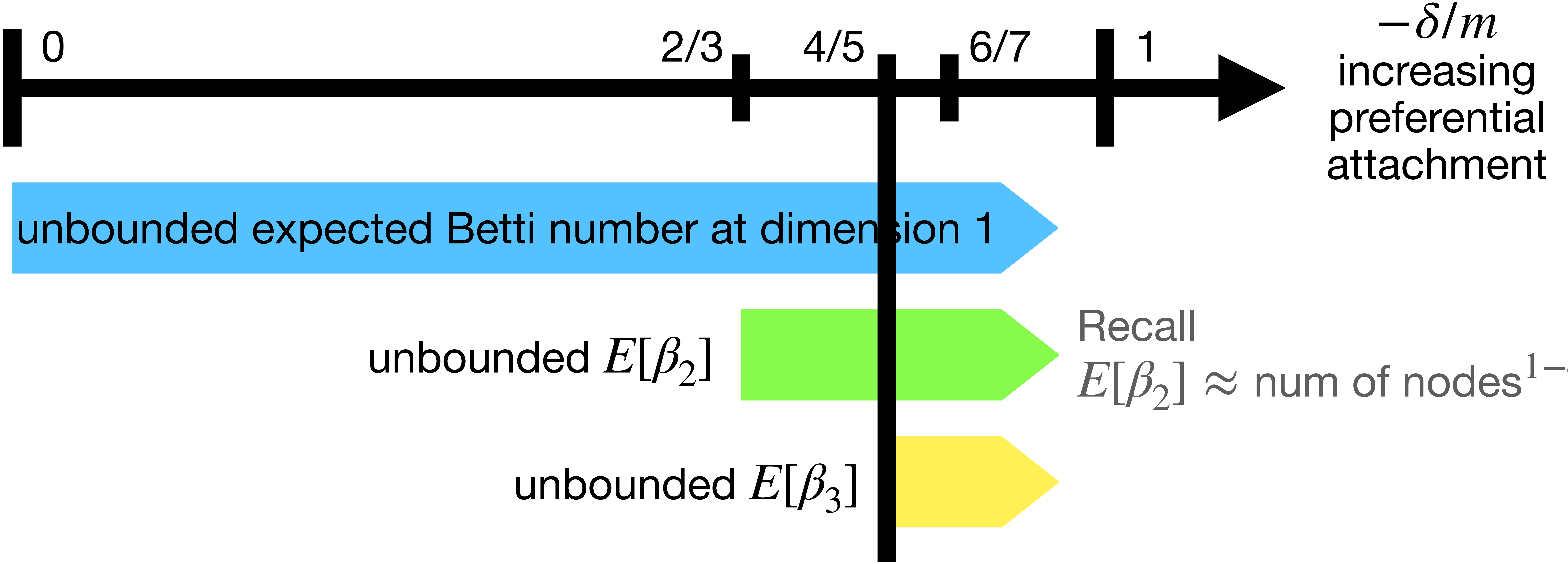
Recall
$E[\beta_2] \approx$ num of nodes^{1-4x}

Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node

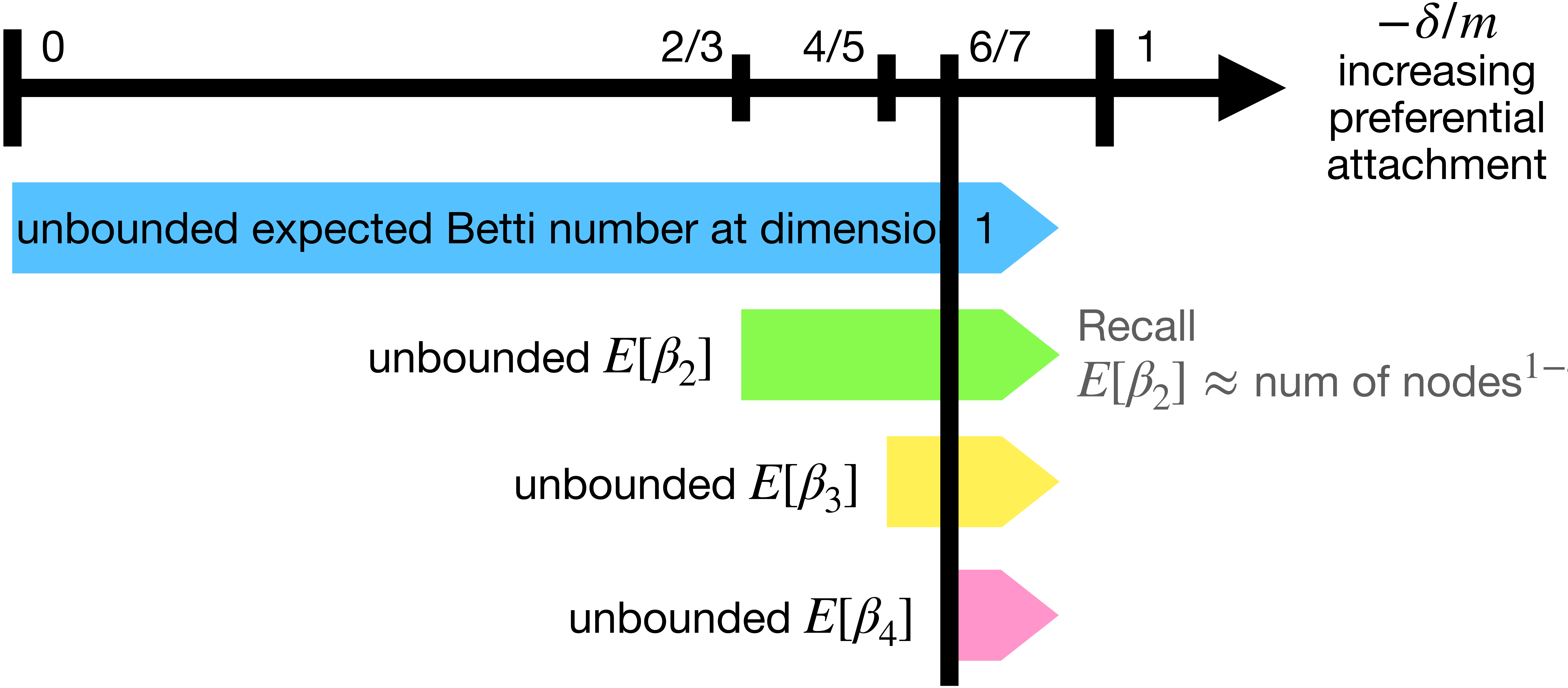


Recall

$E[\beta_2] \approx$ num of nodes^{1-4x}

Phase transition

Recall
P(attaching to v) ∝ degree + δ
m = number of edges per new node



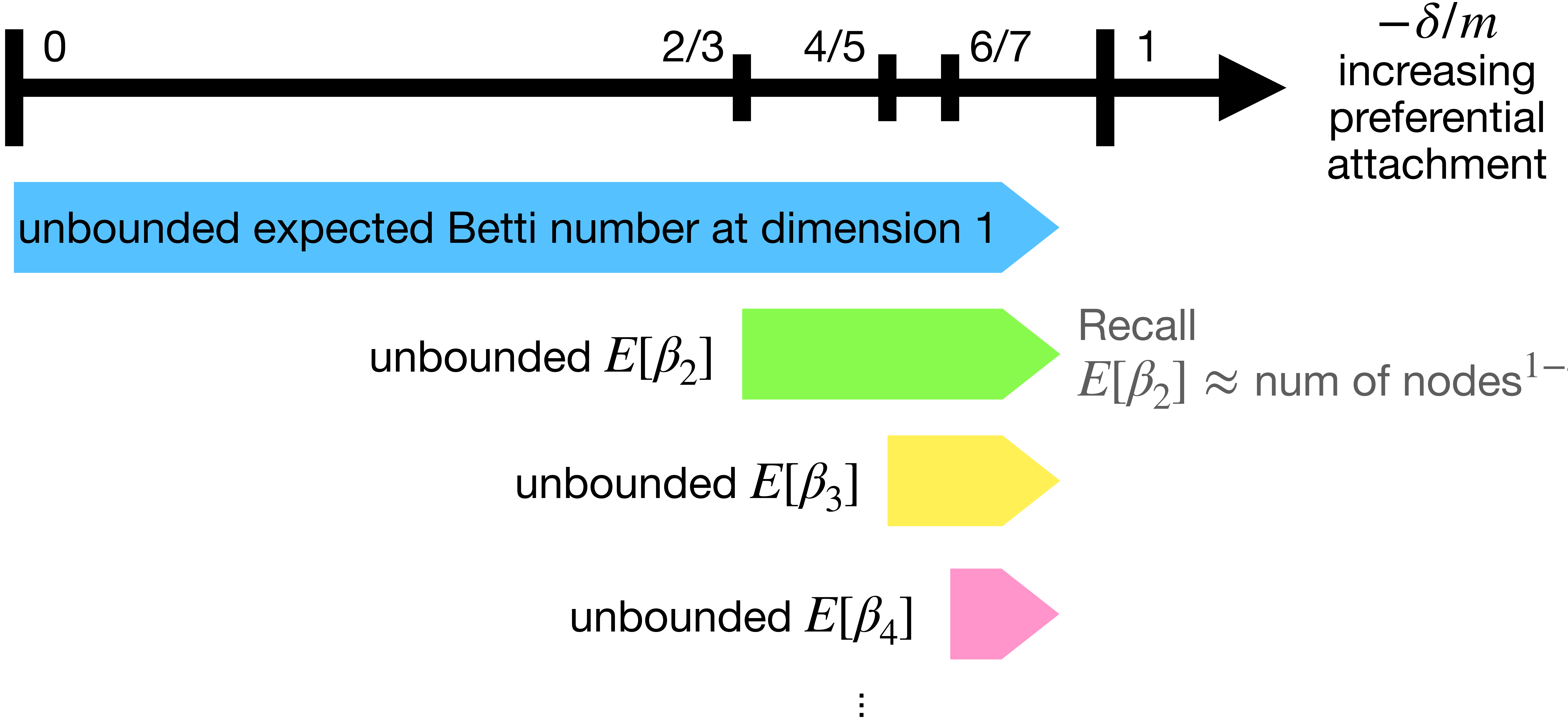
Recall
$E[\beta_2] \approx$ num of nodes^{1-4x}

Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



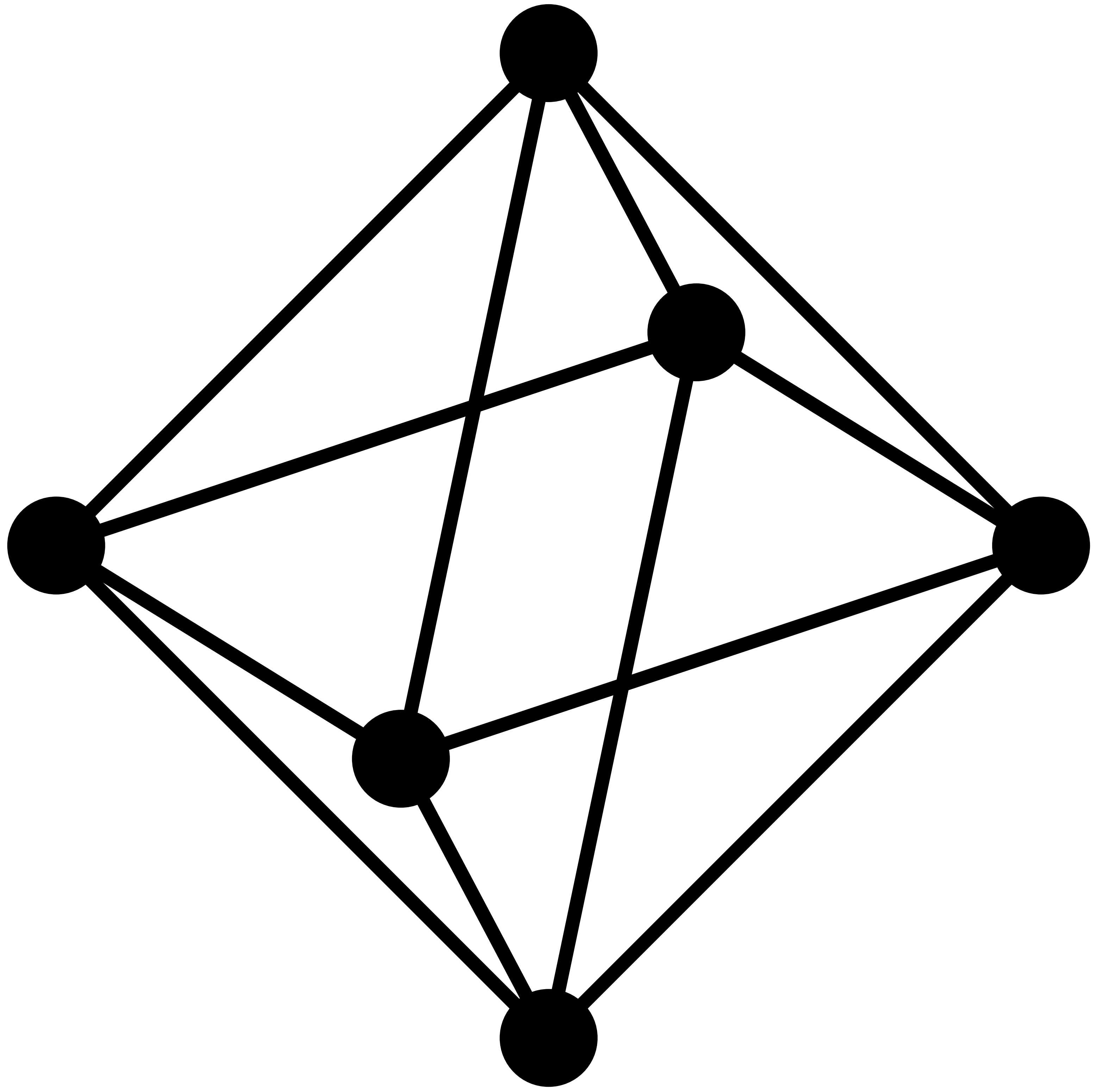
Recall

$E[\beta_2] \approx$ num of nodes$^{1-4\chi}$

Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

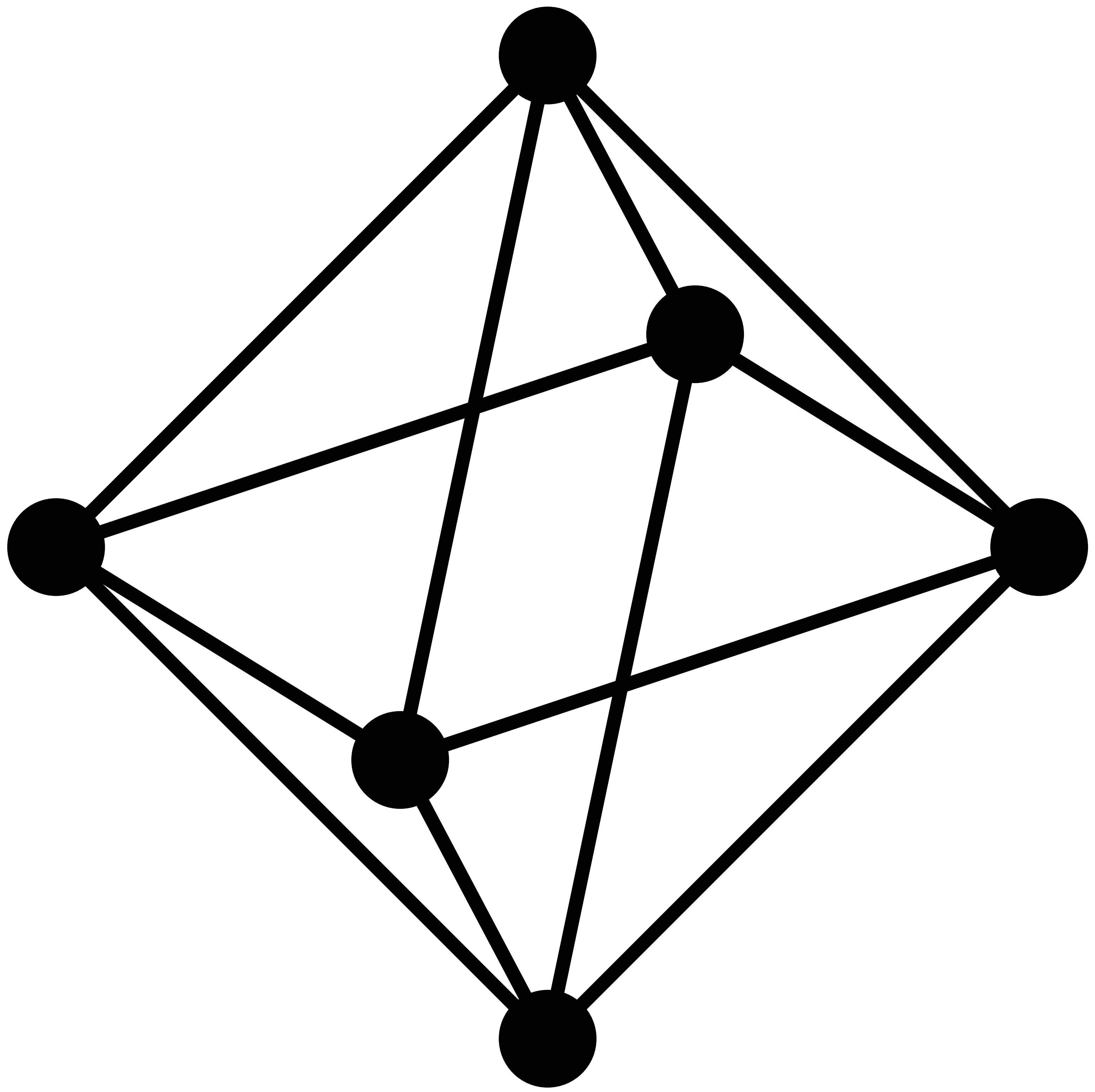
Proof?

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}

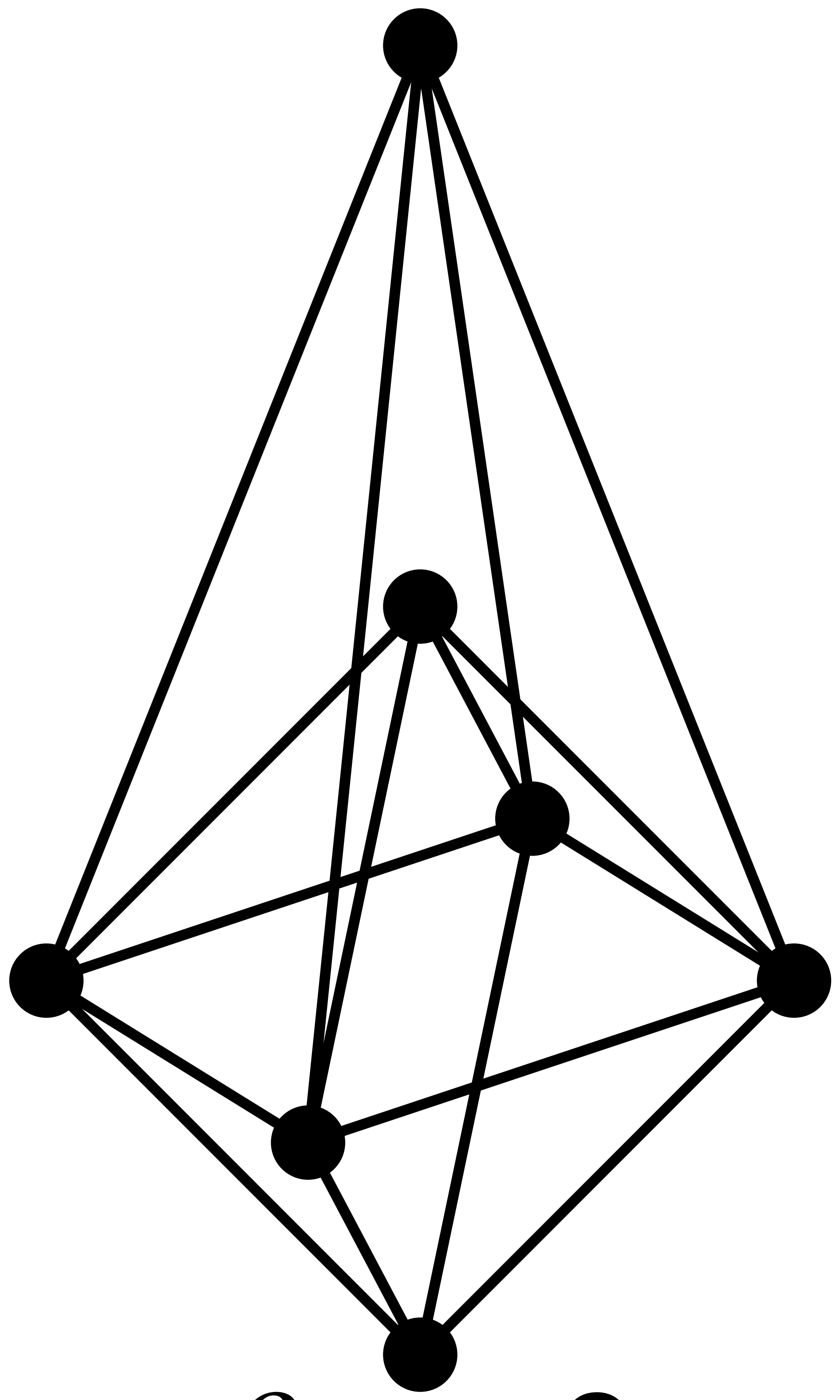


$\beta_2 = 1$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}

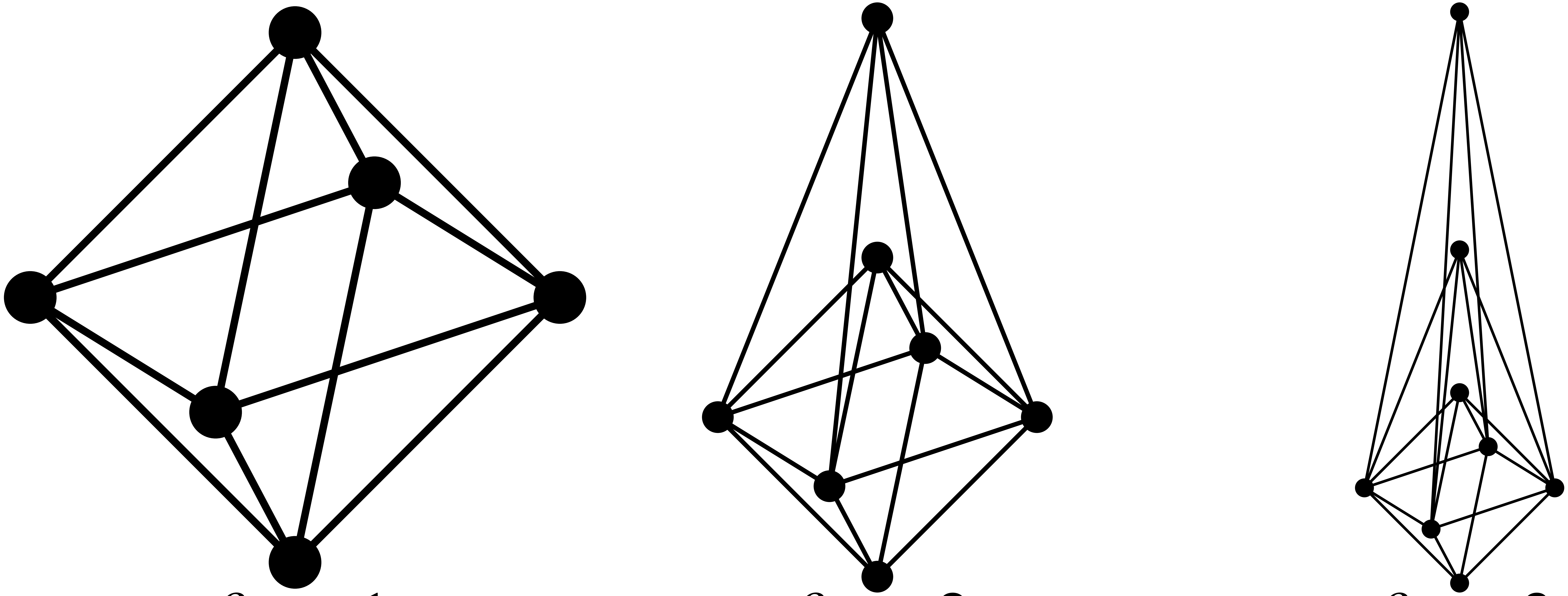


$\beta_2 = 1$



$\beta_2 = 2$

Proof of $E[\beta_2] \approx$ num of nodes^{1-4x}



$\beta_2 = 1$ $\beta_2 = 2$ $\beta_2 = 3$

Subtleties

- Need homological algebra to relate Betti numbers with counts

Subtleties

- Need homological algebra to relate Betti numbers with counts
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]

