DRP: Category Theory for UMAP

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Week 1: Category

Read Definition 1.1.1 of [Riehl, 2017] (`https://math.jhu.edu/~eriehl/context/`) and verify the following items are categories. Get a feeling of how morphisms are like functions but not always so.

1. **Set**, whose objects are sets and morphisms are functions.
2. **Group**, whose objects are groups and morphisms are homomorphisms.
3. Let G be a group. G is a category with one object, which we denote by $*$, and its morphisms are the elements of G. ($*$ is *not* multiplication on G. We are adopting the convention of denoting nameless things by $*$.) Convince yourself that morphisms in this case still behaves like functions.

   (Hint: identify an element $g \in G$ with the map $x \mapsto gx$. The relevant keyword is Cayley theorem.)

   Note that G, on the one hand, forms a category itself, and on the other, is an object in the category **Group**. Much pain of studying category theory comes from having to navigate the different roles the same thing plays in different contexts.

4. Let P be a partially ordered set, i.e. there is a binary relations $\leq$ on P that satisfies

   - $x \leq x$ for every $x \in P$, and
   - $x \leq z$ whenever $x \leq y$ and $y \leq z$.

   P is a category whose objects are elements of P and there is exactly one morphism $x \to y$ for each pair of $x, y \in P$ such that $x \leq y$, and these are all the morphisms. Below are two non-trivial examples of partially ordered sets.

   (a) Let $I = [0, 1] \subseteq \mathbb{R}$. Let $\mathcal{P}$ be the set of finite subsets of I. $\mathcal{P}$ is ordered by inclusion, i.e. $P \leq Q$ if and only if $P \subseteq Q$. Considering this partial order simplifies proofs about Riemann integration when a finite subset of $[0, 1]$ is thought of as a partition of the interval.

(b) Let X be a topological space. The set τ of open subsets of X is again ordered by inclusion, i.e. $U \leq V$ if and only if $U \subseteq V$. τ, thus partially ordered, as a category, is denoted by $\mathbf{Open}(X)$. This example will be extended give a natural example of presheaves later.

A group as a category and a partially ordered set as a category are simple examples of categories for us to check our understanding and intuitions later as the theory develops.

Week 2: Functors

0.1 Functors

Read Definition 1.3.1 of [Riehl, 2017] (`https://math.jhu.edu/~eriehl/context/`) and verify the following items are functors. Get a feeling of how functors are like meta-functions.

1. Let A and B be groups. Recall that each group may be regarded as a single-object category. Show that a homomorphism $f : A \to B$ is a functor between the two single-object categories.

2. Let P and Q be two partially ordered sets, and hence two categories. Show that any order preserving map $f : P \to Q$ (i.e. $f(x) \leq f(y)$ whenever $x \leq y$) is a functor between them.

3. Denote by $\mathbf{VectQ}$ the category of vector spaces over $\mathbb{Q}$, not necessarily finite-dimensional. Consider the forgetful functor $U : \mathbf{VectQ} \to \mathbf{Set}$, which maps each group to its underlying set and homomorphisms to the underlying function.

4. Let S be a set. The free vector space generated by S, which we denote by $\langle S \rangle$ [1], consists of formal finite $\mathbb{Q}$-linear combinations of elements of S, and the addition is defined by matching terms (e.g. $(\frac{3}{2}a + 4b) + (-2a + 5b - c) = -\frac{1}{2}a + 9b - c)$. Every function $f : S \to T$ extends to a linear map $\tilde{f} : \langle S \rangle \to \langle T \rangle$, which is defined by

$$\tilde{f}(\sum q_i a_i) = \sum q_i f(a_i).$$

   (Formally, one may write elements of $\langle S \rangle$ as functions from S to $\mathbb{Q}$ whose values are nonzero for only finitely many elements of S, and identify the function values with the coefficients above. Explicitly, let $q : S \to \mathbb{Q}$ be a function such that $S_q = \{a \in S : q(a) \neq 0\}$ is finite. We identify q with $\sum q(a_i)a_i$, where a_i's are the distinct elements of S_q. Addition and scaling are defined pointwise, i.e. $(p+q)(a) = p(a)+q(a)$, and $(\lambda q)(a) = \lambda(q(a))$.)

[1] $\langle \cdot \rangle$ is a heavily abused notation and means different things in different places, so the notation is not completely canonical.

