Directed Reading Program

Fall 2020

Week 0

Reading [Carlsson, 2009, Otter et al., 2017]

Questions

1. What is the appeal of the topological perspective on data?
2. What does persistence roughly mean?
3. Any applications that interest you? What is it about?

Week 1

Reading

persistence Figure 1 of [Otter et al., 2017] (and/or Section 2.3 of [Carlsson, 2009] strictly before Definition 2.9) for the intuitive idea of persistence

general topology my note in the appendix (Section 1)

complex Chapter 3.I of [Edelsbrunner and Harer, 2010] (We will stick to [Edelsbrunner and Harer, 2010] for several weeks.)

Questions

1. (Homeomorphism)
   (a) Show that $t \mapsto \frac{1}{e^{-t}+1}$ is a homeomorphism from the real line to the open interval $(0, 1)$. (Sketch the graph!) Convince yourself that the real line and the open interval have the same shape.
   (b) Convince yourself that the closed finite circular cone

   $$\{(x, y, z) : z^2 = x^2 + y^2, 0 \leq z \leq 1\}$$

   and the closed disc

   $$\{(x, y) : x^2 + y^2 \leq 1\}$$

   are homeomorphic, and they indeed have the same shape. (Draw them!)

(c) Convince yourself that the closed finite cylinder

$$\{(x, y, z) : x^2 + y^2 = 1, |z| \leq 1\}$$

is homeomorphic to the closed annulus

$$\{(x, y) : 1 \leq x^2 + y^2 \leq 2\}.$$

2. (Non-Homeomorphism) Show that the half-open half-closed interval $[0, 1)$ and the open interval $(0, 1)$ are not homeomorphic.
3. (Simplicial Complex)
   (a) Construct a simplicial complex homeomorphic to the unit circle.
   (b) Construct a simplicial complex that is homeomorphic to the unit sphere.
   (c) Give an example of a non-constant continuous real-valued function on the complexes you constructed

Week 2

Reading

simplicial complex Chapter 3.I of [Edelsbrunner and Harer, 2010]

rigor and intuition (optional) Terence Tao's blog article *There's More to Mathematics than Rigor and Proofs* from `https://terrytao.wordpress.com/career-advice/theres-more-to-mathematics-than-rigour-and-proofs/`

amusement (optional) If you are really frustrated with topology, watch this: `https://www.youtube.com/watch?v=SyD4p8_y8Kw`

Questions

1. (Simplicial Complex)
   (a) Construct a simplicial complex homeomorphic to the unit circle.
   (b) Construct a simplicial complex that is homeomorphic to the unit sphere.
   (c) Give an example of a non-constant continuous real-valued function on the complexes you constructed

Week 3

Reading Chapter 3.II of [Edelsbrunner and Harer, 2010]

Questions

1. (Homotopy Equivalence)

(a) Show that if two spaces are homeomorphic, then they are homotopy equivalent.

(b) Convince yourself that the closed unit ball in $\mathbb{R}^3$

$$\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$$

deformation retracts to the origin (and hence is contractible, i.e. homotopy equivalent to a point), but is not homeomorphic to a point. (Use your intuition; do not try to write down the homotopy immediately.)

(c) Convince yourself that the closed finite cylinder

$$\{(x, y, z) : x^2 + y^2 = 1, |z| \leq 1\}$$

deformation retract to the unit circle.

(d) Convince yourself that the Mobius strip (formed by gluing the shorter sides of a rectangular strip in reverse direction) deformation retract to a circle, and hence is homotopy equivalent to a cylinder, but neither the cylinder nor the Mobius strip is a deformation retract of the other.

2. (Helly's Theorem) Construct a few examples in $\mathbb{R}^2$ to illustrate Helly's Theorem.

3. (Cech and Rips Complex)
   (a) What are Cech and Rips Complexes?
   (b) What are their respective appeal?
   (c) In what way does one approximate the other?