Subtleties

- Need homological algebra to relate Betti numbers with counts
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results in the language of homological algebra

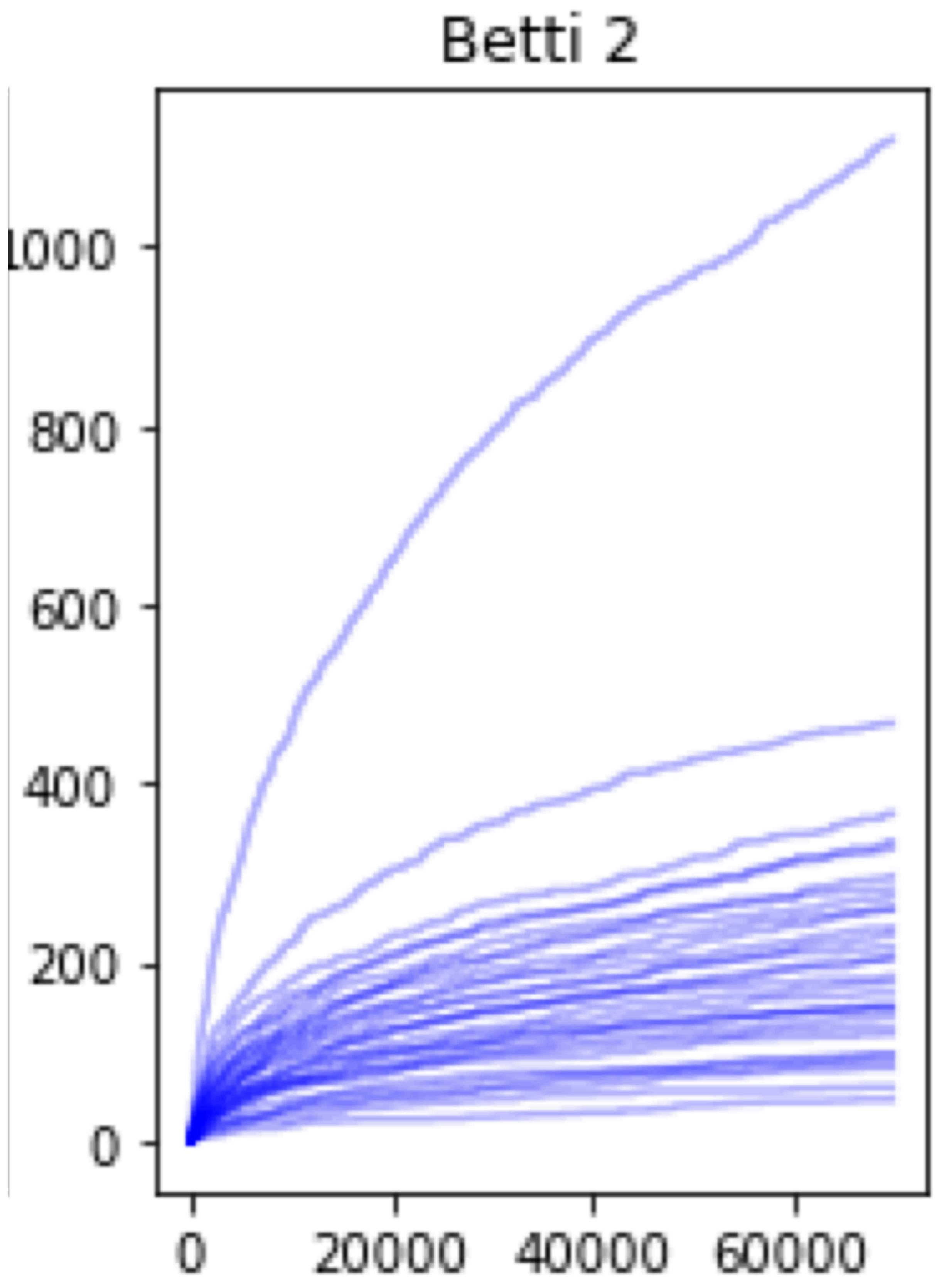
Subtleties

- Need homological algebra to relate Betti numbers with counts
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results in the language of homological algebra
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs

Theorem: $E[\beta_2] \approx$ num of nodes^{1-4x}

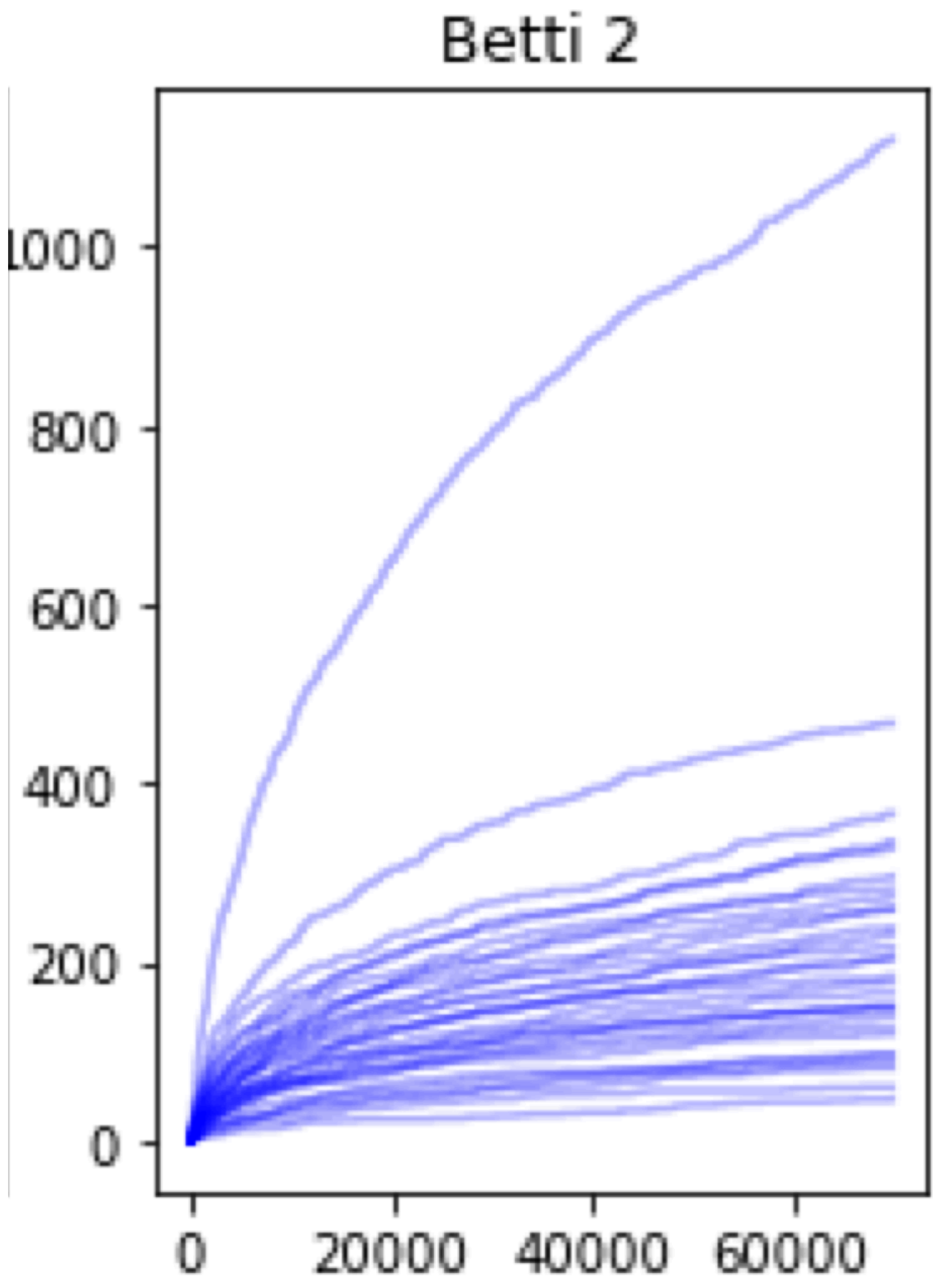
In practice???

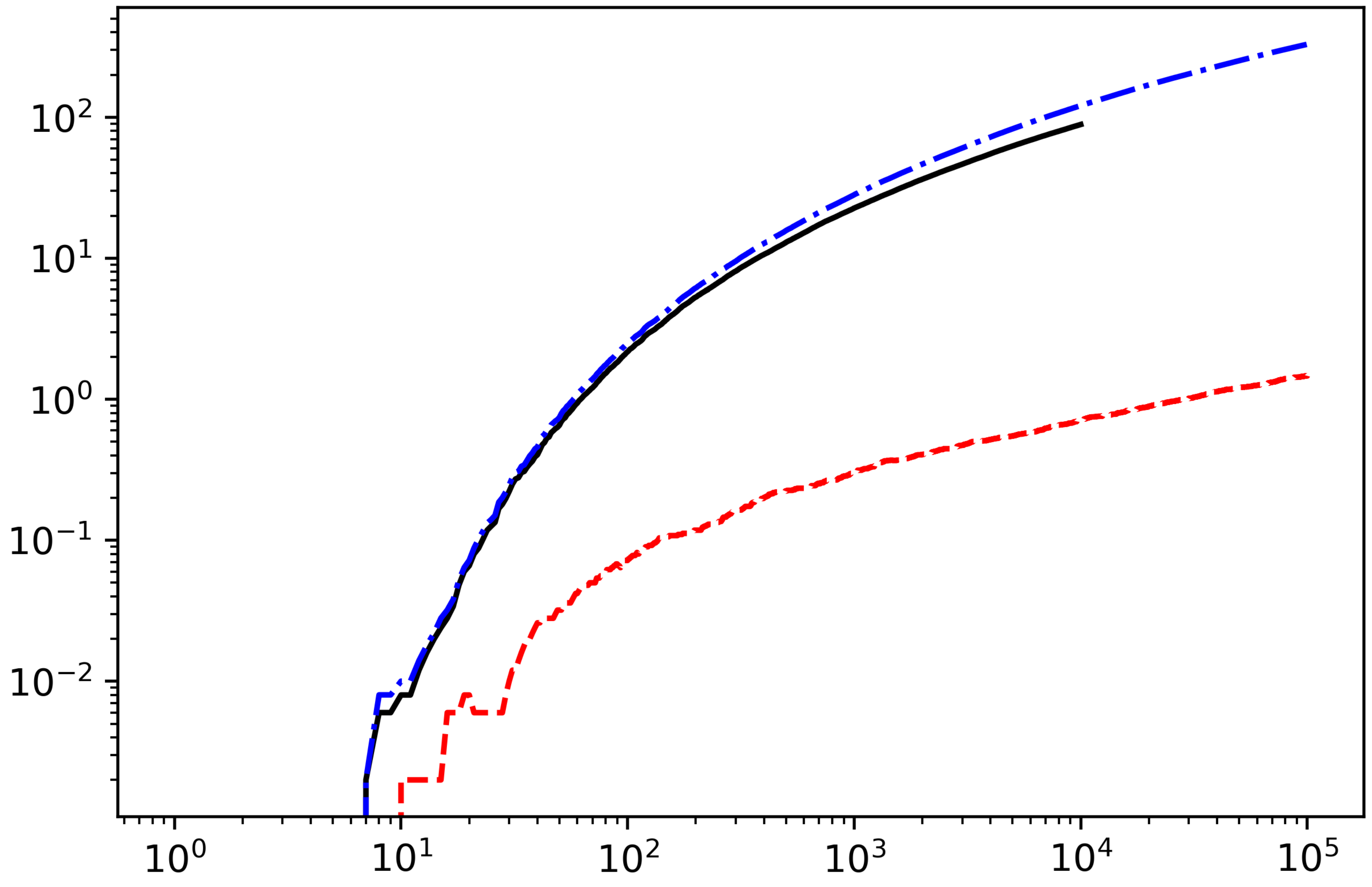
$E[\beta_2] \approx$ **num of nodes**$^{1-4x}$