Consider the functor $F : \mathbf{Set} \to \mathbf{VectQ}$ defined by mapping each set S to $\langle S \rangle$, and each function $f : S \to T$ to its linear extension $\tilde{f}$. Later, we will see that U and F form an *adjunction pair*. Category theory gives us theorems to characterize when a functor belongs to an adjunction pair, and hence allows one to construct the adjoint of the forgetful functor, namely F, without having to actually write out the tedious inner structures $\langle S \rangle$.

5. Let $\mathbf{FinVectQ}$ be the category of finite-dimensional vector spaces over $\mathbb{Q}$ (with linear maps as morphisms). Let $\mathcal{C}$ be the category whose objects are nonnegative integers, whose morphisms from n to m are $\mathbb{Q}$-valued matrices of size $m \times n$, and whose composition is matrix multiplication. Consider the functor $F : \mathcal{C} \to \mathbf{FinVectQ}$ mapping n to $\mathbb{Q}^n$ and a matrix M to the linear transformation $v \mapsto Mv$. Later, we will see that $\mathbf{FinVectQ}$ and $\mathcal{C}$ are equivalent via F.

0.2 Contravariant Functors

The fundamental property of functors is that functors preserve arrows (morphisms): if the domain category of a functor F has a morphism $x \xrightarrow{f} y$, then we have, in the codomain category $Fx \xrightarrow{Ff} Fy$. Sometimes, we would like to consider "functors" G that *reverse* arrows. Instead of having $Gx \xrightarrow{Gf} Gy$, we want / need to have $Fy \xrightarrow{Gf} Gx$. Such functors are called *contravariant* functors, and they can be formally described as functors from the opposite category of the domain. To distinguish ordinary functors, i.e. those that preserve arrows, from contravariant functors, we call ordinary functors *covariant*. See Definitions 1.3.5 and 1.2.1 of [Riehl, 2017] for precise definitions, and verify the following items are contravariant functors.

1. The dual of a vector space V over $\mathbb{Q}$, denoted by V^*, is the vector space of linear maps from V to $\mathbb{Q}$. Consider the contravariant functor $h_{\mathbb{Q}} : \mathbf{FinVectQ} \to \mathbf{FinVectQ}$ that maps each vector space to its dual, and each linear transformation $T : V \to W$ to its adjoint $T^* : W^* \to V^*$, which sends a linear map $w^* : W \to \mathbb{Q}$ to $w^* \circ T$. Note that we could have replaced the codomain $\mathbb{Q}$ by any other fixed vector space V_0 over $\mathbb{Q}$ and consider the functor h_{V_0} that sends each vector space V to the vector space of linear maps from V to V_0. Such functors are examples of *Hom functors*, which we may revisit time allows.

2. Let X be a topological space and $\mathbf{Open}(X)$ be the subsets of X. Consider the contravariant functor $F : \mathbf{Open}(X) \to \mathbf{Set}$ that maps an open set U to $C(U)$, the set of continuous functions from U to $\mathbb{R}$, and an inclusion $U \subseteq V$ to the restriction function $\mathrm{res}_{U,V} : C(V) \to C(U)$, which is defined by $\mathrm{res}_{U,V}(f) = f|U$. Contravariant functors on $\mathbf{Open}(X)$ are called presheaves on X.

Week 3: Natural Transformation

0.3 Natural Transformation

Read Definition 1.4.1 of [Riehl, 2017] (`https://math.jhu.edu/~eriehl/context/`).

1. Let $\varphi, \psi : A \to B$ be homomorphisms between two groups. Recall that group homomorphisms are functors. Show that there is a natural transformation between φ and ψ if and only if they are conjugate to each other, i.e. there exists $b \in B$ such that for every $a \in A$, $\psi(a) = b\varphi(a)b^{-1}$.

2. Let $\varphi, \psi : P \to Q$ be order-preserving maps between two partially ordered sets. Recall that order-preserving maps between partially ordered sets are functors. Show that there is a natural transformation $\eta : \varphi \Rightarrow \psi$ if and only if ψ dominates φ, i.e. $\varphi(x) \leq \psi(x)$ for every $x \in P$.

3. Let X be topological space. Let $F : \mathbf{Open}(X) \to \mathbf{Set}$ be the presheaf of continuous functions, i.e. $F(U)$ is the set of real-valued continuous functions on U. Let $F_s : \mathbf{Open}(X) \to \mathbf{Set}$ be the presheaf of smooth functions, i.e. $F_s(U)$ is the set of smooth real-valued functions on U. Show that the embedding function $i_U : F_s(U) \to F(U)$ defines a natural transformation from F_s to F.