Week 4

Reading

Cech and Rips complex Chapter 3.II of [Edelsbrunner and Harer, 2010]

CW complex (optional) the subsection titled "Cell Complexes" (pages 5 – 8) of Chapter 0 of [Hatcher, 2002] (available at `http://pi.math.cornell.edu/~hatcher/AT/ATpage.html`); consult, if necessary, Pages 44 – 47 of [Hatcher, 2005] for definitions and more intuitive examples of "identification"

Questions

1. (Cech and Rips Complex) Draw Cech complexes and a Rips complexes of the set of points

$$\{(\cos 2\pi(k/100), \sin 2\pi(k/100)) : 1 \leq k \leq 100, k \text{ integer}\}$$

for $r \in \{r_0/4, r_0/2, r_0\}$, where $r_0 = 2\pi/100$. (The distance between consecutive points is slightly less than r_0.)

2. (Cech and Rips Complex)
   (a) What are Cech and Rips Complexes?
   (b) What are their respective appeal?
   (c) In what way does one approximate the other?
3. (Identification; optional) Convince yourself a (hollow) cylinder, the Mobius strip and the torus can be formed (or constructed) by identifying points on a square.
4. (Cell Complex; optional) Convince yourself a cell complex is "just" a simplicial complex where elastic and degenerate polygons other than flat generic triangles are also allowed.
5. (Orientable Surfaces; optional) Convince yourself that the cell complexes of the 1-genus torus (standard torus) and the double torus (2-hole swimming ring) make sense.

Week 5

Reading

Homology the chapter *Introduction* of [Giblin, 2010], everything in Chapter 4.I of [Edelsbrunner and Harer, 2010] strictly before the subsection *Reduced Homology* on page 83

CW complex (optional) the subsection titled "Cell Complexes" (pages 5 – 8) of Chapter 0 of [Hatcher, 2002] (available at `http://pi.math.cornell.edu/~hatcher/AT/ATpage.html`); consult, if necessary, Pages 44 – 47 of [Hatcher, 2005] for definitions and more intuitive examples of "identification"

Questions

1. (Homology) What the heck is homology and why does it describe the number of holes in a space?
2. (Orientation) Try to make sense of the signs in the boundary homomorphism when the coefficient is not in $\mathbb{Z}/2\mathbb{Z}$.
3. (Example) Compute the homology of a triangle (not just the 3 edges, but with the triangular face as well).
4. (Identification; optional) Convince yourself a (hollow) cylinder, the Mobius strip and the torus can be formed (or constructed) by identifying points on a square.
5. (Cell Complex; optional) Convince yourself a cell complex is "just" a simplicial complex where elastic and degenerate polygons other than flat generic triangles are also allowed.
6. (Orientable Surfaces; optional) Convince yourself that the cell complexes of the 1-genus torus (standard torus) and the double torus (2-hole swimming ring) make sense.

Responses The question of whether a space with no holes is always contractible was raised. An affirmative answer exists for cell complexes (a.k.a. CW complex), by Whitehead theorem.

Before we state Whitehead theorem, we need to define what holes are. There are two ways to define holes, namely homology and homotopy, both of which are sequences of groups that describe holes in a space. One may think of holes as non-trivial elements in these groups. We will study homology but not homotopy, partly because of the immense difficulty of computing homotopy groups.

To have a sense of homotopy groups, the first homotopy group, also called the fundamental group, is roughly the equivalence class of maps from the unit circle into the space under homotopy. If all maps from the unit circle are homotopic to the constant map, then the space should have no one-dimensional hole. Higher homotopy groups replace the unit circle $\mathbb{S}^1$ with $\mathbb{S}^n$.

Theorem 1 (Whitehead Theorem, Theorem 4.5 of [Hatcher, 2002])**.** Let X and Y be cell complex (CW complexes). If $f : X \to Y$ be a continuous map that induces an isomorphism between the homotopy groups of X and Y, then f is a homotopy equivalence.