$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

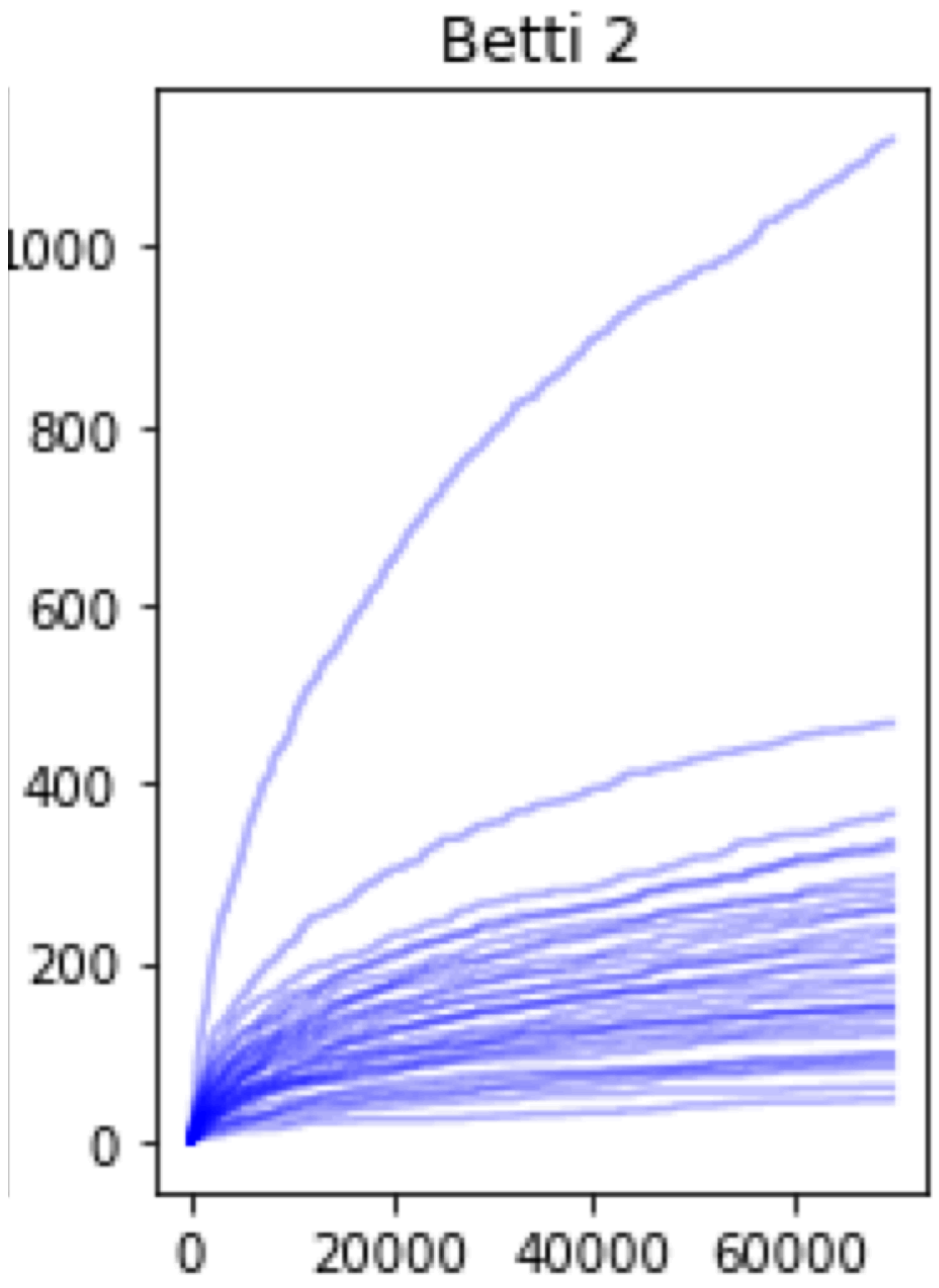
$$\log E[\beta_2] \approx (1-4x)\log(\textbf{num of nodes})$$

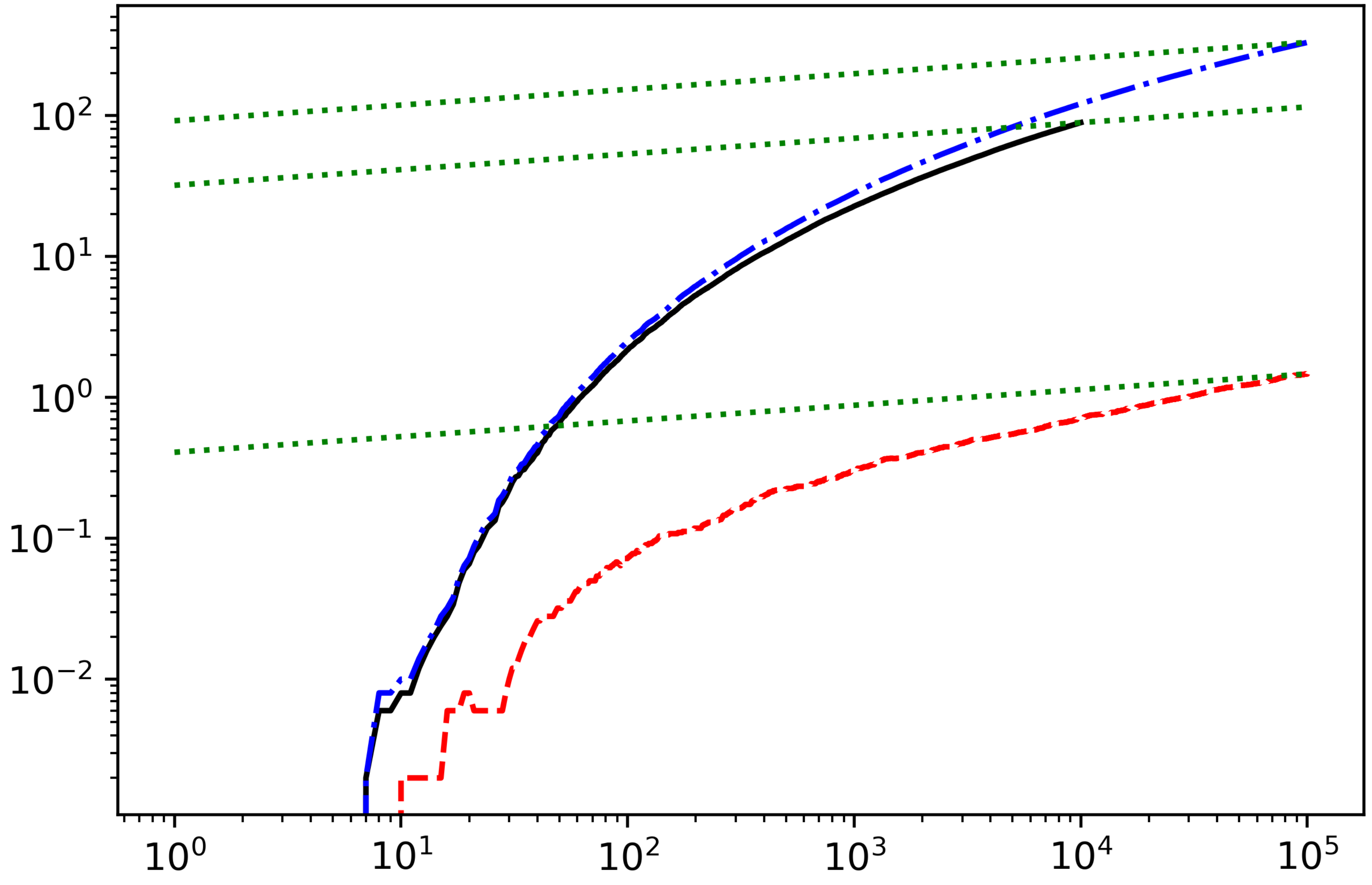




$$E[\beta_2] \approx \textbf{num of nodes}^{1-4x}$$

$$\log E[\beta_2] \approx (1-4x)\log(\textbf{num of nodes})$$





V. What lies ahead

order of magnitude of
expected Betti numbers

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers parameter estimation?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

simplicial preferential attachment?

homotopy connectedness of the infinite complex?

order of magnitude of expected Betti numbers

parameter estimation?

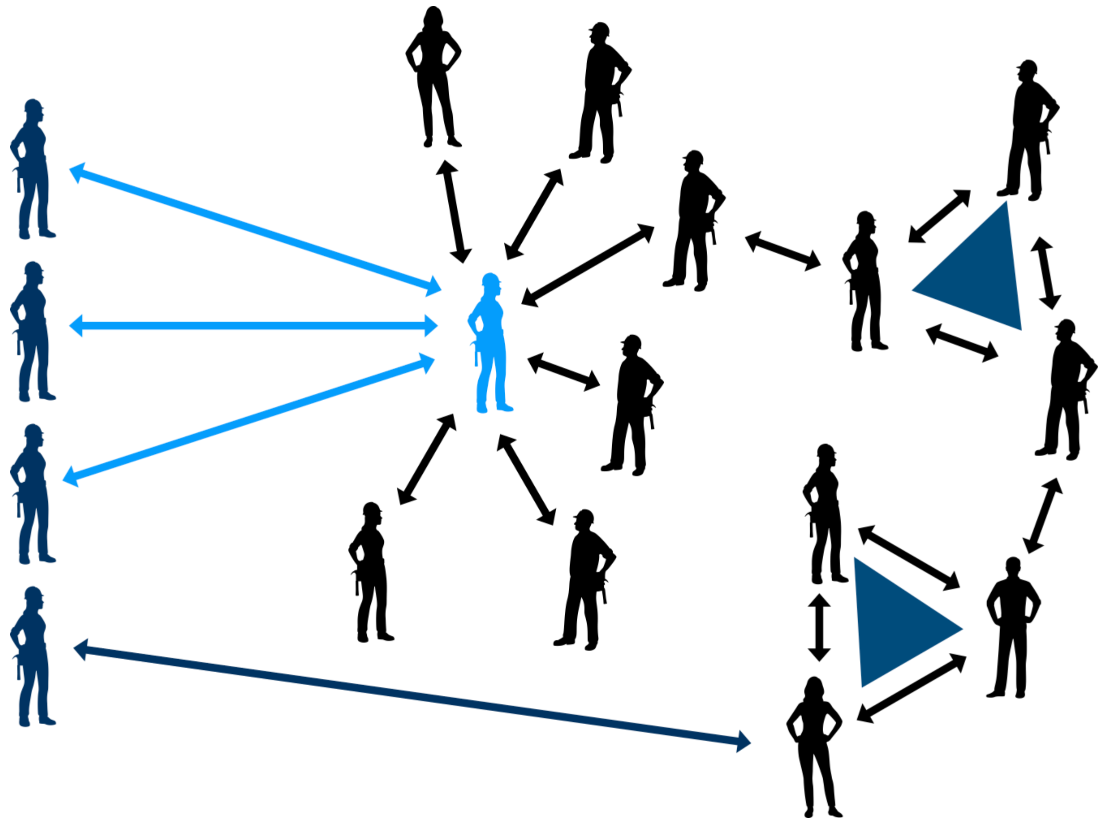
simplicial preferential attachment?

other non-homogeneous complexes?