4. For each finite-dimensional vector space W over $\mathbb{Q}$, let $h^W : \mathbf{FinVectQ} \to \mathbf{FinVectQ}$ be the covariant functor that maps V to the vector space of linear maps from W to V.[2] Let $T : U \to V$ be a linear transformation. Show that T determines a natural transformation from h^V to h^U.

0.4 Equivalence of Categories

Read Definition 1.5.4.

1. Show that two groups are equivalent as categories if and only if they are isomorphic as groups.

2. Characterize equivalent partially ordered sets as categories, i.e. give a necessary and sufficient condition for the categorical equivalence of two partially ordered sets.

3. Let $\mathcal{C}$ be the category whose objects are nonnegative integers, whose morphisms from n to m are matrices of dimension $m \times n$, and whose composition is given by matrix multiplication. Show that C is equivalent to $\mathbf{FinVectQ}$, the category of finite-dimensional vector spaces over $\mathbb{Q}$.

4. Read Definition 1.5.7.

[2] Notice that we use superscript rather than subscript. The convention is that a linear map drops the input from the domain to the codomain. Since we fix the domain this time, we put it on top.

(a) Show that if $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories via $F : \mathcal{C} \to \mathcal{D}$ (and other data), then F is full, faithful and essentially surjective.

(b) Prove Theorem 1.5.9 on your own (i.e. without referring to its proof, nor Lemma 1.5.10).

5. (optional) Show that **Set** (the category of sets) and **Group** (the category of groups) are *not* equivalent categories. (Hint: Consider the cardinality of $\mathbf{Set}(S, \emptyset)$, where $\emptyset$ denotes the empty set.)

0.5 Sheaf (Optional)

This is not really an exercise, but I want to point you to a reference on real-world applications of sheaf theory: Chapter 9 of [Ghrist, 2014] (`https://www2.math.upenn.edu/~ghrist/notes.html`). Don't try to swallow it this week. In fact, throughout this project, we will not make heavy use of sheaf theory other than its formalism. I just want you to have some tangible applications to fall back on in case you feel sick of the abstract nonsense.

Week 4: Adjunction

A category X is said to be locally small if and only if the family $X(x, y)$ of morphisms is a set for every pair of objects x, y. Adjoint functors between two locally small categories are defined as follows.[3]

Definition 1. Let $F : X \to A$ and $G : A \to X$ be functors between two locally small categories. F is said to be the left adjoint of G, and equivalently, G, the right adjoint of F, if and only if

$$A(Fx, a) \cong X(x, Ga), \tag{1}$$

where $\cong$ means the existence of a bijection $\Phi_{x,a} : A(Fx, a) \to X(x, Ga)$ that is natural in the sense that, for every quadruple of morphisms $f : x \to y$, $g : a \to b$, $\varphi : Fx \to a$ and $\psi : Fy \to b$, the diagram

$$\begin{tikzcd} Fx \arrow[r, "\varphi"] \arrow[d, "Ff"] & a \arrow[d, "g"] \\ Fy \arrow[r, "\psi"] & b \end{tikzcd} \tag{2}$$

commutes if and only if the diagram

$$\begin{tikzcd} x \arrow[r, "\Phi_{x,a}(\varphi)"] \arrow[d, "f"] & Ga \arrow[d, "Gg"] \\ y \arrow[r, "\Phi_{x,a}(\psi)"] & Gb \end{tikzcd} \tag{3}$$

[3]We deviate from the definitions in both [Riehl, 2017] and [?]. Instead we have used an equivalent formulation that is more intuitive and is easier to work with. We, however, follow the notations of [?] by denoting categories by capital letters and objects by lower case letters; we also reserve consecutive letters for objects in the same category to avoid using $'$.

does.

Remark.

- The parallel with the adjoint in linear algebra comes from viewing $A(\cdot,\cdot)$ and $X(\cdot,\cdot)$ in (1) as inner products.
- Note that (2) is obtained from (3) by "throwing" G to the left and changing it to F, vice versa.
- It can be shown that the left and right adjoints of a functor are unique up to isomorphism. We will say *the* left adjoint rather than *a* left adjoint below.
- Note that when showing two functors are adjoint to each other by establishing the equivalence of the two commutative diagrams above, the existence of the maps have already been assumed, and only their commutivity needs to be shown.