Therefore, if a group has trivial homotopy groups, then the constant map has to induce an isomorphism, and hence the constant map is a homotopy equivalence with a point.

Counter-examples abound for non-CW complexes. These spaces are too wild to be built from gluing up discs.

Example 2 (Warsaw Circle)**.** Let

$$X_0 = \{(x, y) : 0 < x \leq 1, y = \sin 1/x\} \cup \{(x, y) : x = 0, |y| \leq 1\}$$

and C be a smooth curve from $(0, -1)$ to $(1, \sin 1)$ that does not intersects with X_0 except at these two points. Then $X = X_0 \cup C$ is not contractible, but has trivial homotopy groups.

It is not surprising that X has trivial homotopy groups because image of $\mathbb{S}^n$, which has to be compact, cannot "pass through" the wildly oscillating part near the y-axis. It is not contractible because there is still in a sense "a hole". Proofs, however, are technical.

See the following websites for more details.

- `https://en.wikipedia.org/wiki/Shape_theory_(mathematics)#Warsaw_Circle`
- `https://math.stackexchange.com/questions/1004837/how-to-show-warsaw-circle-is-non-`
- `https://math.ucr.edu/~res/math205B-2012/polishcircle.pdf`

Week 6

Reading

Homology any materials from last week you have not thoroughly understood; the subsection *Induced Maps* on page 83 – 84 of [Edelsbrunner and Harer, 2010], Chapter IV.2 of [Edelsbrunner and Harer, 2010]

Questions

1. any exercise undone last week
2. (Torus; optional) Compute the homology of a torus. (A torus is a square with opposite edges identified. A square can be decomposed into two triangles. This does not give a simplicial complex, but a so-called Δ-complex, which is something more restrictive than a CW complex but more general than a simplicial complex. The simplicial homological computation carries through in this setting.)
3. (Induced map) Consider the hollow tetrahedron S (which is homeomorphic to a sphere) with vertices $v_0, ..., v_3$. Compute the induced map $i : T \to S$ of the inclusion map of the hollow triangle T with vertices v_0, v_1, v_2 (which is homeomorphic to a circle) into the tetrahedron, i.e. $i(v_j) = v_j$ for $j = 0, 1, 2$.
4. (Computation) Convince yourself that computing the Smith normal form amounts to solving linear equations.

Week 7

Reading

Persistent Homology Chapter VII.1 of [Edelsbrunner and Harer, 2010]

Questions

1. any exercise undone last week
2. Compute the persistent homology of the solid tetrahedron (with respect to a sensible filtration). Illustrate Pairing lemma with this example.

Week 8

Reading

Image application [Carlsson et al., 2008]

CNN – a follow-up paper (optional) [Brüel Gabrielsson and Carlsson, 2019]

Questions

1. The paper finds a Klein bottle in the 7^{th}-dimensional sphere $\mathbb{S}^7$. Explain how the sphere arises form image data.
2. The Klein bottle is not revealed directly from the persistent diagram. Instead, the primary circle and secondary circles are found. Explain what they are and how they are different.
3. Explain how the author justifies the primary and secondary circles are in fact on a Klein bottle.

1 Appendix: Crash Course on General Topology

Topology has (at least) two important branches, general topology and algebraic topology. The former concerns convergence and stability under perturbation, and is the foundation of all topology, while the latter concerns shape identification and is the foundation of topological data analysis. This note gives a brief overview of general topology for the study of topological data analysis. In particular, this note aims explaining intuitively the concepts of *open sets*, *continuous maps* and *homeomorphism*.

1.1 General Topology: Convergence and Stability

General topology is about convergence and stability under perturbation. The two concepts are equivalent; convergence sounds more familiar to beginners, but the perspective of stability sometimes make more theoretical sense. Below, we mention some themes of general topology phrased in terms of convergence, and then we establish the equivalence to prepare for the switch to the stability perspective in the next section.