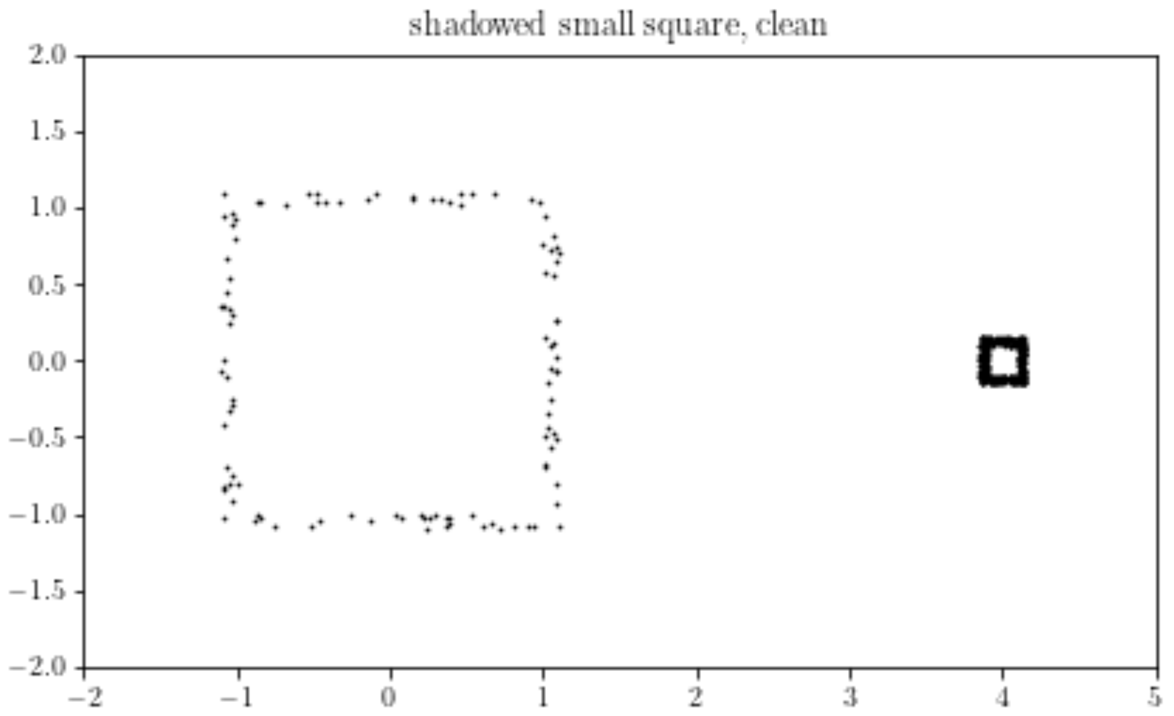
What did we learn today?

- Random topology is cool.
- Preferential attachment graph has interesting topology.
- More interesting things are waiting to be discovered.

Chunyin Siu **cs2323@cornell.edu**
Cornell University



arxiv paper

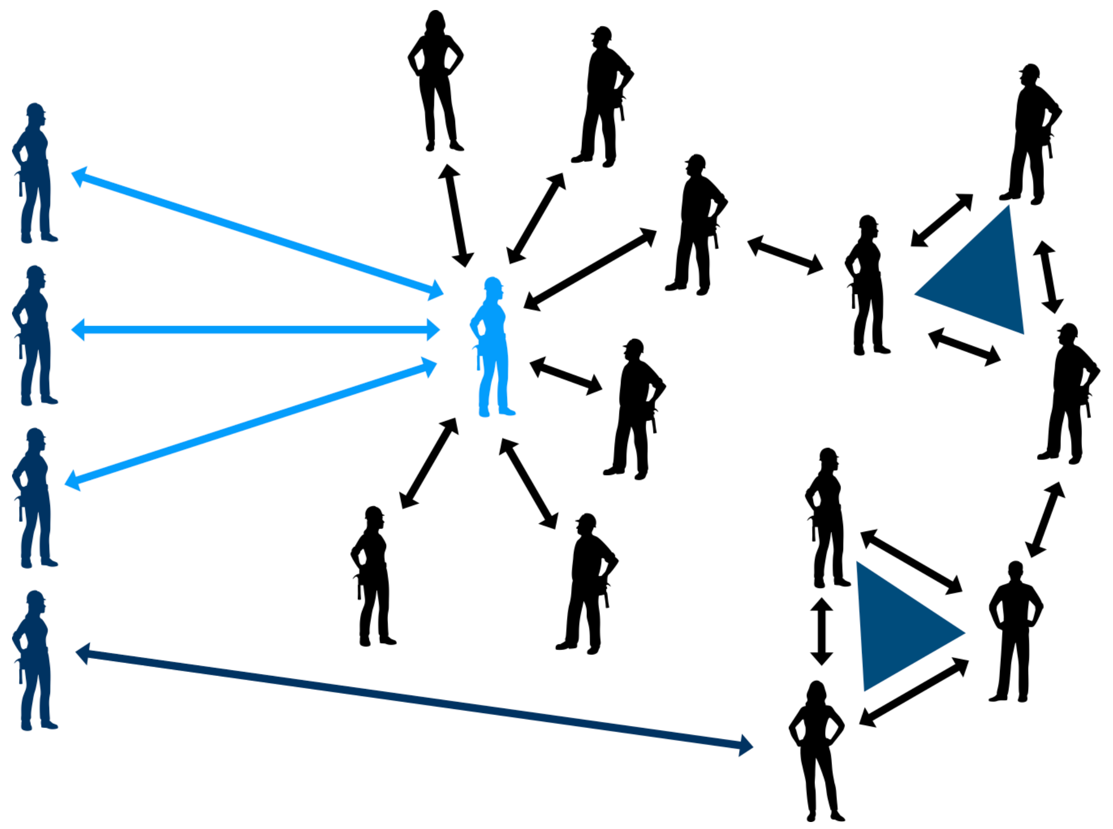


my video about small holes

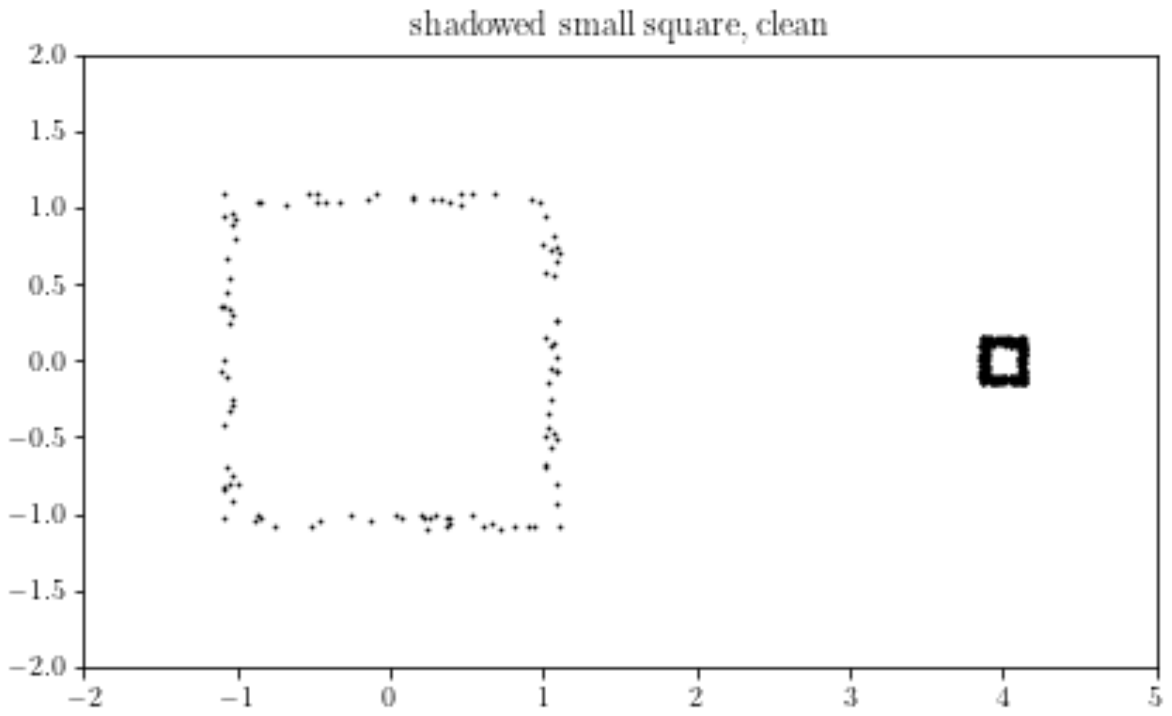
Thank you!