1. Consider $\mathbb{Z}$ and $\mathbb{R}$ as partially ordered sets. Let $i : \mathbb{Z} \to \mathbb{R}$ be the inclusion functor, i.e. $i(n) = n$ for every integer n. Show that the ceiling function $\lceil \cdot \rceil$ and the floor function $\lfloor \cdot \rfloor$ are the left and right adjoints of i. (Hint: If you get stuck, the last remark above may help.)
2. (optional) Consider $\mathbb{R}$ and $[0,1]$ as partially ordered sets. Let $F : \mathbb{R} \to [0,1]$ be the distribution function of a random variable (hence it is a left-continuous monotone function with $\inf F(x) = 0$ and $\sup F(x) = 1$). Show that the quantile function $Q : [0,1] \to \mathbb{R}$ of F, defined by
$$Q(p) = \inf\{x : p \leq F(x)\},$$
is the left adjoint of F.
3. (optional) Let $\varphi : G \to H$ and $\psi : H \to G$ be group homomorphisms. Suppose φ is the left adjoint of ψ. Show that φ and ψ are isomorphisms. (Hint: Construct a natural transformation from id_G to $\psi\varphi$ and one from $\varphi\psi$ to id_H, and recall the characterization of natural transformations from the exercises from the previous week.)
4. Show that the forgetful functor $U : \mathbf{FinVectQ} \to \mathbf{Set}$ is the right adjoint of the free vector space functor $F : \mathbf{Set} \to \mathbf{Ab}$.
5. Show that if $F : X \to A$ and $G : A \to X$ are equivalences of categories, then F is the left adjoint of G. Therefore, adjunction is a weaker form of equivalence.

Week 5: Limit and colimit

Again, we use equivalent definitions that are differently phrased from those in [Riehl, 2017] and [**?**].

Definition 2 (Limit)**.** Let $D : J \to C$ be a functor. An object L in C, together with maps $\varphi_i : D(i) \to L$ for every object i in J, is said to be a limit of D if and only if, for every object Z in C, whenever there exist maps $\psi_i : Z \to D(i)$ such that the outer triangle below commute for every arrow (morphism) $a : i \to j$ in J, there exists a map $\psi : Z \to L$ such that the inner triangles commute.

$$\begin{tikzcd}
& Z \arrow[d, "\psi"] \arrow[ddl, "\psi_i"'] \arrow[ddr, "\psi_j"] & \\
& L \arrow[dl, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\
D(i) \arrow[rr, "D(a)"] & & D(j)
\end{tikzcd} \tag{4}$$

The colimit can be defined by "reversing" all the arrows in (4). Precisely, we have

Definition 3 (Colimit)**.** Let $D : J \to C$ be a functor. An object L in C, together with maps $\varphi_i : L \to D(i)$ for every object i in J, is said to be a colimit of D if and only if, for every object Z in C, whenever there exist maps $\psi_i : D(i) \to C$ such that the outer triangle below commute for every arrow (morphism) $a : i \to j$ in J, there exists a map $\psi : L \to Z$ such that the inner triangles commute.

$$\begin{tikzcd}
& Z & \\
& L \arrow[u, "\psi"] & \\
D(i) \arrow[uur, "\psi_i"] \arrow[ur, "\varphi_i"] \arrow[rr, "D(a)"] & & D(j) \arrow[uul] \arrow[ul, "\psi_j"', "\varphi_i"]
\end{tikzcd} \tag{5}$$

Remark. It can be shown that the limit and colimit, if existent, are unique up to isomorphism.

1. (Limits are like sequential limits.) Let S be a set and $P(S)$ be its power set, i.e. the set of all subsets of S. Then $P(S)$ is a partially ordered set ordered by inclusion. Let $D : \mathbb{Z} \to P(S)$ be a functor between the two categories of partially ordered sets. Show that the limit and colimit of D are $\cap_n D(n)$ and $\cup_n D(n)$ respectively.

2. (Limits are like analytical limits.) Let P be a partially ordered set and $D : J \to P$ be a full functor (i.e. for every pair of objects i, j in J, every morphism between $D(i)$ and $D(j)$ is the image of some morphism $i \to j$ under D). Show that the limit of D is the greatest lower bound of the

image of D ($\{D(i) : i \text{ an object of } J\}$), and the colimit of D is the least upper bound.

3. (Non-linear indexing set; optional) Let $\mathcal{P}$ be the set of finite increasing sequences starting with 0 and ending with 1 in $[0, 1]$. Elements of $\mathcal{P}$ are called partitions, and we think of them as partitions of the unit interval into smaller subintervals. For example,

$$0 \leq 1/5 \leq 2/5 \leq 3/5 \leq 4/5 \leq 1$$

is the partition that breaks up the unit interval into 5 equal pieces. $\mathcal{P}$ is partially ordered by inclusion, i.e. $(t_i)_{i=0}^m \leq (s_j)_{j=0}^n$ if and only if $\{t_i\} \subseteq \{s_j\}$.