Below, (topological) space simply means a collection of points with some structures to describe convergence and stability, just like a vector space is a collection of vectors, for which sum and scaling makes sense. Later we will discuss the shape of a space.

Example 3 (Topological Spaces)**.**

- the real line with ε-δ convergence
- the space of continuous functions from $[0, 1]$ to $\mathbb{R}$ with uniform convergence
- the unit circle with coordinate-wise convergence
- any polygon in the plane with coordinate-wise convergence

Basic issues general topology addresses, phrased in terms of convergence, include

1. How can convergence be defined in spaces more complicated than the real line?

2. Does a map between two spaces preserve convergence properties?

3. Under what assumptions are nice convergence properties guaranteed?

Convergence and stability under perturbation are a pair of dual concepts, in the following sense:

Principle 4 (Open-Closed Duality)**.** A set that is closed under convergence if and only if its complement is stable under perturbation.

This vague informal statement, as well as its name, needs further explanation.

Concept 5 (Closed)**.** A set F is said to be closed under convergence if and only if whenever a sequence (or its generalization) in F converges to a point x, then the point lies in F. The technical name for such a set is *closed set*.

Concept 6 (Open)**.** A set U is said to be stable under perturbation if and only if for any point $x \in U$, small enough perturbation of x will not get it outside of U. The technical name of such a set is *open set*.

Example 7 (Open and Closed Sets)**.**

Closed $[0, 1]$ is a closed set in $\mathbb{R}$ because a sequence in $[0, 1]$ cannot converge to a point outside of $[0, 1]$.

Open $(0, 1)$ is an open set in $\mathbb{R}$ because, a point, say, 0.1 cannot be brought outside of $(0, 1)$ if the perturbation moves points by a distance smaller than 0.01.

Now we explain the duality.

Remark. The duality is often taken as the *definition* of a closed set.

Explanation. Suppose F is closed under convergence. We show F^C is stable under perturbation by contradiction. If F^C is *not* stable under perturbation, then F^C has a point x that can brought outside of F^C, i.e. into F, by arbitrarily small perturbation. Then the images of these increasingly small perturbation form a (generalized) sequence in F that converges to x, contradicting with the fact that F is closed and $x \notin F$.

Conversely, Suppose U is stable against perturbation. We show U^C is closed under convergence by contradiction. If U^C is *not* closed under convergnece, then some (generalized) sequence in U^C converges to a point x outside of U^C, i.e. indside U. Then bringing x to one of a very late point in the sequence is a very small perturbation that brings x into U^C, contradicting with the stability of U. □

From now on, we only discuss stability, for it is a more natural setting for subsequent concepts.

1.2 Map and Shape

Ultimately, we would like to say two spaces have the same shape if there is a *homeomorphism* between them, just like two vector spaces are isomorphic if there is a linear isomorphism between them. Just as a linear isomorphism is a linear map with an inverse linear map, a homeomorphism is a continuous map with a *continuous* inverse. We would like to explain what a continuous map is and why a continuous map with a continuous inverse means the domain and codomain have the same shape.

We start with continuous map, and we link it to stability. We will show that continuous maps may stretch, bend and collapse, but may not tear. Therefore, a continuous map preserves the shape in one direction (but not both because it may collapse, e.g. the whole space may be collapsed to a point). Then, if a continuous map has a continuous inverse, then collapse is also impossible, because collapse amounts to tearing by the *inverse* map.

Concept 8 (Continuous Map)**.** A continuous map is a map that is stable under perturbation, in the sense that a tiny (enough) perturbation of the input leads to a tiny perturbation of the output.

Example 9 (Continuous Map)**.** A differentiable function $f : I \to \mathbb{R}$ on an open interval I is continuous, because the perturbation in the output due to a small perturbation of the input from x_0 to x is approximately $f'(x_0)(x - x_0)$, which is small if $x - x_0$ is small.