Chunyin Siu
Cornell University

c-siu.github.io
cs2323@cornell.edu



arxiv paper



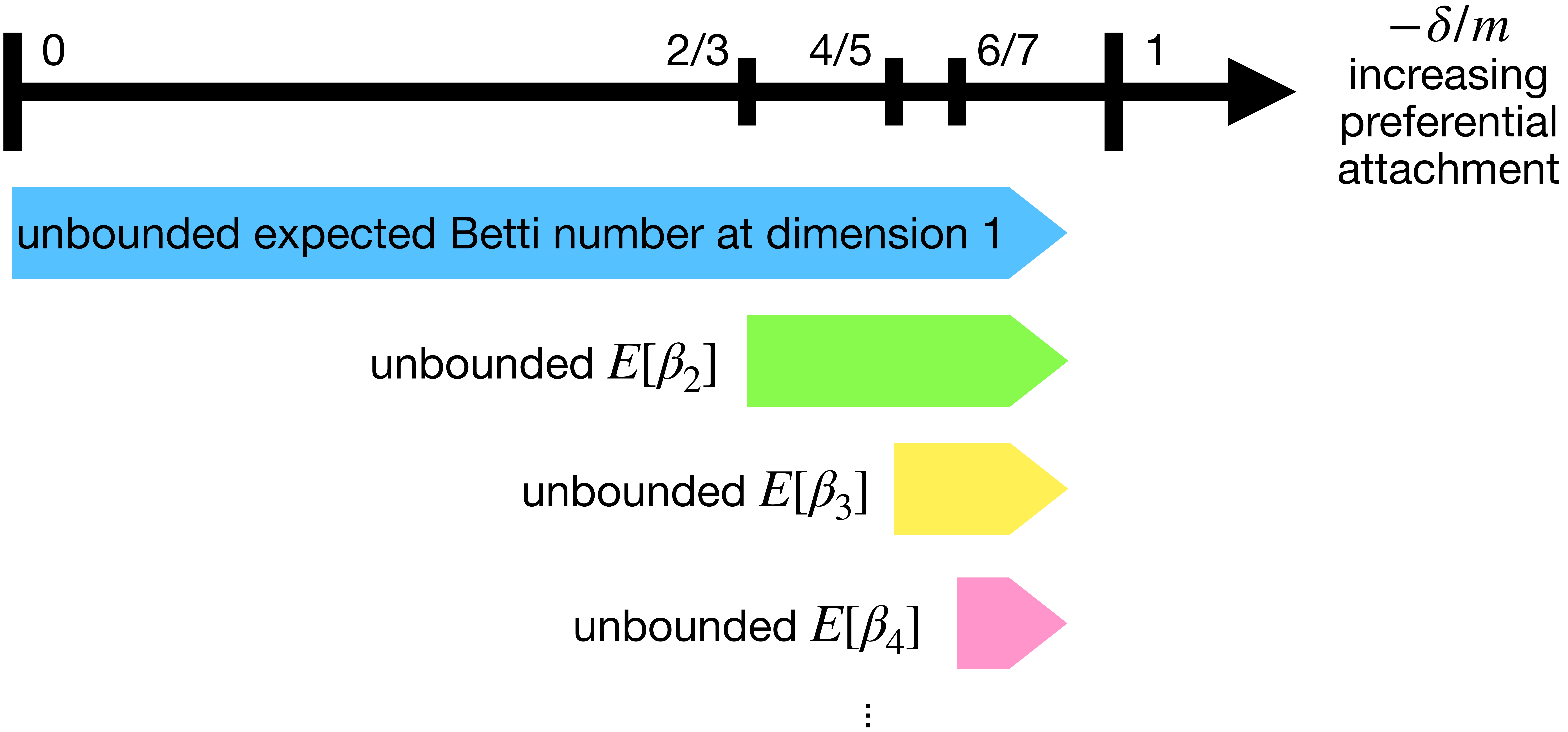
my video about small holes

Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node

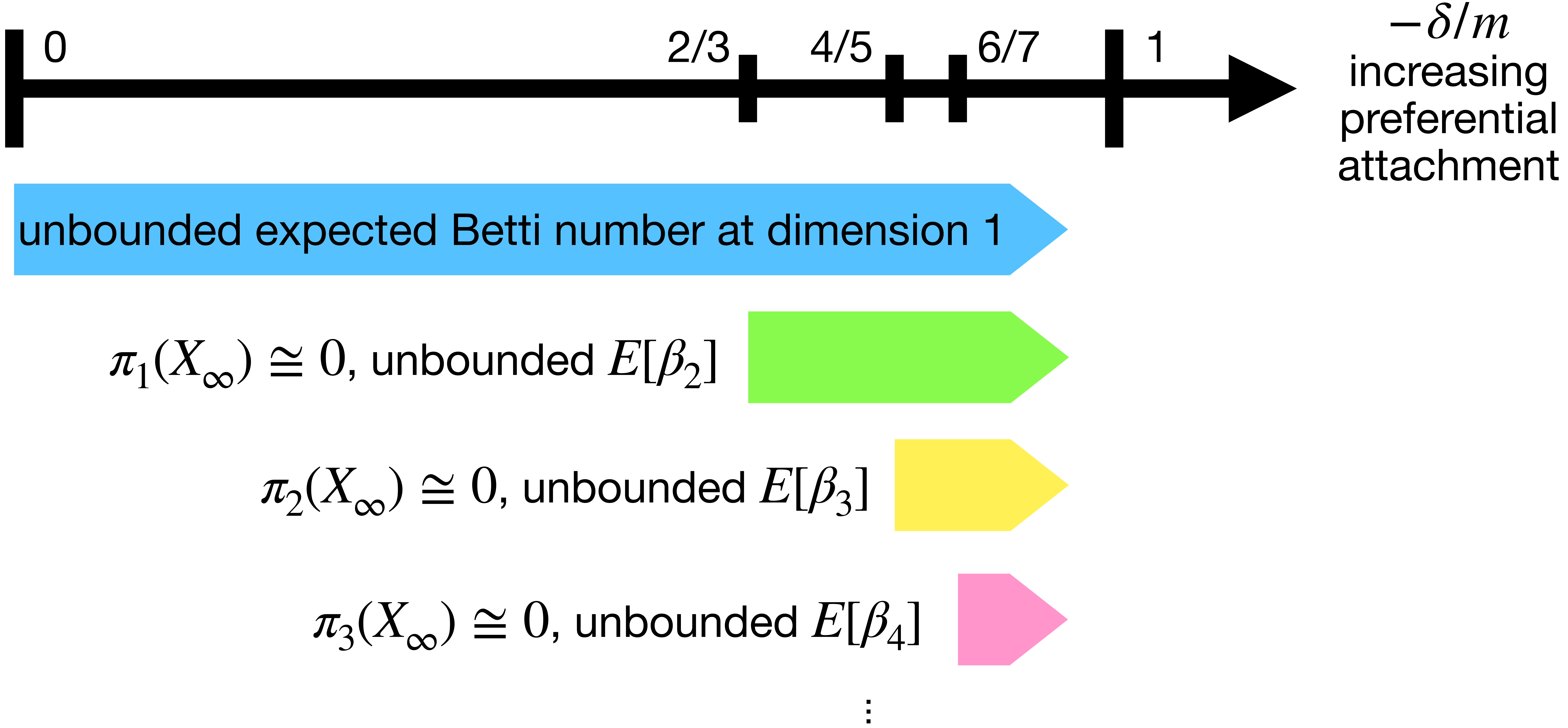


Phase transition

Recall

P(attaching to v) $\propto$ degree + δ

m = number of edges per new node

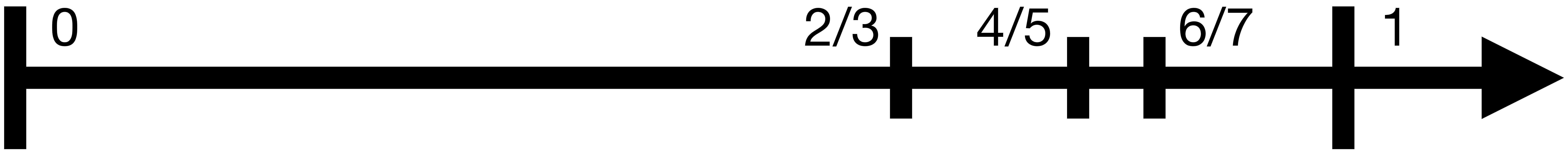


Phase transition

Recall

P(attaching to v) ∝ degree + δ

m = number of edges per new node



$-\delta/m$
increasing preferential attachment

unbounded expected Betti number at dimension 1

$\pi_1(X_\infty) \cong 0$, unbounded $E[\beta_2]$

$\pi_2(X_\infty) \cong 0$, unbounded $E[\beta_3]$

tight?

$\pi_3(X_\infty) \cong 0$, unbounded $E[\beta_4]$

⋮

Subtleties

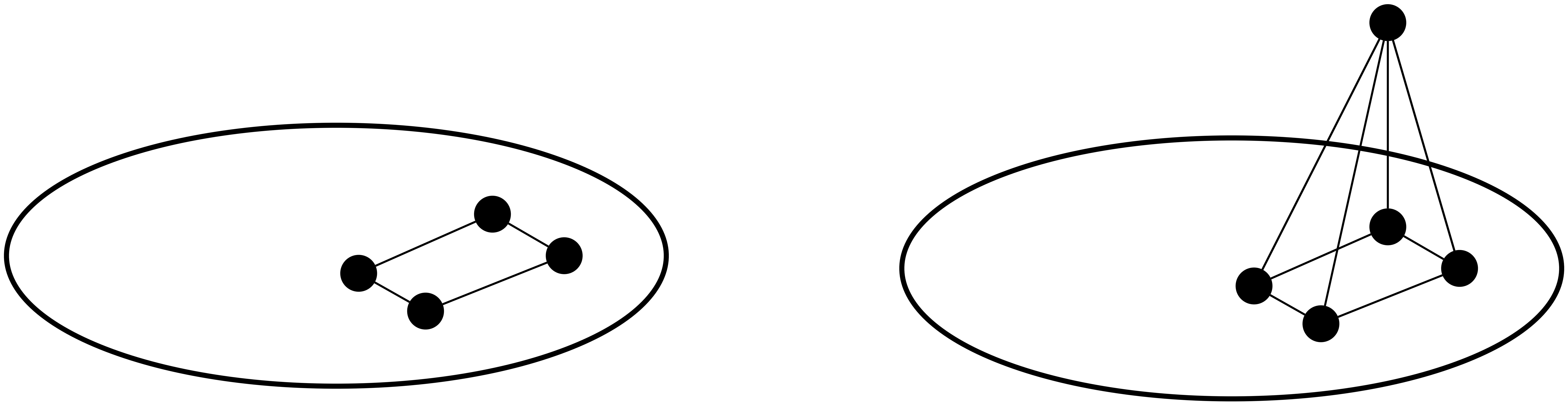
- Need homological algebra to relate Betti numbers with counts

Subtleties

- Need homological algebra to relate Betti numbers with counts
  - adding a vertex = construct mapping cone

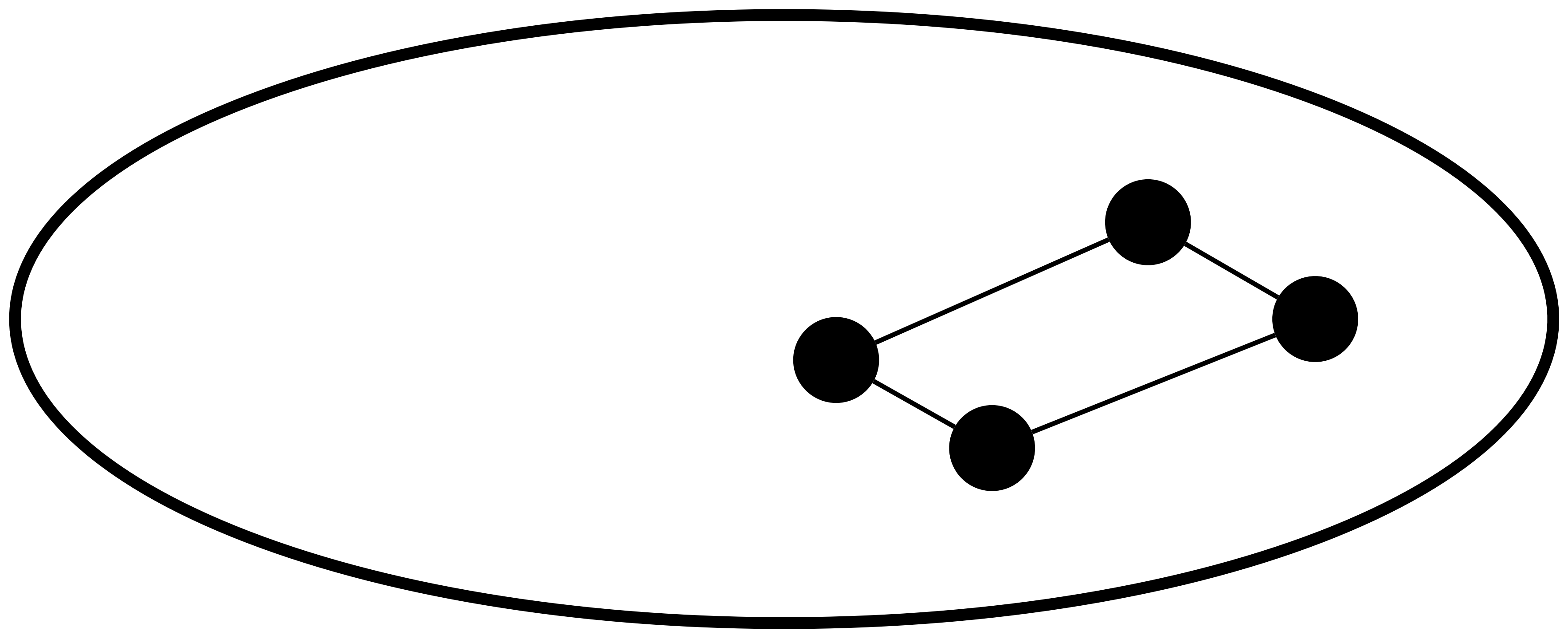
Subtleties

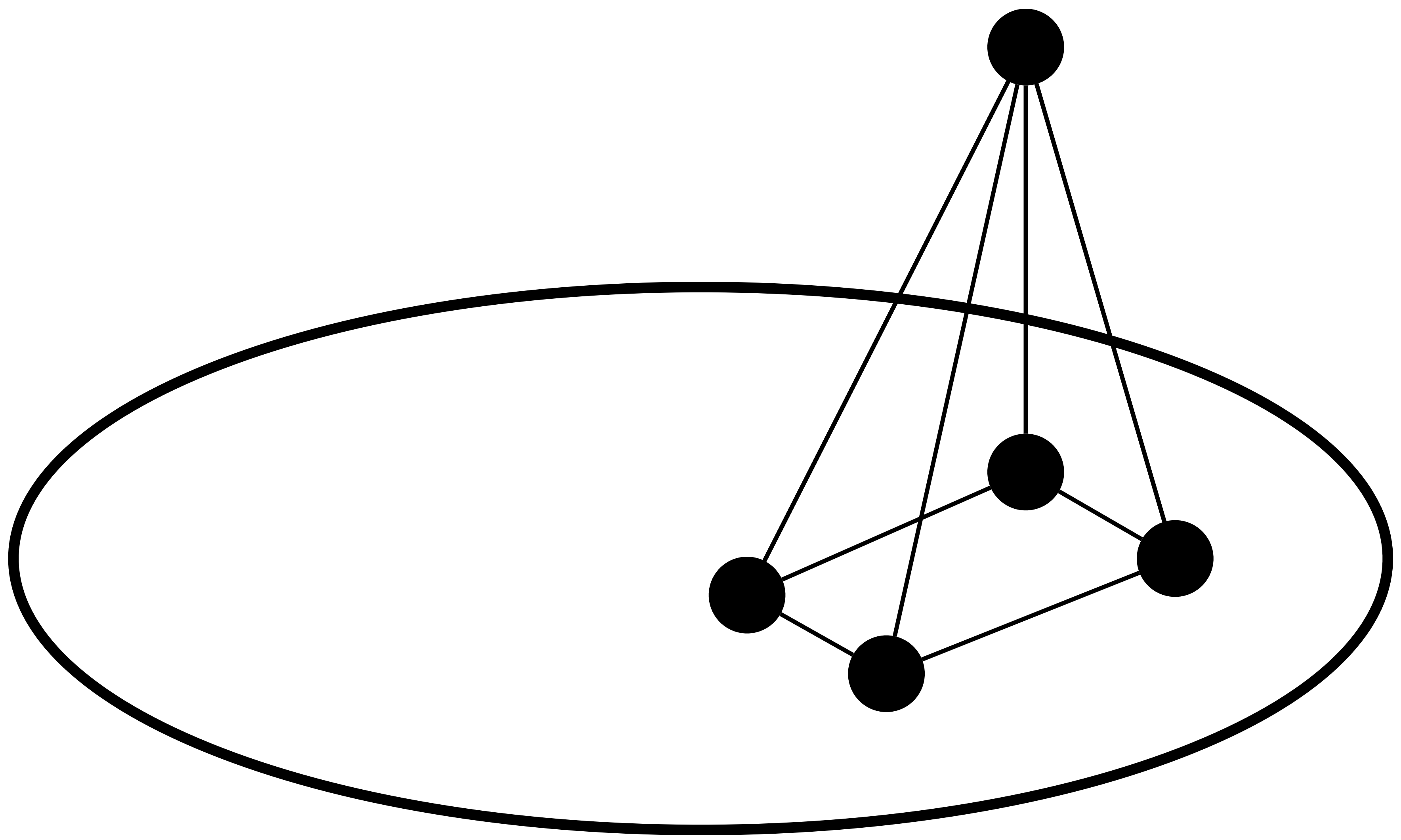
- Need homological algebra to relate Betti numbers with counts
  - adding a vertex = construct mapping cone