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function, and $L : \mathcal{P} \to \mathbb{R}$ be the lower Riemann sum function, i.e.

$$L((t_i)_{i=0}^m) = \sum_{i=0}^{m-1} \max_{x \in [t_i, t_{i+1}]} f(x)(t_{i+1} - t_i).$$

Show that the limit of L is the Riemann integral $I = \int_0^1 f(x)dx$ of f on $[0, 1]$.

4. (Finite limit; limits aggregate data.) Let J be the category with two distinct objects and no non-identity morphisms. Let $D : J \to \mathbf{Set}$ be a functor. Denote by A and B the two image sets under D. Show that the limit of D is the Cartesian product $A \times B$ and the colimit is the disjoint union $A \coprod B$, i.e. the union of A and B, except that we keep track of where the element come from, so each element in the intersection gives rise to two elements in the disjoint union. Formally, this can be described in set language as

$$A \coprod B = \{(a, 0) : a \in A\} \cup \{(b, 1) : b \in B\}.$$

5. (Equalizer and coequalizer; limits aggregate data, again; optional)

Let J be a category with exactly two objects s and t, two morphisms a, b from s to t, and no other non-identity morphisms. Let $D : J \to \mathbf{Set}$ be a functor. Let $X = Ds$, $Y = Dt$, $f = Da$, $g = Db$. Pictorially, we have

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y.$$

Remark. Since all functors can be represented as *diagrams* like the one above, we call them diagrams, and this is why we denote functors by D. The formal definition of diagrams is that a diagram D of shaped J in C is a functor $D : J \to C$. The only difference between a functor and a diagram is the perspective.

(a) Show that the limit of D is $L = \{x \in X : f(x) = g(x)\}$ with $\varphi_s : L \to X$ being the inclusion map, i.e. $\varphi_s(x) = x$, and $\varphi_t : L \to Y$ is f. L is called the equalizer of f and g.

(b) (optional) Let $D : J \to \mathbf{FinVectQ}$ be a functor. Let g be the zero map, i.e. the map that sends all vectors to the zero vector in the codomain. Show that the limit of D is the kernel of f. This is the categorical way of thinking of kernels (or, a semi-categorical way, since we are still defining zero map by the values it takes; a fully categorical approach would be to define $\{0\}$ as the object with exactly one morphism between any object and itself in either direction, and the zero map as a morphism that is the composition $\varphi\psi$ such that the domain of φ and the codomain of ψ are $\{0\}$).

(c) Colimits of functors on J are called coequalizer. We show the quotient set of an equivalence relations is a coequalizer. Let $\sim$ be an equivalence relations on a set A. Let

$$G = \{(a, b) \in A^2 : a \sim b\}$$

be the graph of $\sim$. Let $X = G$, $Y = A$, f and g be the projections to the first and second coordinates. Show that the quotient set $A/\sim$ is the colimit of D.

Remark. Equalizers and coequalizers are important because many constructions boil down to the construction of a giant complicated equalizer or coequalizer.

6. (optional) Show that the limit and colimit are unique up to isomorphism, i.e. if L and K are two (co-) limits, then there exist morphisms $f : L \to K$ and $g : K \to L$ such that $fg = \mathrm{id}_K$ and $gf = \mathrm{id}_L$.[4]

More on Limits and Adjunction

0.6 Existence of Limits

Limits and colimits may not exist.

1. Consider the simplicial category $\mathbf{\Delta}$, whose objects are nonempty finite totally ordered sets and whose morphisms are order-preserving functions. Let J be the discrete two-object category, whose objects are 0 and 1 and whose only morphisms are the identities. Consider the functor $D : J \to \mathbf{\Delta}$ such that $D(0)$ and $D(1)$ are both the singleton. Show that D has no colimit.

[4]We did not show left and right adjoints are unique. Showing it from scratch is tedious. One way to show it efficiently is to show left and right adjoints are in fact limits of appropriate functors.

2. Consider again the simplicial category $\mathbf{\Delta}$ and the two-object category J. Consider the functor $D : J \to \mathbf{\Delta}$ such that each of $D(0)$ and $D(1)$ is the two-element totally ordered set. Show that D has no limit.

3. Consider the category M of metric spaces, whose objects are metric spaces and whose morphisms are non-expansive maps. Show that the functor D defined in question 1, now taking values in M rather than $\mathbf{\Delta}$, again has no colimit.