An elegant way to rephrase continuity is as follows.

Principle 10 (Continuous Map and Stability)**.** A map is continuous if and only if the preimage of an open set V under the map is open.

Remark. This is usually taken as the *definition* of a continuous map.

Explanation. This description holds if and only if small enough perturbation of each point in $f^{-1}(V)$ is still mapped into V. This is possible if and only if each point $y = f(x)$ in V is only slightly perturbed (so that it is still inside V) when x is perturbed lightly enough. □

Now, continuous maps preserve shape to a certain degree. It maps nearby points (then one is a tiny perturbation of the other) to nearby points. Therefore, it may stretch and bend a space, but may not tear or pierce it. However, it does not preserve the whole shape because it may collapse a shape, i.e. far-away points may still be mapped to nearby points.

Example 11 (Collapse and Tearing)**.**

collapse The map from the unit circle to $\mathbb{R}$ defined by projection to the x-axis is continuous, because the x-coordinate does not change much under small perturbation. Note that the map does not preserve the shape as it collapse the circle to the interval, and "annihilates" the hole of the circle.

tearing The step function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 2 \text{ if } x \leq 1/2 \\ 5 \text{ if } x > 1/2 \end{cases}$$

is discontinuous, because any small perturbation of $1/2$ to a slightly larger number leads to a change in the output by $5-2=3$, which is not small. Note that the function f "tears" the interval into two pieces, namely, 2 and 5.

Maps that truly preserve shape are called homeomorphism.

Definition 12. A map $f : X \to Y$ is said to be a homeomorphism if and only if it is bijective, and f and f^{-1} are both continuous. Two spaces are said to be homeomorphic if and only if there exists a homeomorphism between them.

Now, homoemorphism rules out collapse because whenever two far-away points are mapped to nearby points, that means the inverse map maps nearby points to far-away points, contradicting with the continuity of the inverse map.

Example 13 (Homeomorphism)**.** The unit circle $\{(x,y) : x^2+y^2=1\}$ and the boundary of the "unit" square

$$[-1,1]^2 \ \setminus \ (-1,1)^2 = \{(x,y) : \max(|x|,|y|) = 1\}$$

are homeomorphic. For a homeomorphism, consider the "projection" defined by mapping each point p of the unit circle to the intersection of the ray from the origin through p and the square. Note that the circle and the square have the same shape, e.g. they both have a hole, to distinguish them from a θ-shaped space.

Example 14 (Non-Homeomorphism)**.** The map f from $[0,2\pi)$ to the unit circle defined by

$$f(t) = (\cos t, \sin t)$$

is a bijective continuous map, but its inverse is not continuous, as witnessed by the tearing of a small arc centered at $(1,0)$ to two pieces, one near 0 and one near 2π. Note that $[0,2\pi)$ is a different shape from the unit sphere; the former has no whole while the latter has one. One way to show the two spaces are not homeomorphic is to observe that the circle is still one connected piece when any point is removed, but removing a nonzero point, say 1, from $[0,2\pi)$ gives rise to two pieces.

1.3 Further Readings

This note was written becuase I could not find a chapter-length overview of general topology. More lengthy expositions abound across different levels of difficulty. [Arnold, 2011] is a gentle introduction to the intuitive concepts in topology. Prof. Hatcher's note [Hatcher, 2005] strikes a good balance between rigor, intuition (and length!). [Munkres, 2000] is a standard textbook of the field. [Engelking, 1989] is a standard reference of the field.

References

[Arnold, 2011] Arnold, B. (2011). *Intuitive Concepts in Elementary Topology*. Dover Books on Mathematics. Dover Publications.

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[Otter et al., 2017] Otter, N., Porter, M. A., Tillmann, U., Peter, G., and Harrington, H. A. (2017). A roadmap for the computation of persistent homology. *EPJ Data Science*, 6:17.