Subtleties

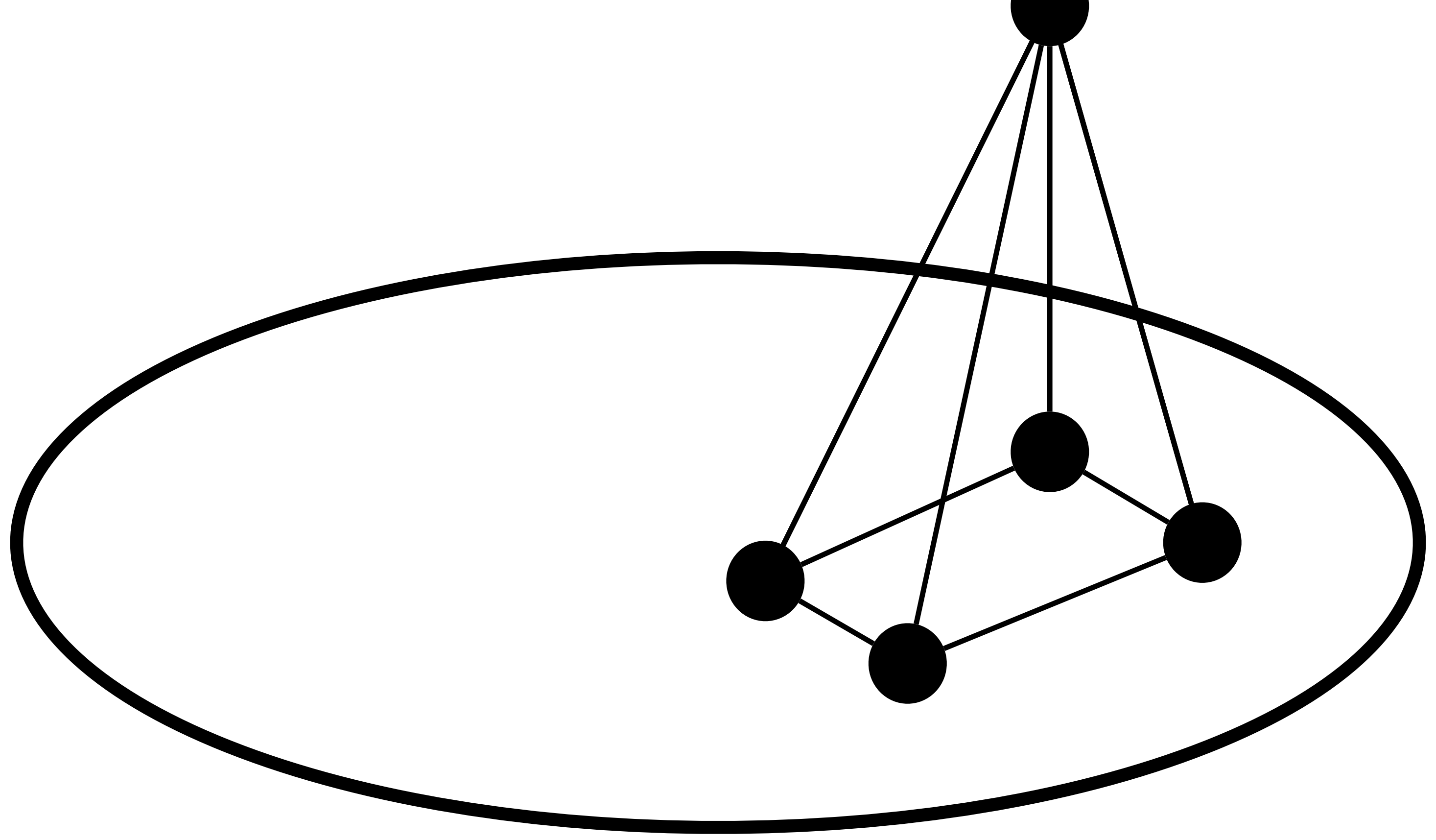
- Need homological algebra to relate Betti numbers with counts
  - adding a vertex = construct mapping cone
  - $\beta_q(\text{new}) \leq \beta_q(\text{old}) + \beta_{q-1}(\text{link})$





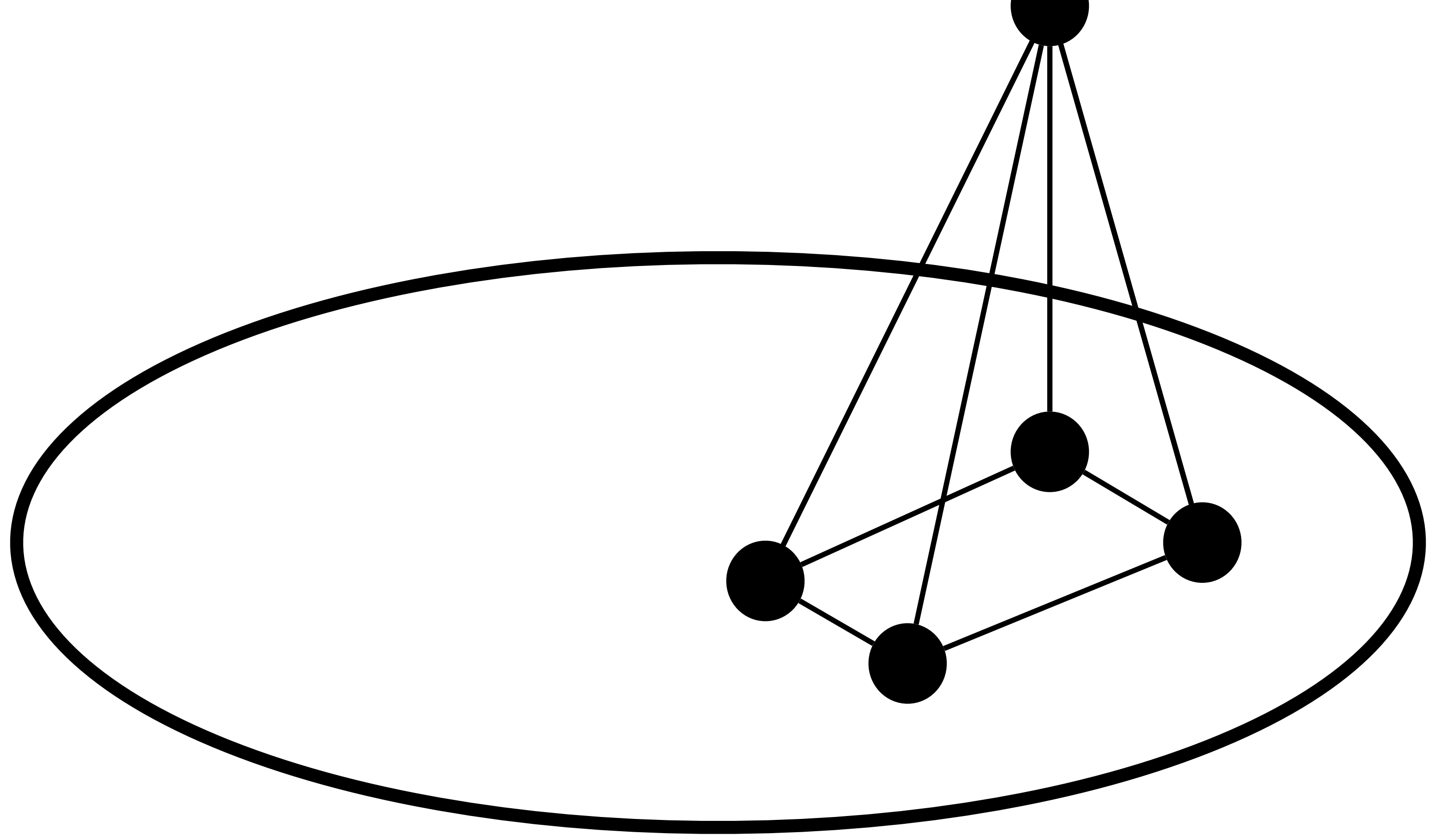
Subtleties

- Need homological algebra to relate Betti numbers with counts
  - β_q(new) $-$ β_q(old) $\leq$ β_{q-1}(link)



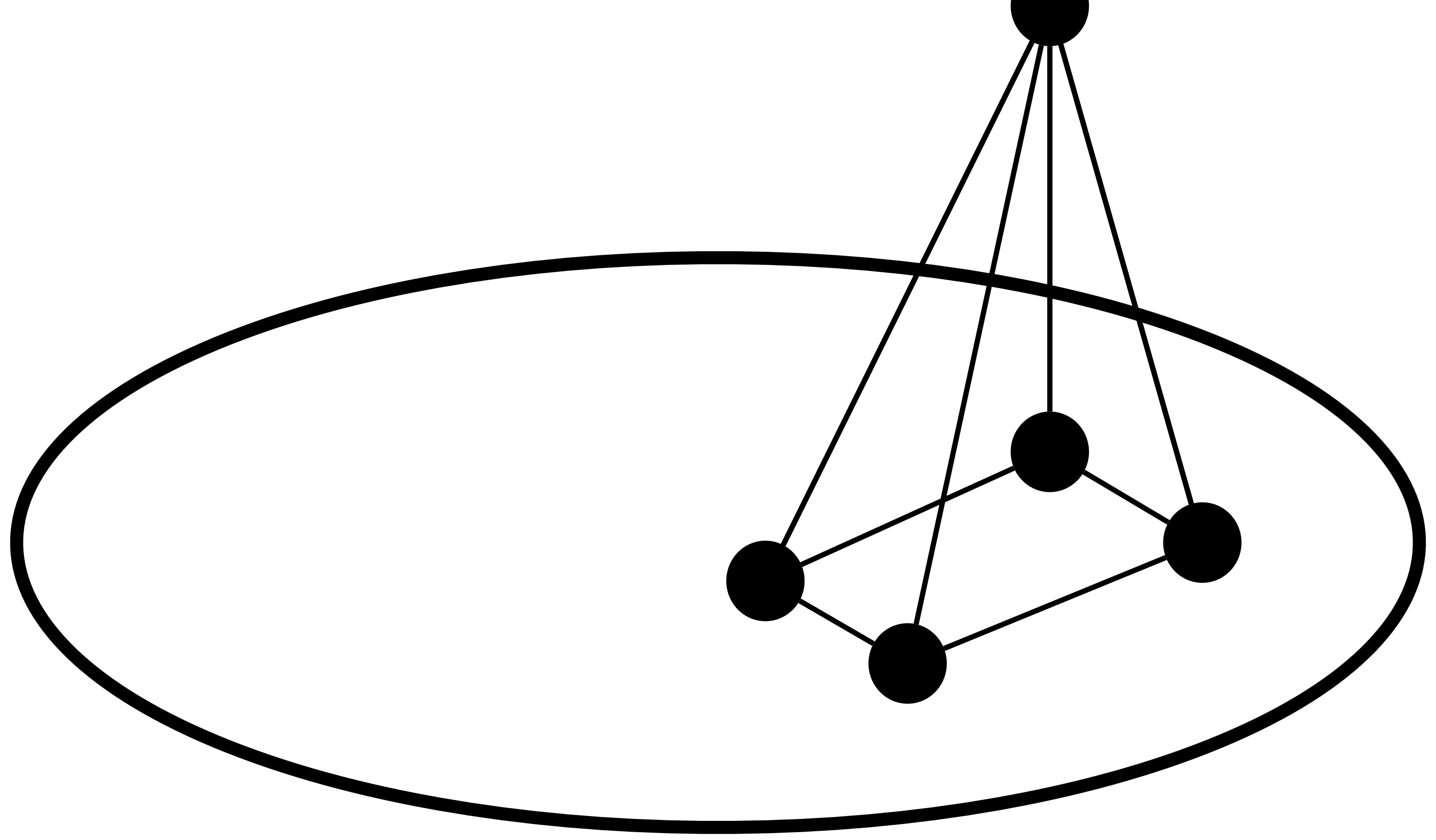
Subtleties

- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]