0.7 Colimit and Density

A subcategory D of a category C is said to be dense if and only if every object in C is a canonical colimit of objects in D (i.e. colimit of a functor whose codomain is D). This is defined in Section X.6 of [MacLane, 1998], but we will *not* go into the technical definition of “canonical”. Instead, we motivate the concept with examples, and content ourselves with a weaker, and hence less useful, but more intuitive notion of “denseness” by ignoring canonicality.

1. We write arbitrary sets as colimits of singletons. Let S be a set. Let $\bar{S}$ be the category whose objects are elements of S and whose only morphisms are the identity morphisms. Let $D_S : \bar{S} \to \mathbf{Set}$ be the functor that maps each element of S to the singleton. Show that S is the colimit of D_S.

   Remark. This functor is in fact the canonical one.

2. We write partially ordered sets as colimits of finite totally ordered sets. Let P be the partially ordered set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ whose partial order is defined by

$$\begin{array}{cccccccc}
0 & & \leq 2 & \leq 3 & \leq 4 & \leq 5 & & \leq 7 \\
& 1 & \leq 2 & & & & & \\
& & & 3 & & & \leq 6 & \leq 7
\end{array}$$

   Let J be category with objects $s_0, s_1, s_2, t_0, t_1, t_2$ and non-identity morphisms $a_i : s_i \to t_i$ and $b_i : s_i \to t_{i+1 \mod 3}$, $i = 0, 1, 2$. Pictorially, we have

$$\begin{tikzcd}
s_0 \arrow[r] \arrow[dr] & t_0 \\
s_1 \arrow[r] \arrow[dr] & t_1 \\
s_2 \arrow[r] \arrow[uur] & t_2
\end{tikzcd}$$

   Let $D : J \to \mathbf{PoSet}$ defined by

$$\begin{aligned}
Dt_0 &= \{0 \leq 2 \leq 3 \leq 4 \leq 5 \leq 7\} \\
Dt_1 &= \{1 \leq 2\} \\
Dt_2 &= \{3 \leq 6 \leq 7\} \\
Ds_i &= Dt_i \cap Dt_{i+1 \mod 3} \text{ for } i = 0, 1, 2
\end{aligned}$$

and the images of arrows under D are induced by inclusion. Show that $P = \operatorname{colim} D$.

3. Let P be a partially ordered set. Let $\mathcal{T}$ be the set of finite totally ordered subsets of P. Note that $\mathcal{T}$ is partially ordered by inclusion, and hence is a category. Let $D : \mathcal{T} \to$ **PoSet** be the functor that maps each totally ordered set to itself as a partially ordered set, and each inclusion arrow to the inclusion map. Show that $P = \operatorname{colim} D$.

   Remark. This colimit is not canonical. The reason is that the indexing category $\mathcal{T}$ is not big enough, and hence the resultant universal mapping property (the commutativity of the diagram that defines the colimit) is not strong enough. However, we will not go into that.

0.8 Adjunction and Limit

1. Show that right adjoints preserve limits. The precise statement to be proven is as follows. Let $G : A \to X$ is the right adjoint to a functor F. Let L be the limit of a functor $D : J \to A$. Then GL is the limit of GD.

2. Let **FinVectZ2** be the category of finite-dimensional vector spaces over the 2-element field $\mathbb{Z}_2 = \{0, 1\}$. (We will not need any properties of $\mathbb{Z}_2$ other than that it is finite, and that it is a field and hence vector spaces over it is well-defined.) Verify the above statement for the forgetful functor U : **FinVectZ2** $\to$ **Set** and the discrete two-object category J, which has exactly two objects and no non-identity morphisms.

3. Define U and J as in the previous question, show that U does not preserve colimits. (Hint: Colimits of $D : J \to$ **Set** are disjoint unions and colimits of $D : J \to$ **FinVectZ2** are direct sums.)

4. Show that left adjoints preserve colimits.

Week 7: Hom Functors

This week and the next, we study two important theorems in category theory. They are inevitably very abstract, since definitions in category theory are abstract in the first place. The proofs themselves are not very insightful either. Read through all exercises before you parse the theorems, and read through both theorems before you dive into their proofs. This will give you an overall picture before the details can get to confuse you. The exercises are arranged in the logical order rather than in ascending order of difficulty. Attempt them in ascending order of difficulty. For instance, apply a theorem before proving it.

1 Hom Functors and Representable Functors

The philosophy of category theory is that everything can be understood through its relationship with other objects, and dissection of a set to examine its elements is unnecessary. Indeed, limits are defined in terms of arrows to and from other objects, rather than "the elements they contain", which is a concept that cannot even be written down in the language of category theory.

Expanding on the relationship-centric philosophy, we may identify an object x with its *Hom functors*, which we define below, and then we may study the object x through its actions on other objects. Later we will see Hom functors are useful in other ways.

Definition 4. Let x be an object of a category X. The covariant Hom functor $h^x : X \to \mathbf{Set}$ represented by x is defined by

$$h^x(a) = X(x, a)$$

for every object a of X, and

$$h^x(a \xrightarrow{f} b) = [(x \xrightarrow{\varphi} a) \mapsto (x \xrightarrow{\varphi} a \xrightarrow{f} b)]$$

for every morphism $a \xrightarrow{f} b$ in X. The contravariant Hom functor represented by x, denoted by $h^x : X \to \mathbf{Set}$ and $h^x : X^{op} \to \mathbf{Set}$ is defined by

$$h_x(a) = X(a, x)$$

for every object a of X, and

$$h_x(a \xrightarrow{f} b) = [(b \xrightarrow{\psi} x) \mapsto (a \xrightarrow{f} b \xrightarrow{\psi} b)]$$

for every morphism $a \xrightarrow{f} b$ in X.

Definition 5. A functor is said to be representable (resp. represented by an object x) if and only if it is naturally isormophic (i.e. a natural transformation consisting of isomorphism) to a Hom functor (resp. represented by x).

1. Consider the category **PoSet** of partially ordered sets (objects are partially ordered sets rather than elements of one partially ordered sets!). Show that the forgetful functor $U : \mathbf{PoSet} \to \mathbf{Set}$ is represented by the one-element partially ordered set $\{0\}$.

2. Consider the category **VectQ** of vector spaces of $\mathbb{Q}$, not necessarily finite dimensional. Show that the forgetful functor $U : \mathbf{VectQ} \to \mathbf{Set}$ is represented by $\mathbb{Q}$.

3. Consider the category **FinVectQ** of finite-dimensional vector spaces over $\mathbb{Q}$. Let F be the functor such that $F(V)$ is the set of linear maps from V to $\mathbb{Q}$. Show that F is representable.

4. Let G be a group and UG be the underlying set of G. Let $\mathbf{G}$ be the one-object category whose morphisms are elements of G, and $*$ be the unique object in $\mathbf{G}$. Let $F : \mathbf{G} \to \mathbf{Set}$ be the functor such that $F* = UG$ and $Fg = (x \mapsto gx)$, where gx denotes the product of g and x in G. Show that F is representable.

5. Characterize Hom functors on a partially ordered set (*not* the category of all partially ordered sets!) by unravelling the definitions.

6. Recall simplicial sets are contravariant functors (Example 1.3.7(vi) of [Riehl, 2017], or Definition 3.10 of [Friedman, 2008]). Identify simplicial sets that are representable contravariant functors.

2 Week 8: Yoneda Lemma and Density Theorem

2.1 Yondea Lemma

Yoneda lemma justifies the idea of identifying an object x with its Hom functors h^x and h_x. For this idea to work, we must ensure that no information is lost when we pass to Hom functor. More precisely, we need naturally isomorphic Hom functors to be represented by isomorphic objects (objects with morphisms between them that compose to the two identities of the two objects). This is a corollary of Yoneda lemma. Density theorem in the following section depends on Yoneda lemma as well.

1. Read the following part of Section 2.2 of [Riehl, 2017]: from Example 2.2.2 to the proof of Theorem 2.2.4 (inclusively).

2. In your own words, write down Theorem 2.2.4 and prove the bijection part of the theorem.

3. In this question, we show that the Yoneda lemma is a generalization of the following trivial linear algebraic fact:

$$e_i \cdot v = v_i \tag{6}$$

   for every $v \in \mathbb{R}^n$, where $\cdot$ denotes the dot product, v_i denotes entry i of the tuple v and e_i denotes the i^{th} standard basis vector, i.e. the tuple whose entry i is 1 and all other entries of which are zero. Note the formal similarity with Yoneda lemma. The moral is that functors (or presheaves) behave like tuples, representable functors behave like standard basis vectors, and Hom behaves like dot product. Throughout this question, denote by $|s|$ the cardinality of a set s and $|S|$ the entrywise cardinality if $S = (s_1, ..., s_n)$ is a tuple of sets, i.e.

$$|S| = (|s_1|, ..., |s_n|).$$

   (a) Let C_n be the category with n objects $1, 2, ..., n$ with no non-identity morphisms. Show that set-valued functors on C_n are n-tuples of sets.