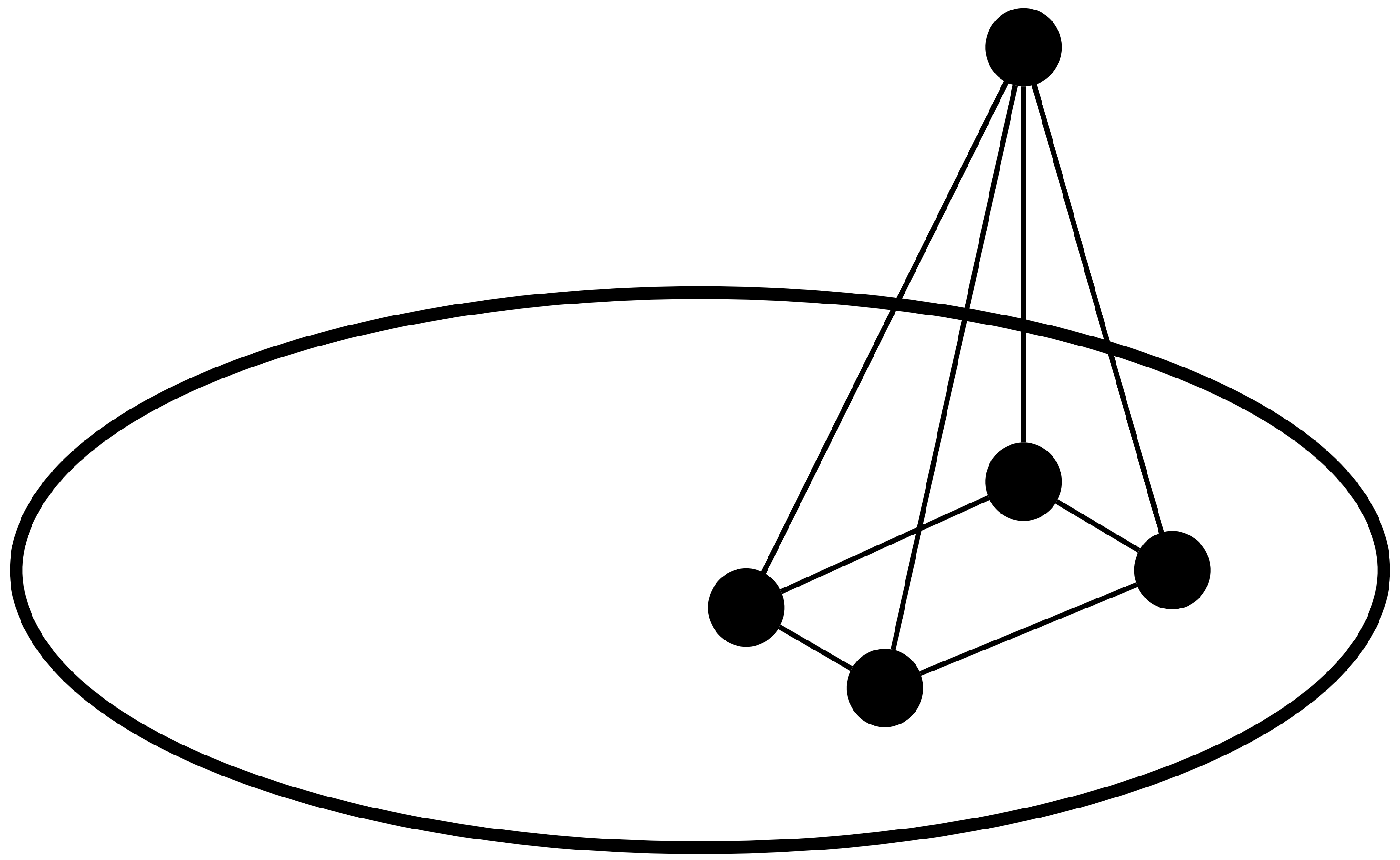
Subtleties

- Need homological algebra to relate Betti numbers with counts
    - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra



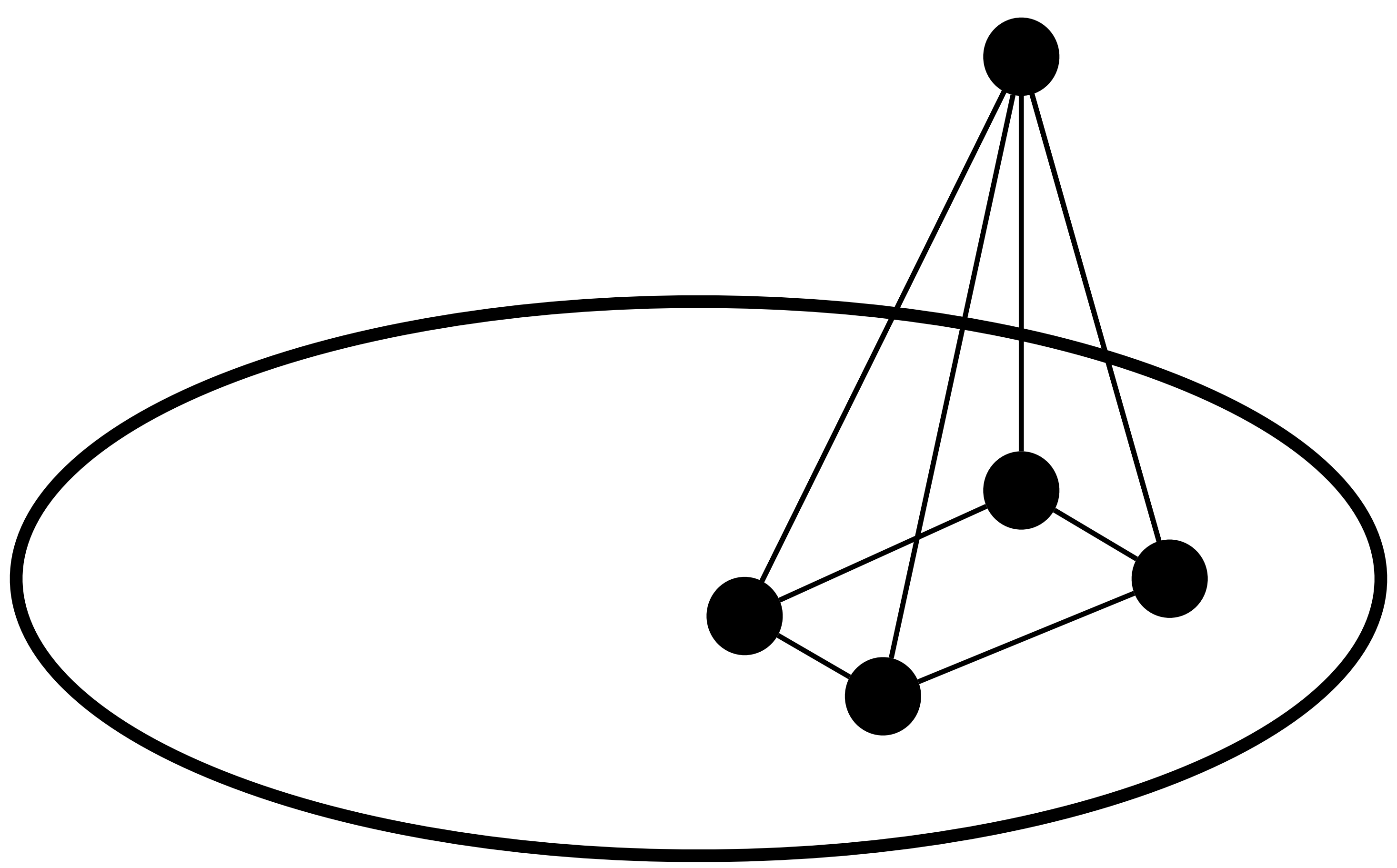
Subtleties

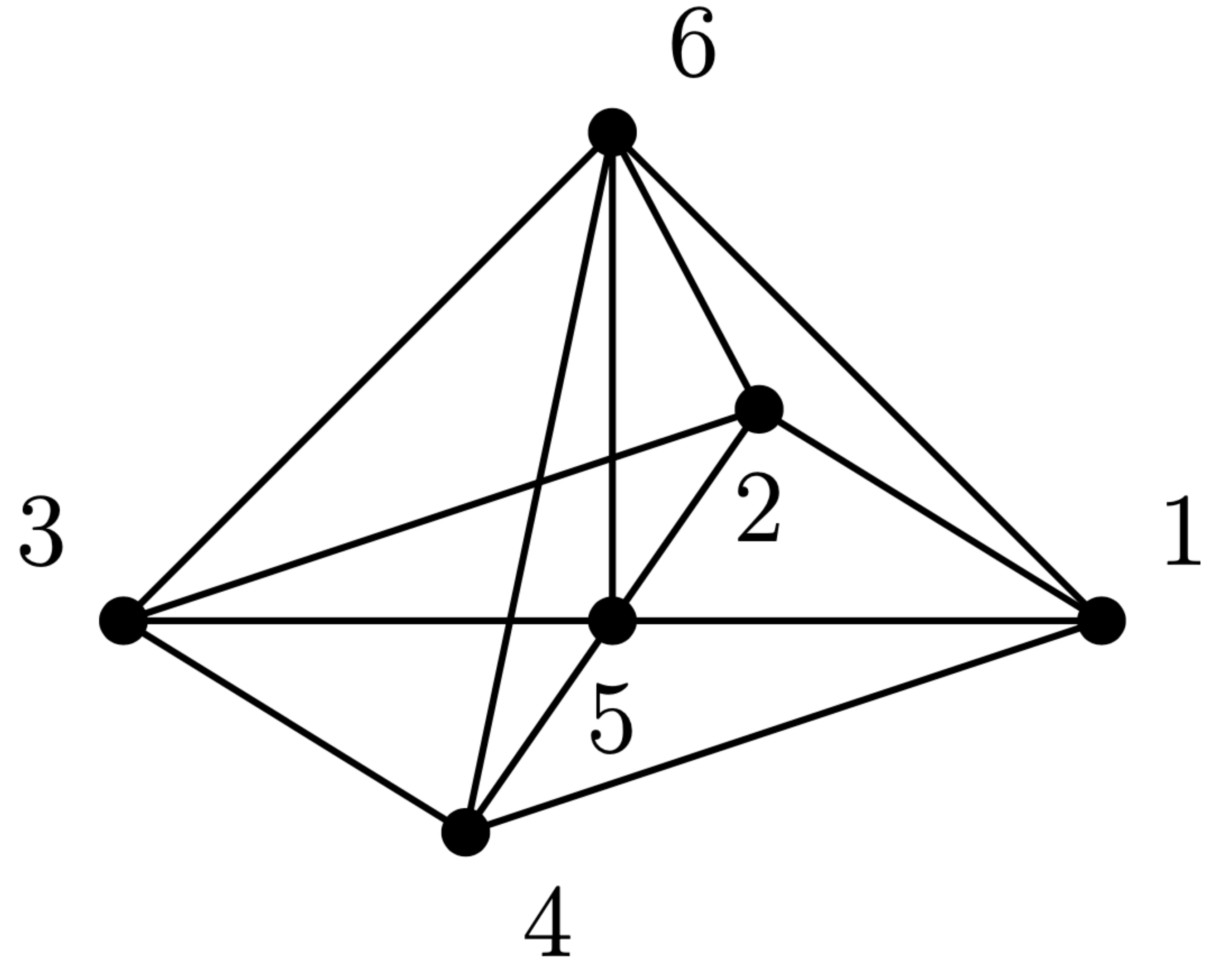
- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \le \beta_{q-1}(\text{link})$
- Identify the "square count" as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra
  - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \le \beta_q(\text{new}) - \beta_q(\text{old}) \le \beta_{q-1}(\text{link})$



Subtleties

- Need homological algebra to relate Betti numbers with counts
  - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra
  - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$





Subtleties

- Need homological algebra to relate Betti numbers with counts
    - $\beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Identify the “square count” as the main term with minimal cycle results in [Gal 2005] and [Kahle 2009]
- Generalize minimal cycle results with homological algebra
    - $1 - \beta_q(\text{link}, S^{q-1}) - \beta_q(\text{link}) \leq \beta_q(\text{new}) - \beta_q(\text{old}) \leq \beta_{q-1}(\text{link})$
- Apply graph counting result in [Garavaglia and Stegehuis 2019] on a large class of subgraphs