   (b) Show that the functor h^i represented by $i \in \mathrm{ob}(C_n)$ satisfies

$$|h^i| = e_i.$$

(c) Let **Z2Vect** be the category of vector spaces over $\mathbb{Z}/2$. (We choose $\mathbb{Z}/2$ for the sake of concreteness. The argument holds for any finite field.) Let $F : \mathbf{Set} \to \mathbf{Z2Vect}$ and $U : \mathbf{Z2Vect} \to \mathbf{Set}$ be the free-forgetful adjunction pair. Recall that F is the left adjoint of U. Show that for set-valued functors S, T on C_n,

$$\dim \operatorname{Hom}(FT, FS) = |T| \cdot |S|.$$

(d) Using Yoneda lemma, show that for every $i \in \operatorname{ob}(C_n)$ and every tuple $S = (s_1, ..., s_n)$ of sets,

$$|\operatorname{Hom}(Fh^i, FS)| = |\operatorname{Hom}(h^i, UFS)| = |(UF)(s_i)|.$$

Hence, show that (6) is true for $v \in \mathbb{Z}^n_{\geq 0}$, where $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers.

(e) Using linearity and continuity, show that (6) is true for $v \in \mathbb{R}^n$.

(f) Show that for tuples $S = (s_1, ..., s_n), T = (t_1, ..., t_n)$ of sets,

$$\log |\operatorname{Hom}(S, T)| = |S| \cdot \log |T|,$$

where the last logarithm is applied entrywise. The moral is that the non-commutativity of Hom makes it fundamentally different from the dot product, despite their similarity.

4. Show that if h^x and h^y are naturally isomorphic (i.e. there exist natural transformations $\alpha : h^x \Rightarrow h^y$ and $\beta : h^y \Rightarrow h^x$ such that for every object a, $\beta_a \alpha_a = \mathrm{id}_{h^x(a)}$ and $\alpha_a \beta_a = \mathrm{id}_{h^y(a)}$), then x and y are isomorphic (i.e. there exist morphisms $f : x \to y$ and $g : y \to x$ such that $gf = \mathrm{id}_x$ and $fg = \mathrm{id}_y$).

5. Let P be a partially ordered set. For $x \in P$, let $D_x = \{y \in P : x \leq y\}$. Show that for every $x, y \in P$, $x = y$ whenever $D_x = D_y$

   (a) using Yoneda lemma (or the previous question, which is a corollary of Yoneda lemma), and

   (b) using a direct argument.

6. (optional) In this question, we show how to construct the left adjoint functor $F : \mathbf{Set} \to \mathbf{VectQ}$ of the forgetful functor $U : \mathbf{VectQ} \to \mathbf{Set}$ from scratch without knowing the answer in advance, as we did in the second exercise. In general, this can be done by appealing to adjoint functor theorems, but for simple cases, this can be done directly.

   (a) Convince yourself that there is no obstruction to the existence of the left adjoint of the forgetful functor (e.g. does the forgetful functor preserve suitable limits?)

(b) Let $\{*\}$ be the singleton, which is unique up to isomorphism in **Set**. Show that if F is the left adjoint of U, then

$$\mathbf{VectQ}(F\{*\}, V) \cong \mathbf{Set}(\{*\}, h^{\mathbb{Q}}(V)) \cong h^{\mathbb{Q}}(V).$$

(Hint: Recall that U is represented by $\mathbb{Q}$.)

(c) Conclude, using Yoneda lemma or one of its corollaries, that $F\{*\} \cong \mathbb{Q}$.

(d) Construct F. (Hint: Recall that every set is a colimit of the singleton, and left adjoints preserve colimits.)

7. (optional) Show that the right adjoint G of a functor $F : X \to A$ is unique up to natural isomorphism. (Hint: Adjunction implies $h_{Ga}(x) \cong h_a(Fx)$.)

2.2 Density Theorem (Optional)

Again, this section is for next week. Further, it will be optional.

Density theorem states that representable functors to **Set** are dense, in the sense that, while not every functor is representable, it is a colimit of representable functors. We saw earlier that every partially ordered set is the colimit of finite totally ordered sets. Here representable functors play the role of finite totally ordered sets.

1. Read Theorem III.7.1 of [MacLane, 1998] and its proof.

2. Convince yourself that if a functor F into a set-valued functor category (i.e. a category whose objects are morphisms from a fixed category to **Set** and morphisms are natural transformations) has a right adjoint (i.e. F is the left adjoint of some functor), then its actions on objects are determined by its actions on representable ones. (Hint: Left adjoints preserve colimits.)

References

[Friedman, 2008] Friedman, G. (2008). An elementary illustrated introduction to simplicial sets.

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